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RATIONAL CURVES ON A COMPLETE INTERSECTION CALABI–YAU VARIETY IN $\mathbb{P}^3 \times \mathbb{P}^3$

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We study rational curves on the Tian-Yau complete intersection Calabi–Yau threefold (CICY) in $\mathbb{P}^3 \times \mathbb{P}^3$. Existence of positive dimensional families of nonsingular rational curves is proved for every degree ≥ 4. The number of nonsingular rational curves of degree $1, 2, 3$ on a general Tian–Yau CICY is finite and enumerated. The number of curves of these degrees are also enumerated for the *special* Tian–Yau CICY. There are two 1-dimensional families of singular rational curves of degree 3 on a general Tian–Yau CICY, making this degree a turning point between finite and infinite number of curves. We also introduce a notion of equivalence of a family of rational curves, and determine the equivalences of the two 1-dimensional families on the Tian–Yau CICY. The equivalences equal the predicted numbers of curves obtained by a power series expansion of the solution of a Picard-Fuchs equation that arises in superconformal field theory.

1. Introduction and basic definitions.

In the 1980's the physicists started considering supersymmetric theories for a 10-dimensional universe. In these theories a Calabi–Yau threefold is attached to every point of the Minkowski time-space. Moreover, certain invariants of this Calabi–Yau threefold are linked to observables in our universe. For example, the number of generations of elementary particles is 3 (electron, muon, tauon) in our universe. The superstring theory yields that the absolute value of the Euler number of the manifold must be twice the number of generations. So physicists were hoping to find relatively easy examples of manifolds with Euler number ± 6. The first example was found by G. Tian and S.-T. Yau ([23], [24]). Their starting point was the following complete intersection Calabi–Yau threefold in $\mathbb{P}^3 \times \mathbb{P}^3$:

$$X = Z\left(\sum x_i^3, \sum x_i y_i, \sum y_i^3\right).$$

We shall call this the *special* Tian–Yau CICY. This variety has Euler number -18. Furthermore, this variety allows a free action by a group G of order 3. Hence, the quotient variety X/G is CY with Euler number -6.

More generally, a Tian–Yau CICY is defined by:

$$X = Z(f_1, f_2, g) \subseteq \mathbb{P}^3_1 \times \mathbb{P}^3_2$$

where f_1, f_2, g are polynomials of bidegrees $(3,0)$, $(0,3)$ and $(1,1)$ respectively, and such that X is nonsingular. By a *general* Tian–Yau variety we shall mean a generic choice of the polynomials f_1, f_2, g. We introduce the following notation: F_1 (resp. F_2) is the cubic surface in $\mathbb{P}^3_1$ (resp. $\mathbb{P}^3_2$) defined by the polynomial of bidegree $(3,0)$ (resp. $(0,3)$). The variety G is defined by the polynomial of bidegree $(1,1)$. In other words:

$$X = F_1 \times F_2 \cap G.$$

All deformations of a general Tian–Yau variety as an abstract variety, are realisable as polynomial deformations of the defining equations. The family of Tian–Yau varieties in $\mathbb{P}^3_1 \times \mathbb{P}^3_2$ is complete in this sense.

After the example of G. Tian and S.-T. Yau was found, an intensive search for more examples of complete intersections in multiprojective space was undertaken ([**2**], [**3**], [**8**], [**9**]). It is proven that no complete intersection Calabi–Yau threefold X in multiprojective space would have $|\chi(X)| = 6$. As in the example given by G. Tian and S.-T. Yau one may look for some group acting freely on the variety. Such groups are of course hard to find and in many cases it is quite easy to prove that such groups cannot exist. In fact, starting with the list of all the CICY threefold types (approximately 10,000) in multiprojective space, the final result was that there were at most 3 types (including the Tian–Yau CICY type) ([**3**]) that possibly could have a free action by a group acting of the desired cardinality. Moreover, to the present day a such group has only been found in the Tian–Yau case.

The quintic in $\mathbb{P}^4$ is the most studied Calabi–Yau threefold. Clemens conjectured that there are only finitely many smooth rational curves on a general quintic threefold for every degree. This has been proven for degrees less than 10 ([**13**], [**12**]). The numbers have been computed for degrees less than or equal to 4 using algebro-geometric techniques ([**13**], [**6**], [**15**]). Using conformal field theory, one is able to predict the number of rational curves of every degree. More precisely, the predicted numbers appear in a power series expansion of the solution of a Picard-Fuchs equation. In the case of the quintic in $\mathbb{P}^4$, the power series looks like:

$$F(q) = 5 + \sum_d n_d d^3 \frac{q^d}{1 - q^d},$$

where n_d is the (conjectured) number of rational curves of degree d. This expansion first appeared in ([**4**]), and started off a new branch of mathematics trying to understand the mathematical implications of mirror symmetry. Recently progress has been made in achiving this goal ([**7**], [**16**]). For a further discussion and a more complete reference list, see ([**17**]).

In general it is hard to determine when the number of curves of a given degree is finite. We address this question in case of a general Tian–Yau CICY. Every rational curve on a Tian–Yau variety has a bidegree, and a degree via the Segre embedding in $\mathbb{P}^{15}$. Furthermore the Hilbert scheme Hilb_X^{dm+1} has a natural partition in open-closed disjoint subschemes $\mathrm{Hilb}_X^{(i,j)m+1}$ with $i+j=d$.

Our main result is the following:

Theorem 1.1. *Let X be a general Tian–Yau CICY. Let $m>3$ be an integer, and $i \in \{0,1,2,3\}$. Then there exist positive dimensional families of nonsingular rational curves of bidegree $(m, m-i)$.*

There exist also positive dimensional families of nonsingular rational curves of bidegrees $(2,2),(3,3),(3,2)$.

