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A Paley–Wiener theorem for the inverse spherical transform is proved for noncompact semisimple Lie groups G which are either of rank one or with a complex structure. Let K be a fixed maximal compact subgroup of G. For each K-bi-invariant function f in the Schwartz space on G, consider the function $\tilde{f}$ defined on a fixed Weyl chamber $\mathfrak{a}^+$ by $\tilde{f}(H) := \Delta(H)\, f(\exp H)$. Here $\Delta(H) := \prod_{\alpha\in\Sigma^+} (\sinh \alpha(H))^{m_\alpha/2}$, where Σ^+ is the set of positive restricted roots and m_α is the multiplicity of the root α. The K-bi-invariant functions f whose spherical transform has compact support are identified as those for which $\tilde{f}$ extends holomorphically and with a specific growth to a certain subset of the complexification $\mathfrak{a}_c$ of $\mathfrak{a}$. The proof of the theorem in the rank-one case relies on the explicit inversion formula for the Abel transform.

Introduction.

The classical Fourier transform is an isomorphism $\mathcal{F}$ of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself. The space $\mathcal{D}(\mathbb{R}^n)$ of compactly supported C^∞ functions on $\mathbb{R}^n$ is dense in $\mathcal{S}(\mathbb{R}^n)$, and the classical Paley–Wiener theorem characterizes its image under $\mathcal{F}$: a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the image under $\mathcal{F}$ of a C^∞ function with support in the Euclidean ball $\{x \in \mathbb{R}^n : |x| \leq r\}$ if and only if it extends to $\mathbb{C}^n$ as an entire function of exponential type r and rapidly decreasing. This is to say that given any integer $N \geq 0$ there exists a constant $\sigma_N > 0$ so that for all $z \in \mathbb{C}^n$

$$|f(z)| \leq \sigma_N (1 + |z|)^{-N} e^{r|\Im z|}.$$

Since $\mathbb{R}^n$ is self-dual, the same theorem also applies to the inverse Fourier transform. So the functions in $\mathcal{S}(\mathbb{R}^n)$ whose image under $\mathcal{F}$ is supported in $\{x \in \mathbb{R}^n : |x| \leq r\}$ are exactly those extending as entire functions on $\mathbb{C}^n$ of exponential type r and rapidly decreasing.

Let G be a noncompact semisimple Lie group with a maximal compact subgroup K. We refer to Section 1 for the notation and the basic definitions. The spherical transform $\mathcal{S}$ is the analogue of the Fourier transform for K-bi-invariant functions on G. Generalizing the notion of rapid decrease used to define $\mathcal{S}(\mathbb{R}^n)$, Harish-Chandra defined the Schwartz space $\mathcal{S}(K\backslash G/K)$ for the K-bi-invariant functions on G. It contains the set $\mathcal{D}(K\backslash G/K)$ of the K-bi-invariant compactly supported C^∞ functions on G as a dense subspace. The spherical transform is an isomorphism of $\mathcal{S}(K\backslash G/K)$ onto the subspace $\mathcal{S}_W(\mathfrak{a}^*)$ of the Weyl group invariants in the Schwartz space over $\mathfrak{a}^*$. In this setting, a Paley–Wiener theorem for the spherical transform has been proved by Helgason [**Hel66**] for G of rank one or with a complex structure. The proof for G arbitrary has been completed by Gangolli [**Gan71**]. As in the classical case, the Helgason–Gangolli Paley–Wiener theorem characterizes the image under $\mathcal{S}$ of the elements of $\mathcal{D}(K\backslash G/K)$ as those functions in $\mathcal{S}_W(\mathfrak{a}^*)$ having an entire extension of exponential type and rapidly decreasing, and the rate of growth is determined by the size of the support. But, unlike the classical case, a Paley–Wiener theorem for the inverse spherical transform cannot be deduced from that of the spherical transform. The following question is therefore quite natural: What are the functions in $\mathcal{S}(K\backslash G/K)$ whose spherical transform has compact support?

This paper provides the answer when G is either of rank one or with a complex structure. The characterization is given in terms of holomorphic extendibility and growth conditions, and the rate of growth is determined by the size of the support of the image. The precise statement is given in Section 2. In the rank-one case the proof of the theorem relies on the explicit formulas for the Abel transform and its inverse as given by Rouvière [**Rou83**]. In the complex case, the theorem is an easy consequence of the explicit expression of the (elementary) spherical functions.

After the fundamental works of Helgason and Gangolli, a number of authors have proved Paley–Wiener type theorems for the spherical or related transforms, e.g. [**Koo75**] and [**Bra96**] in the rank-one case. Not only a Paley–Wiener theorem for the inverse spherical transform has not been considered so far, but also the various estimates required in its proof for the rank-one case are different from those considered in other Paley–Wiener type theorems. The main difficulties are in the proof of the sufficiency of the stated condition, where we also need a detailed analysis of the holomorphic extendibility across given vertical segments of the complex space.

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1. Notation and preliminaries.

In the following, G denotes a connected noncompact real semisimple Lie group with finite center, and K denotes a fixed maximal compact subgroup of G. $\mathfrak{g}$ and $\mathfrak{k}$ ($\subset \mathfrak{g}$) are the Lie algebras of G and K, respectively. $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. The dimension of any maximal abelian subspace of $\mathfrak{p}$ is a constant, called the (real) rank of G. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. $\mathfrak{a}^*$ denotes the (real) dual space of $\mathfrak{a}$. $\mathfrak{g}_c$ is the complexification of $\mathfrak{g}$ and $\mathfrak{a}_c$ is the complexification of $\mathfrak{a}$ in $\mathfrak{g}_c$.

The set of the restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is indicated by Σ. It consists of all $\alpha \in \mathfrak{a}^*$ for which the vector space $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X$ for every $H \in \mathfrak{a}\}$ contains nonzero elements. $m_\alpha := \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ is the multiplicity of the restricted root α. Σ^+ denotes the set of the positive restricted roots corresponding to a choice of a Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$.

The restriction of the exponential map of G to $\mathfrak{a}$ is an analytic diffeomorphism onto the abelian subgroup $A := \exp \mathfrak{a}$. The inverse diffeomorphism is denoted by log. The action on $\mathfrak{a}$ of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{k})$ induces actions of W on $\mathfrak{a}^*$ by duality, on A via the exponential map, and on $\mathfrak{a}_c$ by complex linearity.

Set $\mathfrak{n} := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. $N := \exp \mathfrak{n}$ is a simply connected nilpotent subgroup of G. The map $(k, a, n) \longmapsto kan$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G, and the resulting decompostion $G = KAN$ is called the Iwasawa decomposition of G.

Every element x of G can be written as $x = k_1 a k_2$ for some $k_1, k_2 \in K$ and $a \in A$. a is uniquely determined up to conjugation by elements of W. This property will be referred to as the Cartan decomposition of G, written $G = KAK$.

The Cartan-Killing form B is positive definite on $\mathfrak{p} \times \mathfrak{p}$, so $\langle X, Y \rangle := B(X, Y)$ defines Euclidean structures in $\mathfrak{p}$ and in $\mathfrak{a} \subset \mathfrak{p}$. We extend this inner product to $\mathfrak{a}^*$ by duality, that is we set $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$ if H_γ is the unique element in $\mathfrak{a}$ such that $\langle H_\gamma, H \rangle = \gamma(H)$ for all $H \in \mathfrak{a}$. Set $|H| = \langle H, H \rangle^{1/2}$. For $x \in G$ define $|x| := |H|$ if $x = k_1 \exp H k_2$ with $k_1, k_2 \in K$ and $H \in \mathfrak{a}$.

If $(V, \langle \cdot, \cdot \rangle)$ is a Euclidean space, the Schwartz space on V is the set $\mathcal{S}(V)$ of all rapidly decreasing C^∞ functions on V: a C^∞ function f on V belongs to $\mathcal{S}(V)$ provided for every differential operator D on V with constant coefficients and for every integer $N \geq 0$

$$\tau_{D,N}(f) := \sup_{v \in V} (1 + |v|)^N |Df(v)| < \infty,$$

where $|v| := \langle v, v \rangle^{1/2}$. $\mathcal{S}(V)$ is a Fréchet space in the topology defined by the seminorms $\tau_{D,N}$. $\mathcal{S}_W(\mathfrak{a})$ and $\mathcal{S}_W(\mathfrak{a}^*)$ respectively denote the sets of all rapidly decreasing C^∞ functions on the Eucidean spaces $\mathfrak{a}$ and $\mathfrak{a}^*$ that are

W-invariant. Using the exponential map, the space $\mathcal{S}_W(A)$ of W-invariant rapidly decreasing C^∞ functions on $A = \exp \mathfrak{a}$ can be similarly defined.

A function f on G is said to be K-bi-invariant if $f(k_1 x k_2) = f(x)$ for all $x \in G$ and $k_1, k_2 \in K$. Because of the Cartan decomposition $G = KAK$, a K-bi-invariant function is uniquely determined by its W-invariant restriction to A. Let $\mathbf{D}(G)$ denote the set of the left-invariant differential operators on G. The Schwartz space $\mathcal{S}(K\backslash G/K)$ of K-bi-invariant functions over G is the set of all K-bi-invariant C^∞ functions on G satisfying the following property: For every $D \in \mathbf{D}(G)$ and every integer $N \geq 0$

$$\tau_{D,N}(f) := \sup_{x \in G} (1 + |x|)^N \, \mathbf{d}(x) |Df(x)| \ < \infty.$$

Here $\mathbf{d}$ denotes the K-bi-invariant analytic function on G defined by

(1.1)

$$\mathbf{d}(x) := \prod_{\alpha \in \Sigma^+} \left[\frac{\sinh \alpha(H)}{\alpha(H)} \right]^{\frac{m_\alpha}{2}} \qquad \text{if } x = k_1 \exp H k_2 \text{ with } k_1, k_2 \in K, H \in \mathfrak{a}.$$

The seminorms $\tau_{D,N}$ define a Fréchet topology on $\mathcal{S}(K\backslash G/K)$.

Let $\mathbf{D}_K(G)$ be the set of all left-invariant differential operators on G which are right-invariant under K. The (elementary) spherical functions on G are the K-bi-invariant eigenfunctions φ of every differential operator $D \in \mathbf{D}_K(G)$, normalized by the condition $\varphi(e) = 1$ (e is the identity element in G).

Let dk be the Haar measure on K normalized so that $\int_K \, dk = 1$. Harish-Chandra proved that for $\lambda \in \mathfrak{a}_c^*$ the functions

$$\varphi_\lambda(x) = \int_K e^{(i\lambda - \rho)(H(xk))} \, dk \qquad x \in G, \tag{1.2}$$

exhaust the set of spherical functions on G. Here

$$\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \, \alpha \tag{1.3}$$

and, for $g \in G$, $H(g)$ is the unique element of $\mathfrak{a}$ such that $\exp H(g)$ is the A-component of g in the Iwasawa decomposition $G = KAN$. Moreover, $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda = w.\lambda'$ for some w in the Weyl group W. $\varphi_\lambda(x)$ is a real analytic function of $x \in G$ and a W-invariant entire function of $\lambda \in \mathfrak{a}_c^*$.

The spherical transform, the Abel transform and the Euclidean Fourier transform are respectively the isomorphisms

$$\mathcal{S} : \mathcal{S}(K\backslash G/K) \xrightarrow{\sim} \mathcal{S}_W(\mathfrak{a}^*)$$
$$\mathcal{A} : \mathcal{S}(K\backslash G/K) \xrightarrow{\sim} \mathcal{S}_W(A)$$
$$\mathcal{F} : \mathcal{S}_W(A) \xrightarrow{\sim} \mathcal{S}_W(\mathfrak{a}^*)$$

defined (up to a constant multiple) by

$$\mathcal{S}f(\lambda) := \int_G f(x)\,\varphi_{-\lambda}(x)\,dx, \qquad f \in \mathcal{S}(K\backslash G/K),\ \lambda \in \mathfrak{a}^*,$$
$$\mathcal{A}f(a) := e^{\rho\,(\log a)} \int_N f(an)\,dn, \qquad f \in \mathcal{S}(K\backslash G/K),\ a \in A,$$
$$\mathcal{F}g(\lambda) := \int_A g(a)\,e^{-i\lambda(\log a)}\,da, \qquad g \in \mathcal{S}_W(A),\ \lambda \in \mathfrak{a}^*.$$

The Haar measures dx, dn and da respectively on G, N and A can be normalized so that

$$\mathcal{S} = \mathcal{F} \circ \mathcal{A}. \tag{1.4}$$

Via the Cartan decomposition of G, the spherical transform can be given by integration over the Weyl chamber $\mathfrak{a}^+$: For $f \in \mathcal{S}(K\backslash G/K)$ and $\lambda \in \mathfrak{a}^*$

$$\mathcal{S}f(\lambda) = C \int_{\mathfrak{a}^+} f(\exp H)\,\varphi_{-\lambda}(\exp H)[\Delta(H)]^2\,dH \tag{1.5}$$

where Δ is the function on $\mathfrak{a}^+$ defined by

$$\Delta(H) := \prod_{\alpha\in\Sigma^+} [\sinh \alpha(H)]^{\frac{m_\alpha}{2}} \tag{1.6}$$

and C is a constant depending on the normalization of the measures.

An explicit analytic formula for the inverse Abel transform is not available for arbitrary groups G. The inversion formula for the case $G = SO_0(n,1)$ was given first by Takahashi [**Tak63**]. In 1968, Gangolli solved the complex case [**Gan68**]. For the general rank-one case the explicit formula was determined with different approaches by Eaton [**Eat73**], Koornwinder [**Koo75**], Lohoué and Rychener [**LR82**], and Rouvière [**Rou83**]. These formulas will be described in Sections 3 and 4. For more information on the inversion formulas of the Abel transform we refer to [**Bee88**].

