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In this paper, we introduce a class of Lie algebras which are subalgebras of generalized Cartan type S Lie algebras of characteristic 0. We determine the necessary and sufficient conditions for such Lie algebras to be simple. And we give all derivations of such simple Lie algebras.

1. Introduction.

This paper is a sequel to the paper [7] in which generalized Cartan type S Lie algebras $t^z S(A, T, \varphi)$ over a field F of characteristic 0 were studied. We have tried to make this paper independent of other papers. So in Section 2, we give a description of relevant Lie algebras and some basic facts which will be used in this paper. In Section 3 we introduce a class of Lie algebras which are subalgebras of generalized Cartan type S Lie algebras, and determine the necessary and sufficient conditions for such Lie algebras to be simple. We give all derivations of such simple Lie algebras in Section 4.

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2. Notations and related Lie algebras.

In this section, for the convenience of the reader, we recall the relevant Lie algebra definitions and some basic facts which will be used later in this paper. Throughout this paper we assume that F is a field of characteristic 0, and that A is a nonzero abelian group written additively.

2.1 Generalized Witt algebras.

Let n be a positive integer, and $t_1, \dots, t_n$ independent and commuting indeterminates over F. Denote by P_n and Q_n the polynomial algebra $F[t_1, \dots, t_n]$, and the Laurent polynomial algebra $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ respectively. By $W_n = W_n(F)$ we denote the *Witt algebra*, i.e., the Lie algebra of all formal vector fields

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial t_i} \tag{2.1}$$

with coefficients $f_i \in Q_n$. The bracket in W_n is

$$\left[f\frac{\partial}{\partial t_i}, g\frac{\partial}{\partial t_j}\right] = f\frac{\partial(g)}{\partial t_i}\frac{\partial}{\partial t_j} - g\frac{\partial(f)}{\partial t_j}\frac{\partial}{\partial t_i},$$

where $f, g \in Q_n$, and $i, j \in \{1, 2, \ldots, n\}$. The subalgebra $W_n^+ = W_n^+(F)$ of W_n consisting of all vector fields (2.1) with polynomial coefficients, i.e., $f_i \in P_n$, is known as *the general Lie algebra*, or *the Lie algebra of Cartan type* W. For more details, please refer to [**13**]. It is well known that W_n and W_n^+ are simple Lie algebras.

Let T be a vector space over F. We denote by FA the group algebra of A over F. The elements t^x , $x \in A$, form a basis of this algebra, and the multiplication is defined by $t^x \cdot t^y = t^{x+y}$. We shall write 1 instead of t^0. The tensor product $W = FA \otimes_F T$ is a free left FA-module in the natural way. We denote an arbitrary element of T by ∂ (to remind us of differential operators). For the sake of simplicity, we shall write $t^x\partial$ instead of $t^x \otimes \partial$. We now choose a pairing $\varphi : T \times A \to F$ which is F-linear in the first variable and additive in the second one. For convenience we shall also use the following notations:

$$\varphi(\partial, x) = \langle \partial, x\rangle = \partial(x)$$

for arbitrary $\partial \in T$ and $x \in A$. W becomes a Lie algebra under the following bracket:

$$[t^x\partial_1, t^y\partial_2] := t^{x+y}(\partial_1(y)\partial_2 - \partial_2(x)\partial_1), \tag{2.2}$$

for arbitrary $x, y \in A$ and $\partial_1, \partial_2 \in T$. We refer to this algebra $W = W(A, T, \varphi)$ as a *generalized Witt algebra.*

The subspaces $W_x = t^xT$, $x \in A$, define an A-gradation of W, i.e., W is the direct sum of the W_x's, and $[W_x, W_y] \subset W_{x+y}$ for all $x, y \in A$.

It follows from (2.2) that ad(∂) acts on W_x as a scalar $\partial(x)$. Hence each $\partial \in T$ is ad-semisimple, and T is a torus (i.e., an abelian subalgebra consisting of ad-semisimple elements). In fact T is the only maximal torus of W (see [**3**, Lemma 4.1]). Kawamoto proved in [**11**] that the Lie algebra $W = W(A, T, \varphi)$ is simple if and only if $A \neq 0$ and φ is nondegenerate in the sense that the conditions

$$\langle \partial, x\rangle = 0, \forall\ \partial \in T \ \Rightarrow\ x = 0 \tag{2.3}$$

and

$$\langle \partial, x\rangle = 0, \forall\ x \in A \ \Rightarrow\ \partial = 0 \tag{2.4}$$

hold.

Note that (2.3) implies that A is torsion free. This implies that FA is an integral domain and it implies that the invertible elements of FA have the form at^x, where $a \in F^*, x \in A$.

There is a natural structure of a left W-module on FA, namely the structure is such that

$$t^x\partial \cdot t^y = \partial(y)t^{x+y} \tag{2.5}$$

for $x, y \in A$ and $\partial \in T$. Also we have the natural left FA-module structure on W. These two module structures are related by the identity

$$[fu, gv] = f(u \cdot g)v - g(v \cdot f)u + fg[u, v] \tag{2.6}$$

where $f, g \in FA$ and $u, v \in W$ are arbitrary. The W-module structure on FA gives rise to a homomorphism

$$W \to \text{Der }(FA) \tag{2.7}$$

because each $w \in W$ acts on FA as a derivation. Clearly (2.7) is also a homomorphism of FA-modules. For more details about $W(A, T, \varphi)$, please refer to [**3**].

2.2 Generalized Cartan type W Lie algebras.

Suppose that $W = W(A, T, \varphi)$ denotes a simple generalized Witt algebra. Let I be an index set, $d : I \to T$ an injective map, and write $d_i = d(i)$ for $i \in I$. We say that d is *admissible* if the following two conditions hold:

(Ind) $d_i, i \in I$, are linearly independent;

(Int) $d_i(A) = \mathbf{Z}$ for all $i \in I$.

We assume throughout that an admissible d has been fixed. We set

$$\begin{aligned}
A_d^+ &= \{x \in A : d_i(x) \geq 0,\ \forall i \in I\},\\
A_d^0 &= \{x \in A : d_i(x) = 0,\ \forall i \in I\},\\
A_{d,i} &= \{x \in A : d_i(x) = -1;\ d_j(x) \geq 0,\ \forall j \in I \backslash \{i\}\},\\
A_{d,i}^\# &= \{x \in A : d_i(x) = -1;\ d_j(x) = 0,\ \forall j \in I \backslash \{i\}\},\\
A_d &= A_d^+ \cup \left(\cup_{i \in I} A_{d,i}\right).
\end{aligned}$$

We now introduce some subalgebras of W:

$$W_d^+ = \sum_{x \in A_d^+} W_x;$$

$$W_{d,i} = \left(\sum_{x \in A_{d,i}} Ft^x\right) \cdot d_i,\ i \in I;$$

and

$$W_d = W_d(A, T, \varphi) = W_d^+ + \sum_{i \in I} W_{d,i}.$$

We also introduce the subalgebra FA_d^+ of FA, which is the span of all elements t^x with $x \in A_d^+$. Since W is a left FA-module, we can view W also as a left FA_d^+-module. Then it is easy to see that the subspaces W_d^+ and W_d are FA_d^+-submodules of W.

By restricting the action of W on FA, we can view FA as a left W_d-module, and then FA_d^+ is a W_d-submodule of FA. When d is fixed, and there is no danger of confusion, we shall write

$$A^+,\ A_i,\ A_i^{\#},\ W^+,\ W_i,\ FA^+$$

instead of

$$A_d^+,\ A_{d,i},\ A_{d,i}^{\#},\ W_d^+,\ W_{d,i},\ FA_d^+,$$

respectively. The following Theorem is proved in [**5**].

Theorem 2.1. *The Lie algebra W_d is simple if and only if the following conditions hold:*

(i) *if $\partial \in T$ and $\partial(x) = 0$ for all $x \in A_d$, then $\partial = 0$;*
(ii) *if $x \in A_d$, then $d_i(x) = 0$ for almost all $i \in I$;*
(iii) *$A_i^{\#} \neq \emptyset$ for all $i \in I$.*

The simple Lie algebra W_d is called an algebra of *generalized Cartan type* W. For more details on the Lie algebra W_d, please refer to the papers [**5**] and [**12**].

2.3 Generalized Cartan type S Lie algebras.

It is well known that the classical *divergence* Div:$W_n \to F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ maps $\sum_{i=1}^n\ f_i\ \frac{\partial}{\partial t_i}$ to $\sum_{i=1}^n \frac{\partial(f_i)}{\partial t_i}$. The kernel $\tilde{S}_n$ of Div is a subalgebra of W_n. The algebra $S_n = (\tilde{S}_n)''$ and $S_n^+ = S_n \cap W_n^+$ are called *Lie algebras of Cartan type* S.

Suppose that $W = W(A, T, \varphi)$ is a simple generalized Witt algebra. The *divergence* div: $W \to FA$ is the F-linear map such that

$$\text{div } (t^x \partial) = \partial(x) t^x \tag{2.8}$$

holds for all $x \in A$ and $\partial \in T$. It has the following two properties:

$$\text{div } (fw) = f\text{div } (w) + w \cdot f \tag{2.9}$$

and

$$\text{div } [u, v] = u \cdot \text{div } (v) - v \cdot \text{div } (u) \tag{2.10}$$

where $u, v, w \in W$ and $f \in FA$ are arbitrary. The latter property shows that div is a derivation of W with values in the W-module FA. Since div : $W \to FA$ is a derivation of degree 0, its kernel $\tilde{S} := \ker(\text{div})$ is a homogeneous subalgebra of W:

$$\tilde{S} = \oplus_{x\in A} \tilde{S}_x \ , \ \tilde{S}_x := \tilde{S} \cap W_x. \tag{2.11}$$

For $x \in A$ define the F-linear function $\hat{x} : T \to F$ by $\hat{x}(\partial) = \partial(x)$. The condition (2.3) shows that, if T^* is the dual space of T, the $\mathbf{Z}$-linear map $A \to T^*$ sending $x \mapsto \hat{x}$ is injective. If $T_x := \ker(\hat{x})$, then we have $\tilde{S}_x = t^x T_x$. Hence $\tilde{S}_0 = W_0 = T$ and, for $x \neq 0$, $\tilde{S}_x$ is a hyperplane of W_x. In particular, if $\dim T = 1$, then $\tilde{S} = T$. To avoid trivialities, we shall assume always that $\dim T > 1$.

Let $\bar{S} := (\tilde{S})'$ be the derived algebra of $\tilde{S}$. Note that the notation here is different from that in [7]. We know (see [7]) that

$$\bar{S} = \oplus_{x\neq 0} \tilde{S}_x.$$

More generally, the subspaces $t^z\tilde{S},\ z \in A$, are subalgebras of W and their derived algebras are given by

$$(t^z\tilde{S})' = t^z\bar{S} = \sum_{x\neq z} t^x T_{x-z}. \tag{2.12}$$

If $\dim T \geq 3$, all the subalgebras $t^z\bar{S}$ are simple. If $\dim T = 2$, then $\bar{S}$ itself is simple while the shifted algebras $t^z\bar{S}$, $z \neq 0$, are not. Their derived algebras

$$(t^z\bar{S})' = \sum_{x\neq z,2z} t^x T_{x-z},\ z \neq 0,$$

are simple.