This has the following corollary:

Corollary 1.2. *There exist positive dimensional families of nonsingular rational curves on a general Tian–Yau CICY for every degree n, $n \geq 4$.*

However, this abundance of curves for infinitely many bidegrees, does not extend to all bidegrees, on the contrary, there are infinitely many bidegrees that do not allow any rational curves at all:

Theorem 1.3. *There are no curves of bidegree $(m,1)$ or $(m,0)$ on a general Tian–Yau CICY for $m \geq 4$.*

The number rational curves on a general Tian–Yau CICY is finite only for degrees 1 and 2. An explicit enumeration of rational curves of the various bidegrees shows that these numbers are in agreement with the numbers worked out by S. Hosono, A. Klemm, S. Theisen, S.-T. Yau ([**11**]) and by V.V. Batyrev and D. van Straten ([**1**]) using the Picard-Fuchs equation. The number of *nonsingular* rational curves of degree 3 is finite, but there are also two 1-dimensional families of *singular* rational curves on a general Tian–Yau CICY. We give an algebro-geometric definition of the equivalence of a 1-dimensional family of rational curves, and apply this definition to the two 1-dimensional families of degree 3 curves on a general Tian–Yau CICY.

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2. Preliminaries.

In this section we study the geometry and the rational curves of the variety $G = Z(\sum \alpha_{ij} x_i y_j)$. We start this section with two lemmas.

Lemma 2.1. *Set $G = Z(\sum \alpha_{ij}x_iy_j) \subseteq \mathbb{P}^3_1 \times \mathbb{P}^3_2$ and let L be a line in $\mathbb{P}^3_2$ (resp. $\mathbb{P}^3_1$). Then there exists a unique maximal linear space $V(L)$ in $\mathbb{P}^3_1$ (resp. $\mathbb{P}^3_2$) such that $V(L) \times L$ (resp. $L \times V(L)$) is contained in G.*

Proof. We prove the assertion in the case where the line L is in $\mathbb{P}^3_2$. Assume L is defined by $Z(y_2, y_3) \subseteq \mathbb{P}^3_2$. Set $\bar{G} = G|_{\mathbb{P}^3_1 \times L}$, then $\bar{G}$ is defined in $\mathbb{P}^3_1 \times L$ by the following equation:

$$\sum \alpha_{i0}x_iy_0 + \sum \alpha_{i1}x_iy_1 = 0.$$

Obviously

$$V(L) = Z\left(\sum \alpha_{i0}x_i, \sum \alpha_{i1}x_i\right)$$

is both maximal and unique.

Since we made no assumption on the α_{ij}'s, we can always reduce to the case where L is as above. □

Remark 2.2. In fact we proved more: Every point $a \in \mathbb{P}^3_1$ with the property that $a \times L \subset G$, is contained in $V(L)$. Note also that all cases $\dim V(L) = 1, 2, 3$ occur. The general case is clearly $\dim V(L) = 1$. The definition of $V(L)$ depends on L as well as on G. We are primarily interested in the case when $\dim V(L) = 1$ for all $L \subseteq \mathbb{P}^3_i, \quad i = 1, 2$.

Lemma 2.3. *Let $G = Z(\sum \alpha_{ij}x_iy_j)$. The matrix $[\alpha_{ij}]$ is invertible if and only if $\dim V(L) = 1$ for all $L \subseteq \mathbb{P}^3_i$, $i = 1, 2$.*

Proof. Assume that $[\alpha_{ij}]$ is invertible. Introduce the notation $xAy^t = \sum \alpha_{ij}x_iy_j$, where $x = (x_0, \dots, x_3)$, $y = (y_0, \dots, y_3)$, and

$$A = \begin{pmatrix} \alpha_{00} & \cdots & \alpha_{03} \\ \vdots & & \vdots \\ \alpha_{30} & \cdots & \alpha_{33} \end{pmatrix}.$$

We have to prove that for every line L in $\mathbb{P}^3_1$, $V(L)$ is of minimal dimension.

Consider first the special case where $L = Z(y_2, y_3) \subseteq \mathbb{P}^3_2$. This gives

$$V(L) = Z\left(\sum \alpha_{i1}x_i, \sum \alpha_{i0}x_i\right) \times L \subseteq \mathbb{P}^3_1 \times \mathbb{P}^3_2.$$

Assume that $V(L)$ is not of minimal dimension, i.e., $\dim Z(\sum \alpha_{i1}x_i, \sum \alpha_{i0}x_i) \geq 2$. This implies that $\sum \alpha_{i1}x_i = \lambda \sum \alpha_{i0}x_i$, giving $\alpha_{i1} = \lambda\alpha_{i0}$. In other words, the first two columns are proportional, which contradicts that A is of maximal rank.

The final step is reducing the general situation to the special case considered above. This is done in the following way: Choose any line L in $\mathbb{P}^3_2$. It is possible to change the coordinates on the second factor, such that L is equal to $Z(y'_2, y'_3)$. Call this coordinate change matrix P (i.e.,

$(y'_1, \dots, y'_3)^t = P \cdot (y_1, \dots, y_3)^t$). We make the following change of coordinates on the first factor:

$$x'^t = (A^{-1})^t P^t A^t x^t.$$

This gives $G = Z(\sum \alpha_{ij} x'_i y'_j)$ with respect to the new coordinates, since

$$\sum \alpha_{ij} x_i y_j = xAy^t = (x' APA^{-1}) A(P^{-1} y'^t) = x' A y'^t = \sum \alpha_{ij} x'_i y'_j.$$

The result now follows from the special case considered above.