2. Statement of the results.

As already observed, a K-bi-invariant function is uniquely determined by its W-invariant restriction to A. Via the exponential map, the Schwartz space $\mathcal{S}(K\backslash G/K)$ is therefore identified with a subset of $\mathcal{S}_W(\mathfrak{a})$. In the Paley–Wiener theorems we are going to state, the elements of $f \in \mathcal{S}(K\backslash G/K)$ whose spherical transform has compact support $\operatorname{supp}\mathcal{S}f$ will be characterized in terms of the holomorphic extendibility and growth of the function $H \mapsto \Delta(H)f(\exp H)$ over suitable subsets of the complexification $\mathfrak{a}_c$ of $\mathfrak{a}$.

2.1. The rank-one case. In the rank-one case a further simplification can be made, as we can intrinsically identify $\mathfrak{a}$ with $\mathbb{R}$. Indeed, if G is of rank one, the set Σ^+ of the positive restricted roots consists of at most two elements: α and, possibly, 2α. Let H be the element of $\mathfrak{a}$ satisfying $\alpha(H) = 1$. The choice of H and α and the exponential map allow us to identify $\mathfrak{a}$, $\mathfrak{a}^*$ and A with $\mathbb{R}$. The Weyl group reduces to $\{-1,1\}$ acting on $\mathbb{R}$ by multiplication, so $\mathcal{S}_W(\mathfrak{a})$, $\mathcal{S}_W(\mathfrak{a}^*)$ and $\mathcal{S}_W(A)$ become $\mathcal{S}_+(\mathbb{R})$, the set of even functions in the Schwartz space of $\mathbb{R}$.

Formulas (1.1) and (1.3) respectively become

$$\mathbf{d}(t) = \left(\frac{\sinh t}{t}\right)^{\frac{m_\alpha}{2}} \left(\frac{\sinh(2t)}{2t}\right)^{\frac{m_{2\alpha}}{2}}, \qquad t \in \mathbb{R}, \tag{2.1}$$

$$\rho = \frac{1}{2}\, m_\alpha + m_{2\alpha}. \tag{2.2}$$

One can show that, under the above identifications, $\mathcal{S}(K\backslash G/K)$ corresponds to the set $\mathcal{S}_+^\rho(\mathbb{R})$ of all even C^∞ functions on $\mathbb{R}$ such that for every differential operator D on $\mathbb{R}$ with constant coefficients and for every integer $N \geq 0$

$$\sup_{t\in\mathbb{R}} \,(1+|t|)^N \mathbf{d}(t)|Df(t)| \; < \infty \,.$$

We fix the following constants related to the multiplicities m_α and $m_{2\alpha}$:

$$j := \begin{cases} 1 & \text{if } m_{2\alpha} = 0 \\ 2 & \text{if } m_{2\alpha} \neq 0 \end{cases} \quad \text{and} \quad J := \begin{cases} 2 & \text{if } m_\alpha \text{ is even and } m_{2\alpha} = 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.3}$$

We shall frequently use the property that if 2α is a restricted root, then m_α is even and $m_{2\alpha}$ is odd. The case m_α even and $m_{2\alpha} = 0$ corresponds to the particular situation in which all Cartan subalgebras of G are conjugate ($G \cong SO_0(2n+1,1)$).

We shall use the following notation:

$$\begin{aligned}
&\Re^+ := \{z \in \mathbb{C} : \Re z > 0\} && \\
&\Im_j := i\big(\mathbb{R} \setminus \tfrac{\pi}{j}\mathbb{Z}\big) && (j=1,2)\\
&\Re_j^+ := \Re^+ \cup \Im_j && (j=1,2)\\
&\exists_j := \mathbb{C} \setminus \big\{z : \Re z \leq 0, \Im z \in \tfrac{\pi}{j}\mathbb{Z}\big\} && (j=1,2)\\
&\mathfrak{S}_j := \mathbb{C} \setminus \big\{z : \Im z \in \tfrac{\pi}{j}\mathbb{Z}\big\} && (j=1,2)\\
&S_j := \big\{z \in \mathbb{C} : \Im z \in \big(-\tfrac{\pi}{j}, \tfrac{\pi}{j}\big)\big\} && (j=1,2),
\end{aligned}$$

$\Re z$ and $\Im z$ being respectively the real and the imaginary part of $z \in \mathbb{C}$.

For $z \in S_j \setminus (-\infty, 0]$, set (cf. Formula (1.6))

$$\Delta(z) = (\sinh z)^{\frac{m_\alpha}{2}} (\sinh(2z))^{\frac{m_{2\alpha}}{2}}. \tag{2.4}$$

If $m_{j\alpha}$ is odd, $(\sinh(jz))^{m_{j\alpha}/2}$ stands for $(\sinh(jz))^{(m_{j\alpha}-1)/2}[\sinh(jz)]_+^{1/2}$, where $[\,\cdot\,]_+^{1/2}$ denotes the single-valued holomorphic branch of the square root function determined on $\mathbb{C} \setminus (-\infty, 0]$ by $\sqrt{1} = 1$. If m_α is even and $m_{2\alpha} = 0$, then $\Delta(z) = (\sinh z)^{m_\alpha/2}$ is an entire function on $\mathbb{C}$.

The Paley–Wiener theorem for the inverse spherical transform on rank-one groups is given by the following theorem.

Theorem 2.1. *Let $f \in \mathcal{S}^{\rho}_{+}(\mathbb{R})$, and let $\tilde{f}(t) := \Delta(t) f(t)$ for $t \in (0,\infty)$. Then $\mathcal{S}f$ is compactly supported, with* $\operatorname{supp} \mathcal{S}f \subset [-r,r]$, *if and only if $\tilde{f}(t)$, $t \in (0,\infty)$, extends to a holomorphic function $\tilde{F}$ on $\exists_j$ such that:*

(1) *For every integer $N \geq 0$ there is a constant $\tau_N > 0$ so that for all $z \in \Re^+_j$*

$$|\tilde{F}(z)| \leq \tau_N (1+|\coth z|)^{(j-1)\frac{m_\alpha}{2}} (1+|\coth(jz)|)^{\frac{m_{j\alpha}-J}{2}} (1+|z|)^{-N} e^{r|\Im z|}.$$

(2) *The function F defined by*

$$F(z) := \frac{\tilde{F}(z)}{\Delta(z)} \qquad \text{for } z \in S_j \setminus (-\infty, 0]$$

extends to be an even holomorphic function on the horizontal strip S_j, and $F(t) = f(t)$ for all $t \in \mathbb{R}$.

Observe that the growth estimate of $\tilde{F}$ is given on $\Re^+_j$, not on the entire complex plane. Moreover, a single-valued holomorphic extension of the function f is generally only obtained in the strip S_j. But, when m_α is even and $m_{2\alpha} = 0$, the function f has actually a meromorphic extension to all of $\mathbb{C}$, with poles at most on $i\pi\mathbb{Z} \setminus \{0\}$, with estimated growth on the whole complex plane. Indeed, in this case, $m_\alpha/2$ is a positive integer. The function $F(z) := \tilde{F}(z)(\sinh z)^{-m_\alpha/2}$ is therefore holomorphic on $\exists_j$. Because of 2, $F(z)$ is an even extension of $f(t)$, $t \in \mathbb{R}$, to S_j. Hence it holomorphically extends to $\mathbb{C} \setminus \{i\pi k : k = \pm 1, \pm 2, \dots\}$ by setting $F(z) := F(-z)$ if $\Re z < 0$. The growth condition for $\tilde{F}$ stated in 1 becomes: For every integer $N \geq 0$ there is a constant $\tau_N > 0$ such that for all $z \in \Re^+_j$

$$|\tilde{F}(z)| \leq \tau_N (1+|\coth z|)^{\frac{m_\alpha}{2}-1} (1+|z|)^{-N} e^{r|\Im z|}.$$

It follows, in particular, that the function $(\sinh z)^{m_\alpha - 1} F(z)$ is bounded near the points $i\pi k \quad (k = \pm 1, \pm 2, \dots)$. Thus $(\sinh z)^{m_\alpha - 1} F(z)$ is entire, and $F(z)$ is a meromorphic extension of f to $\mathbb{C}$.

In the case m_α even and $m_{2\alpha} = 0$, Theorem 2.1 can be therefore equivalently stated as follows.

Theorem 2.2. *Suppose m_α is even and $m_{2\alpha} = 0$. Let $f \in \mathcal{S}^{\rho}_{+}(\mathbb{R})$. Then* $\operatorname{supp} \mathcal{S}f \subset [-r,r]$ *if and only if $f(t)$, $t \in \mathbb{R}$, extends to an even meromorphic function F on $\mathbb{C}$ satisfying: For every integer $N \geq 0$ there is a constant $\tau_N > 0$ such that for all $z \in \mathbb{C}$*

$$|\Delta(z) F(z)| \leq \tau_N \, (1+|\coth z|)^{\frac{m_\alpha}{2}-1} \, (1+|z|)^{-N} e^{r|\Im z|}.$$

Theorem 2.1 is proved in Section 3. The proof is based on the relation $\mathcal{S} = \mathcal{F} \circ \mathcal{A}$, applying the classical Paley–Wiener theorem to $\mathcal{F}$ and then using the explicit inversion formulas for the Abel transform $\mathcal{A}$ and for the inverse Abel transform $\mathcal{A}^{-1}$.

Observe that when $m_\alpha = 2$ (i.e. $G \cong SL(2,\mathbb{C}) \cong SO_0(3,1)$), Theorem 2.2 says that $z \longmapsto \Delta(z)F(z)$ is an even entire function of exponential type r and rapidly decreasing. As we shall see shortly, this property generalizes to arbitrary complex groups.

2.2. The complex case. When G admits a complex structure, all the restricted roots α have multiplicity $m_\alpha = 2$. The function Δ in (1.6) becomes

$$\Delta(H) = \prod_{\alpha\in\Sigma^+} \sinh \alpha(H), \qquad H \in \mathfrak{a}. \tag{2.5}$$

Its entire extension to $\mathfrak{a}_c$ will be also denoted by Δ:

$$\Delta(\tilde{H}) := \prod_{\alpha\in\Sigma^+} \sinh \tilde{\alpha}(\tilde{H}), \qquad \tilde{H} \in \mathfrak{a}_c,$$

$\tilde{\alpha}$ being the complex linear extension of $\alpha \in \Sigma^+$ to $\mathfrak{a}_c$.

We define a norm on the complexification $\mathfrak{a}_c = \mathfrak{a} \oplus i\mathfrak{a}$ of $\mathfrak{a}$ by setting

$$|\tilde{H}| := \Big(|\Re\tilde{H}|^2 + |\Im\tilde{H}|^2\Big)^{1/2}.$$

An entire function g on $\mathfrak{a}_c$ is said to be of exponential type $r > 0$ and rapidly decreasing provided for every integer $N \geq 0$ there is a constant $\sigma_N > 0$ such that

$$|g(\tilde{H})| \leq \sigma_N(1 + |\tilde{H}|)^{-N} e^{r|\Im\tilde{H}|}, \qquad \tilde{H} \in \mathfrak{a}_c.$$

For $r > 0$, $B_r := \{\lambda \in \mathfrak{a}^* : |\lambda| \leq r\}$ denotes the closed ball around 0 with radius r in $\mathfrak{a}^*$.

Theorem 2.3. *Suppose G has a complex structure. Let $f \in \mathcal{S}(K\backslash G/K)$. Then $\mathcal{S}f$ is compactly supported, with $\operatorname{supp}\mathcal{S}f \subset B_r$, if and only if $f(\exp H)$, $H \in \mathfrak{a}$, extends to a W-invariant meromorphic function F on $\mathfrak{a}_c$ such that $\tilde{H} \mapsto \Delta(\tilde{H})F(\tilde{H})$ is an entire function on $\mathfrak{a}_c$ of exponential type r and rapidly decreasing.*

Theorem 2.3 is proved in Section 4.

3. Proof of Theorem 2.1.

Because of the classical Paley–Wiener theorem, the Fourier transform of a function $g \in \mathcal{S}_+(\mathbb{R})$ has support in the interval $[-r, r]$ if and only if it extends as an even entire function on $\mathbb{C}$ which is of exponential type r and rapidly decreasing. We indicate the set of such functions by $\mathcal{H}_+^r(\mathbb{R})$.

Under the identifications of A and $\mathfrak{a}^*$ with $\mathbb{R}$ described in Section 2.1, the Euclidean Fourier transform $\mathcal{F} : \mathcal{S}_W(A) \longrightarrow \mathcal{S}_W(\mathfrak{a}^*)$ reduces to the classical Fourier transform on $\mathcal{S}_+(\mathbb{R})$. Because of the relation $\mathcal{S} = \mathcal{F} \circ \mathcal{A}$, our problem is therefore to describe, for every $r > 0$, the subset of $\mathcal{S}_+^\rho(\mathbb{R})$ which is the image of $\mathcal{H}_+^r(\mathbb{R})$ under the inverse Abel transform $\mathcal{A}^{-1}$.

For every $l \geq 0$, let $\mathcal{S}_+^l(\mathbb{R})$ be the set of the even C^∞ functions f on $\mathbb{R}$ such that for every differential operator D on $\mathbb{R}$ with constant coefficients and for every integer $N \geq 0$

$$\sup_{t\in\mathbb{R}} \,(1+|t|)^N \cosh^l t |Df(t)| \; < \infty.$$

Since $(1+|t|)^{-1}\cosh t \leq \sinh t/t \leq \cosh t$, $t \in \mathbb{R}$, this definition is consistent with our previous definition of $\mathcal{S}_+^\rho(\mathbb{R})$ when $l = \rho$ (cf. Formula (2.2)). Moreover, $\mathcal{S}_+^0(\mathbb{R}) = \mathcal{S}_+(\mathbb{R})$.

As shown by Rouvière [**Rou83**], the Abel transform $\mathcal{A} : \mathcal{S}_+^\rho(\mathbb{R}) \longrightarrow \mathcal{S}_+(\mathbb{R})$ can be expressed as a composition of elementary transformations $\mathcal{A}_1$ and $\mathcal{A}_2$.

