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In this paper, we classify all generalized quadrangles weakly embedded of degree 2 in projective space. More exactly, given a (possibly infinite) generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ and a map π from $\mathcal{P}$ (respectively $\mathcal{L}$) to the set of points (respectively lines) of a projective space $\mathbf{P}G(V)$, V a vector space over some skew field (not necessarily finite-dimensional), such that:

(i) π is injective on points,

(ii) if $x \in \mathcal{P}$ and $L \in \mathcal{L}$ with $x \; I \; L$, then x^π is incident with L^π in $\mathbf{P}G(V)$,

(iii) the set of points $\{x^\pi \mid x \in \mathcal{P}\}$ generates $\mathbf{P}G(V)$,

(iv) if $x, y \in \mathcal{P}$ such that y^π is contained in the subspace of $\mathbf{P}G(V)$ generated by the set $\{z^\pi \mid z$ is collinear with x in $\Gamma\}$, then y is collinear with x in Γ,

(v) there exists a line of $\mathbf{P}G(V)$ not in the image of π and which meets $\mathcal{P}^\pi$ in precisely 2 points,

then we show that Γ is a Moufang quadrangle and we can explicitly describe the weak embedding of Γ in $\mathbf{P}G(V)$. This completes the classification of all weak embeddings of arbitrary generalized quadrangles (using the classification of Moufang quadrangles).

1. Introduction.

Weakly embedded polar spaces were introduced by Lefèvre-Percsy, see e.g., [**4**] (although she had a stronger notion of weak embedding, but it was proved to be equivalent with the present one by Thas & Van Maldeghem [**11**, Lemma 2]). In the same paper, she proves that the number of points of a weakly embedded polar space Γ on a secant line (i.e., a line of the projective space not belonging to the polar space and meeting $\mathcal{P}^\pi$ in at least two points) is a constant (and hence does not depend on that line). Following Thas & Van Maldeghem [**11**], we call this constant the *degree* of the weak embedding. In [**3**], Lefèvre-Percsy classifies the finite weakly embedded generalized quadrangles (which are the nondegenerate polar spaces of rank 2) in $\mathbf{P}G(3, q)$. All those thick weak embeddings have automatically degree > 2. In Thas & Van Maldeghem [**12**], all weakly embedded generalized quadrangles in finite projective space are classified. Also, Steinbach & Van Maldeghem [**9**] classify the weakly embedded generalized quadrangles of degree > 2 in arbitrary projective space. In the present paper, we complete the classification of all weakly embedded generalized quadrangles in any projective space by considering the case of degree 2. This has been an open problem for almost twenty years and it is a far-reaching generalization of a result of Dienst [**2**], who classifies all *full* embeddings of generalized quadrangles in arbitrary projective space. A *full* embedding satisfies conditions (i), (ii) and (iii) (of the abstract) and the additional condition that every point in $\mathbf{P}G(V)$ of the image of every line of the quadrangle is also the image of a point of the quadrangle. A direct and elementary argument then shows that a full embedding also satisfies condition (iv). Hence every full embedding is also a weak embedding. Dienst's result says that only the classical Moufang quadrangles turn up with their natural embedding in a (possibly degenerate) polarity, see Tits [**15**]. Asking for a further generalization (i.e., embeddings satisfying only conditions (i), (ii) and (iii) in the abstract above and calling this a *lax* embedding) is not reasonable, as is evidenced by the fact that one can then laxly embed a freely constructed finitely generated generalized quadrangle (in the sense of Tits [**18**]) in some projective space; see Section 8 below. Hence, our result is the best one can do and finishes the general problem. It also provides new and independent proofs for the full case (Dienst [**2**]) and the finite case (Thas & Van Maldeghem [**12**]).

Finding new techniques was essential because, unlike the finite case, there are generalized quadrangles which can be weakly embedded in projective space, but which do not admit a full embedding. In fact, all Moufang quadrangles can, up to duality, be weakly embedded, except for the exceptional ones (see below, also for a list of the generalized quadrangles, Γ say, such that Γ and the dual generalized quadrangle Γ^D is weakly embeddable). Hence the classification of weakly embedded general quadrangles requires methods which are different from those used in the finite or full case. One of the tools we use is the classification of all Moufang quadrangles, recently finished by Tits & Weiss [**20**], but not yet available in the literature. Without invoking this classification, our result remains true if restricted to all *known* generalized quadrangles. But we emphasize the fact that the part of the classification that we use, namely, that every Moufang quadrangle is in a well defined list (to the classes of Moufang quadrangles enumerated in Tits [**16**], one has to add the so-called *exceptional quadrangles of type* F_4, discovered by Richard Weiss in February 1997, and proved to be of exceptional type by Mühlherr & Van Maldeghem [**5**]), is completely finished; the yet unfinished parts in the manuscript of Tits & Weiss [**20**] merely concern the existence problem, which does not affect our proof.

Note that results of Steinbach [7] and Thas & Van Maldeghem [11] treat the same kind of question for polar spaces with some additional conditions. In all cases, the assumptions imply that the polar space is classical (i.e., arises from a vector space with form). In the present paper, we hypothesize an arbitrary generalized quadrangle weakly embedded of degree 2 in arbitrary projective space and prove that it must belong to the class of so-called Moufang quadrangles. Then we have to treat several classes (amongst them the classical cases). In the course of our proof, we slightly improve the result of Steinbach & Van Maldeghem [9] in that we determine when a weak embedding $\pi : \Gamma \to \mathbf{PG}(V)$ is obtained from a full embedding in a subspace of V (defined over a skew subfield of $\mathbb{K}$) by extending the ground field. In Section 8, we put together the results of the present paper with those of Steinbach & Van Maldeghem [9] and list all generalized quadrangles weakly embedded in projective space. We then also show that no further generalization is possible.