We also observe that $\dim V(L) = 1$ for all L, implies that all columns must be linearly independent (by considering the lines $Z(y_2, y_3)$, $Z(y_0, y_3)$, and $Z(y_0, y_2)$). This gives the implication the other way. □

When G is as in the previous proposition, we have a map:

$$l : \mathrm{Grass}\,(1, \mathbb{P}^3) \longrightarrow \mathrm{Grass}\,(1, \mathbb{P}^3)$$

defined by sending L to $V(L)$. This map is obviously bijective, since $l(l(L)) = L$ by definition. In this case, we shall write $l(L)$ for $V(L)$ to signify that it is a line. Moreover, note that then the defining equation can be brought to diagonal form $\sum_{i=0}^3 x_i y_i$ by a suitable change of coordinates on $\mathbb{P}_1^3$ and $\mathbb{P}_2^3$.

Proposition 2.4. *Let $G = Z(\sum \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible, and set $\bar{G} = G|_{\mathbb{P}^3 \times L}$. Then $\bar{G}$ is isomorphic to the blow-up of $\mathbb{P}_1^3$ in $l(L)$.*

Proof. We can, without loss of generality, assume that $\bar{G}$ is defined by

$$Z(x_1 y_2 - x_2 y_1) \subseteq \mathbb{P}^3(x_0, \dots, x_3) \times \mathbb{P}^1(y_1, y_2)$$

(by change of coordinates). Then $l(L)$ is defined by $x_1 = x_2 = 0$. It is enough to check the statement locally, take for instance $x_0 = 1$. Then we have

$$Z(x_1 y_2 - x_2 y_1) \subseteq \mathbb{A}^3 \times \mathbb{P}^1.$$

This is in fact the blow-up of $\mathbb{A}^3$ with center $Z(x_1, x_2)$ ([10], II.7.12.1, p. 163). □

Corollary 2.5. *Let $G = Z(\sum \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible, and let $\check{G} = G|_{H \times L}$, where H is a hyperplane and L is a line. Let $\pi : \bar{G} \longrightarrow \mathbb{P}_1^3$ denote the blow-up of $\mathbb{P}_1^3$ in $l(L)$. Then $\check{G}$ is isomorphic to $\pi^{-1}(H)$. If $l(L) \nsubseteq H$, then $\check{G}$ is isomorphic to H blown up in the point $H \cap l(L)$.*

We end this section by a proposition that we will use extensively in the following sections. For its formulation we need a definition.

Definition 2.6. A rational curve in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$ is of *type* $(\bar{m}, \bar{n})$ if the image of the first (resp. second) projection is of degree m (resp. n).

Proposition 2.7. *Let L be a line in $\mathbb{P}^3_2$, and let C_1 be a rational curve of degree m in $\mathbb{P}^3_1$. Furthermore, let $G = Z(\sum \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible, and denote $G|_{\mathbb{P}^3_1 \times L}$ by $\bar{G}$. Let C be the unique irreducible component of $D = C_1 \times L \cap \bar{G}$ such that $\pi_1(C) = C_1$, where π_1 is the projection map on the first factor. Let $i = lg(C_1 \cap l(L))$. Then C is a rational curve of bidegree $(m, m-i)$ and of type $(\bar{m}, \bar{1})$. Furthermore, every rational curve of bidegree (m,n) and of type $(\bar{m}, \bar{1})$ must arise in this way.*

Proof. The variety $\bar{G}$ is isomorphic to the blow-up of $\mathbb{P}^3_1$ with center $l(L)$, so C is by definition the strict transform of C_1. Moreover, $D = C \cup E_1 \cup \cdots \cup E_i$, where $E_1, \cdots, E_i$ are the exceptional fibers corresponding to the intersection points $p_1, \ldots, p_i$ in $C_1 \cap l(L)$. The curve C is rational ([**10**], V.3.7, p. 389). The degree on the second factor drops by one for each intersection point counted with multiplicity, giving the desired bidegree.

The converse follows by reversal of the above argument. ☐

We have the following important corollary:

Proposition 2.8. *Let $G = Z(\sum \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible. Then there are no nonsingular rational curves of bidegree $(m, 0)$, $m \geq 3$, on G.*

Proof. Assume for contradiction that C is a nonsingular rational curve of bidegree $(m, 0)$, $m \geq 3$ on G, i.e., $C = C_1 \times \{p\}$, where C_1 is a nonsingular rational curve in $\mathbb{P}^3_1$ and p is a point in $\mathbb{P}^3_2$. Fix a line L in $\mathbb{P}^3_2$ passing through p. By the proof Proposition 2.7 $l(L)$ has to be an m-secant to the curve C_1. This is impossible since a nonsingular rational curve of degree m has at most an $(m-1)$ secant for $m \geq 3$. ☐

3. The number of rational curves of degree less than 4 on a general Tian–Yau CICY.

In this section we compute the number of nonsingular rational curves on a general Tian–Yau CICY for degrees less than 4. In the end of this section we describe two 1-dimensional families of singular rational curves of degree 3.