Definition 3.1. [1] Let $l \geq 0$. For $j = 1, 2$, $\mathcal{A}_j$ is the integral operator from $\mathcal{S}_+^{l+\frac{j}{2}}(\mathbb{R})$ to $\mathcal{S}_+^l(\mathbb{R})$ defined by

$$\mathcal{A}_j f(t) := \int_{-\infty}^{+\infty} \Phi f\left([\cosh^j t + x^2]^{1/j}\right)\, dx, \qquad t \in \mathbb{R},$$

where $\Phi f(\cosh t) := f(t)$.

$\mathcal{A}_1$ (resp. $\mathcal{A}_2$) can be interpreted as partial Abel transform associated with one-parameter subgroups of N generated by elements of $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{2\alpha}$).

Theorem 3.2. [2] *Up to a constant multiple,*

$$\mathcal{A} = \mathcal{A}_1^{m_\alpha} \circ \mathcal{A}_2^{m_{2\alpha}}\ .$$

For $j = 1, 2$ define $D_j := \dfrac{1}{\sinh(jt)}\dfrac{d}{dt}$. Then D_j maps $\mathcal{S}_+^l(\mathbb{R})$ into $\mathcal{S}_+^{l+j}(\mathbb{R})$ for every $l \geq 0$. Rouvière proved the following theorem.

Theorem 3.3. [3] *For* $j = 1, 2$

$$D_j \circ \mathcal{A}_j = \mathcal{A}_j \circ D_j \qquad \textit{on } \mathcal{S}_+^{l+\frac{j}{2}}(\mathbb{R}) \tag{3.1}$$

$$\mathcal{A}_j^2 \circ D_j^2 = -\pi D_j \qquad \textit{on } \mathcal{S}_+^{l}(\mathbb{R}). \tag{3.2}$$

[1][**Rou83**], p. 274. See also pp. 283, 286. Rouvière denotes the operator on $\mathcal{S}_+^\rho(\mathbb{R})$ corresponding to $\mathcal{A}$ by F. The relation between our operators $\mathcal{A}_1$ and $\mathcal{A}_2$ and Rouvière's operators F_0 and F_0' is $\mathcal{A}_1 = \sqrt{\pi}F_0$, $\mathcal{A}_2 = \sqrt{\pi}F_0'$.

[2][**Rou83**], Théorème 1, p. 275. See also Théorème 5, p. 283.

[3][**Rou83**], Théorème 1, p. 275, and the computations before it. See also Lemme 6, p. 283, and Lemme 7, p. 286. Observe that Rouvière uses the differential operator $\dfrac{-1}{\sinh(jt)}\dfrac{d}{dt}$.

Up to a constant multiple, $\mathcal{A}^{-1} = \mathcal{A}_2^{m_{2\alpha}} \circ D_2^{m_{2\alpha}} \circ \mathcal{A}_1^{m_\alpha} \circ D_1^{m_\alpha}$.

$\mathcal{A}^{-1}$ is therefore an integro-differential operator, which reduces to a differential operator when m_α is even and $m_{2\alpha} = 0$. According to the various possibilities for m_α and $m_{2\alpha}$, there are three cases:

$$\begin{aligned} \mathcal{A}^{-1} &= D_1^{m_\alpha/2} && (m_\alpha \text{ even}, m_{2\alpha} = 0),\\ \mathcal{A}^{-1} &= \mathcal{A}_1 \circ D_1^{(m_\alpha+1)/2} && (m_\alpha \text{ odd}, m_{2\alpha} = 0),\\ \mathcal{A}^{-1} &= \mathcal{A}_2 \circ D_2^{(m_{2\alpha}+1)/2} \circ D_1^{m_\alpha/2} && (m_\alpha \text{ even}, m_{2\alpha} \text{ odd}). \end{aligned}$$

(All the equalities are given up to constant multiples.)

3.1. Necessity. To prove the necessity of the condition stated in Theorem 2.1, we consider the "complexifications" of the operators $\mathcal{A}_j$ and D_j, and we apply them to the holomorphic extension of the functions in $\mathcal{H}_+^r(\mathbb{R})$ as prescribed by the formulas for $\mathcal{A}^{-1}$.

Let $j > 0$, and let D_j be the differential operator on functions on $\mathbb{C}$ defined by $D_j := \dfrac{1}{\sinh(jz)}\dfrac{d}{dz}$. Observe that D_j maps even (resp. odd) functions into even (resp. odd) functions. The next Proposition 3.7 describes how the (n,m)-th iterate $D_2^n D_1^m$ of D_2, D_1 acts on entire functions which are of exponential type and rapidly decreasing.

Lemma 3.4. *Let $r, s \geq 0$, and let N be a nonnegative integer. Let g be an entire function satisfying for all $z \in \mathbb{C}$*

$$|g(z)| \leq \sigma(1+|z|)^{-N} e^{s|\Re z| + r|\Im z|}$$

for some constant $\sigma > 0$. Then there is a constant $\tilde{\sigma} > 0$ such that for all $z \in \mathbb{C}$

$$|g'(z)| \leq \tilde{\sigma}(1+|z|)^{-N} e^{s|\Re z| + r|\Im z|}.$$

Consequently, for every positive j, $D_j g$ is a meromorphic function, with at most simple poles on $i\frac{\pi}{j}\mathbb{Z}$, satisfying

$$|D_j g(z)| \leq \tilde{\sigma}|\sinh(jz)|^{-1}(1+|z|)^{-N} e^{s|\Re z| + r|\Im z|}.$$

Proof. Fix $z \in \mathbb{C}$. Let γ denote the rectangular contour γ of vertices $\left(\Re z \pm \frac{1}{2\sqrt{2}}, \Im z \pm \frac{1}{2\sqrt{2}}\right)$. For $\zeta \in \gamma$, we have $|\zeta - z| \geq \frac{1}{2\sqrt{2}}$ and $2(1+|\zeta|) \geq 1+|z|$. Cauchy's Formula therefore gives

$$\begin{aligned} |g'(z)| &\leq \frac{1}{2\pi} \int_\gamma \frac{|g(\zeta)|}{|\zeta - z|^2}\,|d\zeta| \\ &\leq \frac{4}{\pi} \int_\gamma |g(\zeta)|\,|d\zeta| \end{aligned}$$

$$\le \frac{4\sigma}{\pi} \int_\gamma (1+|\zeta|)^{-N} e^{s|\Re\zeta|+r|\Im|} \, |d\zeta|$$
$$\le \frac{2^{N+2}\sigma}{\pi} \int_\gamma (1+|z|)^{-N} e^{s(|\Re z|+\frac{1}{2\sqrt{2}})+r(|\Im z|+\frac{1}{2\sqrt{2}})} \, |d\zeta|$$
$$= \tilde\sigma (1+|z|)^{-N} e^{s|\Re z|+r|\Im z|},$$

where $\tilde\sigma := \frac{2^{N+4}}{\sqrt{2}\,\pi} e^{(r+s)/2\sqrt{2}}\sigma$. □

Lemma 3.5. *Let $r \ge 0$, and let $n > 0$ and $N \ge 0$ be integers. Let $l(z)$ be a meromorphic function on $\mathbb{C}$ satisfying the following properties.*

i. $(\sinh z)^{2n-1} l(z)$ *is an entire function.*
ii. *There is a constant $\nu > 0$ such that for all $z \in \mathbb{C} \setminus i\pi\mathbb{Z}$*

$$|l(z)| \le \nu \frac{(1+|\coth z|)^{n-1}}{|\sinh z|^n} (1+|z|)^{-N} e^{r|\Im z|}.$$

Then

1. $(\sinh z)^{2n+1} D_1 l(z)$ *is an entire function.*
2. *There is a constant $\hat\nu > 0$ (depending on r, n, N) such that for all $z \in \mathbb{C} \setminus i\pi\mathbb{Z}$*

$$|D_1 l(z)| \le \hat\nu \frac{(1+|\coth z|)^n}{|\sinh z|^{n+1}} (1+|z|)^{-N} e^{r|\Im z|}.$$

3. $\sinh(2z)(\sinh z)^{2n} D_2 l(z)$ *is an entire function.*
4. *There is a constant $\hat\nu > 0$ (depending on r, n, N) such that for all $z \in \mathbb{C} \setminus i\frac{\pi}{2}\mathbb{Z}$*

$$|D_2 l(z)| \le \hat\nu |\sinh(2z)|^{-1} \left(\frac{1+|\coth z|}{|\sinh z|}\right)^n (1+|z|)^{-N} e^{r|\Im z|}.$$

Proof. Apply Lemma 3.4 to the entire function $g(z) := (\sinh z)^{2n-1} l(z)$, using the following inequalities:

$$\frac{1}{\sqrt{2}}(1+|\coth\zeta|) \le e^{|\Re\zeta|}|\sinh\zeta|^{-1} \le 1+|\coth\zeta|, \qquad \zeta \in \mathbb{C} \setminus i\pi\mathbb{Z}. \tag{3.3}$$

For 3 and 4, observe that $D_2 = \dfrac{1}{2\cosh z} D_1$, which gives

$$\sinh(2z)(\sinh z)^{2n} D_2 l(z) = (\sinh z)^{2n+1} D_1 l(z).$$

□

Lemma 3.6. *Let $r \ge 0$, and let $m, n > 0$ and $N \ge 0$ be integers. Let $l(z)$ be a meromorphic function on $\mathbb{C}$ satisfying the following properties.*

i. $(\sinh(2z))^{2n-1}(\sinh z)^{2m} l(z)$ *is an entire function.*

ii. *There is a constant* $\nu > 0$ *such that for all* $z \in \mathbb{C} \setminus i\frac{\pi}{2}\mathbb{Z}$

$$|l(z)| \ \leq \nu \ \frac{(1+|\coth(2z)|)^{n-1}}{|\sinh(2z)|^n} \left(\frac{1+|\coth z|}{|\sinh z|}\right)^m (1+|z|)^{-N} e^{r|\Im z|}.$$

Then

1. $(\sinh(2z))^{2n+1}(\sinh z)^{2m}\, D_2 l(z)$ *is an entire function.*
2. *There is a constant* $\hat{\nu} > 0$ (*depending on* r, n, m, N) *such that for all* $z \in \mathbb{C} \setminus i\frac{\pi}{2}\mathbb{Z}$

$$|D_2 l(z)| \ \leq \hat{\nu} \ \frac{(1+|\coth(2z)|)^{n}}{|\sinh(2z)|^{n+1}} \left(\frac{1+|\coth z|}{|\sinh z|}\right)^m (1+|z|)^{-N} e^{r|\Im z|}.$$

Proof. Apply Lemma 3.4 to $g(z) := (\sinh(2z))^{2n-1}(\sinh z)^{2m}\, l(z)$, using Inequality (3.3) together with $|\coth z| \leq 2|\coth(2z)| + 1$. □

Proposition 3.7. *Let* g *be an entire function on* $\mathbb{C}$ *which is of exponential type* $r > 0$ *and rapidly decreasing. Let* $j = 1, 2$. *Then for every positive integers* n *and* m:

1. $(\sinh(jz))^{2n-1}(\sinh z)^{2(j-1)m}\, D_j^n D_1^{(j-1)m} g(z)$ *is an entire function.*
2. *For every integer* $N \geq 0$ *there is a constant* $\nu_N > 0$ (*depending also on* j, n, m) *such that for all* $z \in \mathbb{C} \setminus i\frac{\pi}{j}\mathbb{Z}$

$$|D_j^n D_1^{(j-1)m} g(z)| \leq \nu_N \frac{(1+|\coth(jz)|)^{n-1}}{|\sinh(jz)|^n} \left(\frac{1+|\coth z|}{|\sinh z|}\right)^{(j-1)m} \frac{e^{r|\Im z|}}{(1+|z|)^N}.$$

3. *If* g *is even, then* $D_j^n D_1^{(j-1)m} g$ *is even and extends to be holomorphic at* 0.

Proof. Suppose first $j = 1$, and prove 1 and 2 inductively on n. The case $n = 1$ follows from Lemma 3.4 (with $j = 1$ and $s = 0$), and the inductive step is provided by Lemma 3.5, Parts 1 and 2 (with $l(z) = D_1^n g(z)$). Suppose then $j = 2$, and prove 1 and 2 inductively on n for m arbitrarily fixed. The case $n = 1$ is obtained from Lemma 3.5, Parts 3 and 4 (with $l(z) = D_1^m g(z)$). The inductive step is given by Lemma 3.6 (with $l(z) = D_2^n D_1^m g(z)$).

If g is even and holomorphic near 0, then $g'(0) = 0$. Hence $D_j g$ is even and extends to be holomorphic at 0 by setting $D_j g(0) = \frac{1}{j}\, g''(0)$. □

We now want to determine the image under the operators $\mathcal{A}_j$ $(j = 1, 2)$ of the functions $h(t) := D_j^n D_1^{(j-1)m} g(t)$, $t \in \mathbb{R}$, described by Proposition 3.7.

If $h \in \mathcal{S}_+^{l+\frac{j}{2}}(\mathbb{R})$, then $\mathcal{A}_j h$ is a function in $\mathcal{S}_+^{l}(\mathbb{R})$ that can be written as

$$\mathcal{A}_j h(t) = 2 \int_0^\infty \Phi h\big([\cosh^j t + x^2]^{1/j}\big) dx, \qquad t > 0.$$

Substitute the variable $x \in (0,\infty)$ with the variable $w \in (0,\infty)$ defined by the relation $\cosh^j t + x^2 = \cosh^j(t+w)$. Then

$$x^2 = \cosh^j(t+w) - \cosh^j t = \tfrac{2}{j}\sinh\big(t+\tfrac{w}{2}\big)\sinh\left(\tfrac{w}{2}\right),$$

and

$$\mathcal{A}_j h(t) = \sqrt{\tfrac{j}{2}} \int_0^\infty h(t+w)\,\frac{\sinh\left(j(t+w)\right)}{\left[\sinh\left(j\left(t+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)\right]^{1/2}}\,dw, \qquad t>0.$$

Since the map $z \mapsto [\sinh(jz)]_+^{1/2}$ is well defined and holomorphic on $S_j \setminus (-\infty,0]$, we are led to the following definition.