So the eventual determination of all weakly embedded generalized quadrangles of degree 2 requires some knowledge about the classification of Moufang quadrangles. We will introduce notation and repeat some known results in the next section.

2. Definitions and Notation.

2.1. Generalized quadrangles. A *generalized quadrangle* $\Gamma = (\mathcal{P}, \mathcal{L}, \ I\ )$ is a point-line incidence geometry (where $\mathcal{P}$ is the set of points and $\mathcal{L}$ the set of lines) satisfying the following two axioms:

(i) Each point is incident with $t+1$ lines; each line is incident with $s+1$ points; two distinct points are never incident with two distinct lines (here $s, t \geq 1$, possibly infinite).
(ii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ for which $x \ I \ M \ I \ y \ I \ L$.

The pair (s, t) is usually called the *order* of Γ. If $s, t > 1$, then the quadrangle is said to be *thick*. Furthermore, we use standard terminology such as *collinear points, concurrent lines,* etc. Also, there is a *duality* for generalized quadrangles: Every statement has a dual, i.e., if one interchanges the names point and line (and the numbers s and t), then a (usually new) statement is obtained. The dual of Γ is denoted by Γ^D. Further, the line M (respectively the point y) of (ii) is called the *projection* of L onto x (respectively of x onto L). A subquadrangle Γ' of Γ is called an *ideal subquadrangle*, if all lines of Γ incident with a point of Γ' belong to Γ' as well. Dually, one defines the notion of a *full subquadrangle*. In a generalized quadrangle, a line L is called *regular* if for every line M not meeting L, the two lines L and M are contained in a full subquadrangle with two lines per point.

Generalized quadrangles were introduced by Tits in [**14**]. For more information, we refer to the monograph of Payne & Thas [**6**], to Thas [**10**], or Van Maldeghem [**21**] (in the latter also the infinite case is covered).

There is no hope of classifying all generalized quadrangles (the situation is more or less the same as for projective planes), as there are (many variations of) free constructions of such geometries, see e.g., Tits [**18**]. Nevertheless, if one imposes some extra conditions, then classification is possible. Two such conditions are related to our Main Result, namely, the Moufang condition, and the condition of being weakly embedded in a projective space.

2.2. Moufang quadrangles. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a thick generalized quadrangle. We denote by $\Gamma(a)$ the set of elements of Γ incident with the element a (point or line). A *point-elation* is an automorphism of Γ fixing the set $\Gamma(x) \cup \Gamma(y) \cup \Gamma(L)$ elementwise, where x, y, $x \neq y$, are two distinct points incident with the line L. Such a collineation is also called an (x, L, y)-elation. If for some line $M \mathrm{I} x$, $M \neq L$, the group of all (x, L, y)-elations acts transitively on $\Gamma(M) \setminus \{x\}$, then we say that (x, L, y) is a *Moufang path*. Dually, one defines *line-elations* and *Moufang paths* (L, x, M). Let $x, y \in \mathcal{P}$, $L, M \in \mathcal{L}$. If the paths (x, L, y), for all choices of $x \mathrm{I} L \mathrm{I} y$, $x \neq y$ (respectively the paths (L, x, M) for all choices of $L \mathrm{I} x \mathrm{I} M$, $L \neq M$) are Moufang paths, then we say that Γ is a *half-Moufang quadrangle* and that all point-elation groups (respectively line-elation groups) *act transitively*. If all paths (x, L, y) and all paths (L, x, M) are Moufang paths, then we say that Γ is a *Moufang quadrangle*.

The standard examples of Moufang quadrangles are the *classical quadrangles*, i.e., generalized quadrangles related to a vector space with a form. Namely, let W be a (left) vector space over some skew field $\mathbb{L}$ endowed with either a pseudo-quadratic form or a (σ, ϵ)-hermitian form in the sense of Tits [**15**, §8]. Suppose that the from is nondegenerate and of Witt index 2. Define the geometry Γ with points and lines the 1-dimensional and 2-dimensional subspaces of W, where the form vanishes, and symmetrized inclusion as incidence. Then Γ is a generalized quadrangle. (We neglect the case that W is a 4-dimensional vector space endowed with an ordinary quadratic form, in wich Γ is not thick.)

When Γ arises from a $(\sigma, 1)$-quadratic from with $\sigma \neq 1$, then we will call such a quadrangle a *hermitian quadrangle*. When Γ arises from an ordinary quadratic form, we have an *orthogonal quadrangle*. The dual of such a Γ is a *dual hermitian* or a *dual orthogonal quadrangle*, respectively.

Proportional pseudo-quadratic froms give rise to the same generalized quadrangle. We see as follows that a hermitian quadrangle also arises from a $(\sigma', -1)$-quadratic form. Choose $a \in \mathbb{L}$ such that $c := a - a^\sigma \neq 0$. Then qc is a $(\sigma', -1)$-quadratic form, where $t^{\sigma'} = c^{-1} t^\sigma c$ for $t \in \mathbb{L}$. Furthermore, $1 \in \{t + t^{\sigma'} \mid t \in \mathbb{L}\}$.

The *mixed quadrangles* are certain subquadrangles of orthogonal quadrangles defined over a (non-perfect) field of characteristic 2, see Section 7 below. There is a further class of Moufang quadrangles not related to pseudo-quadratic or (σ, ϵ)-hermitian forms, the so-called *exceptional quadrangles.* We will not need an explicit description of these quadrangles (see the reduction in 4.4).