Proposition 3.1. *The numbers $N_{i,j}$ of nonsingular rational curves of bidegree (i,j) on a general Tian–Yau variety are finite for $i + j \leq 3$ and are given by:*

$$\begin{aligned} N_{0,1} = N_{1,0} &= 81 \\ N_{0,2} = N_{2,0} &= 81 \\ N_{1,1} &= 729 \\ N_{1,2} = N_{2,1} &= 2187 \\ N_{3,0} = N_{0,3} &= 0. \end{aligned}$$

Proof. All curves of bidegree $(1,0)$ have to be of the form $L \times \{b\}$ where L is a line on F_1 and b is a point on F_2. The condition that this $(1,0)$ also should be contained in G, gives three possible values for the point b given a line L on F_1. Since F_1 has 27 lines, the number of $(1,0)$ curves is $3 \cdot 27 = 81$. A $(2,0)$ curve is of the form $C_1 \times \{b\} \subseteq H \times \{b\}$, where C is a conic in a hyperplane H in $\mathbb{P}^3_1$. It follows that $H \cap F_1 = C_1 \cup L$, since F_1 is of degree 3. Hence, the number of $(2,0)$ curves has to be equal to the number of $(1,0)$ curves, which is 81. The intersection $L' \times L \cap G$ is obviously an irreducible rational curve of bidegree $(1,1)$, and every rational curve of bidegree $(1,1)$ has to arise in this way. Hence, the number of bidegree $(1,1)$ curves is equal to the number of pairs (L', L) where $L' \subseteq F_1$ and $L \subseteq F_2$. The number of such pairs is $27 \cdot 27 = 729$.

We now want to find the number of rational curves of bidegree $(2,1)$. Denote the lines on F_i by L^k_i where $k \in \{1, \ldots, 27\}$. In the generic situation $l(L^k_2) \cap L^{k'}_1 = \emptyset$ for all $k, k' \in \{1, \ldots, 27\}$. Since there are only finitely many planes H in $\mathbb{P}^3_1$ such that $H \cap F_1$ is the union of three lines, the $l(L^k_2)$ are in the general situation not contained in any of these planes. This gives that for each 1-dimensional family of conics and for each $l(L^k_2)$ we get three planes such that $l(L^k_2)$ intersects a conic contained in the intersection of F_1 and the plane. By Proposition 2.7 each of these cases gives rise to one bidegree $(2,1)$ curve and every bidegree $(2,1)$ curve has to arise this way. This gives the total number of bidegree $(2,1)$ curves: $3 \cdot 27 \cdot 27 = 2187$.

Finally, there are no nonsingular rational curves of bidegree $(3,0)$ or $(0,3)$ by Proposition 2.8. □

Corollary 3.2. *Let N_d denote the number of nonsingular rational curves of degree d on a general Tian–Yau variety. Then*

$$N_1 = 162, \quad N_2 = 891, \quad N_3 = 4374.$$

Remark 3.3. In Section 4 we will prove that there are positive dimensional families of nonsingular rational curves for every degree higher than 3. Hence the list in Corollary 3.2 is complete. The curves of degree 1 and 2 are necessarily nonsingular. However, there exist singular rational curves of degree 3. These have to be of bidegree $(3,0)$, since the $(2,1)$ curves necessarily had to be nonsingular by the proof of Proposition 3.1.

We shall show that a general Tian–Yau variety contains two 1-dimensional families of singular rational curves of degree 3. Suppose C is a rational curve of bidegree $(3,0)$. By Proposition 2.8, the curve C has to be singular. Hence, C has to be a plane rational curve of degree 3, i.e., a nodal or cuspidal cubic curve. After a suitable change of coordinates we may assume that $G = Z(\sum_i x_i y_i)$. The curve C is of the form $C_1 \times \{p\}$, where C_1 is on F_1 and $p = (p_0, p_1, p_2, p_3)$ is a point on F_2. Furthermore, $C_1 \subseteq H_p = Z(\sum_i p_i x_i)$. By Bezout's theorem, $F_1 \cap H_p$ is of degree 3, so $C_1 = F_1 \cap H_p$. Finally, that

$F_1 \cap H_p$ is a singular cubic implies that H_p is a tangent hyperplane, i.e., $H_p \in F_1' \subseteq \left(\mathbb{P}_1^3\right)'$, the dual variety of F_1. We may identify the set of $(3,0)$ curves (not necessary irreducible) on X with the set:

$$\{(H_p, p) \subseteq F_1' \times F_2\} \subseteq \mathbb{P}_1^{3\prime} \times \mathbb{P}_2^3 \cong \mathbb{P}_2^3 \times \mathbb{P}_2^3$$

hence with $F_1' \times F_2 \cap \Delta$, where Δ denotes the diagonal. This set is isomorphic to $F_1' \cap F_2 \in \mathbb{P}_2^3$.

Hence, we can represent the complete family of bidegree $(3,0)$ rational curves as a curve in $\mathbb{P}^3$. We denote this curve by Γ.

Since $\deg F_1' = 12$, $\deg \Gamma = 3 \cdot 12 = 36$. Note that Γ is a (local) complete intersection, so the dualising sheaf is given by:

$$\omega_\Gamma \cong \wedge^2 \mathcal{N}_{\Gamma/\mathbb{P}^3} \otimes \omega_{\mathbb{P}^3}|_\Gamma.$$

Using $\mathcal{N}_\Gamma|_{\mathbb{P}^3} \cong \mathcal{O}_\Gamma(d_1) \oplus \mathcal{O}_\Gamma(d_2)$, where $d_1 = 12$ and $d_2 = 3$ are the degrees of F_1' and F_2 respectively, we get

$$\deg \omega_\Gamma = 2p_a - 2 = (d_1 + d_2 - 4)d_1 d_2 = 396$$

so that the arithmetic genus, p_a, of Γ is 199.

