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Sangwon Seo

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GLOBAL EXISTENCE AND DECREASING PROPERTY OF BOUNDARY VALUES OF SOLUTIONS TO PARABOLIC EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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It is shown that the local solution of parabolic equation with nonlocal boundary condition representing entropy can be extended to whole time domain for weights with large L^1 norms. When the weight is identically zero on some part of the boundary, it is shown that the boundary values can decrease even when the other weights are some large.

1. Introduction.

This paper is concerned with the investigation of large time behavior of solutions to parabolic initial value problem subject to nonlocal boundary condition which describes the entropy in a quasi-static theory of thermoelasticity, namely,

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + \mu u, \quad (x,t) \in \Omega \times (0,T), \\ u(z,t) &= \int_\Omega f(z,y)u(y,t)\,dy, \quad z \in \partial\Omega,\ 0 < t < T, \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where $u_0(x)$ are assumed to be continuous on Ω and μ is a constant. The function $f(z,y)$ is defined for $z \in \partial\Omega$ and $y \in \overline{\Omega}$ and continuous functions in $y \in \overline{\Omega}$ for each $z \in \partial\Omega$. Since for each $z \in \partial\Omega$, $f(z,y)$ plays the weight of integration in (1.1), the function $f(z,y)$ is called weights throughout this paper. Denote $D_T = \Omega \times (0,T)$ and $D_T \cup \Gamma_T = \overline{\Omega} \times [0,T)$. The variable z stands for a generic point of boundary $\partial\Omega$. The large time behavior of the solution u to Problem (1.1) is studied by taking an upper bound for u in Section 2. It is shown that the solution of Problem (1.1) with large weights $f(z,\cdot)$ for each $z \in \partial\Omega$ has an exponential lower bound in Section 3. Moreover, it is shown that the difference between maximum and minimum value on the boundary $\partial\Omega$ can decrease if the weights are zero on a nonempty subset of $\partial\Omega$ in z and zero on the boundary in y, that is, $f(z,y) = 0,\ z \in \Gamma \subset \partial\Omega$ for all $y \in \Omega$ and $f(z,y) = 0,\ y \in \partial\Omega$.

A function $u(x,t)$ is called a subsolution of Problem (1.1) on D_T if $u \in C^{2,1}(D_T) \cap C(D_T \cup \Gamma_T)$ satisfies

$$
\begin{aligned}
&u_t \le \Delta u + \mu u \quad \text{on} \quad D_T,\\
&u(z,t) \le \int_0^1 f(z,y)u(y,t)\,dy, \quad z \in \partial\Omega,\ 0 < t < T, \\
&u(x,0) \le u_0(x), \quad x \in \Omega.
\end{aligned}
\tag{1.2}
$$

A supersolution is defined by reversing inequalities in (1.2). Using the notions of super and subsolution, we state the following:

Theorem 1.1. *Comparison principle: Let $f(z,y)$ be nonnegative and continuous in y on $\overline{\Omega}$ for each $z \in \partial\Omega$. Let also u and v be subsolution and supersolution of Problem (1.1), respectively, and $u(x,0) < v(x,0)$ for $x \in \overline{\Omega}$. Then*

$$u < v \quad \text{in} \quad D_T.$$

As a corollary, $u_0(x) > 0 (< 0)$ implies $u(x,t) > 0 (< 0)$. Moreover, the local existence and uniqueness can be written as:

Theorem 1.2. *Let u_0 in (1.1) be continuous. Then for a small T, there is a unique solution of Problem (1.1) in $C^{2,1}(D_T) \cap C(D_T \cup \Gamma_T)$.*

For the proofs of the comparison principle and the local existence theorem, see [**4**]. Here one can see that the results hold without any restriction to constant μ in the parabolic equation in (1.1). In this paper, we are only interested in the positive solutions of Problem (1.1), and hence we assume that $u_0 > 0$. Throughout this paper, L^1 and L^2 norm of a function on Ω is denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