The classification of Moufang quadrangles by Tits & Weiss [**20**] yields that up to duality every Moufang quadrangle is isomorphic to a hermitian, an orthogonal, a mixed or an exceptional quadrangle.

For later use, we say a generalized quadrangle Γ is a *symplectic quadrangle*, if Γ arises from a 4-dimensional vector space over a commutative field endowed with an alternating form. A symplectic quadrangle is a dual orthogonal quadrangle (associated to a 5-dimensional vector space), but it is convenient to have a separate name for it.

For any (σ, ϵ)-hermitian form f, the *radical* of f is $\mathrm{Rad}(W, f) = \{v \in W \mid f(v, w) = 0 \text{ for all } w \in W\}$.

Let Γ be a generalized quadrangle and p a point in Γ. If a collineation fixes every point collinear with p, then we call that collineation a *central collineation* or a *central elation.* Dually, one defines an *axial elation* or *axial collineation.* Every Moufang quadrangle contains, up to duality, nontrivial central elations. This can easily be deduced from the main result of Tits [**19**].

2.3. Weak embeddings of generalized quadrangles. Let $\mathbf{PG}(V)$ be a projective space, where V is a vector space over some skew field (not necessarily finite-dimensional). The subspace of V generated by vectors $v_1, v_2, \ldots, v_n$ will be denoted by $\langle v_1, v_2, \ldots, v_n\rangle$; we also sometimes write $\mathbb{K}v$ for $\langle v\rangle$, when the vector space is defined over $\mathbb{K}$ and we want to make this clear.

Let Γ be a generalized quadrangle with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation I. Then we say that Γ *is weakly embedded in* $\mathbf{PG}(V)$ if there exists a map π from $\mathcal{P}$ (respectively $\mathcal{L}$) to the set of points (respectively lines) of $\mathbf{PG}(V)$, such that the following conditions are satisfied:

(i) π is injective on points,
(ii) if $x \in \mathcal{P}$ and $L \in \mathcal{L}$ with $x\, I\, L$, then x^π is incident with L^π in $\mathbf{PG}(V)$,
(iii) the set of points $\{x^\pi \mid x \in \mathcal{P}\}$ generates $\mathbf{PG}(V)$,
(iv) if $x, y \in \mathcal{P}$ such that y^π is contained in the subspace of $\mathbf{PG}(V)$ generated by the set $\{z^\pi \mid z \text{ is collinear with } x \text{ in } \Gamma\}$, then y is collinear with x in Γ.

Condition (iv) may be replaced by the (a priori weaker) condition (see Thas & Van Maldeghem [**13**], Corollary 1):

(iv)′ for each point $x \in \mathcal{P}$, the set $\{z^\pi \mid z \text{ is collinear with } x \text{ in } \Gamma\}$ does not generate $\mathbf{PG}(V)$.

Another equivalent statement is: The subspace spanned by $\{z^\pi \mid z$ collinear with x in $\Gamma\}$ is a hyperplane of $\mathbf{PG}(V)$.

The map π is called the *weak embedding*. It will sometimes be convenient to see a weak embedding as an injective morphism from the point-line geometry Γ to the geometry of 1- and 2-dimensional subspaces of a vector space (and then to write $\pi(x)$ instead of x^π for a point x). Also, for a given weak embedding π, we will denote by Γ^π the quadrangle whose points and lines are the images under π of the points and lines of Γ. The quadrangle Γ^π is a subgeometry of $\mathbf{PG}(V)$.

Let π be a weak embedding of Γ. A line of $\mathbf{PG}(V)$ which intersects the set of points of Γ^π in at least two elements, and which is not a line of Γ^π, is called a *secant line*. It has been shown by Lefèvre-Percsy [4] that the number of points of Γ^π on a secant line is a constant, and we call that constant the *degree*. In this paper, we will mainly be concerned with weakly embedded quadrangles of degree 2.

A *full* embedding π of a generalized quadrangle Γ in $\mathbf{PG}(V)$ is a weak embedding such that all points of $\mathbf{PG}(V)$ on a line of Γ^π are also points of Γ^π (this definition has been justified in the Introduction).

3. Main Result.

Main Result. *Let π be a weak embedding of degree 2 of a thick generalized quadrangle Γ in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$ (not necessarily finite-dimensional). Then Γ is a Moufang quadrangle. Up to isomorphism, we have the following cases:*

(1) *Γ is an orthogonal quadrangle (arising from an ordinary quadratic form) and the weak embedding is induced by an injective semi-linear mapping. In other words, there is a (commutative) subfield $\mathbb{F}$ of $\mathbb{K}$, a subspace V_0 of V (viewed as vector space over $\mathbb{F}$) and a quadratic form $Q : V_0 \to \mathbb{F}$ of Witt index 2, such that for each point x of Γ there is a unique point $\mathbb{F}x'$ of the associated quadric with $\pi(x) = \mathbb{K}x'$, and every point of the quadric arises in this way.*

If the radical of the corresponding bilinear form has (vector) dimension at most 1 (which happens for instance whenever $\mathrm{char}\mathbb{K} \neq 2$), then an $\mathbb{F}$-basis of V_0 is a $\mathbb{K}$-basis of V. Hence, in this case, π is obtained from a full embedding in $\mathbf{PG}(V_0)$ by extending the ground field.