We shall refer to the subalgebras $S(A, T, \varphi, z) := t^z\bar{S}$ if $\dim T \geq 3$, and $S(A, T, \varphi, z) := (t^z\bar{S})'$ if $\dim T = 2$, as *Lie algebras of generalized Cartan type S*. The Lie algebras $S(A, T, \varphi, z)$ have the A-gradation:

$$S(A, T, \varphi, z) = \begin{cases} \oplus_{x\in A\setminus\{z\}} t^x T_{x-z}, & \text{if } \dim T > 2, \\ \oplus_{x\in A\setminus\{z,2z\}} t^x T_{x-z}, & \text{if } \dim T = 2. \end{cases}$$

These algebras were studied in papers: [6] when $\dim T = 2$ and $z = 0$, [4] when $\dim T = 2$ and $z \neq 0$, and [7] when $\dim T \geq 3$.

2.4 Generalized Block algebras.

We shall denote by Hom (A, F) the F-vector space of all additive (i.e., $\mathbf{Z}$-linear) maps $A \to F$. We now fix an additive map $\alpha : A \to F$ and a skew-symmetric bi-additive map $\varphi : A \times A \to F$.

Let $L = L(A, \alpha, \varphi)$ be the vector space over F having a basis consisting of all symbols $e_x,\ x \in A$. We make L into a (non-associative) algebra over

F by defining F-bilinear multiplication $L \times L \to L$ by

$$[e_x, e_y] = f(x,y)e_{x+y},\ x, y \in A, \tag{2.13}$$

where

$$f(x,y) = \varphi(x,y) + \alpha(x-y). \tag{2.14}$$

If $\alpha = 0$, then L is a Lie algebra. These Lie algebras were studied in [8]. It was shown by Albert and Frank [1] that, under the assumption that $\alpha \neq 0$, L is a Lie algebra if and only if there exists another additive map $\beta : A \to F$ such that $\varphi = \alpha \wedge \beta$, i.e.,

$$\varphi(x,y) = \alpha(x)\beta(y) - \beta(x)\alpha(y). \tag{2.15}$$

We shall assume throughout that such a β exists, i.e. that L is a Lie algebra. We also assume that

$$K_\alpha \cap K_\beta = 0, \tag{2.16}$$

where K_μ denotes the kernel of μ for any additive map $\mu : A \to F$. Let $L^2 = [L, L]$ be the derived subalgebra of L and Z the center of L. The Lie algebra $\mathcal{L} = L^2/Z$ is simple [4, Theorem 2.5], and we shall also write $\mathcal{L}(A, \alpha, \varphi)$ for $\mathcal{L}$.

We refer to the Lie algebras $\mathcal{L} = \mathcal{L}(A, \alpha, \varphi)$ as *generalized Block algebras.*

Suppose that $\beta \in \mathrm{Hom}\ (A, F)$ can be chosen so that (2.15) holds and

$$\beta(A) = \mathbf{Z}. \tag{2.17}$$

We now define the subset $A_\beta \subset A$ by

$$A_\beta = \{x \in A : \beta(x) \geq -1\}\ , \tag{2.18}$$

and denote by L_β the subspace of L with a basis consisting of all e_x with $x \in A_\beta$. It follows that L_β is a subalgebra of L.

We shall denote by $\mathcal{L}_\beta$ or $\mathcal{L}_\beta(A, \alpha, \varphi)$ the quotient algebra L_β/Z. The Lie algebra $\mathcal{L}_\beta$ is simple. For more details about $\mathcal{L}(A, \alpha, \varphi)$ and $\mathcal{L}_\beta(A, \alpha, \varphi)$, please refer to [4] and [8].

3. Lie algebra $S_d(A, T, \varphi, z)$.

Now we are ready to introduce our main object, the Lie algebra $S_d(A, T, \varphi, z)$. We assume that a generalized Witt algebra $W = W(A, T, \varphi)$, a generalized Cartan type W Lie algebra $W_d = W_d(A, T, \varphi) \subset W$, a generalized Cartan type S Lie algebra $S(A, T, \varphi, z) \subset W$ are given, and we also assume throughout the paper that all of these algebras are simple. Our hypotheses here imply that $\dim T \geq 2$. We define

$$S_d(A, T, \varphi, z) := W_d \cap S(A, T, \varphi, z).$$

Then $S_d = S_d(A, T, \varphi, z)$ is a subalgebra of W. We shall also call the algebra S_d a *generalized Cartan type* S *Lie algebra*. It is clear that S_d has an A-gradation with the following components of degree $x \in A$:

$$(S_d)_x = \begin{cases} 0, & \text{for } \ x = z \\ (t^x T_{x-z}) \cap W_d, & \text{for } \ x \in A \setminus \{z\} \end{cases}$$

if $\dim T > 2$,

$$(S_d)_x = \begin{cases} 0, & \text{for } \ x = z \ \text{ or } \ x = 2z \\ (t^x T_{x-z}) \cap W_d, & \text{for } \ x \in A \setminus \{z, 2z\} \end{cases}$$

if $\dim T = 2$. It follows that $(S_d)_x = 0$ for all $x \notin A_d$.

For convenience, by $\langle U_1 \rangle$ we denote the subspace of the vector space V generated by $U_1 \subset V$.

Theorem 3.1. *Suppose that the Lie algebras* $W = W(A, T, \varphi)$, $W_d = W_d(A, T, \varphi) \subset W$, *and* $S = S(A, T, \varphi, z) \subset W$ *are given, and that all of them are simple. Then* $S_d(A, T, \varphi, z)$ *is simple if and only if the following conditions hold:*

(a) $d_i(z) = -1$ *for all* $i \in I$;
(b) I *is finite.*

Proof. For simplicity we write $L = S_d(A, T, \varphi, z)$, so $L_x = (S_d)_x$ for $x \in A$.

($\Rightarrow$) Suppose that L is simple. We shall first show that (a) is true. For contradiction we assume that there exists an element in I, say 1, such that $d_1(z) \neq -1$. If $t^x d_1 \in W_1 \cap L$ (for definition of W_1 see Section 2.2), we know that $d_1(x) = -1$ and $d_1(x - z) = 0$, so $-1 = d_1(x) = d_1(z) \neq -1$. It is a contradiction. Then $W_1 \cap L = 0$, i.e., $L_x = 0$ for all $x \in A_1$. Theorem 2.1(iii) assures that we can choose $y \in A$ with $-y \in A_1^{\#}$. So $y \in A_d^+$. If $z \neq y$, then $L_y = t^y T_{y-z} \neq 0$. Let $J = \oplus_{x\,:\,d_1(x) > 1} L_x$. If $z = y$, we see that $L_{3y} \neq 0$, let $J = \oplus_{x\,:\,d_1(x) > 3} L_x$. It is easy to verify that J is a nonzero proper ideal of L in both cases, which contradicts the simplicity of L. Consequently (a) holds.

On the other side we have $-z \in A_d^+$. From Theorem 2.1(ii) it follows that I is finite. So (b) is true.

($\Leftarrow$) Suppose (a) and (b) hold. We write $I = \{1, 2, \cdots, n\}$. We know that $L_x \neq 0$ if and only if $x \in A_d \setminus \{z\}$. Fix $\{u_1, u_2, \cdots, u_n\} \subset A$ such that $d_i(u_j) = \delta_{i,j}$. Let $A(d) = \mathbf{Z}u_1 \oplus \mathbf{Z}u_2 \oplus \cdots \oplus \mathbf{Z}u_n$, and $A'(d) = \{x \in A | d_i(x) = 0 \ \forall \ i \in I\}$. Then it follows from (b) that $A = A(d) \oplus A'(d)$.

Case 1. Suppose that $\dim T = 2$ and $|I| = 1$, say $I = \{1\}$. Choose $d_2 \in T$ such that $T = Fd_1 \oplus Fd_2$. Denote $e_x = t^x(d_1(x - z)d_2 - d_2(x - z)d_1)$ for

$x \in A_d$. It follows that $\{e_x | x \in A_d \setminus \{z\}\}$ is a basis of $L = S_d(A, T, \varphi, z)$. For any $x, y \in A_d$ we have

$$\begin{aligned}
[e_x, e_y] &= [t^x(d_1(x-z)d_2 - d_2(x-z)d_1), t^y(d_1(y-z)d_2 - d_2(y-z)d_1)] \\
&= t^{x+y}\big((d_1(x-z)d_2(y) - d_2(x-z)d_1(y)) \\
&\qquad \cdot t^x(d_1(x-z)d_2 - d_2(x-z)d_1) \\
&\quad - (d_1(y-z)d_2(x) - d_2(y-z)d_1(x)) \\
&\qquad \cdot t^y(d_1(y-z)d_2 - d_2(y-z)d_1)\big) \\
&= t^{x+y}\big(d_1(x-z)d_2(y-z) - d_2(x-z)d_1(y-z)\big) \\
&\qquad \cdot (d_1(x+y-z)d_2 - d_2(x+y-z)d_1) \\
&= \begin{vmatrix} d_1(x-z) & d_1(y-z) \\ d_2(x-z) & d_2(y-z) \end{vmatrix} e_{x+y}.
\end{aligned}$$

Denote $\alpha(x) = -d_2(x) - d_2(z)d_1(x),\ \beta(x) = d_1(x)$ for $x, y \in A$. It is clear that $\beta(A) = \mathbf{Z}$. By Section 2.4 we know that $L \simeq \mathcal{L}_\beta(A, \alpha, \beta)$. Consequently L is simple.

Case 2. Suppose that $\dim T = 2$ and $|I| = 2$, say $I = \{1, 2\}$. Since φ is nondegenerate, we get that $A = A(d) = d_1(A) \otimes d_2(A) \simeq \mathbf{Z} \otimes \mathbf{Z}$. We may assume that $d_i(x) = x_i$ for $x = (x_1, x_2) \in \mathbf{Z} \otimes \mathbf{Z}$. Thus $A_d = \Big((\mathbf{Z}_+ - 1) \otimes (\mathbf{Z}_+ - 1)\Big) \setminus \{(-1, -1)\}$. Same as Case 1, we define $e_x,\ x \in A_d$. Then we also have

$$[e_x, e_y] = \begin{vmatrix} d_1(x-z) & d_1(y-z) \\ d_2(x-z) & d_2(y-z) \end{vmatrix} e_{x+y},$$

i.e.,

$$[e_x, e_y] = \begin{vmatrix} x_1+1 & y_1+1 \\ x_2+1 & y_2+1 \end{vmatrix} e_{x+y},\ \forall\, x, y \in A_d.$$

It is well known that the Lie algebra $S_2^+ \subset W_2^+$ has basis $\{t_1^{x_1} t_2^{x_2}(t_2 \frac{\partial}{\partial t_1} - t_1 \frac{\partial}{\partial t_2}) | (x_1, x_2) \in A_d\}$. It is easy to verify that the following linear map is an isomorphism of Lie algebras:

$$S_d \to S_2^+,\ e_x \mapsto t_1^{x_1} t_2^{x_2}\left(t_2 \frac{\partial}{\partial t_1} - t_1 \frac{\partial}{\partial t_2}\right).$$

Thus S_d is also simple in this case.