2. Global existence.

Under the convexity assumption on the domain Ω and the following assumptions on the weights;

$$\|f(z,\cdot)\|_1 < 1 \quad \text{for each} \quad z \in \partial\Omega, \tag{2.1}$$

it was known ([**3**]) that the maximum modulus $\max_{x\in\overline{\Omega}} |v(x,t)|$ of solution v to the parabolic equation,

$$v_t = \Delta v + \nu v, \quad \nu \le 0,$$

subject to the nonlocal boundary conditions in (1.1) decreases, moreover, it was shown the exponential decay of the solution; there are some constants c_1 and $\gamma > 0$ such that

$$\max_{x\in\overline{\Omega}} |v(x,t)| \le c_1 e^{-\gamma t}. \tag{2.2}$$

On the other hand, the author of [**2**] set up an example of one dimensional problem with $\mu = 0$ and weights that violate (2.1), such that its maximum modulus of solution increases exponentially. In this paper, basic assumption on the weights is that

$$f(z, y) \geq 0 \quad \text{for} \quad y \in \overline{\Omega},$$

for each $z \in \partial\Omega$. One of the main concern is to find an upper bound of solutions in (1.2) with nonnegative weights in the integrals of boundary conditions which guarantees the global existence. For Problem (1.1) with arbitrary nonnegative weights, the author could not find other results about global existence. For convenience, let the domain $\Omega = \{x = (x_1, x_2) \in R^2 : |x| < 1\}$. The results derived here can be extended to higher dimension without difficulties.

Theorem 2.1. *For the solution u to Problem (1.1) with nonnegative weights f, there is a sufficiently smooth function $S(x,t)$ such that*

$$u(x,t) < S(x,t), \quad t > 0. \tag{2.3}$$

The choice of the function $S(x,t)$ depends on the weights and $\sup_{x\in\overline{\Omega}} u_0(x)$.

Proof. Let $S = p + q$ with p and q such that

$$p(x,t) = Ae^{\kappa t + a(r-1)} \quad \text{and} \quad q(x,t) = Ae^{\kappa t - a(r+1)}, \quad x \in \overline{\Omega},\ t > 0, \tag{2.4}$$

where $r = |x| = \left(x_1^2 + x_2^2\right)^{1/2}$. The positive constants A, a and κ will be chosen later. Then p and q satisfy

$$\begin{aligned} p(z,t) &= Ae^{\kappa t}, \quad q(z,t) = Ae^{\kappa t}e^{-2a} \quad \text{for each} \quad z \in \partial\Omega, \\ \|p(\cdot,t)\|_2 &= \sqrt{2\pi}\left(\frac{1}{2a} - \frac{1}{4a^2} + \frac{e^{-2a}}{4a^2}\right)^{1/2} Ae^{\kappa t}, \\ \|q(\cdot,t)\|_2 &= \sqrt{2\pi}\left\{\frac{e^{-2a}}{4a^2} - \left(\frac{1}{2a} + \frac{1}{4a^2}\right)e^{-4a}\right\}^{1/2} Ae^{\kappa t}. \end{aligned}$$

If a satisfies

$$\sqrt{2\pi}\left\{\left(\frac{1+2a}{4a^2}\right)^{1/2} + \left(\frac{1}{2a}\right)^{1/2}\right\} \max_{z\in\partial\Omega}\{\|f(z,\cdot)\|_2\} \leq 1, \tag{2.5}$$

then one has

$$\{\|p(\cdot,t)\|_2 + \|q(\cdot,t)\|_2\}\,\|f(z,\cdot)\|_2 < Ae^{\kappa t} \quad \text{for each} \quad z \in \partial\Omega. \tag{2.6}$$

Note that

$$\begin{aligned}
&\frac{\partial S}{\partial r}(0,t) = 0, \quad t > 0,\\
ar^{-1}&\left\{e^{a(r-1)} - e^{-a(r+1)}\right\} \le ar^{-1}e^{a(r-1)},\\
&\lim_{r\to 0}\left\{ar^{-1}e^{a(r-1)}\right\} = a^2e^{-a},\\
&e^{a(r-1)} + e^{-a(r+1)} \ge 2e^{-a}.
\end{aligned}$$

Since $ar^{-1}e^{a(r-1)}$ is decreasing for $r < 1/a$ and increasing for $r > 1/a$, one has $ar^{-1}e^{a(r-1)} \le \max\{a^2e^{-a}, a\}$. Choose the constant a so large that (2.5) holds and,

$$a^2e^{-a} < a, \quad ar^{-1}e^{a(r-1)} \le a,$$

and take an integer $k \ge 2$ so that $2a^ke^{-a} > a$. Now let

$$\kappa \ge a^2 + a^k + \mu.$$

Then, by the choice of κ, k and a, we see that $S = p + q$ satisfies

$$\begin{aligned}
S_t &= \kappa S\\
&\ge a^kS + a^2S + \mu S\\
&= Ae^{\kappa t}a^k\left\{e^{a(r-1)} + e^{-a(r+1)}\right\} + a^2S + \mu S\\
&\ge Ae^{\kappa t}a^k(2e^{-a}) + a^2S + \mu S\\
&\ge Ae^{\kappa t}a + a^2S + \mu S\\
&\ge Ae^{\kappa t}ar^{-1}e^{a(r-1)} + a^2S + \mu S\\
&\ge Ae^{\kappa t}ar^{-1}\left(e^{a(r-1)} - e^{-a(r+1)}\right) + a^2S + \mu S\\
&= r^{-1}\frac{\partial S}{\partial r} + \frac{\partial^2 S}{\partial r^2} + \mu S\\
&= \Delta S + \mu S \quad \text{on} \quad D_T,
\end{aligned}$$