(2) *Γ is a mixed quadrangle and the weak embedding is induced by an injective semi-linear mapping. In other words, there is a (commutative) subfield $\mathbb{F}$ of $\mathbb{K}$, a subspace V_0 of V (viewed as vector space over $\mathbb{F}$) and a quadratic form $Q : V_0 \to \mathbb{F}$ of Witt index 2, such that for each point x of Γ there is a unique point $\mathbb{F}x'$ of the associated quadric with $\pi(x) = \mathbb{K}x'$. Not every point of the quadric arises in this way.*

(3) Γ *is the unique generalized quadrangle of order* $(2,2)$ (*symplectic quadrangle* $W(2)$ *over* $\mathbf{GF}(2)$) *and the weak embedding is the universal weak embedding in the sense of* Thas & Van Maldeghem [**12**], *with* $\operatorname{char}\mathbb{K} \neq 2$.

In particular this means that Γ can never be a hermitian quadrangle, nor can Γ be isomorphic or dual to an exceptional quadrangle.

The Main Result will follow from the reduction in 4.4, and from Lemma 5.1, Lemma 5.5, Lemma 6.3, Lemma 6.4 and Lemma 7.2.

In the case where Γ is an orthogonal quadrangle weakly embedded (of degree 2) in $\mathbf{PG}(V)$, let Γ be associated to a vector space W over the field $\mathbb{L}$ and to the ordinary quadratic form q on W. In Case (1) of the Main Result, there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and an injective semi-linear mapping $\varphi : W \to V$ (with respect to α) such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all points $\mathbb{L}w$ of Γ (which means that the weak embedding $\pi : \Gamma \to \mathbf{PG}(V)$ is induced by a semi-linear mapping). In particular, Γ is fully embedded in the projective space $\mathbf{PG}(\varphi(W))$, where $\varphi(W)$ is a vector space over the (commutative) subfield $\alpha(\mathbb{L})$ of $\mathbb{K}$.

A central tool in the classification is the notion of regular lines (as defined above) in generalized quadrangles. Indeed, it is easily seen (as we will show in Lemma 4.2) that every line of a weakly embedded quadrangle of degree 2 is regular. Once the Moufang condition is proved, this will facilitate considerably the rest of the proof.

The paper is organized as follows. In the next section we reduce the problem to Moufang quadrangles and discuss which classes of Moufang quadrangles have regular lines. To prove the Moufang condition, we show that any generalized quadrangle weakly embedded of degree 2 admits axial elations, see Lemma 4.2. (Compare the approach in Steinbach & Van Maldeghem [**9**] and Steinbach [**8**], where the weakly embedded generalized quadrangles admit central elations.) In Section 5, we handle the case of orthogonal quadrangles (Cases (1) and (3) of the Main Result). In Section 6, we show that the only weak embeddings of degree 2 of the dual hermitian quadrangles are given by semi-linear mappings of orthogonal quadrangles. In Section 7, we deduce from Steinbach & Van Maldeghem [**9**, (6.1.3)] that for any weakly embedded mixed quadrangle Γ of degree 2, Γ^π is part of the null set of a quadratic form of Witt index 2 in a subspace V_0 of V (over a subfield $\mathbb{F}$ of $\mathbb{K}$). This is Case (2) of the Main Result. In Section 8, we state the complete classification of generalized quadrangles weakly embedded in projective space independent of the degree of the weak embedding. Then we show that a further generalization to lax embeddings is not possible. Furthermore, we mention a corollary on generalized quadrangles Γ with the property that both Γ and the dual Γ^D are weakly embeddable in projective space.

We remark that part of the proof of the Main Result is contained in other papers. Indeed, the classification of weak embeddings of generalized quadrangles arising from a vector space with a form has been done in Steinbach [**7**] and Steinbach & Van Maldeghem [**9**], independently of the degree of the weak embedding. The latter reference also covers the mixed case. It is only when it became clear to us that for degree 2, the Moufang condition can be proved, that a complete classification came into reach. This reduction to the Moufang quadrangles is the crux of the proof. It is based on a lemma of Steinbach [**8**].

4. Reduction to Moufang quadrangles.

For any point p of a generalized quadrangle Γ weakly embedded in $\mathbf{PG}(V)$, we denote by ξ_p the unique hyperplane of $\mathbf{PG}(V)$ spanned by the points x^π, with x collinear with p. A *special linear transformation* of V is an element of $\mathrm{SL}(V)$ (the subgroup of the group of all invertible linear mappings from V to V, which is generated by the transvections).

The following lemma is due to Steinbach [**8**, Proposition 2.1]. We phrase it a little differently, according to our needs.

Lemma 4.1 ([**8**]). *Let Γ be a generalized quadrangle weakly embedded of degree 2 in $\mathbf{PG}(V)$, where V is a vector space over some skew field $\mathbb{K}$. Let L be any line of Γ. Except for the universal weak embedding of $W(2)$, there is a unique subspace ξ_L of codimension 2 contained in all ξ_p with p on L.*

The subspace ξ_L in Lemma 4.1 is $\xi_L = \xi_p \cap \xi_q$, where p and q are different points on L.

Lemma 4.2. *Let Γ be a generalized quadrangle weakly embedded of degree 2 in $\mathbf{PG}(V)$, where V is a vector space over some skew field $\mathbb{K}$. Then all lines of Γ are regular. Also, Γ is a half-Moufang quadrangle. More exactly, for every line L of Γ, and every pair (x, x') of collinear points of Γ such that xx' is concurrent with L and x, x' not on L, there exists an axial elation with axis L mapping x to x'. Moreover, this elation is induced by a special linear transformation of V.*

Proof. First, if the weak embedding is the universal one of $W(2)$ (with $\mathrm{char}\mathbb{K} \neq 2$), then it is proved in Van Maldeghem [**21**, Section 8.6] that all collineations of Γ are induced by special linear transformations of V. Alternatively, this also follows from Thas & Van Maldeghem [**13**]. Hence from now on, we may assume that for any line L of Γ, there is a unique subspace ξ_L of codimension 2 contained in all ξ_y, for all points y of L. To simplify notation, we identify Γ with Γ^π in this proof.