Case 3. Suppose that $\dim T > 2$.

If $|I| = \dim T = n$, as we did in Case 2 we can deduce that $L \simeq S_n^+$, the special algebra of rank n. Thus in this case S_d is simple. Next we assume that $\dim T > |I| = n > 0$.

Let J be a nonzero ideal of L. It is suffices to show that $J = L$. Choose a nonzero element $u \in J$, say

$$u = \sum_{i=1}^{m} t^{x_i}\partial_i, \ x_i \in A_d \setminus \{z\}, \partial_i \in T_{x_i - z} \tag{3.1}$$

with m is minimal. Then $x_1, \cdots, x_m$ are distinct and $\partial_i \neq 0$.

Claim 1. *We have that $m = 1$.*

Otherwise we suppose $m > 1$. Since $T_z \subset L$, from the minimality of m it follows that

$$(\hat{x}_1)|_{T_z} = (\hat{x}_2)|_{T_z} = \cdots = (\hat{x}_m)|_{T_z}.$$

Thus

$$\hat{x}_i - \hat{x}_j \in F\hat{z}, \ \forall\, i, j \in \{1, 2, \cdots, m\}. \tag{3.2}$$

Subclaim. *We can choose such an element u in (3.1) such that $d_1(x_1) = -1$.*

Suppose $d_1(x_1) \geq 0$. If $\partial_1 \in Fd_1$, let $y_1 \in A_1^{\#}$, then

$$u' = [t^{y_1}d_1, u] = \sum_{i=1}^{m} t^{y_1 + x_i}(d_1(x_i)\partial_i - \partial_i(y_1)d_1) \in J \setminus \{0\}$$

and $d_1(y_1 + x_1) < d_1(x_1)$. If $\partial_1 \notin Fd_1$, then $A_{d,1} \not\subset \ker(\partial_1)$, Otherwise we can show that $\partial_1(A_d) = 0$, it is impossible. We choose $y_1 \in A_{d,1} \setminus \ker(\partial_1)$. We deduce that

$$u' = [t^{y_1}d_1, u] = \sum_{i=1}^{m} t^{y_1 + x_i}(d_1(x_i)\partial_i - \partial_i(y_1)d_1) \in J \setminus \{0\},$$

and also $d_1(y_1 + x_1) < d_1(x_1)$. After finitely many steps of this kind process, we get a nonzero element u in (3.1) such that $d_1(x_1) = -1$. Our subclaim follows.

Without loss of generality we may assume that $\partial_1 = d_1$. From (3.2) it follows that there exists $\lambda_i \in F^*$ such that

$$\hat{x}_i = \lambda_i \hat{z} + \hat{x}_1, \ \forall\, i \in \{2, \cdots, m\}.$$

Since $x_i \in A_d$, we have $d_1(x_i) \geq -1$, then $-\lambda_i \in \mathbf{N}$, the set of natural numbers. Thus $d_1(x_2) \geq 0$. If $\partial_2 \in Fd_1$, then

$$[t^{x_1}d_1, t^{x_2}\partial_2] = t^{x_1 + x_2}(d_1(x_i)\partial_i - \partial_i(y_1)d_1) \neq 0.$$

It follows that

$$u' = [t^{x_1}d_1, u] = \sum_{i=2}^{m} t^{x_1 + x_i}(d_1(x_i)\partial_i - \partial_i(y_1)d_1) \in J \setminus \{0\}.$$

This contradicts the minimality of m. Consequently $\partial_2 \notin Fd_1$. Choose $z_1 \in A_{d,1} \setminus \ker(\partial_2)$, then $[t^{x_1}d_1, t^{z_1}d_1] = 0$ and $[t^{z_1}d_1, t^{x_2}\partial_2] = t^{z_1+x_2}(d_1(x_2)\partial_2 - \partial_2(z_1)d_1) \neq 0$. Thus

$$u' = [t^{z_1}d_1, u] = \sum_{i=2}^{m} t^{z_1+x_i}(d_1(x_i)\partial_i - \partial_i(z_1)d_1) \in J \setminus \{0\}.$$

It again contradicts the minimality of m. Therefore $m = 1$. Claim 1 follows.

Claim 2. *We have $T_z \subset J$.*

From Claim 1 we know that there exists a nonzero element $t^x\partial_0 \in J$, where $x \in A_d$, $\partial_0 \in T_{x-z}$.

Subcase 1. Suppose $x \in A^{\#}_{d,i}$ for some $i \in I$, say $x \in A^{\#}_{d,1}$. Then $\partial_0 \in Fd_1$. Thus $t^x d_1 \in J$. For any $\partial \in T_{-x-z}$, from

$$[t^x d_1, t^{-x}\partial] = \partial - \partial(x)d_1 \in J,$$

and $d_1 \notin T_{-x-z}$, we know that $\langle d_1, \partial - \partial(x)d_1 | \partial \in T_{-x-z}\rangle = T$. Hence $\langle \partial - \partial(x)d_1 | \partial \in T_{-x-z}\rangle = T_{-z} = T_z$. therefore $T_z \subset J$.

Subcase 2. Suppose $x \in A^{+}_{d}$. Define $d(x) := \sum_{i=1}^{n} d_i(x)$. If $d_1(x) > 0$, choose $y_1 \in A^{\#}_{d,1}$. Since $d_1(x-z) \neq 0$ and $\partial_0(x-z) = 0$, we know that d_1 and ∂_0 are linearly independent. From the computation

$$[t^{y_1}d_1, t^x\partial_0] = t^{x+y_1}(d_1(x)\partial_0 - \partial_0(y_1)d_1) \neq 0,$$

we get a nonzero element $t^{x+y_1}(d_1(x)\partial_0 - \partial_0(y_1)d_1) \in J$ with $d(x+y_1) = d(x) - 1$. By repeatedly using this method, after finitely many steps we deduce that there exists a nonzero element $t^y\partial \in J$ with $y \in A^0_d$. If $A^{\#}_{d,i} \subset \ker(\partial)$ for all $i \in I$, we infer that $\partial(A_d) = 0$. It contradicts Theorem 2.1(i). Thus there exists an $i \in I$ such that $A^{\#}_{d,i} \not\subset \ker(\partial)$, say $A^{\#}_{d,1} \not\subset \ker(\partial)$. Choose $z_1 \in A^{\#}_{d,1} \setminus \ker(\partial)$, then

$$[t^{z_1}d_1, t^y\partial] = -t^{z_1+y}\partial(z_1)d_1 \in J \setminus \{0\}.$$

By Subcase 1 we obtain again that $T_z \subset J$. Similarly $T_z \subseteq J$ for $x \in A_{d,i}$. Thus Claim 2 is proved.

If $x \in A_d$ and $\hat{x} \notin F\hat{z}$, then $T_z \not\subset \ker\hat{x}$. Choose $\partial \in T_z \setminus \ker\hat{x}$. From $[\partial, t^x\partial'] = \partial(x)t_x\partial' \in J$ we know that $t^x T_{x-z} \subset J$.

If $y \in A_d \setminus \{0\}$ and $\hat{y} \in F\hat{z}$, it follows that $y = -kz$ for some positive integer k. Since $\{\hat{x} | x \in A^{\#}_{d,1}\} \not\subset F\hat{z}$, we choose $x_1 \in A^{\#}_{d,1}$ with $\hat{x}_1 \notin F\hat{z}$. Since $y \neq z$, then $d_i(y) > 0$ for all $i \in I$. Note that $t^{y-x_1}T_{y-x_1-z} \subset J$. Then, for $t^{x_1}d_1, t^{y-x_1}\partial_1 \in L$, we have

$$[t^{y-x_1}\partial_1,\, t^{x_1}d_1] = -t^y(d_1(y-x_1)\partial_1 - \partial_1(x)d_1) \in J.$$

Because $\langle d_1, d_1(y-x_1)\partial_1 - \partial_1(x)d_1 | \partial_1 \in T_{y-x_1-z}\rangle = T$. Then $\langle d_1(y-x_1)\partial_1 - \partial_1(x)d_1 | \partial_1 \in T_{y-x_1-z}\rangle = T_{y-z} = T_z$, hence $L_y \in J$. Therefore $J = L$. This completes the proof of this theorem. □

Note that $z \notin A_d$ if $|I| > 1$, and $z \in A_d$ if $|I| = 1$. The following corollary follows directly from the above theorem.

Corollary 3.2. *Suppose that $S_d = S_d(A, T, \varphi, z)$ is simple. Then $(S_d)_x \neq 0$ if and only if $x \in A_d \setminus \{z\}$.*

4. Derivations of $S_d(A, T, \varphi, z)$.

In this section we assume that the Lie algebra $S_d(A, T, \varphi, z)$ is simple, and we shall determine all the derivations of $S_d(A, T, \varphi, z)$. From the proof of Theorem 3.1 we know that:

(a) If $I = \emptyset$, $S_d(A, T, \varphi, z) = S(A, T, \varphi, z)$, which was thoroughly studied in [7];

(b) If $|I| = \dim T = n$, $S_d(A, T, \varphi, z) \simeq S_n^+$, which was studied in many references, for example [12];

(c) If $\dim T = 2$ and $|I| = 1$, $S_d(A, T, \varphi, z) \simeq \mathcal{L}_\beta(A, \alpha, \beta)$ for some suitable $\alpha, \beta \in \hom(A, F)$, which was thoroughly studied in [8].

So from now on we **always assume** that $0 < |I| < \dim T$ and $\dim T \geq 3$. Write $L = S_d(A, T, \varphi, z)$, $I = \{1, 2, \cdots, n\}$ and $S_d^+ = \sum_{x \in A_d^+} (S_d)_x$. Recall that L has an A-gradation with the following components of degree $x \in A$:

$$L_x = \begin{cases} 0, & \text{for} \quad x = z \\ (t^x T_{x-z}) \cap W_d, & \text{for} \quad x \in A \setminus \{z\}. \end{cases}$$

A derivation D of L is called *homogeneous of degree* $x \in A$ if $D(L_y) \subset L_{x+y}$ for all $y \in A$.

From direct computation we can easily obtain the following lemma.

Lemma 4.1. *Every $D \in \mathrm{Der}(L)$ has the form*

$$D = \sum_{y \in A} D_y, \tag{4.1}$$

where D_y is a derivation of L of degree y, such that for each $u \in L$ there are only finitely many $y \in A$ with $D_y(u) \neq 0$.

First we construct some derivations of S_d. For any additive function $\mu : A \to F$ the linear map

$$D_\mu(X) = \mu(x)X, \ \forall \ X \in (S_d)_x$$

is a derivation of S_d of degree 0.