Definition 3.8. For $j=1,2$, let $\mathcal{A}_j^c$ denote the integral transform given, for all functions h for which it is well defined, by

$$\mathcal{A}_j^c h(z) := \sqrt{\tfrac{j}{2}} \int_0^\infty h(z+w)\,\frac{\sinh(j(z+w))}{\left[\sinh\left(j\left(z+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)\right]_+^{1/2}}\,dw,$$
$$z \in S_j \setminus (-\infty,0].$$

To study the operator $\mathcal{A}_j^c$ we need the following theorem.

Theorem 3.9. [4] *Let U be an open subset of $\mathbb{C}$, and let $\Psi(z,w)$ be a continuous function on $U \times (0,\infty)$. Assume:*

i. *For every $w \in (0,\infty)$, $\Psi(z,w)$ is holomorphic in U.*
ii. *For every compact subset K of U there exists a function $M_K(w)$ which is integrable in $(0,\infty)$ and such that for all $z \in K$ and $w \in (0,\infty)$*

$$|\Psi(z,w)| \le M_K(w).$$

Then $\psi(z) := \int_0^\infty \Psi(z,w)\,dw$ is holomorphic on U.

Lemma 3.10. *Let h be an even holomorphic function on S_j with the following property: For every $\delta \in (0,\frac{\pi}{j})$ there exists a constant C_δ such that for all z with $|\Im z| \le \delta$*

$$|h(z)| \le C_\delta |\sinh(jz)|^{-1/2}(1+|z|)^{-2}.$$

Then $\mathcal{A}_j^c h(z)$, $z \in S_j \setminus (-\infty,0]$, is well-defined and it extends to an even holomorphic function on S_j, which we also denote by $\mathcal{A}_j^c h$. Moreover, $\mathcal{A}_j^c h(t) = \mathcal{A}_j h(t)$ for all $t \in \mathbb{R}$.

[4][**Lan93**], Lemma 1.1, Chapter XV, p. 392, and [**LR70**], p. 368, for the M-test.

Proof. Set

$$\Psi(z,w) := h(z+w)\, \frac{\sinh(j(z+w))}{\left[\sinh\left(j\left(z+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)\right]_+^{1/2}}.$$

By assumption, $\Psi(z,w)$ is holomorphic in $z \in S_j \setminus (-\infty,0]$ for every fixed $w \in (0,\infty)$. For every integer $m \geq 2$, let $\tilde{S}_m := \{z \in \mathbb{C} : |\Im z| \leq \frac{\pi}{j} - \frac{1}{m}, \Re z \geq -m\} \setminus \{z \in \mathbb{C} : \Re z < 0, |\Im z| < 1/m\}$. Let $\chi_A(w)$ denote the characteristic function of $A \subset (0,\infty)$. Since

$$\left|\frac{\sinh(j(z+w))}{\sinh\left(j\left(z+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)}\right| \leq \left|\coth\left(j\left(z+\frac{w}{2}\right)\right)\right| + \coth\left(j\frac{w}{2}\right),$$

there is a constant $C>0$ so that for $(z,w) \in \tilde{S}_m \times (0,\infty)$

(3.4)
$$|\Psi(z,w)| \leq C\left[\chi_{(0,m]}(w)\left(\coth\left(j\frac{w}{2}\right)\right)^{1/2} + \chi_{(m,\infty)}(w)\big(1+(w-m)\big)^{-2}\right].$$

Theorem 3.9 thus guarantees that $\mathcal{A}_j^c h(z)$ is holomorphic on $S_j \setminus (-\infty,0]$.

We now prove that $\mathcal{A}_j^c h$ is even by showing that $\mathcal{A}_j^c h(iy) = \mathcal{A}_j^c h(-iy)$ for all $y \in \left(0,\frac{\pi}{j}\right)$. Let L_y denote the horizontal half-line in $\Re_j^+$ from iy to infinity. The change of variables $u = iy + w$ gives

$$\mathcal{A}_j^c h(iy) = \sqrt{\tfrac{j}{2}} \int\limits_{L_y} h(u)\, \frac{\sinh(ju)}{[d_j(iy,u)]_+^{1/2}}\, du$$

where

$$d_j(iy,u) := \sinh\left(j\left(\tfrac{u+iy}{2}\right)\right)\sinh\left(j\left(\tfrac{u-iy}{2}\right)\right) = \tfrac{1}{2}[\cosh(ju) - \cos(jy)].$$

By assumption h is holomorphic in the horizontal strip S_j. $[d_j(iy,\cdot)]_+^{1/2}$ is holomorphic in the domain D_y obtained from S_j by removing the vertical segments $\left(-i\frac{\pi}{j}, -iy\right]$ and $\left[iy, i\frac{\pi}{j}\right)$. Therefore the function

$$f(y,u) := h(u)\frac{\sinh(ju)}{[d_j(iy,u)]_+^{1/2}}$$

is a holomorphic function of $u \in D_y$.

Let $R>1$, and let $\gamma^R = \bigcup_{k=1}^5 \gamma_k^R$ be the closed curve in D_y pictured in Figure 1 (γ_5^R is the quarter of circle centered at iy with radius y/R).

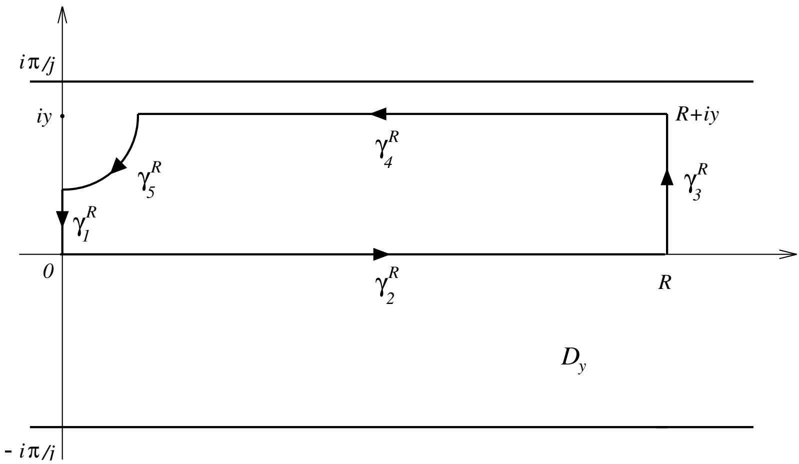

Figure 1. The contour of integration γ.

By Cauchy's Theorem

$$\int_{\gamma^R} f(y,u)\,du = 0.$$

Because of the growth condition of h, when $u = R + it$, $t \in (0, y)$,

$$\begin{aligned} |f(y,u)| &\le C_y |\sinh(ju)|^{-1/2}(1+|u|)^{-2}|\sinh(ju)||d_j(iy,u)|^{-1/2} \\ &\le C'_y(1+R)^{-2}. \end{aligned}$$

Hence $\lim_{R\to\infty}\int_{\gamma_3^R} f(y,u)\,du = 0$. If $u \in D_y$ is close to iy, then there is a constant $C_0 = C_0(y) > 0$ such that $|f(y,u)| \le C_0\left|\frac{u-iy}{2}\right|^{-1/2}$. Hence $f(y,\cdot)$ is integrable along the segment $(0, iy)$ on the imaginary axis with

$$\lim_{R\to\infty}\int_{\gamma_1^R} f(y,u)\,du = -i\int_0^y f(y,it)\,dt.$$

Also, if $u = iy + \frac{y}{R}e^{i\theta}$, $\theta \in (-\pi/2, 0)$, then

$$\int_{\gamma_5^R} |f(y,u)|\,|du| \le C_0 \int_{-\pi/2}^{0} \left|\frac{ye^{i\theta}}{2R}\right|^{-1/2} \frac{y}{R}\,d\theta = C_0\pi\sqrt{\frac{y}{2R}} \longrightarrow 0 \text{ as } R \to \infty,$$

so $\lim_{R\to\infty}\int_{\gamma_5^R} f(y,u)\,du = 0$. Since $\lim_{R\to\infty}\int_{\gamma_4^R} f(y,u)\,du = -\sqrt{\frac{2}{j}}\,\mathcal{A}_j^c h(iy)$, then

$$\mathcal{A}_j^c h(iy) = \sqrt{\tfrac{j}{2}}\,[I_1(y) + I_2(y)]$$

where

$$I_1(y) = -i\int_0^y f(y,it)\,dt = -i\int_0^y h(it)\,\frac{\sinh(ijt)}{[d_j(iy,it)]_+^{1/2}}\,dt,$$

$$I_2(y) = \int_0^\infty f(y,t)\,dt = \int_0^\infty h(t)\,\frac{\sinh(jt)}{[d_j(iy,t)]_+^{1/2}}\,dt.$$

Consider now $-y$ and the closed curve $\Gamma^R = \bigcup_{k=1}^5 \Gamma_k^R$ in D_y which is symmetric to γ^R with respect to the real axis (cf. Figure 2).

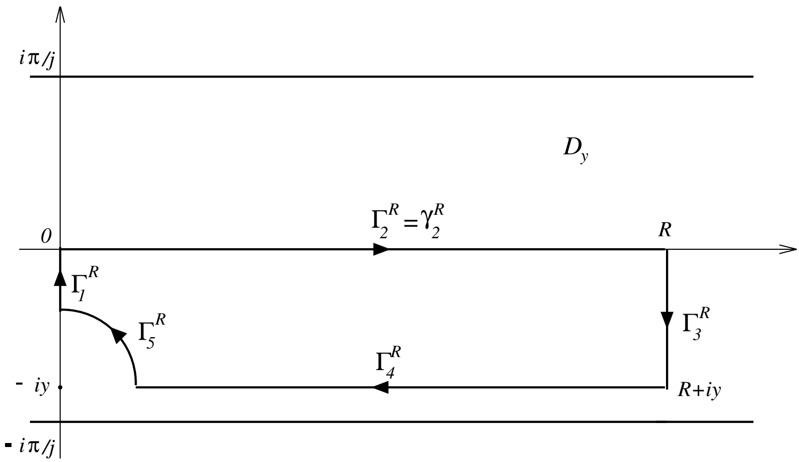

Figure 2. The contour of integration Γ.

Computations analogous to those made above show

$$\mathcal{A}_j^c h(-iy) = -\sqrt{\tfrac{j}{2}}\,\lim_{R\to\infty}\int_{\Gamma_4^R} f(-y,u)\,du = \sqrt{\tfrac{j}{2}}\,[I_1(-y)+I_2(-y)]$$

where

$$I_1(-y) = \lim_{R\to\infty}\int_{\Gamma_1^R} f(-y,u)\,du = i\int_{-y}^0 f(-y,it)\,dt$$

$$= i\int_{-y}^0 h(it)\,\frac{\sinh(ijt)}{[d_j(-iy,it)]_+^{1/2}}\,dt,$$

$$I_2(-y) = \lim_{R\to\infty}\int_{\Gamma_2^R} f(-y,u)\,du = \int_0^\infty f(-y,t)\,dt = \int_0^\infty h(t)\,\frac{\sinh(jt)}{[d_j(-iy,t)]_+^{1/2}}\,dt.$$

Since $d_j(-iy,t) = d_j(iy,t)$, then $I_2(-y) = I_2(y)$. Since h is even and $d_j(iy,it) = d_j(-iy,-it)$, then

$$I_1(y) = -i\int_0^y h(it)\,\frac{\sinh(ijt)}{[d_j(iy,it)]_+^{1/2}}\,dt = i\int_{-y}^0 h(it)\,\frac{\sinh(ijt)}{[d_j(-iy,it)]_+^{1/2}}\,dt = I_1(-y).$$

Thus $\mathcal{A}_j^c h(-iy) = \mathcal{A}_j^c h(iy)$. $\mathcal{A}_j^c h$ is even, so we can extend it to $S_j \setminus \{0\}$ by setting $\mathcal{A}_j^c h(z) := \mathcal{A}_j^c h(-z)$ if $\Re z < 0$. Moreover, (3.4) shows that $\mathcal{A}_j^c h$ remains bounded on $\tilde{S}_1 \cap \Re_j^+$ and hence that it holomorphically extends to S_j. Finally, $\mathcal{A}_j^c h$ and $\mathcal{A}_j h$ are continuous even functions of $t \in \mathbb{R}$: Since they agree on $(0,\infty)$, they must agree on all $\mathbb{R}$. □

To extend $\mathcal{A}_j h$ outside S_j, we need to make its integrand single-valued. The key observation is that the map

$$z \longmapsto \left[\frac{\sinh(jz)}{\sinh\left(j\left(z+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)}\right]_+^{1/2}$$

is well-defined and holomorphic on $\exists_j$ for every fixed $w \in (0,\infty)$.

Definition 3.11. For $j = 1,2$, let $\mathcal{A}_j^+$ denote the integral operator given, for all functions h for which it is well defined, by

$$\begin{aligned}&\mathcal{A}_j^+ h(z)\\ &:= \sqrt{\tfrac{j}{2}}\int_0^\infty h(z+w)\left[\frac{\sinh(jz)}{\sinh\left(j\left(z+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)}\right]_+^{1/2}\sinh(j(z+w))\,dw,\\ &\qquad\qquad z \in \exists_j.\end{aligned}$$

The next proposition determines the holomorphic extension of $\mathcal{A}_j h$ when $h = D_j^n D_1^{(j-1)m} g$ is given by Proposition 3.7.