Let M be any line of Γ not concurrent with L. Let $\mathbf{PG}(3, \mathbb{K})$ be the 3-dimensional projective space generated by L and M. Then $\mathbf{PG}(3, \mathbb{K})$ intersects Γ in a full subquadrangle Γ' weakly embedded in $\mathbf{PG}(3, \mathbb{K})$. Let

$t+1$ be the number of lines of Γ' through a point of Γ', then considering $u^\perp \cap v^\perp$ for two noncollinear points of Γ', we obtain a line of $\mathbf{PG}(3,\mathbb{K})$, which is not a line of Γ', meeting Γ' in precisely $t+1$ points. Since the degree of the weak embedding is 2, we obtain $t+1=2$ and L is a regular line.

Let a be the intersection of the lines xx' and L. Then $a \neq x, x'$, since x, x' are not on L. Let b be any other point of Γ on L. Let L' be any line of Γ through b, with $L' \neq L$. Let y and y' be the unique points of Γ on L' collinear with x and x', respectively. Notice that y and y' are not contained in ξ_a. Hence there exists a unique collineation θ_1 of $\mathbf{PG}(V)$ fixing all points of ξ_a, stabilizing all subspaces through b, and mapping y to y'. Similarly, there is a unique collineation θ_2 of $\mathbf{PG}(V)$ fixing all points of ξ_b, stabilizing all subspaces through a, and mapping x to x'. We put $\theta = \theta_1\theta_2$. Clearly θ fixes all points of $\xi_L(=\xi_a \cap \xi_b)$, it maps x to x', and it maps y to y'. Moreover, every subspace of $\mathbf{PG}(V)$ containing L is stabilized by θ, since it is stabilized by both θ_1 and θ_2.

Now let z be any point of Γ. We show that z^θ is a point of Γ. This is clear if z lies on L. Now suppose that z is incident with xy. Since the line L is regular, there is a unique point z' on $x'y'$ collinear in Γ with z and the line zz' meets L in, say, the point c of Γ. Since θ fixes the plane $\langle L, z\rangle$, it maps the intersection z of xy and $\langle L, z\rangle$ onto the intersection z' of $x'y'$ and $\langle L, z\rangle = \langle L, z'\rangle$.

Notice that the restriction of θ to ξ_c fixes all points in the hyperplane ξ_L of ξ_c (see Lemma 4.1); hence it must fix all subspaces of ξ_c through some point of ξ_c. Clearly this point must belong to zz'; also it must belong to L, since every subspace through L is stabilized. Hence this point is c. We conclude that θ leaves invariant every line of Γ meeting L.

Now let z be collinear with x, but z not on ax. We may suppose that z is not collinear with y. Let again $c \neq a$ be the projection of z onto L. Also, let w be the projection of d onto xz, where d is a point of Γ on L different from a and c, and let w' be the projection of x' onto dw. By the regularity of lines, the line $x'w'$ meets the line cz in, say, the point z'. Since wx is the unique line of $\mathbf{PG}(V)$ through x meeting both cz and dw, and since $w'x'$ is the unique line of $\mathbf{PG}(V)$ through x' meeting both cz and dw, the image of wx under θ is $w'x'$. It follows that $w^\theta = w'$ and $z^\theta = z'$.

Now let z be arbitrary, but not collinear with either x or a. We may assume that z is not on L. Let N_z be the projection of L onto z. Let N be any line on z different from N_z and from the projection of ax onto z. Let w be the projection of x onto N. Then w is collinear with x and not on ax, hence w^θ belongs to Γ by the previous step. Let N_a be the projection of N onto a. As before, N^θ is the unique line of $\mathbf{PG}(V)$ through w^θ meeting both N_a and N_z (which are both fixed by θ). By the regularity of L, N^θ is a line of Γ and it meets N_z in the point z' of Γ, with clearly $z' = z^\theta$.

Now let z be collinear with a. Then we interchange the roles of (a, x, x') and (b, y, y'), and we conclude that also in this case z^θ belongs to Γ.

Hence θ preserves Γ and induces clearly an axial elation in Γ with axis L. The lemma is proved. □

Lemma 4.3. *Let Γ be a generalized quadrangle weakly embedded of degree 2 in $\mathbf{PG}(V)$, for some skew field $\mathbb{K}$. Then Γ is a Moufang quadrangle and the little projective group of Γ is induced by $\mathbf{PSL}(V)$.*

Proof. We can copy word for word the proof of Lemma 4.0.2 of Steinbach & Van Maldeghem [**9**]. □

Reduction 4.4. We have shown that, in order to prove the Main Result, we have to classify the weak embeddings of degree 2 of Moufang quadrangles. Candidates must have regular lines. Using the classification of Moufang quadrangles as carried out by Tits & Weiss [**20**], Van Maldeghem discusses the regularity of points and lines of Moufang quadrangles in [**21**], Table 5.1. It follows that the only Moufang quadrangles with regular lines are the mixed quadrangles, the orthogonal quadrangles, and the duals of some hermitian quadrangles. The classification of weakly embedded mixed and orthogonal quadrangles has been treated in general (for every degree) in Steinbach & Van Maldeghem [**9**]. Sections 5 and 7 take care of them. In Section 6, we look at the dual hermitian case.