The following Lemmas 4.2-4.5 will be useful in the sequel.

Lemma 4.2. *Let $x, y \in A \setminus \{0\}$. Then $T_x = T_y$ if and only if $\hat{x} \in F\hat{y}$.*

This lemma is obvious.

Lemma 4.3. *Let $x_1, x_2 \in A_d \setminus \{z\}$. If one of the following conditions holds:*
(a) *$x_2 \in A_d^+$ and $x_1 \in A_{d,i}$ for certain $i \in I$,*
(b) *$x_1, x_2 \in A_d^+$ and $T_{x_1-z} \neq T_{x_2-z}$,*
then

$$[L_{x_1}, L_{x_2}] = L_{x_1+x_2}. \tag{4.2}$$

Proof. Suppose (a) is satisfied. If $d_i(x_2) > 0$, then $d_i \in T_{x_1-z} \setminus (T_{x_2} \cup T_{x_2-z})$, hence $Fd_i + \langle d_i(x_2)\partial - \partial(x_1)d_i | \partial \in T_{x_2-z}\rangle = T$. We deduce that $\langle d_i(x_2)\partial - \partial(x_1)d_i | \partial \in T_{x_2-z}\rangle = T_{x_1+x_2-z}$. From

$$[t^{x_1}d_i, t^{x_2}\partial] = t^{x_1+x_2}(d_i(x_2)\partial - \partial(x_1)d_i),\ \forall\ \partial \in T_{x_2-z},$$

we see that (4.2) follows.

If $d_i(x_2) = 0$, then $x_1 + x_2 \in A_{d,i}$. If $x_1 + x_2 = z$ further, it follows from $(S_d)_z = 0$ that (4.2) follows. Suppose $x_1 + x_2 \neq z$, by Lemma 4.2 then $T_{x_2-z} \neq T_{x_1}$. Choose $\partial \in T_{x_2-z} \setminus T_{x_1}$, so $[t^{x_1}d_i, t^{x_2}\partial] = -\partial(x_1)t^{x_1+x_2}d_i$. Since $(S_d)_{x_1+x_2} = Ft^{x_1+x_2}d_i$, hence (4.2) follows again.

Suppose (b) is satisfied. We have

$$[t^{x_1}\partial_1, t^{x_2}\partial_2] = t^{x_1+x_2}(\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1) \tag{4.3}$$

for all $\partial_1 \in T_{x_1-z}$, $\partial_2 \in T_{x_2-z}$.

If $x_1 = 0$, it follows from $T_{x_1-z} \neq T_{x_2-z}$ that $\hat{x}_2 \notin F\hat{z}$. Then $T_{x_2} \neq T_z$. Choose $\partial_1 \in T_{x_2} \setminus T_z$. Then (4.2) follows from (4.3). Now we assume that $x_1, x_2 \in A_d^+ \setminus \{0\}$.

We claim that $T_{x_1-z} \neq T_{x_2}$ or $T_{x_1} \neq T_{x_2-z}$. Otherwise from $T_{x_1-z} = T_{x_2}$ and $T_{x_1} = T_{x_2-z}$, by Lemma 4.2 we obtain

$$d_1(x_2)(x_1 - z) = (d_1(x_1) + 1)x_2,\ d_1(x_1)(x_2 - z) = (d_1(x_2) + 1)x_1.$$

Then $x_1 = -d_1(x_1)z$ and $x_2 = -d_1(x_2)z$, it contradicts $T_{x_1-z} \neq T_{x_2-z}$. Hence our claim holds.

We assume that $T_{x_1-z} \neq T_{x_2}$. If $T_{x_1-z} \setminus T_{x_2} \subset T_{x_2-z}$, then we deduce that $T_{x_1-z} = T_{x_2}$, it is impossible. So $T_{x_1-z} \setminus (T_{x_2} \cup T_{x_2-z}) \neq \emptyset$. Choose $\partial_1 \in T_{x_1-z} \setminus (T_{x_2} \cup T_{x_2-z})$, then $F\partial_1 + \langle \partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1 | \partial_2 \in T_{x_2-z}\rangle = T$. Hence $\langle \partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1 | \partial_2 \in T_{x_2-z}\rangle = T_{x_1+x_2-z}$. From (4.3) it follows that (4.2) holds. □

Lemma 4.4. *For a fixed $i \in I$ and a fixed $x_0 \in A_d \setminus \{z\}$ with $d_i(x_0) = 1$, the subspace*

$$(S_d)_{x_0} + \sum_{x \in A_d,\ d_i(x) \leq 0} (S_d)_x$$

generates S_d as a Lie algebra.

Proof. Denote by M the subalgebra generated by the above subspace. We shall show that $(S_d)_x \subset M$ for $x \in A_d$ by induction on $k = d_i(x)$. By definition of M this is true for $x \in A_d$ with $d_i(x) < 1$.

Claim 1. *We can assume that $x_0 \in A_d^0$.*

If $d_j(x_0) = -1$ for one $j \in I \setminus \{i\}$, choose $u_j \in A_{d,j}^{\#}$, by Lemma 4.3 we deduce that $[(S_d)_{x_0}, (S_d)_{-u_j}] = (S_d)_{x_0-u_j} \subset M$. Note that $d_j(x_0 - u_j) = 0$. Then we may assume that $x_0 \in A_d^+$.

If $d_j(x_0) > 0$ for some $j \in I \setminus \{i\}$, also using $u_j \in A_{d,j}^{\#}$, by Lemma 4.3 we obtain that $[(S_d)_{x_0}, (S_d)_{u_j}] = (S_d)_{x_0+u_j} \subset M$. Note that $d_j(x_0 + u_j) = d_j(x_0) - 1$. Thus after finitely many such steps we may assume that $x_0 \in A_d^0$. This is our Claim 1.

Claim 2. *There exists $y_0 \in A_d^0 \setminus \{0\}$ such that $(S_d)_{x_0}, (S_d)_{x_0+y_0} \subset M$.*

Since $1 < |I| < \dim T$ we know that $A_d^0 \neq 0$. Choose $y' \in A_d^0 \setminus \{0\}$. If $T_{x_0-z} = T_{y'-z}$, then $x_0 - z = 2y' - 2z$, i.e., $x_0 - 2y' + z = 0$, we set $y_0 = 2y'$. If $T_{x_0-z} \neq T_{y'-z}$ we set $y_0 = y'$. Thus $y_0 \in A_d^0 \setminus \{0\}$ and $T_{x_0-z} \neq T_{y'-z}$. Since $(S_d)_{x_0}, (S_d)_{y_0} \subset M$ and $[(S_d)_{x_0}, (S_d)_{y_0}] = (S_d)_{x_0+y_0}$ (Lemma 4.3), then $(S_d)_{x_0}, (S_d)_{x_0+y_0} \subset M$. Claim 2 follows.

Suppose that $x \in A_d$ with $d_i(x) = 1$. Note that $T_{x_0-z} = T_{x-x_0-z}$ implies $x - x_0 - z = 2(x_0 - z)$, i.e., $x - 3x_0 - z = 0$, and that $T_{x_0+y_0-z} = T_{x-x_0-y_0-z}$ implies $x - 3x_0 - 3y_0 + z = 0$. Then $T_{x_0-z} \neq T_{x-x_0-z}$ or $T_{x_0+y_0-z} \neq T_{x-x_0-y_0-z}$. Without loss of generality we assume that $T_{x_0-z} \neq T_{x-x_0-z}$. By Lemma 4.3 we know that $[(S_d)_{x-x_0}, (S_d)_{x_0}] = (S_d)_x$. By noting that $(S_d)_{x-x_0}, (S_d)_{x_0} \subset M$, we get that $(S_d)_x \subset M$. So $(S_d)_x \subset M$ for $x \in A_d$ with $d_i(x) \leq 1$.

Suppose that $(S_d)_{x_0} \subset M$ for all $x_0 \in A_d$ with $d_i(x_0) = k$ for a fixed $k \geq 1$. Consider $x \in A_d$ with $d_i(x) = k+1$. Similar to the above argument we know that $T_{x_0-z} \neq T_{x-x_0-z}$ or $T_{x_0+y_0-z} \neq T_{x-x_0-y_0-z}$, say $T_{x_0-z} \neq T_{x-x_0-z}$, By Lemma 4.3 we obtain that $[(S_d)_{x-x_0}, (S_d)_{x_0}] = (S_d)_x$. By noting that $(S_d)_{x-x_0}, (S_d)_{x_0} \subset M$, we get that $(S_d)_x \subset M$.

By induction we obtain that $(S_d)_x \subset M$ for all $x \in A_d$. This completes the proof of Lemma 4.4. □

Lemma 4.5. (a) *Suppose that D_1, D_2 are derivations of a Lie algebra $\mathbf{g}$, and that $M \subset \mathbf{g}$ generates $\mathbf{g}$. If $D_1|_M = D_2|_M$, then $D_1 = D_2$.*

(b) *Suppose that $D_1, D_2 \in \mathrm{Der}(S_d)$ are homogeneous of degree $y \in A_d$. If $D_1(u) = D_2(u)$ for all $u \in (S_d)_x$ with $x \in A_d^+$, i.e., $D_1|_{S_d^+} = D_2|_{S_d^+}$, then $D_1 = D_2$.*

Proof. (a) is obvious.

(b) It suffices to show that $D_1(t^x d_1) = D_2(t^x d_1)$ for $x \in A_{d,1}$. If $x + y \notin A_d \backslash \{z\}$, we see that $D_1(t^x d_1) = D_2(t^x d_1) = 0$. Suppose that $x+y \in A_d \backslash \{z\}$, and that $D_1(t^x d_1) = t^{x+y}\partial \neq 0$ and $D_2(t^x d_1) = t^{x+y}\partial'$. If $\partial \neq \partial'$ we choose $x' \in A_d^+$ with $x' + x \in A_d^+$ such that $\partial(x') - \partial'(x') \neq 0$. For any $\partial_1 \in T_{x'-z}$, we have

$$[t^{x'}\partial_1, t^x d_1] = t^{x'+x}(\partial_1(x)d_1 - d_1(x')\partial_1).$$

Then $[t^{x'}\partial_1, t^{x+y}\partial] = [t^{x'}\partial_1, t^{x+y}\partial']$, i.e.,

$$\partial_1(x+y)\partial - \partial(x')\partial_1 = \partial_1(x+y)\partial' - \partial'(x')\partial_1,$$

so, we deduce that

$$\partial_1(x+y)(\partial - \partial') = (\partial(x') - \partial'(x'))\partial_1, \ \forall\ \partial_1 \in T_{x'-z}.$$

This is impossible since $\dim T_{x'-z} > 1$. Thus we get a contradiction. Consequently $\partial = \partial'$, i.e., $D_1(t^x d_1) = D_2(t^x d_1)$. □

Now we are ready to describe the homogeneous derivations D_y in (4.1).

Proposition 4.6. *If $y \notin A_d$, then every homogeneous derivation D of S_d of degree y is* 0.