Finally, we want to determine the singularities of the curve Γ. The dual surface F_1' has a double curve of degree $\frac{1}{2}d(d-1)(d-2)(d^3-d^2-d-12) = 27$ (see e.g. [**18**]), where $d = \deg F_1 = 3$. The nodes on Γ are precisely the intersection points between F_2 and the double curve on F_1'. Let δ denote the number of nodes. Then

$$\delta = 3 \cdot 27 = 81.$$

The surface F_1' also has a cuspidal edge of degree $4d(d-1)(d-2) = 24$ ([**18**]). The cusps on Γ are the intersection points between F_2 and this curve. Let κ denote the number of cusps, then

$$\kappa = 3 \cdot 24 = 72.$$

Hence the geometric genus, p_g, of Γ is given by $p_g = p_a - \delta - \kappa = 46$.

In the end of this section we will give a definition of the equivalence, that apply to our two 1-dimensional families. We start out by reviewing the definition given by S. Katz of the equivalence of a family of rational curves ([**14**]). Let X be a Calabi–Yau threefold and let H be a k-dimensional nonsingular family of nonsingular rational curves on X. Consider

$$\begin{array}{ccc} \mathbb{D} & \longrightarrow & H \times X \\ {\scriptstyle\pi}\big\downarrow & & \\ H & & \end{array}$$

where $\mathbb{D}$ is the total space of the family and π is the projection on the first factor. Let $\mathfrak{N}_{\mathbb{D}/H \times X}$ denote the normal bundle of $\mathbb{D}$ in $H \times X$. The equivalence, $e(H)$, of the family H is then defined to be:

$$e(H) = \deg\ c_k(R^1\pi_\star\mathfrak{N}_{\mathbb{D}/H\times X}).$$

We are interested in determining the equivalence of a family of mildly singular curves, say local complete intersection curves with at most node and cusp singularities. Furthermore, we shall allow the base space of the family to have the same kind of mild singularities. Because of the singularities, we can not apply Katz' notion of equivalence directly. It is easy to see that $R^1\pi_\star\mathfrak{N}_{\mathbb{D}/H\times X}$ is not a vector bundle in general.

Let C be a local complete intersection curve in X. Let $\mathcal{I}$ be its defining ideal. The sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free, hence so is the normal sheaf $\mathcal{N}_{C/X} = (\mathcal{I}/\mathcal{I}^2)'$. The adjunction formula says:

$$\wedge^2\mathcal{N}_{C/X} \otimes \omega_X|_C \cong \omega_C$$

where ω_C is the dualising sheaf. Since X is Calabi–Yau, this gives:

$$\wedge^2\mathcal{N}_{C/X} \cong \omega_C.$$

Furthermore, since $\mathcal{N}_{C/X}$ is a rank 2 bundle, we have the following perfect pairing $\mathcal{N}_{C/X} \cong \mathcal{N}'_{C/X} \otimes \wedge^2\mathcal{N}_{C/X}$. Hence

$$\mathcal{N}_{C/X} \cong \mathcal{N}'_{C/X} \otimes \omega_C.$$

By Serre Duality we get:

$$H^1(\mathcal{N}_{C/X}) \cong H^{1-1}(\mathcal{N}'_{C/X} \otimes \omega_C)' \cong H^0(\mathcal{N}_{C/X})'. \tag{5}$$

Consider a family $\pi : \mathbb{D} \to H$ of local complete intersection curves with at most cusps and nodes on X. The relative version of the isomorphism (5) is

$$R^1\pi_\star\mathfrak{N}_{\mathbb{D}/H\times X} \cong \mathrm{Hom}\,(\pi_*\mathfrak{N}_{\mathbb{D}/H\times X}, \mathcal{O}_H). \tag{6}$$

Assuming that (6) holds and that the Kodaira-Spencer map

$${\Omega^1_H}' \longrightarrow \pi_*\mathfrak{N}_{\mathbb{D}/H\times X}$$

is an isomorphism (e.g., if H is a component of the Hilbert scheme), we obtain Kodaira-Spencer

$$R^1\pi_\star\mathfrak{N}_{\mathbb{D}/H\times X} \cong \mathrm{Hom}\,(\pi_\star\mathfrak{N}_{\mathbb{D}/H\times X}, \mathcal{O}_H) \cong \mathrm{Hom}\,({\Omega^1_H}', \mathcal{O}_H) = {\Omega^1_H}''.$$

In the cases we are interested in, $R^1\pi_\star\mathfrak{N}_{\mathbb{D}/H\times X}$ is not necessarily isomorphic to ${\Omega^1_H}''$, and ${\Omega^1_H}''$ is not a vector bundle. When $H = \Gamma$ is a curve, we can modify ${\Omega^1_H}''$ so as to obtain a vector bundle on the normalisation of Γ, and it is this bundle we shall use to define the equivalence of Γ.

We want to associate a number to our family of $(3,0)$-curves. This family is not a component of the Hilbert scheme of curves. However, it parametrises all equivalence classes of maps from $\mathbb{P}^1 \longrightarrow X$ of degree 3, when we identify maps with the same image. We would like to define the equivalence using a vector bundle on Γ.

We shall associate a vector bundle to ${\Omega^1_\Gamma}''$ in a natural way. The curve Γ is singular and ${\Omega^1_\Gamma}''$ is not isomorphic to Ω^1_Γ. However, the canonical map $\Omega^1_\Gamma \longrightarrow {\Omega^1_\Gamma}''$ is surjective. (This is easily seen by local computations at cusps and nodes of Γ.) Let $\widetilde{\Gamma}$ denote the normalisation of Γ and let ψ be the natural map:

$$\psi : \widetilde{\Gamma} \longrightarrow \Gamma.$$

We have the following exact sequence of sheaves on $\widetilde{\Gamma}$:

$$\psi^\star\Omega^1_\Gamma \longrightarrow \Omega^1_{\widetilde{\Gamma}} \longrightarrow \Omega^1_{\widetilde{\Gamma}/\Gamma} \longrightarrow 0. \tag{7}$$