5. Weakly embedded orthogonal quadrangles of degree 2.

In this section, we handle the weakly embedded orthogonal quadrangles of degree 2. Let Γ be a thick orthogonal quadrangle with natural embedding in the projective space $\mathbf{PG}(W)$, where W is a vector space over the commutative field $\mathbb{L}$. So Γ arises from an ordinary quadratic form q on W. Let there be given a weak embedding $\pi : \Gamma \to \mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$. Then π is induced by a semi-linear mapping (with respect to the embedding $\alpha : \mathbb{L} \to \mathbb{K}$) by Steinbach [**7**], Steinbach & Van Maldeghem [**9**, (5.1.1)] (apart from the universal weak embedding of $W(2)$).

We show that there is a (commutative) subfield $\mathbb{F}$ of $\mathbb{K}$ and a subspace V_0 of V (viewed as vector space over $\mathbb{F}$) such that the point set of Γ^π is the point set of a projective quadric of Witt index 2 in $\mathbf{PG}(V_0)$.

Lemma 5.1. *Let Γ be an orthogonal quadrangle arising from the vector space W over the field $\mathbb{L}$, endowed with the ordinary quadratic form q. Assume that Γ is weakly embedded of degree 2 in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$ (not the universal weak embedding of $W(2)$).*

Then there exists a commutative subfield $\mathbb{F}$ of $\mathbb{K}$, a subspace V_0 of V (over $\mathbb{F}$) and a quadratic form $Q : V_0 \to \mathbb{F}$ of Witt index 2, such that for each point

x of Γ there is a point $\mathbb{F}x'$ of the associated quadric with $\pi(x) = \mathbb{K}x'$. Every point of the quadric arises in this way.

Proof. By Steinbach [**7**], Steinbach & Van Maldeghem [**9**, (5.1.1)], the weak embedding π is induced by a semi-linear mapping. Hence there exists an embedding $\alpha : \mathbb{L} \to \mathbb{K}$ and a semi-linear mapping $\varphi : W \to V$ such that $\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$, for all points $\mathbb{L}w$ of Γ.

Since the degree of π is 2, we may conclude as follows that φ is injective. By Steinbach & Van Maldeghem [**9**, Subsection 5.3], we know that $\ker\varphi \subseteq \mathrm{Rad}(W, f)$, where f is the bilinear form associated to the quadratic form q defining Γ. If $0 \neq r \in \mathrm{Rad}(W, f)$ with $\varphi(r) = 0$, then there exist vectors $x, y \in W$ such that $\mathbb{L}x$, $\mathbb{L}y$ and $\mathbb{L}(x + y + r)$ are (noncollinear) points of Γ. But then $\pi(\mathbb{L}(x + y + r))$ is a third point on the secant line spanned by $\pi(\mathbb{L}x)$ and $\pi(\mathbb{L}y)$, a contradiction.

We consider the subspace $V_0 := \varphi(W)$ of V (over the commutative subfield $\mathbb{F} := \alpha(\mathbb{L})$ of $\mathbb{K}$). We define a quadratic form $Q : V_0 \to \mathbb{F}$ of Witt index 2, by $Q(\varphi(w)) := \alpha(q(w))$, for $w \in W$. This yields an orthogonal space isomorphic to W. Now Lemma 5.1 is obvious. $\square$

The proof of Lemma 5.1 shows that the semi-linear mapping φ inducing π is injective. Hence for each $\mathbb{L}$-basis $\mathcal{B}$ of W, the set $\varphi(\mathcal{B})$ is an $\mathbb{F}$-basis of V_0. Our next aim is to decide when an $\mathbb{F}$-basis of V_0 is a $\mathbb{K}$-basis of V in Lemma 5.1.

General setting 5.2. We start with a more general setting (to prove the next lemma in full generality). Let W be a (left) vector space over some skew field $\mathbb{L}$ endowed with one of the following nondegenerate forms of Witt index 2:

(a) a pseudo-quadratic form q on W (with associated (σ, ϵ)-hermitian form f),
(b) a (σ, ϵ)-hermitian form f on W with $\Lambda_{\min} := \{c - \epsilon c^\sigma \mid c \in \mathbb{L}\} = \{c \in \mathbb{L} \mid \epsilon c^\sigma = -c\} =: \Lambda_{\max}$.

By Γ we denote the associated generalized quadrangle. Let $\pi : \Gamma \to \mathbf{PG}(V)$ be a weak embedding, where V is a vector space over the skew field $\mathbb{K}$. We assume that π is induced by a semi-linear mapping $\varphi : W \to V$ (with respect to the embedding $\alpha : \mathbb{L} \to \mathbb{K}$). (By Steinbach [**7**], Steinbach & Van Maldeghem [**9**, (5.1.1)] this is true up to few exceptions.) We set $\mathbb{F} := \alpha(\mathbb{L})$ and $V_0 := \varphi(W)$.

Recall that for any point p of the generalized quadrangle Γ, we denote by ξ_p the unique hyperplane of $\mathbf{PG}(V)$ spanned by the points $\pi(x)$, with x collinear with p. A vector $w \in W$, on which the form vanishes (i.e., $q(w) = 0$ or $f(w, w) = 0$, respectively), is called *singular*. This happens if and only if $\mathbb{L}w$ is a point of Γ.

Lemma 5.3. *In the general setting of 5.2 the following holds: If $\mathbb{L}p$ is a point of Γ and $w \in W$ such that $\varphi(w)$ is contained in $\xi_{\mathbb{L}p}$, then $f(w,p) = 0$.*

Proof. If w is singular, then the claim follows from the weak embedding axiom 2.3 (iv). Hence we may assume that w is nonsingular and $w \notin \mathrm{Rad}(W,f)$. Then there exist singular points $\mathbb{L}x$, $\mathbb{L}y$ with $f(x,y) = 1$ such that $w \in \mathbb{L}x + \mathbb{L}y$. We may assume $f(x,p) \neq 0$ and $f(y,p) \neq 0$. Indeed, if $f(x,p) = 0$, then $\varphi(x)$ and $\varphi(w)$, and therefore also $\varphi(y)$, are contained in $\xi_{\mathbb{L}p}$. This yields $f(y,p) = 0$ by the weak embedding axiom and hence $f(w,p) = 0$.