Proof. We shall divide the proof into three cases.

Case 1. Suppose $d_i(y) \leq -3$ for some $i \in I$.

From Corollary 3.2 we see that $D((S_d)_x) = 0$ for all $x \in A_d$ with $d_i(x) \leq 1$. By Lemmas 4.4 and 4.5 it follow that $D = 0$.

Case 2. Suppose $d_1(y) = -2$ and $d_i(y) \geq -2$ for all $i \in I$.

Then $D((S_d)_x) = 0$ for $x \in A_d$ with $d_1(x) \leq 0$. If $|I| = 1$, since $(S_d)_z = 0$ then $D((S_d)_{z-y}) \subset (S_d)_z = 0$ and $d_1(z - y) = 1$. By Lemma 4.4 it follows that $D = 0$.

Suppose $|I| > 1$. Assume that $\{u_1, u_2, \cdots, u_n\} \subset A$ such that $d_i(u_j) = -\delta_{i,j}$. If $d_i(y) \leq 0$ for some $i \in I \setminus \{1\}$, it follows from $d_1(u_i - u_1 + y) = -1, d_i(u_i - u_1 + y) < 0$ that $D((S_d)_{u_i-u_1}) \subset (S_d)_{u_i-u_1+y} = 0$. By Lemmas 4.4 and 4.5, we also have $D = 0$.

Suppose $|I| > 1$ and $d_i(y) > 0$ for all $i \in I \setminus \{1\}$. Choose $v_1 \in A_1^{\#}, v_2 \in A_2^{\#}$. Let $D(t^{v_2-v_1}d_2) = \lambda t^{v_2-v_1+y}d_1$ for some $\lambda \in F$. From $[t^{v_2-v_1}d_2, d_1 - d_2] = -2t^{v_2-v_1}d_2$ and $D(d_1 - d_2) = 0$ we get that $\lambda[t^{v_2-v_1+y}d_1, d_1 - d_2] = -2\lambda t^{v_2-v_1+y}d_1$. Then $\lambda(d_1(y) - d_2(y)) = 0$. Since $d_1(y) - d_2(y) < -3$ we obtain that $\lambda = 0$, i.e., $D((S_d)_{v_i-v_1}) = 0$. By Lemmas 4.4 and 4.5, it follows that $D = 0$.

Case 3. Suppose that $|I| \geq 2$, that $d_i(y) \geq -1$ for all $i \in I$ and that $d_1(y) = d_2(y) = -1$. We first show that $D((S_d)_x) = 0$ for all $x \in A_d$ with $d_1(x) \leq 0$. If $x + y \notin A_d$ then $D((S_d)_x) \subset (S_d)_{x+y} = 0$. In particular, $D(T_z) = 0$. Suppose $x \in A_d$ with $d_1(x) = 0$ and $x + y \in A_d$. then $x + y \in A_1$ and $d_2(x) \geq 1, d_i(x) \geq 0, \forall\, i > 1$. If $y \neq z$ then $\hat{y} \notin F\hat{z}$. Choose $\partial_1 \in T_z \backslash T_y$.

Let $D(t^x\partial) = \lambda t^{x+y}d_1$ where $\lambda \in F$. Applying D to $[\partial_1, t^x\partial] = \partial_1(x)t^x\partial$, we obtain that $\lambda\partial_1(y) = 0$. thus $\lambda = 0$. Consequently $D((S_d)_x) = 0$ in this case.

Suppose that $y = z$. Recall that $x, x + y(= x + z) \in A_d$, $d_1(x) = 0$ and $d_i(x) > 0$ for all $i \in I \setminus \{1\}$. Choose $u_j \in A_j^{\#}$, $\forall\, j \in I$. For any $x_0 \in A_d^0 \setminus \{0\}$ we have

$$[t^{x_0}\partial, t^{-u_1}\partial_1] = -t^{x_0-u_1}(\partial(u_1)\partial_1 + \partial_1(x_0)\partial), \tag{4.4}$$

where $\partial \in T_{x_0-z}, \partial_1 \in T_{u_1+z}$. Let $D(t^{-u_1}\partial_1) = \lambda_{\partial_1}t^{z-u_1}d_2$, where $\lambda_{\partial_1} \in F$. Since $\hat{x}_0, \hat{z}, \hat{u}_1$ are linearly independent, we can choose $\partial \in T_{x_0-z} \cap T_{z-u_1} \setminus T_z$. Then $\langle \partial(u_1)\partial_1 + \partial_1(x_0)\partial | \partial_1 \in T_{u_1+z}\rangle = T_{x_0-u_1-z}$ since $\partial \notin T_{u_1+z}$ and $\langle \partial, \partial(u_1)\partial_1 + \partial_1(x_0)\partial | \partial_1 \in T_{u_1+z}\rangle = T$. Applying D to (4.4) we obtain that

$$\begin{aligned} -D(t^{x_0-u_1}(\partial(u_1)\partial_1 + \partial_1(x_0)\partial)) &= [t^{x_0}\partial, \lambda_{\partial_1}t^{z-u_1}d_2] \\ &= \lambda_{\partial_1}\partial(z-u_1)t^{z-u_1}d_2 = 0, \end{aligned}$$

for all $\partial_1 \in T_{u_1+z}$. Then $D((S_d)_{x_0-u_1}) = 0$. Similarly we can get $D((S_d)_{x_0-u_2}) = 0$. By $[(S_d)_{-x_0}, (S_d)_{x_0-u_2}] = (S_d)_{-u_2}$ we have $D((S_d)_{-u_2}) = 0$. By induction on $k = d_2(x) \geq 2$, and using $[(S_d)_{-u_2}, (S_d)_{x+u_2}] = (S_d)_x$, we can obtain that $D((S_d)_x) = 0$ for $x \in A_d$ with $d_1(x) = 0$.

Next we claim that $D((S_d)_{-u_1}) = 0$ for $u_1 \in A$.

If $y - u_1 \notin A_d$ we have $D((S_d)_{-u_1}) \subset (S_d)_{y-u_1} = 0$.

Suppose that $y - u_1 \in A_d$. Then $d_i(y) \geq 0$ for all $i \geq 3$. If $y = z$, from the above argument we know that $D((S_d)_{-u_1}) = 0$. Suppose also that $y \neq z$. Choose $\partial_1 \in T_z \setminus T_y$. Let $D(t^{-u_1}\partial) = \lambda_\partial t^{y-u_1}d_2$ for $\partial \in T_{z+u_1}$, where $\lambda_\partial \in F$. Then by $[\partial_1, t^{-u_1}\partial] = -\partial_1(u_1)t^{-u_1}\partial$ we obtain that $[\partial_1, \lambda_\partial t^{y-u_1}d_2] = -\lambda_\partial\partial_1(u_1)t^{-u_1}d_2$. Thus $\lambda_\partial\partial_1(y) = 0$. Since $\partial_1(y) \neq 0$ we infer that $\lambda_\partial = 0$. Consequently $D((S_d)_{-u_1}) = 0$ also. Therefore our claim is true. By Lemmas 4.4 and 4.5 we conclude that $D = 0$ in this case. Hence we have proved that $D = 0$ when $y \notin A_d$. □

Proposition 4.7. *Suppose that $y \in A_d \setminus \{0\}$, and that $D \in \mathrm{Der}(S_d)$ is homogeneous of degree y.*

(a) *If $y \neq z$, there exists $t^y\partial_0 \in (S_d)_y$ such that $D = \mathrm{ad}\,(t^y\partial_0)$.*
(b) *If $y = z$, we have $|I| = 1$ and $D \in F \cdot \mathrm{ad}\,(t^y d_1)$.*

Proof. For any $x \in A_d \setminus \{z\}$, we define the linear map $D_x : T_{x-z} \to T_{x+y-z}$ (or $D_x : Fd_i \to T_{x+y-z}$ if $x \in A_i$) by $D(t^x\partial) = t^{x+y}(D_x\partial)$. By applying D to

$$[t^{x_1}\partial_1, t^{x_2}\partial_2] = t^{x_1+x_2}(\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1)$$

where $x_1, x_2 \in A_d \backslash \{z\}$ and $\partial_1 \in T_{x_1-z}$, $\partial_2 \in T_{x_2-z}$, we obtain that

$$\begin{aligned} &\langle D_{x_1}\partial_1, x_2\rangle\partial_2 - \langle D_{x_2}\partial_2, x_1\rangle\partial_1 + \partial_1(x_2+y)D_{x_2}\partial_2 - \partial_2(x_1+y)D_{x_1}\partial_1 \\ &\quad = D_{x_1+x_2}(\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1) \end{aligned} \tag{4.5}$$

holds for $\partial_1 \in T_{x_1-z},\ \partial_2 \in T_{x_2-z}$, and $x_1, x_2 \in A_d\backslash\{z\}$.

Case 1. Suppose that $\hat{y}$ and $\hat{z}$ are linearly independent. By setting $x_2 = 0$ in (4.5) we obtain

$$\partial_1(y)D_0(\partial_2) = \partial_2(y)D_{x_1}(\partial_1) + \langle D_0(\partial_2), x_1\rangle\partial_1. \tag{4.6}$$

By setting here $x_2 = 0$ in (4.6) we obtain that

$$\partial_1(y)D_0(\partial_2) = \partial_2(y)D_0(\partial_1).$$

Choose $\partial_2 \in T_z \setminus T_y$, and denote $\partial_0 = \partial_2(y)^{-1}D_0(\partial_2)$. Then we have $\partial_0 \in T_{y-z}$ and

$$D_0(\partial_1) = -\partial_1(y)\partial_0,\ \forall \partial_1 \in T_z. \tag{4.7}$$

Hence we can rewrite (4.6) as

$$\partial_2(y)(D_{x_1}(\partial_1) - \partial_1(y)\partial_0 + \partial_0(x_1)\partial_1) = 0.$$

Thus we deduce that

$$D_{x_1}(\partial_1) = \partial_1(y)\partial_0 - \partial_0(x_1)\partial_1.$$

It follows that $D = -\text{ad}\,(t^y\partial_0)$. Note that by now we have not known that $t^y\partial_0 \in S_d$ yet. If $\partial_0 = 0$ or $y \in A_d^+$, from $\partial_0 \in T_{y-z}$ then $t^y\partial_0 \in S_d$. If $\partial_0 \neq 0$ and $y \in A_i$ for some $i \in I$, since $A_i \not\subset \ker(\partial_0)$, choose $x_0 \in A_i \setminus \ker(\partial_0)$. Since $D((S_d)_{x_0}) = 0$ and $D = \text{ad}\,(t^y\partial_0)$, we deduce that

$$[t^y\partial_0, t^{x_0}d_i] = t^{x_0+y}(\partial_0(x_0)d_i + \partial_0) = 0.$$

Thus $\partial_0 = -\partial_0(x_0)d_i$. Hence $t^y\partial_0 \in S_d$.