and by the inequality (2.6), we obtain for $z \in \partial\Omega$,

$$\begin{aligned}
S(z,t) = p(z,t) + q(z,t) &= A\left(e^{\kappa t} + e^{\kappa t}e^{-2a}\right)\\
&> Ae^{\kappa t}\\
&> \|f(z,\cdot)\|_2(\|p(\cdot,t)\|_2 + \|q(\cdot,t)\|_2)\\
&\ge \int_\Omega f(z,y)(p+q)\ dy\\
&= \int_\Omega f(z,y)S(y,t)\ dy, \quad 0 < t < T.
\end{aligned}$$

Note that above inequalities hold for arbitrary positive constant A. After choosing a and κ, let A satisfy $2Ae^{-a} > \sup_{x\in\overline{\Omega}} u_0(x)$. Then one has

$$S(x,0) = A\left\{e^{a(r-1)+e^{-a(r+1)}}\right\} \geq \sup_{x\in\overline{\Omega}} u_0(x).$$

Hence $S(x,t)$ is a supersolution to (1.1), and thus inequality (2.3) holds by Theorem 1.1. $\square$

By Theorem 2.1, we have a supersolution for any $T > 0$. Hence the local solution u on D_T from Theorem 1.2 is bounded in $\overline{D}_T$ for arbitrary $T > 0$, and thus u can be extended to the whole time domain.

3. Decreasing property of boundary values.

In this section, boundary behavior of the solution to Problem (1.1) is studied. The difference of largest and smallest boundary values grows exponentially (inequality (3.4)). When the weights are identically zero on some part of boundary, it is shown that the difference can be nonincreasing in Theorem 3.2.

From now on, it is assumed that $\mu \geq 0$. A lower bound of the solution of Problem (1.1) with weights satisfying for each $z \in \partial\Omega$,

$$f(z,y) \geq 0,\ y \in \overline{\Omega} \quad \text{and} \quad \|f(z,\cdot)\|_1 > 1, \tag{3.1}$$

can be obtained by the following Theorem:

Theorem 3.1. *If $\mu \geq 0$ and u is a solution to Problem (1.1) and the weight f satisfies (3.1), then there are positive constants c_2 and γ such that*

$$u(x,t) \geq c_2 e^{\gamma t} \quad on \quad D_T. \tag{3.2}$$

Proof. Let $v = u^{-1}$, where u is the positive solution to Problem (1.1). By denoting $F(z) = \|f(z,\cdot)\|_1$, we see that for arbitrary $T > 0$,

$$\begin{aligned} v_t &= u^{-2}\left(-\Delta u - \mu u\right) \\ &= \Delta v - 2\frac{|\nabla u|^2}{u^3} - \mu v \\ &\leq \Delta v - \mu v \quad \text{on} \quad D_T, \end{aligned}$$

and for $t > 0$,

$$\begin{aligned} v(z,t) &= \left(\int_\Omega f(z,y)u(y,t)\ dy\right)^{-1} \\ &\leq F^{-1}(z)\int_\Omega \frac{f(z,y)}{F(z)}u^{-1}(y,t)\ dy. \end{aligned}$$

Since $\|f(z,\cdot)\|_1/F^2(z) < 1$ for each $z \in \partial\Omega$ and $\mu \geq 0$, one has $v \leq c_1 e^{-\gamma t}$ for some positive constants c_1 and γ by (2.2) and Theorem 1.1. Therefore, the solution u to Problem (1.1) satisfies (3.2) with $c_2 = c_1^{-1}$. □

For weights satisfying (3.1), consider maximum and minimum of boundary values;

$$M(t) = \max_{z\in\partial\Omega} u(z,t), \quad m(t) = \min_{z\in\partial\Omega} u(z,t).$$

Let $M(t) = u(\alpha,t), m(t) = u(\beta,t)$ for some $\alpha,\beta \in \partial\Omega$. Then, if

$$f(\alpha,y) > f(\beta,y) \quad \text{for each} \quad y \in \Omega, \tag{3.3}$$

the difference $\delta(t) = M(t) - N(t)$ satisfies

$$\begin{aligned}\delta(t) &= \int_\Omega \{f(\alpha,y) - f(\beta,y)\}u(y,t)\ dy \\ &\geq c\{\|f(\alpha,\cdot)\|_1 - \|f(\beta,\cdot)\|_1\}e^{\gamma t},\end{aligned} \tag{3.4}$$

for some positive constant c by Theorem 3.1. Hence the difference of the boundary values become exponentially large if (3.1) and (3.3) are satisfied.