Set $H_1 = \langle p, p' \rangle$ with p' singular and $f(p,p') = 1$. Then $W = H_1 \perp {H_1}^{\perp}$ with ${H_1}^{\perp}$ spanned by singular points. We write $\mathbb{L}x = \mathbb{L}(\lambda p + p' + a_1)$, $\mathbb{L}y = \mathbb{L}(\mu p + p' + a_2)$ with $\lambda, \mu \in \mathbb{L}$ and $a_1, a_2 \in {H_1}^{\perp}$. Then there are $0 \neq c, d \in \mathbb{L}$ with $w = c(\lambda p + p' + a_1) + d(\mu p + p' + a_2)$. Applying φ, this yields that $(\alpha(c) + \alpha(d))\varphi(p')$ is contained in $\xi_{\mathbb{L}p}$. Thus $c + d = 0$ and $f(w,p) = 0$. □

Lemma 5.4. *In the general setting of 5.2 the following holds: If $\dim \mathrm{Rad}(W,f) \leq 1$, $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in W$ such that $\varphi(w_1), \ldots, \varphi(w_n)$ are linearly independent over $\mathbb{F}$, then $\varphi(w_1), \ldots, \varphi(w_n)$ are linearly independent over $\mathbb{K}$. In particular, an $\mathbb{F}$-basis of V_0 is a $\mathbb{K}$-basis of V in this case.*

Proof. We use induction on n, the case $n = 1$ is trivial. Let $n \geq 2$ and $w_1, \ldots, w_n \in W$ such that $\varphi(w_1), \ldots, \varphi(w_n)$ are linearly independent over $\mathbb{F}$. Then $w_1, \ldots, w_n$ are linearly independent over $\mathbb{L}$. There exists a singular point p in W with $\langle w_1, \ldots, w_n \rangle \not\subseteq p^{\perp}$. (Otherwise $\langle w_1, \ldots, w_n \rangle \subseteq \mathrm{Rad}(W,f)$, since W is spanned by its singular points. But this is a contradiction to the assumption on $\dim \mathrm{Rad}(W,f)$.) Write $\langle w_1, \ldots, w_n \rangle = \langle u_1, \ldots, u_n \rangle$ where $u_1, \ldots, u_{n-1} \in p^{\perp}$ and $u_n \notin p^{\perp}$. Then $X := \langle \varphi(w_1), \ldots, \varphi(w_n) \rangle_{\mathbb{K}} = \langle \varphi(u_1), \ldots, \varphi(u_n) \rangle_{\mathbb{K}}$. Further, $X \not\subseteq \xi_p$ by Lemma 5.3. We have $\langle \varphi(u_1), \ldots, \varphi(u_{n-1}) \rangle_{\mathbb{K}} \subseteq X \cap \xi_p$ and the latter is a hyperplane of X. Since $\langle \varphi(w_1), \ldots, \varphi(w_n) \rangle_{\mathbb{F}} = \langle \varphi(u_1), \ldots, \varphi(u_n) \rangle_{\mathbb{F}}$, we see that $\varphi(u_1), \ldots, \varphi(u_n)$ are linearly independent over $\mathbb{F}$. Hence by induction $\varphi(u_1), \ldots, \varphi(u_{n-1})$ are linearly independent over $\mathbb{K}$ and $n - 1 \leq \dim(X \cap \xi_p) \leq n - 1$. But then $X = (X \cap \xi_p) \oplus \langle \varphi(u_n) \rangle_K$ is n-dimensional. □

For weakly embedded orthogonal quadrangles, we have shown:

Lemma 5.5. *Let Γ be an orthogonal quadrangle arising from the vector space W over the field $\mathbb{L}$, endowed with the ordinary quadratic form q (with associated bilinear form f). Let $\pi : \Gamma \to \mathbf{PG}(V)$ be a weak embedding of degree 2, where V is a vector space over the skew field $\mathbb{K}$. Assume that π is induced by the semi-linear mapping $\varphi : W \to V$ (with respect to $\alpha : \mathbb{L} \to \mathbb{K}$). If $\mathrm{Rad}(W,f)$ has dimension at most 1 (which is true whenever $\mathrm{char}\mathbb{K} \neq 2$), then for any $\mathbb{L}$-basis $\mathcal{B}$ of W, the set $\varphi(\mathcal{B})$ is a $\mathbb{K}$-basis of V.*

Hence weak embeddings of degree 2 of orthogonal quadrangles are as described in Cases (1) and (3) of the Main Theorem. The next example shows that we cannot drop the assumption on the dimension of the radical $\mathrm{Rad}(W, f)$.

Example 5.6. Let $\mathbb{L}$ be a nonperfect field of characteristic 2. We consider the symplectic quadrangle over $\mathbb{L}$ and pass to an isomorphic quadrangle Γ arising from a quadratic form in vector space dimension $4 + \dim_{\mathbb{L}^2} \mathbb{L}$, see Cohen [**1**, (3.23), (3.27)]. If we extend scalars to the algebraic closure $\mathbb{K}$ of $\mathbb{L}$, then Γ is weakly embedded in vector space dimension 5.