Proposition 3.12. *Let $j = 1,2$ and let n,m be positive integers. Suppose $h \in \mathcal{S}_+^{jn+(j-1)m}(\mathbb{R})$ extends to an even meromorphic function on $\mathbb{C}$ with the following properties.*

i. $(\sinh(jz))^{2n-1}(\sinh z)^{2(j-1)m}\,h(z)$ *is an entire function.*
ii. *For every integer $N \geq 0$ there is a constant $\nu_N > 0$ such that for all $z \in \mathbb{C} \setminus i\frac{\pi}{j}\mathbb{Z}$*

$$|h(z)| \leq \nu_N \frac{(1+|\coth(jz)|)^{n-1}}{|\sinh(jz)|^n}\left(\frac{1+|\coth z|}{|\sinh z|}\right)^{(j-1)m}(1+|z|)^{-N}e^{r|\Im z|}.$$

iii. *h is holomorphic at $z = 0$.*

Then

1. *$\mathcal{A}_j^+ h$ is holomorphic in $\exists_j$, and $\mathcal{A}_j^+ h(t) = (\sinh(jt))^{1/2}\mathcal{A}_j h(t)$ for all $t \in (0,\infty)$.*
2. *For every integer $N \geq 0$ there is $\sigma_N > 0$ (depending also on n, m, j) such that for all $z \in \Re_j^+$*

$$|\mathcal{A}_j^+ h(z)| \leq \sigma_N \left(\frac{1+|\coth(jz)|}{|\sinh(jz)|}\right)^{n-1}\left(\frac{1+|\coth z|}{|\sinh z|}\right)^{(j-1)m}(1+|z|)^{-N}e^{r|\Im z|}.$$

3. *$\mathcal{A}_j^c h$ is an even holomorphic extension of $\mathcal{A}_j h(t)$, $t \in \mathbb{R}$, to S_j.*
4. *$\mathcal{A}_j^c h(z) = \dfrac{\mathcal{A}_j^+ h(z)}{[\sinh(jz)]_+^{1/2}}$ on $S_j \setminus (-\infty, 0]$.*

Proof. Let

$$\Psi(z,w) = h(z+w)\left[\frac{\sinh(jz)}{\sinh\left(j\left(z+\frac{w}{2}\right)\right)\sinh\left(j\frac{w}{2}\right)}\right]_+^{1/2}\sinh(j(z+w)).$$

Because of Condition i on h, Ψ is continuous on $\exists_j \times (0,\infty)$ and holomorphic in $\exists_j$ for every fixed $w \in (0,\infty)$. If $z = t+iy \in \Re^+$ and $w \in (0,\infty)$, then $|\sinh(j(z+w))| \geq \sinh(jt)$, and $|\coth(j(z+w))| \leq \coth(jt)$. Condition ii (for $N=0$) therefore gives the following estimate:

$$\begin{aligned}
&|\Psi(z,w)| \\
&\leq \nu_0\left(\frac{1+|\coth(j(z+w))|}{|\sinh(j(z+w))|}\right)^{n-1}\left(\frac{1+|\coth(z+w)|}{|\sinh(z+w)|}\right)^{(j-1)m}\frac{e^{r|\Im z|}}{\left[\sinh\left(j\frac{w}{2}\right)\right]^{1/2}} \\
&\leq \nu_0\left(\frac{1+\coth(jt)}{\sinh(jt)}\right)^{n-1}\left(\frac{1+\coth t}{\sinh t}\right)^{(j-1)m}\frac{e^{ry}}{\left[\sinh\left(j\frac{w}{2}\right)\right]^{1/2}} \\
&= \left(\text{function bounded on compact subsets of } \Re^+\right)\cdot\left[\sinh\left(j\tfrac{w}{2}\right)\right]^{-1/2}.
\end{aligned}$$

If $z = t+iy \in \mathfrak{S}_j$, then $\sin(jy) \neq 0 \neq \sin y$. Condition ii on h (with $N=0$) gives, for some constant $\nu_0' > 0$,

$$\begin{aligned}
&|\Psi(z,w)| \\
&\leq \nu_0'|\sin(jy)|^{2(1-n)}|\sin y|^{2(1-j)m}\left(1+\left|\frac{\sinh(jt)}{\sin(jy)}\right|\right)^{1/2}\frac{e^{ry}}{\left[\sinh\left(j\frac{w}{2}\right)\right]^{1/2}} \\
&= (\text{function bounded on compact subsets of } \mathfrak{S}_j)\cdot\left[\sinh\left(j\tfrac{w}{2}\right)\right]^{-1/2}.
\end{aligned}$$

Since $w \mapsto \left[\sinh\left(j\frac{w}{2}\right)\right]^{-1/2}$ is integrable on $(0,\infty)$, Theorem 3.9 implies that $\mathcal{A}_j^+ h$ is holomorphic on $\exists_j = \Re^+ \cup \mathfrak{S}_j$.

If $(z,w) \in \Re_j^+ \times (0,\infty)$, we have $|z+w| \geq |z|$, $|\sinh(j(z+w))| \geq |\sinh(jz)|$, and $|\coth(j(z+w))| \leq \sqrt{2} + |\coth(jz)|$. The growth condition for h then implies: For every integer $N \geq 0$ and $z \in \Re_j^+$

$$\begin{aligned}
&|\mathcal{A}_j^+ h(z)| \\
&\leq \sqrt{\tfrac{j}{2}}\, \nu_N \left(\frac{1+\sqrt{2}+|\coth(jz)|}{|\sinh(jz)|}\right)^{n-1} \left(\frac{1+\sqrt{2}+|\coth z|}{|\sinh z|}\right)^{(j-1)m} \\
&\qquad \cdot (1+|z|)^{-N} e^{r|\Im z|} \int_0^\infty \left[\sinh\left(j\tfrac{w}{2}\right)\right]^{-1/2} dw \\
&\leq \sigma_N \left(\frac{1+|\coth(jz)|}{|\sinh(jz)|}\right)^{n-1} \left(\frac{1+|\coth z|}{|\sinh z|}\right)^{(j-1)m} (1+|z|)^{-N} e^{r|\Im z|},
\end{aligned}$$

with

$$\sigma_N := 3^{[n-1+(j-1)m]/2} \sqrt{\tfrac{j}{2}}\, \nu_N \int_0^\infty \left[\sinh\left(j\tfrac{w}{2}\right)\right]^{-1/2} dw.$$

Property 3 is a consequence of Lemma 3.10. In fact, h is holomorphic and even on S_j, and if $|\Re z| \geq 1$ and $|\Im z| < \pi/j$, then

$$\begin{aligned}
|h(z)| &\leq \nu_2 \frac{(1+|\coth(jz)|)^{n-1}}{|\sinh(jz)|^n} \left(\frac{1+|\coth z|}{|\sinh z|}\right)^{(j-1)m} (1+|z|)^{-2} e^{r|\Im z|} \\
&\leq \nu_2' \, |\sinh(jz)|^{-1/2} (1+|z|)^{-2}.
\end{aligned}$$

Finally, Property 4 follows immediately because $\mathcal{A}_j^c h$ and $\dfrac{\mathcal{A}_j^+ h}{[\sinh(jz)]_+^{1/2}}$ are both holomorphic on $S_j \setminus (-\infty, 0]$ and agree with $\mathcal{A}_j h$ on $(0,\infty)$. $\square$

Proof of Theorem 2.1 (Necessity). Let $g \in \mathcal{H}_+^r(\mathbb{R})$ and let $f := \mathcal{A}^{-1} g$. Suppose first that m_α is even and $m_{2\alpha} = 0$ (that is $j = 1$ and $J = 2$). Then (up to a constant multiple) $f = D_1^{m_\alpha/2} g$, and Proposition 3.7 (with $j = 1$ and $n = m_\alpha/2$) proves that f extends to an even meromorphic function F on $\mathbb{C}$ satisfying the condition stated in Theorem 2.2.

Suppose now that $J = 1$, i.e. that either m_α is odd (so $j = 1$) or m_α is even and $m_{2\alpha}$ is odd ($j = 2$). Then (up to a constant multiple)

$$f = \mathcal{A}^{-1} g = \mathcal{A}_j D_j^n D_1^{(j-1)m} g$$

with $n = (m_{j\alpha}+1)/2$ and $m = m_\alpha/2$. Because of Proposition 3.7, we can apply Proposition 3.12 to the function $h = D_j^n D_1^{(j-1)m} g$.

For $z \in \exists_j$, set

$$\tilde{F}(z) := (\sinh z)^{(j-1)m_\alpha/2} (\sinh(jz))^{(m_{j\alpha}-1)/2} \mathcal{A}_j^+ h(z).$$

Observe that the exponents $(j-1)m_\alpha/2$ and $(m_{j\alpha}-1)/2$ are nonnegative integers, so $\tilde{F}(z)$ is holomorphic on $\exists_j$. Moreover, for $t \in (0,\infty)$,

$$\begin{aligned}\tilde{F}(t) &= (\sinh t)^{(j-1)m_\alpha/2}(\sinh(jt))^{(m_{j\alpha}-1)/2}\mathcal{A}_j^+ h(t)\\ &= (\sinh t)^{(j-1)m_\alpha/2}(\sinh(jt))^{(m_{j\alpha}-1)/2}(\sinh(jt))^{1/2}\mathcal{A}_j h(t)\\ &= (\sinh t)^{m_\alpha/2}(\sinh(2t))^{m_{2\alpha}/2} f(t).\end{aligned}$$

The growth condition of $\mathcal{A}_j^+ h$ on $\Re_j^+$ given by Proposition 3.12 determines the growth estimate 2 for $\tilde{F}$.

Let $(\sinh(jz))^{m_{j\alpha}/2} := (\sinh(jz))^{(m_{j\alpha}-1)/2}[\sinh(jz)]_+^{1/2}$ if $m_{j\alpha}$ is odd. For $z \in S_j \setminus (-\infty, 0]$,

$$\begin{aligned}F(z) :&= \frac{\tilde{F}(z)}{(\sinh z)^{m_\alpha/2}(\sinh(2z))^{m_{2\alpha}/2}}\\ &= \frac{(\sinh z)^{(j-1)m_\alpha/2}(\sinh(jz))^{(m_{j\alpha}-1)/2}\mathcal{A}_j^+ h(z)}{(\sinh z)^{m_\alpha/2}(\sinh(2z))^{m_{2\alpha}/2}}\\ &= \frac{\mathcal{A}_j^+ h(z)}{[\sinh(jz)]_+^{1/2}}\\ &= \mathcal{A}_j^c h(z).\end{aligned}$$

Condition 1 then follows from the equality $\mathcal{A}_j^c h(t) = \mathcal{A}_j D_j^n D_1^{(j-1)m} g(t) = f(t)$ $(t \in \mathbb{R})$ and from Proposition 3.12. □

3.2. Sufficiency. Before completing the proof of Theorem 2.1, we give, following Rouvière, the explicit form of the Abel transform $\mathcal{A}f$ of a function $f \in \mathcal{S}_+^\rho(\mathbb{R})$. Let dX (resp. dX') denote the Lebesgue measure on $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{2\alpha}$) corresponding to the Euclidean structure induced by the inner product $(X,Y) := -B(X,\theta Y)$, where B is the Cartan-Killing form and θ is the Cartan involution of $\mathfrak{g}$. Via $SU(2,1)$-reduction, Rouvière proved the following theorem.

Theorem 3.13. [5] *Let $f \in \mathcal{S}_+^\rho(\mathbb{R})$. Then there is a constant C so that*

$$\mathcal{A}f(t) = C \int\limits_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} \Phi f\left(\left[(\cosh t + |X|^2)^2 + |X'|^2\right]^{1/2}\right) dX\, dX', \qquad t \in \mathbb{R},$$

where $\Phi f(\cosh t) := f(t)$. When $m_{2\alpha} = 0$, disregard the variable X' and the integration over $\mathfrak{g}_{2\alpha}$.

For a fixed normalization of the Haar measure dn of N, the constant C can be explicitely determined as a function of the multiplicities m_α and $m_{2\alpha}$.

[5] [**Rou83**], p. 272,(8). See also p. 283.

For our purposes it is more appropriate to have a different expression for $\mathcal{A}f(t)$, $t \in (0,\infty)$. We first pass to spherical coordinates on $\mathfrak{g}_\alpha$ and on $\mathfrak{g}_{2\alpha}$, and then perform a change of variables in the integral that takes Φf back to f. Finally, we replace f by $\tilde{f}$. The cases $m_{2\alpha} = 0$ and $m_{2\alpha} \neq 0$ are kept separated.

Case $m_{2\alpha} = 0$: For all $t \in (0,\infty)$

$$\begin{aligned}
\mathcal{A}f(t) &= C \int_{\mathfrak{g}_\alpha} \Phi f(\cosh t + |X|^2)\, dX \\
&= 2\, C\pi^{m_\alpha/2}\Gamma\big(\tfrac{m_\alpha}{2}\big)^{-1} \int_0^\infty \Phi f(\cosh t + r^2)\, r^{m_\alpha - 1}\, dr \\
&= C' \int_0^\infty f(t+w) \left[\sinh\!\big(t+\tfrac{w}{2}\big) \sinh\big(\tfrac{w}{2}\big)\right]^{\frac{m_\alpha}{2}-1} \sinh(t+w)\, dw \\
&= C' \int_0^\infty \tilde{f}(t+w) \left[\frac{\sinh\!\big(t+\frac{w}{2}\big)\sinh\big(\frac{w}{2}\big)}{\sinh(t+w)}\right]^{\frac{m_\alpha}{2}-1} dw,
\end{aligned}$$

where $C' = 2\left(\frac{\pi}{2}\right)^{m_\alpha/2} \Gamma\left(\frac{m_\alpha}{2}\right)^{-1} C$.