Let Ω denote the image of $\psi^\star\Omega^1_\Gamma$ in $\Omega^1_{\widetilde{\Gamma}}$, i.e.,

$$0 \longrightarrow \Omega \longrightarrow \Omega^1_{\widetilde{\Gamma}} \longrightarrow \Omega^1_{\widetilde{\Gamma}/\Gamma} \longrightarrow 0. \tag{8}$$

The sheaf Ω can be considered as a "modification" of ${\Omega^1_\Gamma}''$ in the following way. Consider the following commutative diagram:

$$\begin{tikzcd} \psi^\star\Omega^1_\Gamma \arrow[r] \arrow[d] & \psi^\star{\Omega^1_\Gamma}'' \arrow[d] \\ \Omega^1_{\widetilde{\Gamma}} \arrow[r, "\sim"] & {\Omega^1_{\widetilde{\Gamma}}}'' \end{tikzcd}.$$

We define the *equivalence* $e(\Gamma)$ of Γ as the (degree of the) first Chern class of the image of $\psi^\star{\Omega^1_\Gamma}''$ in ${\Omega^1_{\widetilde{\Gamma}}}''$. Note that $\psi^\star\Omega^1_\Gamma \longrightarrow \psi^\star{\Omega^1_\Gamma}''$ is surjective and $\Omega^1_{\widetilde{\Gamma}} \cong {\Omega^1_{\widetilde{\Gamma}}}''$, hence the image is isomorphic to Ω.

We get

$$\begin{aligned} e(\Gamma) = c_1(\Omega) &= c_1(\Omega^1_{\widetilde{\Gamma}}) - \deg \Omega^1_{\widetilde{\Gamma}/\Gamma} \\ &= 2p_g(\widetilde{\Gamma}) - 2 - \#\text{cusps}. \end{aligned} \tag{9}$$

Since the geometric genus is 46 and the number of cusps is 72, we get,

$$e(\Gamma) = 2 \cdot 46 - 2 - 72 = 18.$$

Remark 3.4. This number is equal to the predicted number of curves calculated by S. Hosono, A. Klemm, S. Theisen and S.-T. Yau ([**11**], p. 521) and by D. van Straten and V. V. Batyrev ([**1**]). A general hypersurface of bidegree $(3, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ also has a 1-dimensional family of singular rational curves of bidegree $(3, 0)$. Applying the above definition of equivalence to this family gives the number 162. This agrees with the conjectured number in [**11**].

4. Rational Curves of higher degree on a general Tian–Yau CICY.

In this section we study rational curves of degree higher than 3 on a general Tian–Yau CICY. We consider certain linear systems on $\mathbb{P}^2$, and use them to prove the existence of positive dimensional families of curves of every degree greater than 3 on a general Tian–Yau CICY. In the first part of the section we refine this study, and give results concerning existence of rational curves of various bidegrees.

We want to give a constructive proof of Theorem 1.1, and we start out by two preliminary lemmas.

Lemma 4.1. *Fix a point p in $\mathbb{P}^2$ and let $d \geq 3$. The linear system of curves of degree d, with a point of order $(d-1)$ at p, is of dimension $2d$, and a generic member is an irreducible rational curve.*

Proof. The dimension of the linear system of curves of degree d is $\binom{d+2}{2} - 1$. The condition that a curve has a given point p as a multiple point of order $(d-1)$, is equivalent to the vanishing of the $(d-1)$ first partial derivatives at p. This gives $1 + \cdots + (d-1)$ conditions on the coefficients, and the first statement follows. To prove prove the second statement it suffices to show that there exists an irreducible rational curve in the linear system. One can construct one in the following way: Let

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$

$$f(u, v) = (u^d, u^{d-1}v, \ldots, v^d).$$

Let $C = f(\mathbb{P}^1) \subseteq \mathbb{P}^d$. Choose $d-1$ points on C. These $=$ points span a linear subspace L of dimension $(d-2)$. Let $L' \subset L$ be a linear subspace with the following properties: dim $L' = d-3$ and $L' = \cap C = \emptyset$, and let $\pi : \mathbb{P}^d \longrightarrow \mathbb{P}^2$ be the projection from the linear subspace L'. Then $\hat{C} = \pi(C)$ is a curve with the desired properties. $\square$

Lemma 4.2. *Let F be a nonsingular cubic surface in $\mathbb{P}^3$. For every natural number $m \geq 3$, there exists a 2-dimensional family of nonsingular rational curves of degree m on F.*

Proof. The hypersurface F is isomorphic to $\mathbb{P}^2$ blown up in six points $p_0, \ldots,$ p_5. Consider the linear system σ^0 of curves of degree $d \geq 3$, and with a multiple point of order $(d-1)$ at p_0 in $\mathbb{P}^2$. We denote a generic curve of σ^0 by C_0. The strict transform of C_0 is a rational curve C_1 of degree $2d+1$. Since the dimensions of the linear systems considered down on $\mathbb{P}^2$ is at least 6 by the preceding lemma, the statement is proved for odd degrees $7, 9, 11, \ldots$. For even degrees we take a sublinear system σ^1 of σ^0, by demanding the curve to pass through p_1 once. The strict transform of a generic curve is a rational curve of degree $2d$. The dimensions of these families of curves are at least 5. In the same manner we can take curves that in addition to the requirements above also pass though p_2 and so on. In each case the dimension drops by no more than one. Hence, we have inclusions $\sigma^0 \supset \sigma^1 \supset \ldots \supset \sigma^t \supset \ldots \supset \sigma^5$. (The linear system σ^t constist of curves of degree d passing through the points $p_1, \ldots, p_t$.) This gives the desired results for the remaining degrees $3, 4, 5$. In the case $m = 3$ (corresponding to $d = 3$ and $t = 4$) the dimension is at least equal to 2. ☐

Now we give a constructive proof of Theorem 1.1.