Case 2. Suppose that $y = z$. Then $|I| = 1$ and $z \in A_1^{\#}$. Since $D_0((S_d)_0) \subset (S_d)_z = 0$, we know that $D_0 = 0$.

Claim 1. *For $x_1 \in A_d \setminus \{z\}$ with $\hat{x}_1 \notin F\hat{z}$, there exists a constant $a_{x_1} \in F$ such that*

$$D_{x_1}\partial = a_{x_1}\partial,\ \forall\ \partial \in T_{x_1} \cap T_z. \tag{4.8}$$

If $x_1 \in A_d \setminus A_d^+$, clearly (4.8) is true. Next we suppose that $x_1 \in A_d^+$. By setting $\partial_1 = \partial_2 = \partial \in T_{x_1} \cap T_z$ and $x_2 = -z$ in (4.5), we obtain that

$$-\langle D_{x_1}\partial, z\rangle\partial - \langle D_{-z}\partial, x_1\rangle\partial = 0,$$

and so

$$\langle D_{-z}\partial, x_1\rangle = -\langle D_{x_1}\partial, z\rangle \tag{4.9}$$

holds. On the other hand, for $x_2 = -z,\ \partial_2 = \partial \in T_{x_1} \cap T_z$, and arbitrary $\partial_1 \in T_{x_1-z}$, (note that we allow $x_1 \in A_1$ here), (4.5) gives that

$$\langle D_{x_1}\partial_1, z\rangle\partial + \langle D_{-z}\partial, x_1\rangle\partial_1 = \partial_1(z)D_{x_1-z}\partial. \tag{4.10}$$

By evaluating both sides at x_1 and using $\partial_1(x_1) = \partial_1(z)$, we obtain that

$$\partial_1(z)[\langle D_{x_1-z}\partial, x_1\rangle - \langle D_{-z}\partial, x_1\rangle] = 0.$$

As $\hat{x}_1 \notin F\hat{z}$, we can choose $\partial_1 \in T_{x_1-z} \setminus T_z$, and so

$$\langle D_{x_1-z}\partial, x_1\rangle = \langle D_{-z}\partial, x_1\rangle, \ \forall\ \partial \in T_{x_1} \cap T_z. \tag{4.11}$$

By substituting $x_1 + z$ for x_1 in (4.9), and using $\langle D_{-z}\partial, z\rangle = 0$, we infer that

$$-\langle D_{x_1-z}\partial, x_1\rangle = \langle D_{-z}\partial, x_1\rangle$$

holds for $\partial \in T_{x_1} \cap T_z$. By comparing this equation with (4.11), we conclude that

$$\langle D_{-z}\partial, x_1\rangle = 0, \ \forall\ x_1 \in A_d,\ \partial \in T_{x_1} \cap T_z. \tag{4.12}$$

Now (4.10) gives that

$$\partial_1(z) D_{x_1-z}\partial = \langle D_{x_1}\partial_1, z\rangle\partial$$

for $\partial_1 \in T_{x_1-z}$, $\partial \in T_{x_1} \cap T_z$, $x_1 \in A_d$, with $\hat{x}_1 \notin F\hat{z}$. By choosing $\partial_1 \in T_{x_1-z} \setminus T_z$ and setting $a_{x_1-z} = \frac{\langle D_{x_1}\partial_1, z\rangle}{\partial_1(z)}$, then we have $D_{x_1-z}\partial = a_{x_1-z}\partial$, thus

$$D_{x_1}\partial = a_{x_1}\partial, \ \forall\ x_1 \in A_d, \partial \in T_{x_1} \cap T_z.$$

Hence our first claim is proved.

Claim 2. *If $x_1, x_2 \in A_d^+$ with $\hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \notin F\hat{z}$, then*

$$a_{x_1+x_2} = a_{x_1} + a_{x_2}. \tag{4.13}$$

In order to prove this claim we shall consider first the case where $\hat{x}_1, \hat{x}_2$, and $\hat{z}$ are linearly independent. Then we can choose $\partial_1 \in (T_{x_1} \cap T_z)\setminus T_{x_2}$ and $\partial_2 \in (T_{x_2} \cap T_z)\setminus T_{x_1}$. It follows that $\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1$ is a nonzero vector in $T_{x_1+x_2} \cap T_z$. By using the first claim, (4.5) gives that

$$(a_{x_1+x_2} - a_{x_1} - a_{x_2})[\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1] = 0, \tag{4.13$'$}$$

and so (4.13) holds in this case.

Now assume that $\hat{x}_1, \hat{x}_2$ and $\hat{z}$ are linearly dependent. Since $\dim T \geq 3$ we can choose $w \in A$ such that $\hat{x}_1, \hat{w}$ and $\hat{z}$ are linearly independent. By using the case already established, we have

$$a_{x_1} = a_{x_1+w} + a_{-w} = a_{x_1} + a_w + a_{-w},$$

and

$$a_{x_1+x_2} = a_{x_1+w} + a_{x_2-w} = a_{x_1} + a_w + a_{x_2} + a_{-w},$$

and conclude again that (4.13) holds. Hence our second claim is proved.

Now let $x \in A_d^+$ with $\hat{x} \in F\hat{z}$ and set $a_x = a_{x+v} - a_v$ where $v \in A_d^+$ with $\hat{v} \notin F\hat{z}$. By our second claim, $a_{x+v} - a_v$ is independent of the choice of v.

With a_x now defined for all $x \in A_d^+$, it is easy to see that (4.13) is valid for all $x_1, x_2 \in A_d^+$.

We shall now remove the restriction $\hat{x}_1 \notin F\hat{z}$ in (4.8). Thus assume that $\hat{x}_1 \in F\hat{z}$ and $x_1 \in A_d^+$. We choose $x_2 \in A_d^+$ so that $\hat{x}_2 \notin F\hat{z}$, and let $\partial_1 \in T_z$ and $\partial_2 \in T_{x_2} \cap T_z$. By using the first claim and $\partial_2(x_1) = \partial_1(y) = 0$, the equation (4.5) gives $\langle D_{x_1}\partial_1, x_2\rangle\partial_2 = a_{x_1}\partial_1(x_2)\partial_2$. Hence $\langle (D_{x_1} - a_{x_1})\partial_1, x_2\rangle = 0$ for all $x_2 \in A_d^+$ with $\hat{x}_2 \notin F\hat{z}$, and so (4.8) holds also for $\hat{x}_1 \in F\hat{z}$.

For $x \in A_d^+$, we define linear maps $D'_x : T_{x-z} \to T$ by $D'_x\partial = D_x\partial - a_x\partial$. By Claim 1, $T_x \cap T_z$ is contained in the kernel of D'_x. In particular, if $\hat{x} \in F\hat{z}$ and $x \neq z$, then $D'_x = 0$. If $\hat{x} \notin F\hat{z}$, then $T_x \cap T_z$ is a hyperplane in T_{x-z}, and so the vector

$$\partial_x := \frac{D'_x\partial}{\partial(y)} \tag{4.14}$$

is independent of the choice of $\partial \in T_{x-z} \setminus T_z$ (note that $y = z$). Thus we have

$$D'_x\partial = \partial(y)\partial_x, \ \forall\ x \in A_d^+, \partial \in T_{x-z}. \tag{4.15}$$

If $\hat{x} \in F\hat{z}$, then ∂_x is not defined but (4.15) is also valid because $D'_x = 0$ and $\partial(y) = 0$ for $\partial \in T_{x-z} = T_z$.

By substituting $D'_{x_1} + a_{x_1}$ for D_{x_1} and making similar substitutions for D_{x_2} and $D_{x_1+x_2}$ in (4.5), we obtain that

$$\begin{aligned}
&\langle D'_{x_1}\partial_1, x_2\rangle\partial_2 - \langle D'_{x_2}\partial_2, x_1\rangle\partial_1 + \partial_1(x_2 + y)D'_{x_2}\partial_2 \\
&\quad - \partial_2(x_1 + y)D'_{x_1}\partial_1 + a_{x_2}\partial_1(x)\partial_2 - a_{x_1}\partial_2(x)\partial_1 \\
&= D'_{x_1+x_2}(\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1)
\end{aligned}$$

holds for $x_1, x_2 \in A_d^+$, $\partial_1 \in T_{x_1-z}$ and $\partial_2 \in T_{x_2-z}$.

By using (4.15) and similar expressions for D'_{x_2} and $D'_{x_1+x_2}$, the last equation can be rewritten as follows

$$\begin{aligned}
&\partial_1(y)[a_{x_2} + \partial_{x_1}(x_2)]\partial_2 - \partial_2(y)[a_{x_1} + \partial_{x_2}(x_1)]\partial_1 \\
&= [\partial_1(x_2)\partial_2(y) - \partial_2(x_1)\partial_1(y)]\partial_{x_1+x_2} \\
&\quad + \partial_1(y)\partial_2(x_1 + y)\partial_{x_1} - \partial_2(y)\partial_1(x_2 + y)\partial_{x_2}.
\end{aligned} \tag{4.16}$$

Claim 3. *The vector ∂_x for $x \in A_d^+$ with $\hat{x} \notin F\hat{z}$ are independent of x.*

Suppose $x_1, x_2 \in A_d^+$ with $\hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \notin F\hat{z}$. For $\partial_1 \in T_{x_1} \cap T_z \setminus \{0\}$ and $\partial_2 \in T_{x_2-z} \setminus T_z$, (4.16) gives

$$\partial_1(x_2)(\partial_{x_2} - \partial_{x_1+x_2}) = [a_{x_1} + \partial_{x_2}(x_1)]\partial_1. \tag{4.17}$$

If $\hat{x}_1, \hat{x}_2, \hat{z}$ are linearly dependent, we see that $\partial_1(x_2) = 0$. Thus we obtain from (4.17) that $a_{x_1} = -\partial_{x_2}(x_1)$.