Case $m_{2\alpha} \neq 0$: For all $t \in (0,\infty)$

$$\begin{aligned}
&\mathcal{A}f(t) \\
&= C \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} \Phi f\left(\left[(\cosh t + |X|^2)^2 + |X'|^2\right]^{1/2}\right) dX\, dX' \\
&= \frac{4\, C\, \pi^{(m_\alpha + m_{2\alpha})/2}}{\Gamma\left(\frac{m_\alpha}{2}\right)\Gamma\left(\frac{m_{2\alpha}}{2}\right)} \int_0^\infty\!\!\int_0^\infty \Phi f\left(\left[(\cosh t + r^2)^2 + s^2\right]^{1/2}\right) r^{m_\alpha-1} s^{m_{2\alpha}-1} dr\, ds \\
&= C'' \int_0^\infty\!\!\int_0^\infty f(t+w+v) \left[\sinh\!\big(t+\tfrac{w}{2}\big)\sinh\big(\tfrac{w}{2}\big)\right]^{\frac{m_\alpha}{2}-1} \sinh(t+w) \\
&\qquad \cdot \left[\sinh(2(t+w)+v)\sinh v\right]^{\frac{m_{2\alpha}}{2}-1} \sinh(2(t+w+v))\, dw\, dv \\
&= C'' \int_0^\infty\!\!\int_0^\infty \tilde{f}(t+w+v) \left[\frac{\sinh\!\big(t+\frac{w}{2}\big)\sinh\big(\frac{w}{2}\big)}{\sinh(t+w+v)}\right]^{\frac{m_\alpha}{2}-1} \\
&\qquad \cdot \left[\frac{\sinh(2(t+w)+v)\sinh v}{\sinh(2(t+w+v))}\right]^{\frac{m_{2\alpha}}{2}-1} \frac{\sinh(t+w)}{\sinh(t+w+v)}\, dw\, dv,
\end{aligned}$$

where $C'' = 2\left(\frac{\pi}{2}\right)^{m_\alpha/2} \pi^{m_{2\alpha}/2} \left[\Gamma\left(\frac{m_\alpha}{2}\right)\Gamma\left(\frac{m_{2\alpha}}{2}\right)\right]^{-1} C$.

In the above integrals, the variable $r \in (0,\infty)$ has been replaced by the variable $w \in (0,\infty)$ defined by the relation

$$\cosh t + r^2 = \cosh(t+w),$$

and the variable $s \in (0,\infty)$ has been replaced by the variable $v \in (0,\infty)$ defined by the relation

$$\cosh^2(t+w) + s^2 = \cosh^2(t+w+v).$$

The constants C' and C'' do not affect the result we want to prove. We therefore disregard them.

The idea to prove the sufficiency of the conditions in Theorem 2.1 is the following. The hypothesis on f imposed by Theorem 2.1 involve the holomorphic extension $\tilde{F}$ to $\exists_j$ of the function $\tilde{f}(t) := \Delta(t)f(t)$, $t \in (0,\infty)$. The function $\tilde{f}$ also appears in the integrand of $\mathcal{A}f(t)$, $t \in (0,\infty)$. We formally extend $\mathcal{A}f$ to $\mathcal{A}F$ on $\exists_j$ by replacing $\tilde{f}(t)$ by $\tilde{F}(z)$, and the variable t by the variable z in the remaining hyperbolic sines. A little extra care is required when dealing with square roots. The growth condition for $\tilde{F}$ on $\Re_j^+$ is used to prove that $\mathcal{A}F$ is holomorphic on (some open neighborhood of) $\Re_j^+$. Condition 2 in Theorem 2.1 is employed to show that $\mathcal{A}F$ is even, which allows us to extend it to $\mathbb{C} \setminus i\frac{\pi}{j}\mathbb{Z}$. Finally, the growth condition is used again, to prove either that $\mathcal{A}F$ is bounded near each point in $i\frac{\pi}{j}\mathbb{Z}$ (and hence it is entire) or that $\mathcal{A}F$ is rapidly decreasing with exponential growth r.

Definition 3.14. Let $f \in \mathcal{S}_+^\rho(\mathbb{R})$ satisfy the conditions stated in Theorem 2.1. For $z \in \exists_j$, formally define

$$\mathcal{A}F(z) := \int_0^\infty \tilde{F}(z+w)\,[s_1(z,w)]^{\frac{m_\alpha}{2}-1}\,dw, \qquad \text{if } m_{2\alpha} = 0,$$

$$\mathcal{A}F(z) := \int_0^\infty\int_0^\infty \tilde{F}(z+w+v)\,[s_2(z,w,v)]^{\frac{m_\alpha}{2}-1}\,[s_3(z,w,v)]^{\frac{m_{2\alpha}}{2}-1} \cdot \frac{\sinh(z+w)}{\sinh(z+w+v)}\,dw\,dv, \qquad \text{if } m_{2\alpha} \neq 0,$$

where

$$s_1(z,w) := \frac{\sinh\bigl(z+\frac{w}{2}\bigr)\sinh\bigl(\frac{w}{2}\bigr)}{\sinh(z+w)},$$

$$s_2(z,w,v) := \frac{\sinh\bigl(z+\frac{w}{2}\bigr)\sinh\bigl(\frac{w}{2}\bigr)}{\sinh(z+w+v)},$$

$$s_3(z,w,v) := \frac{\sinh(2(z+w)+v)\sinh v}{\sinh(2(z+w+v))},$$

and $[*]^{\frac{m_{j\alpha}}{2}-1} := [*]^{\frac{m_{j\alpha}-1}{2}}[1/*]_+^{1/2}$ if $m_{j\alpha}$ is odd.

Remark 3.15. By definition, the function $\mathcal{A}F$ extends, up to a constant multiple, $\mathcal{A}f(t)$, $t \in (0,\infty)$. Observe that the square roots appearing in the formula when $m_{j\alpha}$ is odd are well-defined single-valued holomorphic functions of $z \in \exists_j$ for all $v, w \in (0,\infty)$.

Lemma 3.16. *Define*

$$s^*(x) := \begin{cases} x^{-1} & \text{if } x \in (0,1) \\ 1 & \text{if } x \in [1,\infty). \end{cases}$$

Then there is a constant $C > 0$ such that for every $\zeta = a + ib$ with $a \geq 0$ and for every $x \in (0,\infty)$

$$\max\left\{ \left|\frac{\sinh(\zeta+2x)}{\sinh(\zeta+x)\sinh x}\right|, 1 + |\coth(\zeta+x)| \right\} \leq C\, s^*(x).$$

Lemma 3.17. *Let $\Psi(z,w)$ and $\Psi(z,w,v)$ denote respectively the integrands of $\mathcal{A}F(z)$ when $m_{2\alpha} = 0$ and $m_{2\alpha} \neq 0$. Then, for every integer $N \geq 0$ there are constants $\eta_N, \mu_N > 0$ such that for all $z \in \Re_j^+$ and $v, w \in (0,\infty)$*

$$|\Psi(z,w)| \leq \eta_N (s^*(w))^{(2-J)/2}(1+|z+w|)^{-N} e^{r|\Im z|},$$
$$|\Psi(z,w,v)| \leq \mu_N (s^*(w))^{3/4}(s^*(v))^{3/4}(1+|z+w+v|)^{-N} e^{r|\Im z|}.$$

Proof. Observe first that there is a constant $C > 0$ so that for all $(\zeta, x) \in \Re_j^+ \times (0,\infty)$ and $j = 1, 2$

$$\left|\frac{\sinh(j\zeta+x)\sinh x}{\sinh(j\zeta+2x)}\right| (1 + |\coth(j\zeta+2x)|) \leq C. \tag{3.5}$$

When $m_\alpha > 1$ and $m_{2\alpha} = 0$, the exponent $(m_\alpha/2) - 1$ of the function $s_1(z,w)$ in Definition 3.14 is positive. The growth condition for $\tilde{F}$ gives: For every integer $N \geq 0$ there is a constant $\tau_N > 0$ such that

$$\begin{aligned}
&|\Psi(z,w)| \\
&\leq \tau_N |s_1(z,w)|^{(m_\alpha/2)-1}\big(1+|\coth(z+w)|\big)^{(m_\alpha-J)/2}(1+|z+w|)^{-N}e^{r|\Im z|} \\
&\leq \tau_N \left[|s_1(z,w)|\big(1+|\coth(z+w)|\big)\right]^{(m_\alpha/2)-1}\big(1+|\coth(z+w)|\big)^{(2-J)/2} \\
&\qquad \cdot (1+|z+w|)^{-N}e^{r|\Im z|} \\
&\leq \eta_N (s^*(w))^{(2-J)/2}(1+|z+w|)^{-N}e^{r|\Im z|}.
\end{aligned}$$

If $m_\alpha = 1$ and $m_{2\alpha} = 0$, then $J = 1$ and

$$\begin{aligned} |\Psi(z,w)| &= \left|\tilde{F}(z+w)\right| \left|\frac{\sinh(z+w)}{\sinh\left(z+\frac{w}{2}\right)\sinh\left(\frac{w}{2}\right)}\right|^{1/2} \\ &\le \tau'_N (1+|z+w|)^{-N} e^{r|\Im z|} (s^*(w/2))^{1/2} \\ &\le \eta_N (s^*(w))^{1/2} (1+|z+w|)^{-N} e^{r|\Im z|}. \end{aligned}$$

Suppose now $m_{2\alpha} > 1$. In this case the exponents of both functions $s_2(z,w,v)$ and $s_3(z,w,v)$ are positive. The growth condition for $\tilde{F}$ gives: For every integer $N \ge 0$ there is a constant $\tau_N > 0$ such that

$$\begin{aligned} &|\Psi(z,w,v)| \\ &\le \tau_N |s_2(z,w,v)|^{(m_\alpha/2)-1} |s_3(z,w,v)|^{(m_{2\alpha}/2)-1} \left|\frac{\sinh(z+w)}{\sinh(z+w+v)}\right| \\ &\quad \cdot (1+|\coth(z+w+v)|)^{m_\alpha/2} (1+|\coth(2(z+w+v))|)^{(m_{2\alpha}-1)/2} \\ &\qquad \cdot (1+|z+w+v|)^{-N} e^{r|\Im z|} \\ &\le \tau'_N \left[\, |s_2(z,w,v)| \, (1+|\coth(z+w)|) \,\right]^{(m_\alpha/2)-1} (1+|\coth(z+w+v)|) \\ &\qquad \cdot \left[|s_3(z,w,v)| \, (1+|\coth(2(z+w+v))|)\right]^{(m_{2\alpha}/2)-1} \\ &\qquad \cdot (1+|\coth(2(z+w+v))|)^{1/2} (1+|z+w+v|)^{-N} e^{r|\Im z|} \\ &\le \tau''_N \, (1+|\coth(z+w+v)|) \, (1+|\coth(2(z+w+v))|)^{1/2} \\ &\qquad \cdot (1+|z+w+v|)^{-N} e^{r|\Im z|} \qquad\qquad \text{by Inequality (3.5)} \\ &\le \mu_N (s^*(w))^{3/4} (s^*(v))^{1/4} (s^*(2v))^{1/2} (1+|z+w+v|)^{-N} e^{r|\Im z|} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 3.16} \\ &\le \mu_N (s^*(w))^{3/4} (s^*(v))^{3/4} (1+|z+w+v|)^{-N} e^{r|\Im z|}. \end{aligned}$$

When $m_{2\alpha} = 1$, the exponent of $s_3(z,w,v)$ is $-1/2$. Hence

$$\begin{aligned} &|\Psi(z,w,v)| \\ &= |\tilde{F}(z+w+v)| |s_2(z,w,v)|^{(m_\alpha/2)-1} \\ &\qquad \cdot \left|\frac{\sinh(2(z+w+v))}{\sinh(2(z+w)+v)\sinh v}\right|^{1/2} \left|\frac{\sinh(z+w)}{\sinh(z+w+v)}\right| \\ &\le \tau'_N |s_2(z,w,v)|^{(m_\alpha/2)-1} \, (1+|\coth(z+w+v)|)^{m_\alpha/2} \, (s^*(v))^{1/2} \\ &\qquad \cdot (1+|z+w+v|)^{-N} \, e^{r|\Im z|} \\ &\le \tau''_N \left[\, |s_2(z,w,v)| \, (1+|\coth(z+w+v)|) \,\right]^{(m_\alpha/2)-1} \\ &\qquad \cdot (1+|\coth(z+w+v)|)(s^*(v))^{1/2} \, (1+|z+w+v|)^{-N} \, e^{r|\Im z|} \\ &\le \mu_N (s^*(w))^{3/4} (s^*(v))^{1/4} (s^*(v))^{1/2} (1+|z+w+v|)^{-N} e^{r|\Im z|} \end{aligned}$$

$$\text{by Inequality (3.5) and Lemma 3.16}$$
$$\leq \mu_N(s^*(w))^{3/4}(s^*(v))^{3/4}(1+|z+w+v|)^{-N}e^{r|\Im z|}.$$

□

Proposition 3.18. *Let f and $\mathcal{A}F$ be as in Definition 3.14. Then $\mathcal{A}F$ is holomorphic in $\Re^+$ and continuous on $\Re_j^+$. Moreover, for every integer $N \geq 0$ there is a constant $\sigma_N \geq 0$ such that for all $z \in \Re_j^+$*

$$|\mathcal{A}F(z)| \leq \sigma_N(1+|z|)^{-N}e^{r|\Im z|}.$$

Proof. Let $\Psi(z,w)$ and $\Psi(z,w,v)$ be as in Lemma 3.17. The assumption on $\tilde{F}$ and Remark 3.15 ensure that they are holomorphic functions of $z \in \exists_j$ and continuous functions on $\exists_j \times (0,\infty)$ and $\exists_j \times (0,\infty) \times (0,\infty)$, respectively. The estimates in Lemma 3.17, Theorem 3.9 and the Dominated Convergence Theorem prove that the function $\mathcal{A}F$ is holomorphic in $\Re^+$ and continuous in $\Re_j^+$.