Proof of the theorem. Let $X = F_1 \times F_2 \cap G$ be a general Tian–Yau CICY, and let L be one of the 27 lines on F_2. Let q_1, q_2, q_3 be the points of intersection of $l(L)$ and F_1. Furthermore, fix a blowing down of the exceptional divisors $\pi : F_1 \longrightarrow \mathbb{P}^2$, and let $\bar{q}_i = \pi(q_i)$ for $i = 1, 2, 3$. We shall use the linear systems of curves in $\mathbb{P}^2$ considered in Lemma 4.1 and in Lemma 4.2.

Consider first $m \geq 3$ and $i = 0$. By Lemma 4.2 we have linear systems σ_t with t base points $p_1, \ldots, p_t$. A general member of this linear system does not pass through any of the $\bar{q}_i$, i.e., it gives rise to a rational curve of bidegree (m, m) on X by Proposition 2.7. Since these linear systems are all positive dimensional, we get positive dimensional families of bidegree (m, m), for $m > 2$, on X.

In order to prove the statement in the case $m \geq 3$ and $i = 1$, we take sublinear systems σ^{t_1} of the σ^t considered above, by assigning the basepoint q_1. The dimension of σ^{t_1} is $\dim \sigma^t - 1$. Lemma 4.2 then gives $\dim \sigma^{t_1} \geq 1$, and the result follows.

For $i = 2$ we take sublinear systems of σ^{t_1}, by assigning $\bar{q}_2$ as an additional base point. By the same reasoning as above this gives positive dimensional families, using Prop. 2.7, Lemma 4.1 and Lemma 4.2 for $m > 3$.

Finally, the case $i = 3$ is treated analogously by considering sublinear systems of σ^t by assigning $\bar{q}_1, \bar{q}_2, \bar{q}_3$ as base points. Using Prop. 2.7, Lemma 4.1 and Lemma 4.2 we obtain positive dimensional families of bidegree $(m, m-3)$ curves for $m > 3$.

In the proof of Proposition 3.1 we gave all rational curves of bidegree $(2, 1)$. They were realised as degenerations of a 1-dimensional family of bidegree $(2, 2)$ rational curves of type $(\bar{2}, \bar{1})$, constructed from a pencil of planes in $\mathbb{P}^3$ containing a line in F_1. (In fact, this shows that there are exactly 27 1-dimensional families of bidegree $(2, 2)$ and of type $(\bar{2}, \bar{1})$.) ☐

Theorem 4.3. *A general Tian–Yau CICY contains no nonsingular rational curves of bidegree* $(m, m - i)$ *and of type* $(\bar{m}, \bar{1})$*, for* $m \geq i \geq 4$.

Proof. An i-secant of a curve when $i \geq 4$, has to be contained in F_1, by Bezout's theorem. In other words, it has to be one of the 27 lines, but for a general Tian–Yau CICY, none of the 27 $l(L)$'s are among the 27 lines on F_1. ☐

Remark 4.4. Note that Theorem 1.3 follows directly from the proof of Theorem 4.3.

The last theorem states the non-existence of rational curves of certain bidegrees on a general Tian–Yau CICY. However, there may exist nongeneral Tian–Yau CICYs with rational curves of these bidegrees.

Proposition 4.5. *Let $d \geq 3$ and let $t \in \{0, 1, 2, 3, 4, 5\}$. There exist varieties of Tian–Yau CICY type with positive dimensional families of nonsingular rational curves of bidegree $(2d + 1 - t, d + 2 - t)$.*

Proof. In Lemma 4.1 and Lemma 4.2 we constructed the linear systems of curves σ^t. The strict transforms of these curves have a $(d - 1)$-secant, E_0, the exceptional divisor corresponding to p_0. Furthermore, the degree of the strict transform of a general member of σ^t is $2d+1-t$. If F_i is a nonsingular cubic surface in $\mathbb{P}^3_i$, denote by L^k_i, $k \in \{1, \ldots, 27\}$, the 27 lines on F_i. Now, choose a pair of cubic surfaces F_1, F_2 and a $G = Z(\sum \alpha_{ij}x_iy_j)$, such that the matrix $[\alpha_{ij}]$ is invertible, and with the property that there exists a pair of lines L^j_1 and L^n_2 such that $l(L^n_2) = L^j_1$. Applying Proposition 2.7 gives the desired result. ☐

Remark 4.6. All of these curves are of type $(\bar{m}, \bar{1})$, except for the case $d = 3, t = 5$, which gives a bidegree $(2, 0)$ curve.

Clemens' conjecture states that a general quintic threefold in $\mathbb{P}^4$ contains a finite number of smooth rational curves for every degree. The conjecture has been proven for degrees less than 10 ([**13**], [**12**]). Corollary 1.2 states the existence of positive dimensional families of nonsingular rational curves for every degree ≥ 4 on a general Tian–Yau CICY.

5. Curves of degree 1, 2 and 3 on the special Tian–Yau CICY.

In this section we are going to study rational curves of degree less than 4 on the special Tian–Yau variety.