This increasing property fails when some weights are identically zero, that is, for some nonempty set $\Gamma \subset \partial\Omega$, $f(z,\cdot) \equiv 0$ for $z \in \Gamma$. To see this, assume that, for each $z \in \partial\Omega$, $f(z,y) \in C_y^2(\Omega)$ and that

$$f(z,y) \equiv 0 \quad \text{for each} \quad z \in \partial\Omega,\ y \in \partial\Omega. \tag{3.5}$$

By the assumption (3.5) and using integration by parts, we see that, for $\xi \in \Gamma^c$,

$$\begin{aligned}&\frac{du(\xi,t)}{dt} \\ &= -\int_{\partial\Omega} \nabla_n f(\xi,\omega)u(\omega,t)\ d\omega + \int_\Omega \{\Delta_y f(\xi,y) + \mu f(\xi,y)\}\, u(y,t)\ dy \\ &= \int_\Omega \left[\Delta_y f(\xi,y) + \mu f(\xi,y) - \int_{\partial\Omega} f(\omega,y)\nabla_n f(\xi,\omega)\ d\omega\right] u(y,t)\ dy \\ &= \int_\Omega I_\xi(f;y)u(y,t)\ dy.\end{aligned} \tag{3.6}$$

Therefore, if $I_\xi(f;y) \geq 0$ for each $y \in \overline{\Omega}$, then the boundary value $u(\xi,t)$ increases, and if $I_\xi(f;y) \leq 0$, then the boundary value decreases. Since we are interested in the weights for which the value $u(\xi,t)$ decreases, consider the nonnegative function g on Ω such that

$$\Delta g = \lambda g \quad \text{in} \quad \Omega, \tag{3.7}$$

and

$$g = 0 \quad \text{on} \quad \partial\Omega. \tag{3.8}$$

Define the characteristic function on $\partial\Omega = \Gamma \cup \Gamma^c$ by

$$\chi(z) = \begin{cases} 0 & \text{for } \; z \in \Gamma \\ 1 & \text{for } \; z \in \Gamma^c, \end{cases} \tag{3.9}$$

and let

$$h(z, y) = \chi(z) g(y). \tag{3.10}$$

Theorem 3.2. *If* Γ *is sufficiently large in the sense that*

$$\int_\Gamma \nabla_n g(\omega)\, d\omega \quad \textit{is sufficiently large,} \tag{3.11}$$

then, for sufficiently small $\mu \geq 0$*, the boundary values of the solution to Problem* (1.1) *with weights given by* (3.10) *are nonincreasing.*

Proof. For $\xi \in \partial\Omega$, $I_\xi(h, y)$ satisfies

$$\begin{aligned} I_\xi(h; y) &= \chi(\xi)\Delta g(y) + \mu\chi(\xi)g(y) - \int_{\partial\Omega} \chi(\omega)\chi(\xi)g(y)\nabla_n g(\omega)\, d\omega \\ &= \chi(\xi)\Delta g(y) + \mu\chi(\xi)g(y) - \chi(\xi)\int_{\Gamma^c} g(y)\nabla_n g(\omega)\, d\omega \\ &= \chi(\xi)\left\{\lambda + \mu - \int_{\Gamma^c} \nabla_n g(\omega)\, d\omega\right\} g(y). \end{aligned} \tag{3.12}$$

If the nonnegative constant μ is sufficiently small, then we can set $I_\xi(h; y)$ as nonpositive by taking Γ as sufficiently large in the sense of (3.11) while $\|h(z, \cdot)\|_1$ are large for $z \in \Gamma^c$. This can be done because the eigenvalue λ is fixed negative number and g is a smooth function. Therefore, by the identities (3.6), we get the conclusion. $\square$

On the other hand, since $g(y)$ is nonnegative for every $y \in \Omega$, $\nabla_n g(\omega)$ in (3.12) is nonpositive. Thus, if

$$\mu > -\lambda,$$

then $I_\xi(h; y) > 0$ on arbitrary proper subset Γ^c of $\partial\Omega$. Therefore, $u(\xi, t)$ is increasing for $\xi \in \Gamma^c$.

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Seoul National University
Seoul 151-742
Korea
E-mail address: sangwon@math.snu.ac.kr