To determine the growth of $\mathcal{A}F$ on $\Re_j^+$, observe that if M is an even integer ≥ 4, $l \in (0,1)$, and $\Re\zeta \geq 0$, then there is a constant $C > 0$ so that

$$\int_0^\infty (s^*(x))^l(1+|\zeta+x|)^{-M}dx \leq C\,(1+|\zeta|)^{-M+1}. \tag{3.6}$$

Indeed, there exist constants C_1 and C_2 such that

$$\int_0^1 (s^*(x))^l(1+|\zeta+x|)^{-M}dx \leq (1+|\zeta|)^{-M}\int_0^1 x^{-l}\,dx \leq C_1(1+|\zeta|)^{-M}$$

and[6]

$$\begin{aligned}\int_1^\infty (s^*(x))^l(1+|\zeta+x|)^{-M}dx &= \int_1^\infty (1+|\zeta+x|)^{-M}dx \\ &\leq \int_0^\infty (1+|\zeta|^2+x^2)^{-M/2}dx \\ &= \frac{1\cdot 3\cdot 5\cdots(M-3)}{2\cdot 4\cdot 6\cdots(M-2)}\,\frac{\pi}{2}\,(1+|\zeta|^2)^{-(M-1)/2} \\ &\leq C_2(1+|\zeta|)^{-M+1}.\end{aligned}$$

[6] [**Dwi61**], Formula 856.21. The formula can be applied because of the assumption that $M/2$ is an integer ≥ 2.

For every integer $N \geq 0$, choose

$$M := \begin{cases} N+4+2(j-1) & \text{if } N \text{ is even} \\ N+3+2(j-1) & \text{if } N \text{ is odd.} \end{cases}$$

Then M is an even integer ≥ 4.

If $m_{2\alpha} = 0$ $(j=1)$, the estimate in Lemma 3.17 yields

$$\begin{aligned} |\mathcal{A}F(z)| &\leq \eta_M e^{r|\Im z|} \int_0^\infty (s^*(w))^{(2-J)/2}(1+|z+w|)^{-M}\,dw \\ &\leq \eta_M'(1+|z|)^{-M+1}e^{r|\Im z|} \qquad \text{by (3.6)} \\ &\leq \sigma_N(1+|z|)^{-N}e^{r|\Im z|}. \end{aligned}$$

If $m_{2\alpha} \neq 0$ $(j=2)$, set $\Psi_1(z,v) = \int_0^\infty \Psi(z,w,v)\,dw$. Then

$$\begin{aligned} |\Psi_1(z,v)| &\leq \mu_M(s^*(v))^{3/4}e^{r|\Im z|}\int_0^\infty (s^*(w))^{3/4}(1+|z+w+v|)^{-M}\,dw \\ &\leq \mu_M'(s^*(v))^{3/4}(1+|z+v|)^{-M+1}e^{r|\Im z|} \qquad \text{by (3.6)} \\ &\leq \mu_M'(s^*(v))^{3/4}(1+|z+v|)^{-M+2}e^{r|\Im z|}, \end{aligned}$$

and, since $M-2$ is again an even integer ≥ 4,

$$\begin{aligned} |\mathcal{A}F(z)| &\leq \int_0^\infty |\Psi_1(z,v)|\,dv \\ &\leq \mu_M' e^{r|\Im z|}\int_0^\infty (s^*(v))^{3/4}(1+|z+w|)^{-M+2}\,dv \\ &\leq \mu_M''(1+|z|)^{-M+3}e^{r|\Im z|} \qquad \text{by (3.6)} \\ &\leq \sigma_N(1+|z|)^{-N}e^{r|\Im z|}. \end{aligned}$$

□

We now prove that, under the above assumptions, $\mathcal{A}F(iy)$ is a real analytic function of y on the interval $I_k := \left(k\frac{\pi}{j}, (k+1)\frac{\pi}{j}\right)$ for every integer k. Thus $\mathcal{A}F$ extends holomorphically across each vertical segment iI_k. The proof is an application of the classical criterion for which a C^∞ function g is real analytic on an open interval $I \subset \mathbb{R}$ if and only if for every compact $K \subset I$ there is a constant $M > 0$ such that

$$\left|\frac{d^h g}{dy^h}(y)\right| \leq M^{h+1}h!$$

for all $y \in K$ and all integers $h \geq 0$.

Lemma 3.19. *Let $j = 1, 2$, $z \in \mathfrak{S}_j$ and $a, b, c \in [0, \infty)$ with $b \neq 0$. Define*

$$s(j, z, a, b, c) := \frac{\sinh(jz + a + b)\sinh b}{\sinh(jz + a + 2b + c)}.$$

Then there are functions $l(j, z)$ and $m(j, z)$ which are bounded on compact subsets of $\mathfrak{S}_j$ such that

$$|s(j, z, a, b, c)| \leq l(j, z)$$
$$|s(j, z, a, b, 0)|^{-1} \leq m(j, z) s^*(b)$$

for all j, z, a, b, c (s^* is the function defined in Lemma 3.16).

Lemma 3.20. *Let f and $\mathcal{A}F$ be as in Proposition 3.18. Then $\mathcal{A}F(iy)$ is a real analytic function of $y \in I_k := \left(k\frac{\pi}{j}, (k+1)\frac{\pi}{j}\right)$ for every integer k.*

Proof. Observe first that if $s_1(z, w)$, $s_2(z, w, v)$, $s_3(z, w, v)$ are as in Definition 3.14 and if $s(j, z, a, b, c)$ is as in Lemma 3.19, then

$$s(1, z, 0, w/2, 0) = \frac{\sinh\left(z + \frac{w}{2}\right)\sinh\left(\frac{w}{2}\right)}{\sinh(z + w)} = s_1(z, w)$$
$$s(1, z, 0, w/2, v) = \frac{\sinh\left(z + \frac{w}{2}\right)\sinh\left(\frac{w}{2}\right)}{\sinh(z + w + v)} = s_2(z, w, v)$$
$$s(2, z, 2w, v, 0) = \frac{\sinh(2(z + w) + v)\sinh v}{\sinh(2(z + w + v))} = s_3(z, w, v).$$

Moreover, for $z = t + iy \in \mathfrak{S}_j$ and $w, v \in (0, \infty)$ we have

$$\left|\frac{\sinh(z + w)}{\sinh(z + w + v)}\right| \leq \frac{\cosh t}{|\sin y|}.$$

Set

$$S_1(z, w) := [s_1(z, w)]^{(m_\alpha/2)-1}$$
$$S_2(z, w, v) := [s_2(z, w, v)]^{(m_\alpha/2)-1}[s_3(z, w, v)]^{(m_{2\alpha}/2)-1}\frac{\sinh(z + w)}{\sinh(z + w + v)}.$$

Then S_1 and S_2 are holomorphic functions of $z = t + iy \in \mathfrak{S}_j$. Suppose first $m_\alpha \neq 1$ and $m_{2\alpha} \neq 1$. If $l(j, z)$ is the function in Lemma 3.19, we have

$$|S_1(z, w)| \leq [l(1, z)]^{(m_\alpha/2)-1} \tag{3.7}$$

$$|S_2(z, w, v)| \leq [l(1, z)]^{(m_\alpha/2)-1}[l(2, z)]^{(m_{2\alpha}/2)-1}\frac{\cosh t}{|\sin y|}. \tag{3.8}$$

Observe that the right-hand sides of (3.7) and (3.8) are bounded on the compact subsets of $\mathfrak{S}_j$ and do not depend on w, v.

Let k be an arbitrarily fixed integer. For simplicity, the dependence on the choice of k will be omitted in the following notation. For every $\delta \in (0, \pi/4j)$, consider the open half-strip

$$S_{j,\delta} := \left\{ z \in \mathbb{C} : \Re z > 0, \Im z \in \left(k\tfrac{\pi}{j} + 2\delta, (k+1)\tfrac{\pi}{j} - 2\delta \right) \right\}$$

and its left edge

$$I_{j,\delta} = i\left(k\tfrac{\pi}{j} + 2\delta, (k+1)\tfrac{\pi}{j} - 2\delta\right).$$

Let $Q_{j,\delta}$ be the open rectangle $\{z \in S_{j,\delta} : \Re z < 2\}$. Then the closed rectangle $R_{j,\delta}$ of vertices $(2+\delta)+i\left(\frac{\pi}{j}k+\delta\right)$, $(2+\delta)+i\left(\frac{\pi}{j}(k+1)-\delta\right)$, $-\delta+i\left(\frac{\pi}{j}(k+1)-\delta\right)$ and $-\delta + i\left(\frac{\pi}{j}k + \delta\right)$ contains $Q_{j,\delta}$ and is entirely contained in $\mathfrak{S}_j$.

Let K be the supremum of the right-hand sides of (3.7) and (3.8) over $z = t + iy \in R_{j,\delta}$. By Cauchy's Inequalities, for every integer $h \geq 0$,

$$\left| \frac{\partial^h}{\partial y^h} S_1(iy, w) \right| \leq \frac{K}{\delta^h} h! \qquad \text{and} \qquad \left| \frac{\partial^h}{\partial y^h} S_2(iy, w, v) \right| \leq \frac{K}{\delta^h} h! \tag{3.9}$$

for all $iy \in I_{j,\delta}$.

Suppose $z \in S_{j,\delta} \backslash Q_{j,\delta}$. Then the circle Γ centered at z with radius δ is entirely contained in the subset $D := \left\{ \zeta \in \mathbb{C} : \Re\zeta \geq 1, \Im\zeta \in \left(\frac{\pi}{j}k, \frac{\pi}{j}(k+1)\right) \right\}$ of $\Re_j^+$. The growth estimate of $\tilde{F}$ on $\Re_j^+$ with $N = 2j$ gives for all $\zeta \in D$

$$\begin{aligned} &|\tilde{F}(\zeta)| \\ &\leq \tau_{2j}(1 + |\coth\zeta|)^{(j-1)m_\alpha/2}(1 + |\coth(j\zeta)|)^{(m_{j\alpha}-J)/2}(1 + |\zeta|)^{-2j} e^{r|\Im\zeta|} \\ &\leq \tau'(1 + |\zeta|)^{-2j} \end{aligned}$$

for some constant τ' (depending on k). For $\zeta \in \Gamma$, $|\zeta| \geq |z| - \delta \geq |z| - 1 \geq |z|/2$, so $(1 + |\zeta|)^{-2j} \leq 2^{2j}(1 + |z|)^{-2j}$. Applying Cauchy's Integral formula, we obtain, for all $h \geq 0$,

$$\left|\tilde{F}^{(h)}(z)\right| \leq h! \int_\Gamma \frac{|\tilde{F}(\zeta)|}{|\zeta - z|^{h+1}} \frac{|d\zeta|}{2\pi} \leq h!\, 2^{2j} \frac{\tau'}{\delta^h} (1 + |z|)^{-2j}.$$

Therefore, for some constant τ'',

$$\left| \frac{\partial^h}{\partial y^h} \tilde{F}(iy + w + (j-1)v) \right| \leq h!\, \frac{\tau''}{\delta^h} (1 + w^2 + (j-1)v^2)^{-j} \tag{3.10}$$

for all integers $h \geq 0$, $iy \in I_{j,\delta}$ and $w, v \in (0, \infty)$ with $w + (j-1)v \geq 2$. Since $\tilde{F}$ is holomorphic on $\exists_j$, we can conclude that for every integer $h \geq 0$ there is a constant $\tilde{M} > 0$ such that for all $iy \in I_{j,\delta}$ and $w, v \in (0, \infty)$

$$\left| \frac{\partial^h}{\partial y^h} \tilde{F}(iy + w + (j-1)v) \right| \leq \frac{\tilde{M}}{\delta^h} h!\, (1 + w^2 + (j-1)v^2)^{-j}. \tag{3.11}$$

Let $\Psi(z,w)=\tilde{F}(z+w)S_1(z,w)$ and $\Psi(z,w,v)=\tilde{F}(z+w+v)S_2(z,w,v)$. Then for every integer $h\geq 0$ and all $iy\in I_{j,\delta}$, $w,v\in(0,\infty)$

$$\begin{aligned}\left|\frac{\partial^h}{\partial y^h}\Psi(iy,w)\right| &\leq \sum_{n=0}^{h}\binom{h}{n}\left|\frac{\partial^{h-n}}{\partial y^{h-n}}\tilde{F}(iy+w)\right|\left|\frac{\partial^n}{\partial y^n}S_1(iy,w)\right|\\ &\leq \sum_{n=0}^{h}\frac{h!}{n!(h-n)!}\left(\frac{\tilde{M}}{\delta^{h-n}}(h-n)!\right)(1+w^2)^{-1}\left(\frac{K}{\delta^n}n!\right)\\ &\leq (h+1)h!\,\frac{\tilde{M}K}{\delta^h}(1+w^2)^{-1}\\ &\leq h!\left(\frac{2}{\delta}\right)^h\tilde{M}K(1+w^2)^{-1}\end{aligned}$$

and, similarly,

$$\left|\frac{\partial^h}{\partial y^h}\Psi(iy,w,v)\right|\leq h!\left(\frac{2}{\delta}\right)^h\tilde{M}K(1+w^2+v^2)^{-2}.$$

If $m_\alpha=1$ or $m_{2\alpha}=1$, then for all $z=t+iy\in\mathfrak{S}_j$ and $w,v\in(0,\infty)$

$$\begin{aligned}|S_1(z,w)|&=|s_1(z,w)|^{-1/2}\leq m(1,z)^{1/2}(s^*(w/2))^{1/2},\\ |S_2(z,w,v)|&=|s_2(z,w,v)|^{(m_\alpha/2)-1}|s_3(z,w,v)|^{-1/2}\left|\frac{\sinh(z+w)}{\sinh(z+w+v)}\right|\\ &\leq (l(1,z))^{(m_\alpha/2)-1}(m(2,z))^{1/2}\frac{\cosh t}{|\sin y|}(s^*(v))^{1/2}\end{aligned}$$

where $l(j,z)$ and $m(j,z)$ are as in Lemma 3.19. If K is an upper bound for $(2m(1,z))^{1/2}$ and $(l(1,z))^{(m_\alpha/2)-1}(m(2,z))^{1/2}\cosh t|\sin y|^{-1}$ over all $z=t+iy\in R_{j,\delta}$, then

$$\left|\frac{\partial^h}{\partial y^h}S_1(iy,w)\right|\leq\frac{K}{\delta^h}h!(s^*(w))^{1/2}\quad\text{and}\quad\left|\frac{\partial^h}{\partial y^h}S_2(iy,w,v)\right|\leq\frac{K}{\delta^h}h!(s^*(v))^{1/2}$$

for all $iy\in I_{j,\delta}$ and $w,v\in(0,\infty)$. Computations as above therefore give

$$\begin{aligned}\left|\frac{\partial^h}{\partial y^h}\Psi(iy,w)\right|&\leq h!\left(\frac{2}{\delta}\right)^h\tilde{M}K(s^*(w))^{1/2}(1+w^2)^{-1}\\ \left|\frac{\partial^h}{\partial y^h}\Psi(iy,w,v)\right|&\leq h!\left(\frac{2}{\delta}\right)^h\tilde{M}K(s^*(v))^{1/2}(1+w^2+v^2)^{-2}.\end{aligned}$$

Differentiation under integral sign then proves that, for any multiplicities m_α and $m_{2\alpha}$, $\mathcal{A}F(iy)$ is C^∞ on each $I_{j,\delta}$ and that, for some constant $M>0$,

$$\left|\frac{d^h}{dy^h}\mathcal{A}F(iy)\right|\leq M^{h+1}h!\,.$$

Since the sets $I_{j,\delta}$ cover iI_k, $\mathcal{A}F(iy)$ is a real analytic function of $y \in I_k$. $\square$

Our last step is to prove that $\mathcal{A}F$ is even. We need the following lemma.