Proposition 5.1. *The numbers $N_{i,j}$ of nonsingular rational curves of bidegree (i, j) on a general Tian–Yau variety are finite for $i + j \leq 3$ and are given by:*

$$\begin{aligned} N_{0,1} = N_{1,0} &= 81 \\ N_{0,2} = N_{2,0} &= 81 \\ N_{1,1} &= 567 \\ N_{1,2} = N_{2,1} &= 972 \\ N_{3,0} = N_{0,3} &= 0. \end{aligned}$$

Proof. The number of rational curves of bidegree $(1, 0)$ and $(2, 0)$ are computed in the same way as in Proposition 3.1. The number of $(1, 1)$ curves is equal to the number of irreducible intersections $L' \times L \cap G$, where $L' \subseteq F_1$ and $L \subseteq F_2$, by Proposition 2.7. We have 729 pairs of lines to consider. Computation gives 567 irreducible intersections, hence 567 (1,1) curves. The number of rational curves of bidegree $(2,1)$ is a little more complicated to obtain. First, notice that there is no rational curves of bidegree $(2,1)$ and of type $(\bar{1},\bar{1})$. Assume for contradiction that there is one, and denote it by C. Let $L_i = \pi_i(C)$. Choose a plane H in $\mathbb{P}^3_1$ such that $L_1 \subseteq H$. Since $H \times L_2 \cap G$ is the blow-up of H in $l(L_2)$ and $L_1 \neq l(L_2)$, $D = L_1 \times L_2 \cap G$ is of dimension one. Assume that $L_1 \cap l(L_2) = \emptyset$, then $L_1 \cong D$. This implies that C's degree on the first factor is at most 1. Assume that $L_1 \cap L_2 \neq \emptyset$, then D consists of a curve $\bar{C}$ the strict transform of L_1, and an exceptional divisor E. The curve C has to be contained in $\bar{C}$, since it is dominant on the first factor. This implies that $C = \bar{C}$. This implies that C's degree on the first factor is 1.

In view of Proposition 2.7, to give a rational curve of bidegree $(2,1)$ is equivalent to giving a conic C in F_1 and a line L in F_2 such that $l(L)$ intersect C in one point. Any conic C in $\mathbb{P}^3$ is contained in a unique hyperplane H, so $F_1 \cap H = C \cup L'$, where L' is a line in F_1. Conversely, every line L' determines a pencil of planes. The lines on the Fermat cubic F_2 are:

$$y_{n_0} + \alpha^i y_{n_1} = y_{n_2} + \alpha^j y_{n_3} = 0,$$

where the $n_i \in \{0,1,2,3\}$ and are all different. Let L be $y_0 + \alpha^i y_1 = y_2 + \alpha^j y_3 = 0$, then $l(L) \cap F_1 = (1, \alpha^i, -\alpha^l, -\alpha^{j+l})$ for $l \in \{0,1,2\}$.

Consider one line in F_1, say: $L' = Z(x_0 + x_3 = x_1 + \alpha x_3)$. The pencil is then given by: $H_a = Z(ax_0 + \alpha^2 x_1 + x_2 + aX_3)$ (where we allow $a = \infty$). Let C_a be defined by: $H_a \cap F_1 = C_a \cup L$.

Demanding a point in $l(L) \cap F_1$ to be in H_a, gives the following condition on a: $a(1-\alpha^{j+l}) = \alpha^l - \alpha^{i-2}$. There are 27 cases to consider: i,j,l may take values in $\{0,1,2\}$. A case by case study gives that only 12 of these give a rational curve of bidegree $(2,1)$, i.e., where $l(L)$ intersects an irreducible C_a once. This accounts for 9 lines on F_1 (9 different pairs (i,j)). A similar study of the remaining 18 lines, gives 24 rational curves of the desired bidegree (out of 54 candidates). Hence the number of rational curves of bidegree $(2,1)$ is 36. Note that all these curves are mapped to $L \subseteq F_2$ by the second projection. By symmetry the total number of rational curves of bidegree $(2,1)$ is $27 \cdot 36 = 972$. □

Corollary 5.2. *Let N_d denote the number of nonsingular rational curves of degree d on a general Tian–Yau variety. Then*

$$N_1 = 162, \quad N_2 = 729, \quad N_3 = 1944.$$

Remark 5.3. The numbers of curves of bidegree $(0,1)$, $(1,0)$ and $(1,1)$ have been calculated previously, using similar techniques ([**5**], [**19**]).

Corollary 5.4. *There exist positive dimensional families of rational curves on the special Tian–Yau CICY for every degree n, $n \geq 4$.*

Proof. This is a corollary of the proof of Corollary 1.2. The construction of curves relied on the use of Proposition 2.7, i.e. on the fact that $G = Z(\sum_{ij} \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible, which is clearly satisfied in this case. The last necessary ingredient in the proof is that none of the $l(L)$, where L is one of the 27 lines of F_2, are tangent to the surface F_2. This is easily checked for the special Tian–Yau CICY. The rest of the proof is identical to the proof of Theorem 1.1. □

Remark 5.5. By comparing Corollary 5.2 with Corollary 3.2, we see that the number of curves of degree 2 and 3 on the special Tian–Yau variety are different from the corresponding numbers for the general Tian–Yau variety. For example for degree 2, the difference stems from the number of $(1,1)$ curves. The difference between the numbers of $(1,1)$ curves on the general and the special is $729 - 567 = 162$. This difference is explained by the fact that we have 162 reducible intersections of type $L_1 \times L_2 \cap G$ (where $L_i \subseteq F_i$) on the special Tian–Yau variety, and none on the general Tian–Yau variety. A reducible intersection $L_1 \times L_2 \cap G$ gives one $(1,0)$ curve and one $(0,1)$ curve. It is easy to see that every rational curve of bidegree $(1,0)$ is contained in precisely two distinct intersections of the type $L_1 \times L_2 \cap G$. Hence, we get 81 $(1,0)$ curves. By Proposition 5.1 we know that these are in fact the only ones.

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