Assume that $\hat{x}_1, \hat{x}_2, \hat{z}$ are linearly independent. Then $\partial_1 \in T_{x_1} \cap T_z$ can be chosen so that $\partial_1(x_2) \neq 0$. By evaluating both sides of (4.17) at x_1, we obtain that $\partial_1(x_2)(\partial_{x_2}(x_1) - \partial_{x_1+x_2}(x_1)) = 0$. Since $\partial_1(x_2) \neq 0$ we deduce that

$$\partial_{x_2}(x_1) = \partial_{x_1+x_2}(x_1),\ \forall\ x_1, x_2 \in A_d^+,\ \text{with } \hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \notin F\hat{z}. \tag{4.18}$$

Symmetrically we have

$$\partial_{x_1}(x_2) = \partial_{x_1+x_2}(x_2),\ \forall\ x_1, x_2 \in A_d^+,\ \text{with } \hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \notin F\hat{z}.$$

By evaluating (4.17) at x_2 we get

$$a_{x_1} = \partial_{x_2}(x_2 - x_1) - \partial_{x_1+x_2}(x_2),$$

i.e.,

$$a_{x_1} = \partial_{x_2}(x_2 - x_1) - \partial_{x_1}(x_2). \tag{4.19}$$

By evaluating (4.17) at z, we find

$$\partial_{x_1+x_2}(z) = \partial_{x_2}(z) = \partial_{x_1}(z).$$

By evaluating (4.16) at z, we find that

$$\partial_1(z)\partial_2(z)[\partial_{x_1}(x_2) - \partial_{x_2}(x_1) + a_{x_2} - a_{x_1}] = 0.$$

Since we can choose $\partial_1 \in T_{x_1 - z}$ and $\partial_2 \in T_{x_2 - z}$ such that $\partial_1(z)\partial_2(z) \neq 0$, we infer that

$$a_{x_1} - a_{x_2} = \partial_{x_1}(x_2) - \partial_{x_2}(x_1).$$

By replacing x_1 with $x_1 + x_2$ and using Claim 2, we obtain the equation

$$a_{x_1} = \partial_{x_1+x_2}(x_2) - \partial_{x_2}(x_1 + x_2) = \partial_{x_1}(x_2) - \partial_{x_2}(x_1 + x_2).$$

Combining this with (4.19), we deduce that $\partial_{x_1}(x_2) = \partial_{x_2}(x_2)$. Considering (4.19) also we have $a_{x_1} = -\partial_{x_2}(x_1)$. So $\partial_{x_1}(x_2)$ is independent of x_1, consequently ∂_{x_1} is independent of x_1. Then Claim 3 is proved.

Denote $-\partial_x$ ($x \in A_d^+$ with $\hat{x} \notin F\hat{z}$) by ∂_0. So we have

$$a_x = \partial_0(x),\ \forall\ x \in A_d^+,\ \text{with } \hat{x} \notin F\hat{z}.$$

By the definition of a_{x_1} for $x \in A_d^+$ with $\hat{x} \notin F\hat{z}$, we also have

$$a_x = \partial_0(x),\ \forall\ x \in A_d^+. \tag{4.20}$$

Combining this with (4.15), we deduce that

$$D_x\partial = -\partial(y)\partial_0 + \partial_0(x)\partial,\ \forall\ x \in A_d^+, \partial \in T_{x-z}. \tag{4.21}$$

So $D|_{S_d^+} = \text{ad}\ (t^y\partial_0)|_{S_d^+}$. By Lemma 4.5 we see that $D = \text{ad}\ (t^y\partial_0)$.

Choose $x \in A_d^0\backslash\{0\}$, and $\partial \in T_{x-z}\backslash T_z$. We see that $D((S_d)_x) \subset (S_d)_{x+z} = Ft^{x+z}d_1$, i.e., $D_x\partial = \lambda_\partial d_1$ for some $\lambda_\partial \in F$. Combining this with (4.21), we infer that

$$\lambda_\partial d_1 = -\partial(y)\partial_0 + \partial_0(x)\partial.$$

Since $d_1 \notin T_{x-z}$ and $\dim T_{x-z} \geq 2$ we conclude from the above equation that $\partial_0(x) = 0$. Furthermore we deduce that $\partial_0 = \lambda_\partial \partial(y)^{-1}d_1$, i.e., $\partial_0 \in Fd_1$.

Case 3. Suppose that $y = -\lambda z$, where $\lambda \in \mathbf{N} = \{1, 2, 3, \dots\}$.

By setting $x_2 = 0$ in (4.5) we obtain

$$\partial_1(y)D_0(\partial_2) = \langle D_0(\partial_2), x_1\rangle\partial_1,$$

for $\partial_1 \in T_{x_1-z}, \partial_2 \in T_z$. Since $\dim T_{x_1-z} \geq 2$ we deduce that $\langle D_0(\partial_2), x_1\rangle = 0$ for all $x \in A_d$. It follows that $D_0(\partial_2) = 0$ for $\partial_2 \in T_z$, i.e., $D_0 = 0$.

Now we show that Claim 1 is also true in this case.

First suppose that $x_1 \in A_d^+$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$. Since $\hat{x} \notin F\hat{z}$ and $-x_1 - z \in A_d^+$, we can choose $\partial_2 \in T_{-x_1-z}\backslash T_z$. By setting $x_2 = -x_1$ and $\partial_1 = \partial \in T_{x_1} \cap T_z$ in (4.5) and using $D_0 = 0$ we obtain that

$$\partial_2(x_1 + y)D_{x_1}(\partial) + \langle D_{x_1}(\partial), x_1\rangle\partial_2 + \langle D_{-x_1}(\partial_2), x_1\rangle\partial = 0.$$

By evaluating the above equation at $x_1 + y - z$ and by using $\langle D_{x_1}(\partial), x_1 + y - z\rangle = 0$, $\partial(x_1) = \partial(y) = \partial(z) = 0$, we conclude that $\langle D_{x_1}(\partial), x_1\rangle = 0$, and consequently

$$\partial_2(x_1 + y)D_{x_1}(\partial) = -\langle D_{-x_1}(\partial_2), x_1\rangle\partial,$$

holds for all $\partial \in T_{x_1} \cap T_z$ and $\partial_2 \in T_{-x_1-z}$. Since $\partial_2(x_1+y) = -(\lambda+1)\partial_2(z) \neq 0$, then (4.8) holds for $x_1 \in A_d^+$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$.

We shall show (4.8) for $x_1 \in A_d^+$ by induction on $d(x_1) := \sum_{i\in I} d_i(x_1)$.

This has been proved for all x_1 with $d(x_1) \leq 1$. Now suppose (4.8) holds for all x_1 with $d(x_1) \leq k$ (≥ 1). Consider $x_2 \in A_d^+$ with $d(x_2) = k + 1$. Choose $x_0 \in A_d^+$ with $d(x_0) = 1$ and with $\hat{x}_0, \hat{x}_2, \hat{z}$ being linearly independent. By inductive hypothesis we have

$$D_{x_0}\partial' = a_{x_0}\partial', \ \forall\ \partial' \in T_{x_0} \cap T_z,$$

$$D_{x_2-x_0}\partial = a_{x_2-x_0}\partial, \ \forall\ \partial \in T_{x_2-x_0} \cap T_z.$$

By replacing x_1 with x_0, x_2 with $x_2 - x_0$, ∂_1 with ∂_0', ∂_2 with ∂ respectively in (4.5), we obtain that

$$D_{x_2}(\partial_0'(x_2)\partial - \partial(x_0)\partial_0') = (a_{x_0} + a_{x_2-x_0})(\partial_0'(x_2)\partial - \partial(x_0)\partial_0') \tag{4.22}$$

for all $\partial_0' \in T_{x_0} \cap T_z$, $\partial \in T_{x_2-x_0} \cap T_z$. It suffices to show that

$$\langle \partial_0'(x_2)\partial - \partial(x_0)\partial_0' | \partial_0' \in T_{x_0} \cap T_z, \partial \in T_{x_2-x_0} \cap T_z\rangle = T_{x_2} \cap T_z.$$

Choose $\partial_0' \in (T_{x_0} \cap T_z)\backslash T_{x_2}$, then

$$F\partial_0' + \langle \partial_0'(x_2)\partial - \partial(x_0)\partial_0' | \partial \in T_{x_2-x_0} \cap T_z\rangle = F\partial_0' + T_{x_2-x_0} \cap T_z$$

is codimension 1 in T. Hence the subspace

$$\langle \partial_0'(x_2)\partial - \partial(x_0)\partial_0' | \partial \in T_{x_2-x_0} \cap T_z\rangle \subset T_{x_2} \cap T_z$$

is codimension 2 in T. Therefore (4.22) holds. Thus there exists $a_{x_2} \in F$ such that $D_{x_2}\partial = a_{x_2}\partial$ for all $\partial \in T_{x_2} \cap T_z$. Consequently Claim 1 is true.

Exactly the same as that in Case 2, Claim 2 is true in this case also.

Now same as in Case 2, we define a_x for $x \in A_d^+$ with $\hat{x} \in F\hat{z}$ and we can remove the restriction $\hat{x} \notin F\hat{z}$ in (4.8), then we can also define the linear map D_x', the vector ∂_x in (4.14) for $x \in A_d^+$ with $\hat{x} \notin F\hat{z}$, and same as before we also get equations (4.15), (4.16).

Now we claim that Claim 3 is true in this case also. The proof is exactly the same as it was in Case 2.

We also denote $-\partial_x$ ($x \in A_d^+$ with $\hat{x} \notin F\hat{z}$) by ∂_0. So we have

$$a_x = \partial_0(x), \ \forall\, x \in A_d^+, \text{ with } \hat{x} \notin F\hat{z}.$$

The same as in Case 2, Equations (4.20) and (4.21) are true. So $D|_{S_d^+} = \text{ad}\,(t^y\partial_0)|_{S_d^+}$. By Lemma 4.5 we see that $D = \text{ad}\,(t^y\partial_0)$. By definition of ∂_0 we know that $t^y\partial_0 \in (S_d)_y$.

By now we have completed the proof of Proposition 4.7. □

Proposition 4.8. *Suppose that $D \in \text{Der}(S_d)$ is homogeneous of degree 0. Then there exists a $\nu \in \text{hom}(A, F)$ such that $D = D_\nu$.*

Proof. For any $x \in A_d^+$, we define the linear map $D_x : T_{x-z} \to T_{x-z}$ (or $D_x : Fd_i \to Fd_i$ if $x \in A_i$) by $D(t^x\partial) = t^{x+y}(D_x\partial)$. If $\partial \in T_{x_1-z}$, then $\partial(x_1) = \partial(z)$ and $\langle D_{x_1}\partial, x_1\rangle = \langle D_{x_1}\partial, z\rangle$. As $x = 0$, the equation (4.5) takes the form

$$\begin{aligned}(4.23)\qquad &\langle D_{x_1}\partial_1, x_2\rangle\partial_2 - \langle D_{x_2}\partial_2, x_1\rangle\partial_1 + \partial_1(x_2)D_{x_2}\partial_2 - \partial_2(x_1)D_{x_1}\partial_1 \\ &= D_{x_1+x_2}(\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1)\end{aligned}$$

where $\partial_1 \in T_{x_1-z}$, $\partial_2 \in T_{x_2-z}$, and $x_1, x_2 \in A_d^+$. By setting $v = 0$ in (4.23), we obtain that $\langle D_0\partial_2, x_1\rangle\partial_1 = 0$. Hence $D_0 = 0$.

We claim that (4.8) holds for $x_1 \in A_d\backslash\{z\}$ and $\partial \in T_{x_1} \cap T_z$.
If $x_1 \in A_d\backslash A_d^+$, (4.8) holds clearly. Next suppose that $x \in A_d^+$.