Lemma 3.21. *Let $j = 1, 2$, and let $n > 0$ and $m \geq 0$ be integers. Suppose h is an even holomorphic function on S_j with the following property. For every $\delta \in (0, \pi/j)$ there is a constant $C_\delta > 0$ such that for all z with $|\Im z| \leq \delta$*

$$|h(z)| \leq C_\delta |\sinh(jz)|^{-n/2} |\sinh z|^{-(j-1)m/2} (1 + |z|)^{-2[n+1+(j-1)m]}.$$

Then h satisfies the hypothesis of Lemma 3.10.

Moreover, for every $\delta \in (0, \pi/j)$ there is a constant $C'_\delta > 0$ such that whenever $|\Im z| \leq \delta$

$$|\mathcal{A}_j^c h(z)| \leq C'_\delta |\sinh(jz)|^{-(n-1)/2} |\sinh z|^{-(j-1)m/2} (1 + |z|)^{-2[n+(j-1)m]}.$$

Proof. Since h is holomorphic on S_j, the estimate on $|\Im z| \leq \delta$ describes the growth of $h(z)$ only for large values of $|\Re z|$, where

$$|\sinh(jz)|^{-n/2} |\sinh z|^{-(j-1)m/2} \leq |\sinh(jz)|^{-1/2}.$$

It is therefore clear that h satisfies the hypothesis of Lemma 3.10.

For $z \in S_j \setminus (-\infty, 0]$ and $w \in (0, \infty)$, let

$$\Psi(z, w) := h(z + w) \frac{\sinh(j(z + w))}{\left[\sinh\left(j\left(z + \frac{w}{2}\right)\right) \sinh\left(j \frac{w}{2}\right)\right]_+^{1/2}}.$$

Because of Lemma 3.16, if $|\Im z| \leq \delta$ and $\Re z \geq 0$, then

$$\begin{aligned}
|\Psi(z, w)| &\leq C_\delta \left| \frac{\sinh(j(z + w))}{\sinh\left(j \frac{w}{2}\right) \sinh\left(j \frac{w}{2}\right)} \right|^{1/2} |\sinh(j(z + w))|^{-(n-1)/2} \\
&\qquad \cdot |\sinh(z + w)|^{-(j-1)m/2} (1 + |z + w|)^{-2[n+1+(j-1)m]} \\
&\leq \hat{C}_\delta (s^*(w))^{1/2} |\sinh(jz)|^{-(n-1)/2} |\sinh z|^{-(j-1)m/2} \\
&\qquad \cdot (1 + |z + w|)^{-2[n+1+(j-1)m]}.
\end{aligned}$$

Formula (3.6) therefore implies: For $|\Im z| \leq \delta$ and $\Re z \geq 0$

$$\begin{aligned}
|\mathcal{A}_j^c h(z)| &\leq \hat{C}_\delta |\sinh(jz)|^{-(n-1)/2} |\sinh z|^{-(j-1)m/2} \\
&\qquad \cdot \int_0^\infty (s^*(w))^{1/2} (1 + |z + w|)^{-2[n+1+(j-1)m]} \, dw \\
&\leq C'_\delta |\sinh(jz)|^{-(n-1)/2} |\sinh z|^{-(j-1)m/2} (1 + |z|)^{-2[n+(j-1)m]}
\end{aligned}$$

for some constant C'_δ. Since $\mathcal{A}_j^c h(z)$ is even, this estimate holds also for $\Re z \leq 0$. $\square$

Proposition 3.22. *In the assumptions of Proposition 3.18, $\mathcal{A}F$ is an even holomorphic function on a neighborhood of $\Re_j^+$.*

Proof. The growth condition for $\tilde{F}$ and the fact that $1+|\coth(jz)|$ is bounded for large $|\Re z|$ imply that the even holomorphic function F satisfies the hypothesis for h in Lemma 3.21 with $n = m_{j\alpha}$ and $m = m_\alpha$. The same lemma also ensures that the application of $\mathcal{A}_2^c$ to F $m_{2\alpha}$-times and of $\mathcal{A}_1^c$ to $(\mathcal{A}_2^c)^{m_{2\alpha}}\, F$ m_α-times is legittimate and gives an even holomorphic function on S_j.

F holomorphically extends $f(t)$, $t \in \mathbb{R}$, to S_j. Hence $(\mathcal{A}_1^c)^{m_\alpha}\,(\mathcal{A}_2^c)^{m_{2\alpha}}\, F(z)$ holomorphically extends $\mathcal{A}_1^{m_\alpha}\mathcal{A}_2^{m_{2\alpha}} f(t)$, $t \in \mathbb{R}$, to S_j. Up to constant multiples, $\mathcal{A}F(z)$ holomorphically extends $\mathcal{A}f(t)$, $t \in (0,\infty)$ to some neighborhood U of $\Re_j^+$, and, because of Theorem 3.2, $\mathcal{A}f(t) = \mathcal{A}_1^{m_\alpha}\mathcal{A}_2^{m_{2\alpha}} f(t)$ on $\mathbb{R}$. Thus, up to a constant, $\mathcal{A}F(z)$ must agree with $(\mathcal{A}_1^c)^{m_\alpha}\,(\mathcal{A}_2^c)^{m_{2\alpha}}\, F$ on $S_j \cap U$, and, therefore, it is even. $\square$

Proof of Theorem 2.1 (Sufficiency). Proposition 3.18 and Lemma 3.20 proved that $\mathcal{A}F$ is holomorphic in a neighborhood of $\Re_j^+$. Because of Proposition 3.22, $\mathcal{A}F$ has to be even, so we can extend it holomorphically to $\mathbb{C}\setminus i\frac{\pi}{j}\mathbb{Z}$ by setting $\mathcal{A}F(z) := \mathcal{A}F(-z)$ if $\Re z < 0$. The growth condition proved in Proposition 3.18 therefore holds on all $\mathbb{C}\setminus i\frac{\pi}{j}\mathbb{Z}$. In particular, $\mathcal{A}F$ is bounded near each point in $i\frac{\pi}{j}\mathbb{Z}$, and therefore it extends to be entire. By continuity, the growth condition can be extended to $\mathbb{C}$ to become: For every integer $N \geq 0$ there is a constant $\sigma_N > 0$ such that for all $z \in \mathbb{C}$

$$|\mathcal{A}F(z)| \leq \sigma_N (1+|z|)^{-N} e^r |\Im z|.$$

Thus: If f satisfies the conditions stated in Theorem 2.1, then $\mathcal{A}f(t)$, $t \in \mathbb{R}$, extends to be an even entire function $\mathcal{A}F$ on $\mathbb{C}$ which is of exponential type r and rapidly decreasing. $\square$

4. Proof of Theorem 2.3.

The study of the spherical transform on complex groups is greatly simplified by an explicit formula for the elementary spherical functions. Let $(\mathfrak{a}_c^*)' := \{\lambda \in \mathfrak{a}_c^* : \langle \alpha, \lambda\rangle \neq 0 \text{ for all } \alpha \in \Sigma\}$, where $\langle\cdot,\cdot\rangle$ denotes the $\mathbb{C}$-bilinear extension to $\mathfrak{a}_c^*$ of the inner product in $\mathfrak{a}^*$ induced by the Cartan-Killing form. Then, for every $\lambda \in (\mathfrak{a}_c^*)'$ and $H \in \mathfrak{a}$

$$\Delta(H)\,\varphi_\lambda(\exp H) = 2^{-|\Sigma^+|}\,\frac{\pi(\rho)}{\pi(i\lambda)} \sum_{w\in W} (\det w) e^{iw\lambda(H)} \tag{4.1}$$

where Δ is given by (2.5), W is the Weyl group, $|\Sigma^+|$ denotes the cardinality of Σ^+, and π is the polynomial function on $\mathfrak{a}_c^*$ defined by

$$\pi(\nu) := \prod_{\alpha\in\Sigma^+} \langle \alpha, \nu\rangle. \tag{4.2}$$

Formula (4.1), due to Harish-Chandra, relates the spherical transform of a function $f \in \mathcal{S}(K\backslash G/K)$ to the Fourier transform of the function $H \longmapsto \Delta(H)f(\exp H)$ on $\mathfrak{a}$. Indeed, since

$$\Delta(wH) = (\det w)\Delta(H)$$

for all $H \in \mathfrak{a}$ and $w \in W$, we obtain from Formula (1.5): Up to a constant multiple, for all $\lambda \in (\mathfrak{a}^*)' := (\mathfrak{a}_c^*)' \cap \mathfrak{a}^*$,

$$\begin{aligned}
\mathcal{S}f(\lambda) &= \int_{\mathfrak{a}^+} f(\exp H)\varphi_{-\lambda}(\exp H)[\Delta(H)]^2\, dH \\
&= \frac{1}{|W|}\int_{\mathfrak{a}} \Delta(H)f(\exp H)\Delta(H)\varphi_{-\lambda}(\exp H)\, dH \\
&= \frac{2^{-|\Sigma^+|}}{|W|}\,\frac{\pi(\rho)}{\pi(-i\lambda)}\sum_{w\in W}\int_{\mathfrak{a}} (\det w)\Delta(H)f(\exp H)e^{-iw\lambda(H)}\, dH \\
&= 2^{-|\Sigma^+|}\,\frac{\pi(\rho)}{\pi(-i\lambda)}\int_{\mathfrak{a}} \Delta(H)f(\exp H)e^{-i\lambda(H)}\, dH.
\end{aligned}$$

Therefore, up to a constant depending only on $|\Sigma^+|$ and on the normalization of the measures,

$$\pi(-i\lambda)\mathcal{S}f(\lambda) = \mathcal{F}(\Delta(f\circ\exp))(\lambda) \tag{4.3}$$

for all $\lambda \in (\mathfrak{a}^*)'$. By continuity, (4.3) holds for all $\lambda \in \mathfrak{a}^*$.

Proof of Theorem 2.3. Since $\pi(-i\lambda)$ is a polynomial in λ, Formula (4.3) together with the classical Paley–Wiener theorem prove that $\mathcal{S}f$ is compactly supported, with $\operatorname{supp}\mathcal{S}f \subset B_r$, if and only if $\Delta(H)f(\exp H)$, $H \in \mathfrak{a}$, extends to an entire function $\tilde{F}$ on $\mathfrak{a}_c$ which is rapidly decreasing and of exponential type r. Set $F(\tilde{H}) := \frac{\tilde{F}(\tilde{H})}{\Delta(\tilde{H})}$, $\tilde{H} \in \mathfrak{a}_c$. Then F is a meromorphic function on $\mathbb{C}$ with singularities at most on the set

$$\{\tilde{H} \in \mathfrak{a}_c : \Delta(\tilde{H}) = 0\} = \bigcup_{\alpha\in\Sigma^+}\{\tilde{H} \in \mathfrak{a}_c : \alpha(\Re\tilde{H}) = 0 \text{ and } \alpha(\Im\tilde{H}) \in i\mathbb{Z}\}.$$

Since $\tilde{F}$ agrees with $\Delta(H)f(\exp H)$ on $\mathfrak{a}$, F extends to be holomorphic on the set $\{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for some } \alpha \in \Sigma^+\}$ by setting $F(H) := f(\exp H)$. So F extends $f(\exp H)$, $H \in \mathfrak{a}$. In particular, F must be W-invariant. □

We conclude this section with a remark on the Abel transform. For complex groups an explicit formula for the inverse Abel transform is available.

As proved by Gangolli[7] , if $f \in \mathcal{S}(K\backslash G/K)$, then, up to a constant multiple,

$$f(\exp H) = \prod_{\alpha\in\Sigma^+} \frac{1}{\sinh \alpha(H)}\, \partial(H_\alpha)\, \mathcal{A}f(\exp H) \tag{4.4}$$

where $H_\alpha \in \mathfrak{a}$ is uniquely determined by the condition $\alpha(H) = \langle H_\alpha, H\rangle$ for all $H \in \mathfrak{a}$, and $\partial(H_\alpha)$ is the corresponding differential operator on $\mathfrak{a}$.

Theorem 1.2 can be also proved using Formula (4.4) to characterize the functions f whose Abel transform extends to a rapidly decreasing entire function of exponential type r. However, doing so, we would not free ourselves from the use of the explicit espression for the elementary spherical functions. In fact, the only known general procedure to get (4.4) is to take the inverse Fourier transform of both sides of (4.3), using the property $\mathcal{S} = \mathcal{F} \circ \mathcal{A}$. Note that this is not the case for the rank-one groups. In fact, the explicit formulas for $\mathcal{A}$ and $\mathcal{A}^{-1}$ we used has been determined by Rouvière (and others) directly, without assuming any knowledge of the elementary spherical functions on the group.

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