First suppose that $x_1 \in A_d^+$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$. Since $\hat{x} \notin F\hat{z}$ and $-x_1 - z \in A_d^+$, we can choose $\partial_2 \in T_{-x_1-z}\backslash T_z$. By setting $x_2 = -x_1$ and $\partial_1 = \partial \in T_{x_1} \cap T_z$ in (4.23) and using $D_0 = 0$, we obtain that

$$\partial_2(x_1)D_{x_1}(\partial) + \langle D_{x_1}(\partial), x_1\rangle\partial_2 + \langle D_{-x_1}(\partial_2), x_1\rangle\partial = 0.$$

By evaluating the above equation at x_1-z and by using $\langle D_{x_1}(\partial), x_1-z\rangle = 0$, $\partial(x_1) = \partial(z) = 0$, we conclude that $\langle D_{x_1}(\partial), x_1\rangle = 0$, and consequently

$$\partial_2(x_1)D_{x_1}(\partial) = -\langle D_{-x_1}(\partial_2), x_1\rangle\partial,$$

holds for all $\partial \in T_{x_1} \cap T_z$ and $\partial_2 \in T_{-x_1-z}$. Since $\partial_2(x_1) = -\partial_2(z) \neq 0$, then (4.8) holds for $x_1 \in A_d^+$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$.

We shall show (4.8) for $x_1 \in A_d^+$ by induction on $d(x_1) := \sum_{i\in I} d_i(x_1)$.

This has been proved for all x_1 with $d(x_1) \leq 1$. Now suppose (4.8) holds for all x_1 with $d(x_1) \leq k$ (≥ 1). Consider $x_2 \in A_d^+$ with $d(x_2) = k+1$. Choose $x_0 \in A_d^+$ with $d(x_0) = 1$ and with $\hat{x}_0, \hat{x}_2, \hat{z}$ being linearly independent. By inductive hypothesis we have

$$D_{x_0}\partial' = a_{x_0}\partial', \ \forall\ \partial' \in T_{x_0} \cap T_z,$$

$$D_{x_2-x_0}\partial = a_{x_2-x_0}\partial, \ \forall\ \partial \in T_{x_2-x_0} \cap T_z.$$

By replacing x_1 with x_0, x_2 with $x_2 - x_0$, ∂_1 with ∂_0', ∂_2 with ∂ respectively in (4.23), we obtain that

$$D_{x_2}(\partial_0'(x_2)\partial - \partial(x_0)\partial_0') = (a_{x_0} + a_{x_2-x_0})(\partial_0'(x_2)\partial - \partial(x_0)\partial_0')$$

for all $\partial_0' \in T_{x_0} \cap T_z$, $\partial \in T_{x_2-x_0} \cap T_z$. It suffices to show that

$$\langle \partial_0'(x_2)\partial - \partial(x_0)\partial_0' | \partial_0' \in T_{x_0} \cap T_z, \partial \in T_{x_2-x_0} \cap T_z\rangle = T_{x_2} \cap T_z.$$

Choose $\partial_0' \in (T_{x_0} \cap T_z)\backslash T_{x_2}$, then

$$F\partial_0' + \langle \partial_0'(x_2)\partial - \partial(x_0)\partial_0' | \partial \in T_{x_2-x_0} \cap T_z\rangle = F\partial_0' + T_{x_2} \cap T_z$$

is codimension 1 in T. Hence the subspace

$$\langle \partial_0'(x_2)\partial - \partial(x_0)\partial_0' | \partial \in T_{x_2-x_0} \cap T_z\rangle \subset T_{x_2} \cap T_z$$

is codimension 2 in T. Therefore (4.23) holds. Thus there exists $a_{x_2} \in F$ such that $D_{x_2}\partial = a_{x_2}\partial$ for all $\partial \in T_{x_2} \cap T_z$. Consequently our claim about (4.8) is true.

Exactly the same as that in Case 2 in the proof of Proposition 4.7, Claim 2 in the proof of Proposition 4.7 is true in this case also.

Now as in Case 2 of the proof of Proposition 4.7, we define a_x for $x \in A_d\backslash\{z\}$ with $\hat{x} \in F\hat{z}$ and we can remove the restriction $\hat{x} \notin F\hat{z}$ in (4.8). Then we have obtained that:

(a) For any $x \in A_d \setminus \{z\}$, there exists a constant $a_x \in F$ such that

$$D_x\partial = a_x\partial, \ \forall\ \partial \in T_x \cap T_z. \tag{4.8$'$}$$

(b) For all $x_1, x_2 \in A_d^+$,

$$a_{x_1+x_2} = a_{x_1} + a_{x_2}.$$

Now we claim that, for any $x \in A_d \setminus \{z\}$,

$$D_x\partial = a_x\partial, \ \forall\ \partial \in T_{x-z}. \tag{4.24}$$

If $\hat{x} \in F\hat{z}$, this follows from (4.8′) since $T_x \cap T_z = T_x \cap T_z = T_z$. If $x \in A_d \setminus (A_d^+ \cup \{z\})$, (4.24) is clear. Next we suppose $x \in A_d^+$ with $\hat{x} \notin F\hat{z}$. We shall show this by induction on $d(x) := \sum_{i \in I} d_i(x)$.

If $d(x) = 0$, by replacing D with $D + D_\mu$ for a suitable $\mu \in \hom(A, F)$, we may assume that $a_x \neq 0$. For $\partial \in T_{x-z} \backslash T_z$ let $D(t^x \partial) = a_x t^x \partial'$. For any $i \in I$, $u_i \in A_i$, appling D to $[t^x \partial, t^{u_i} d_i] = t^{x+u_i} \partial(u_i) d_i$, we obtain that

$$[a_x t^x \partial', t^{u_i} d_i] + [t^x \partial, a_{u_i} t^{u_i} d_i] = (a_x + a_{u_i}) t^{x+u_i} \partial(u_i) d_i.$$

We deduce that $a_x(\partial - \partial')(u_i) d_i = 0$. Thus $\partial(u_i) = \partial'(u_i)$ for all $u_i \in A_i$ and any $i \in I$. So we obtain that $\partial = \partial'$.

Suppose (4.24) holds for any $x \in A_d \setminus \{z\}$ with $d(x) \leq k$ where $k \geq 0$. Consider $x_1 \in A_d^+ \backslash (\mathbf{Z} z)$ with $d(x_1) = k + 1$. Suppose $t^{x_1} \partial \in (S_d)_{x_1}$ with $\partial \in T_{x-z} \backslash T_z$. For any $i \in I$ and any $u_i \in A_i$, we know that $D|_{(S_d)_{x_1+u_i}} = a_{x_1+u_i}|_{(S_d)_{x_1+u_i}}$. By replacing D with $D + D_\mu$ for a suitable $\mu \in \hom(A, F)$, we may assume that $a_{x_1} \neq 0$. Let $D(t^{x_1} \partial) = a_{x_1} t^{x_1} \partial'$. Appling D to $[t^{x_1} \partial, t^{u_i} d_i] = t^{x_1+u_i} (\partial(u_i) d_i - d_i(x_1) d_i)$, we obtain that

$$a_{x_1} [t^{x_1} \partial', t^{u_i} d_i] + [t^{x_1} \partial, a_{u_i} t^{u_i} d_i] = (a_{x_1} + a_{u_i}) t^{x_1+u_i} (\partial(u_i) d_i - d_i(x_1) d_i).$$

Then we infer that $d_i(x_1)(\partial - \partial') = (\partial - \partial')(u_i) d_i$. If $(\partial - \partial')(u_i) \neq 0$, we deduce that $d_i \in T_{x_1 - z}$. This contradicts the fact that $x_1 \in A_d^+$. So we deduce that $\partial(u_i) = \partial'(u_i)$ for all $u_i \in A_i$, any $i \in I$. Thus $\partial = \partial'$.

By induction we see that (4.24) is true. Define $\nu \in \hom(A, F)$ so that $\nu(x) = a_x$ for all$x \in A_d^+$. Then we see that $D|_{S_d^+} = \mathrm{ad}\ (t^y \partial_0)|_{S_d^+}$. By Lemma 4.5 we conclude that $D = D_\nu$. □

We now summarize the results on derivations of $S_d(A, T, \varphi, z)$ obtained in this section.

Theorem 4.9. *Every $D \in \mathrm{Der}(S_d(A, T, \varphi, z))$ has the form $D = \sum_{y \in A} D_y$ for degree y derivations D_y, such that for each $u \in S_d(A, T, \varphi, z)$ there only finitely many $y \in A$ with $D_y(u) \neq 0$, where*

(a) $D_y = \mathrm{ad}\ (t^y \partial_0)$ *for some* $t^y \partial_0 \in (S_d)_y$ *if* $y \in A_d \setminus \{0, z\}$*;*
(b) $D_y = a\mathrm{ad}\ (t^y d_1)$ *for some* $a \in F$ *if* $y = z \in A_d$*;*
(c) $D_y = D_\nu$ *for some* $\nu \in \hom(A, F)$ *if* $y = 0$*.*

As in [**3**, Proposition 3.3], we also have that the sum $D = \sum_{y \in A} D_y$ in the above Theorem is finite if $\dim T < \infty$.

References

[1] A.A. Albert and M.S. Frank, *Simple Lie algebras of characteristic p*, Univ. e Politec. Torino Rend. Sem. Mat., **14** (1954-55), 117-139.

[2] R. Block, *On torsion-free abelian groups and Lie algebras*, Proc. Amer. Math. Soc., **9** (1958), 613-620.

[3] D. Ž. Đoković and K. Zhao, *Derivations, isomorphisms, and second cohomology of generalized Witt algebras*, Trans. Amer. Math. Soc., **350**(**2**) (1998), 643-664.

[4] ———, *Derivations, isomorphisms, and second cohomology of generalized Block algebras*, Algebra Colloquium, **3** (1996), 245-272.

[5] ———, *Generalized Cartan type W Lie algebras in characteristic* 0, J. Alg., **195** (1997), 170-210.

[6] ———, *Some infinite dimensional simple Lie algebras in characteristic* 0 *related to those of Block*, J. Pure and Applied Algebras, **127**(**2**) (1998), 153-165.

[7] ———, *Generalized Cartan type S Lie algebras in characteristic* 0, J. Alg., **193** (1997), 144-179.

[8] ———, *Some infinite dimensional simple subalgebras of generalized Block algebras*, J. Alg., **192** (1997), 74-101.

[9] ———, *Second cohomology of generalized Cartan type S Lie algebras in characteristic zero*, J. Pure and Applied Algebras, **136** (1999), 101-126.

[10] I. Kaplansky, *Seminar on simple Lie algebras*, Bull. Amer. Math. Soc., **60** (1954), 470-471.

[11] N. Kawamoto, *Generalizations of Witt algebras over a field of characteristic zero*, Hiroshima Math. J., **16** (1986), 417-426.

[12] J.M. Osborn, *New simple infinite dimensional Lie algebras of characteristic* 0, J. Alg., **185** (1996), 820-835.

[13] ———, *Derivations and isomorphisms of Lie algebras of characteristic* 0, Studies in Advanced Mathematics, **4** (1997), 95-108.

[14] A.N. Rudakov, *Group of automorphisms of infinite-dimensional simple Lie algebras*, Math. USSR-Izvestija, **3**(**4**) (1965), 707-722.

[15] I.M. Singer and S. Sternberg, *The infinite groups of Lie and Cartan,* I: *The transitive groups*, J. Analyse Math., **15** (1965), 1-114.

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