6. Weakly embedded dual hermitian quadrangles of degree 2.

In this section, we handle the case of weakly embedded dual hermitian quadrangles of degree 2. We prove that there are exactly two possibilities, occurring from exceptional isomorphisms between certain dual hermitian and orthogonal quadrangles.

Notation 6.1. Let Δ be a hermitian quadrangle with *natural embedding* in the projective space $\mathbf{PG}(W)$, where W is a (left) vector space over the skew field $\mathbb{L}$. This means that Δ consists of the singular points and lines with respect to a $(\sigma, -1)$-quadratic form q with $1 \in \Lambda_{\min}$. (Recall that for $\epsilon = -1$, we have $\Lambda_{\min} = \{c + c^\sigma \mid c \in \mathbb{L}\}$ and $\Lambda_{\max} = \{c \in \mathbb{L} \mid c = c^\sigma\}$.)

If $\Lambda_{\min} = \Lambda_{\max}$ (which can only fail when $\mathrm{char}\mathbb{L} = 2$ and σ fixes the center $Z(\mathbb{L})$ of $\mathbb{L}$ elementwise), then Δ coincides with the polar space arising from the $(\sigma, -1)$-hermitian form f associated to q. If $x, y \in W$ with $q(x) = 0$, $q(y) = 0$ and $f(x, y) = 1$, then (x, y) is called a *hyperbolic pair*.

Remark 6.2. For any generalized quadrangle Γ weakly embedded of degree 2 in a projective space $\mathbf{PG}(V)$ (with weak embedding π), we use the following method to calculate image points: Recall from Lemma 4.2 that each line of Γ is regular. We write the nine points of a 3×3-grid of points of Γ in a 9-tuple consisting of the first, second and third row. Here $\cdot$ stands for a point without name. If $(x_1, x_2, x_3; y_1, y_2, \cdot; z_1, z_2, z_3)$ is such a 3×3-grid, then $\pi(z_3) = \langle \pi(z_1), \pi(z_2) \rangle \cap \langle \pi(x_3), \pi(y_1), \pi(y_2) \rangle$.

Lemma 6.3. *Let Γ be a generalized quadrangle weakly embedded of degree 2 in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$. Assume that Γ^D is a hermitian quadrangle with natural embedding in the projective space $\mathbf{PG}(W)$, where W is a vector space over the skew field $\mathbb{L}$ (see Notation 6.1).*

If the vector space dimension of W is 4, then $\Lambda_{\min}$ is a field and

(i) *$\mathbb{L}$ is a commutative separable quadratic extension of Λ_{min} (here $\Lambda_{\min} = \Lambda_{\max}$), or*

(ii) *$\mathbb{L}$ is a quaternion division ring over $\Lambda_{\min}$ with σ its standard involution (here $\Lambda_{\min} \subset \Lambda_{\max}$ in characteristic 2).*

In both cases Γ *is isomorphic to the orthogonal quadrangle associated to the vector space* $\mathbb{L}\times{\Lambda_{\min}}^4$ *(over* $\Lambda_{\min}$*) with the quadratic form* $(x_0, x_1, x_2, x_3, x_4) \mapsto x_0{x_0}^\sigma - x_1x_3 + x_2x_4$*,* σ *as above.*

Notice that $x_0{x_0}^\sigma = x_0 \cdot 1 \cdot {x_0}^\sigma \in \Lambda_{\min}$, since $1 \in \Lambda_{\min}$. The associated bilinear form is $b(x, y) = x_0{y_0}^\sigma + y_0{x_0}^\sigma - x_1x_3 - y_1y_3 + x_2x_4 + y_2y_4$ with trivial radical. The lemma reduces the dual hermitian case with $\dim W = 4$ to certain orthogonal quadrangles considered in the previous section.

Proof. We apply the method of Remark 6.2. The following is inspired by the proof of Tits [**15**, (10.2)]. By Tits [**15**, (10.5), (10.9)], the result follows from the assertion that all maps from $\Lambda_{\min}$ to $\Lambda_{\min}$ of the form $\lambda \mapsto c^\sigma\lambda c$, where $0 \neq c \in \mathbb{L}$, commute with each other.

The expression $c^\sigma\lambda c$ occurs in the following construction. We write $W = \langle v_1, w_1\rangle \perp \langle v_2, w_2\rangle$ with hyperbolic pairs (v_i, w_i), $i = 1, 2$. For each point x and each line L of the hermitian quadrangle Γ^D, we set $\rho_x(L) := \langle x, L\cap x^\perp\rangle$. For $\lambda \in \Lambda_{\min}$ and $0 \neq c \in \mathbb{L}$, we obtain

$$\begin{array}{rcl} \langle v_1, \lambda v_2 + w_2\rangle & \overset{\rho_{cw_1+v_2}}{\mapsto} & \langle cw_1 + v_2, c^{-\sigma}v_1 + \lambda v_2 + w_2\rangle \\ & \overset{\rho_{w_2}}{\mapsto} & \langle w_2, -\lambda cw_1 + c^{-\sigma}v_1\rangle \\ & \overset{\rho_{w_1+v_2}}{\mapsto} & \langle w_1 + v_2, c^{-\sigma}w_2 - \lambda cw_1 + c^{-\sigma}v_1\rangle \\ & \overset{\rho_{v_1}}{\mapsto} & \langle v_1, c^\sigma\lambda cv_2 + w_2\rangle \\ & =: & \tau_c(\langle v_1, \lambda v_2 + w_2\rangle). \end{array}$$

In view of the above, we have to show that

$$(*)\quad \tau_d(\tau_c(\langle v_1, \lambda v_2 + w_2\rangle)) = \tau_c(\tau_d(\langle v_1, \lambda v_2 + w_2\rangle)) \qquad \text{for } \lambda \in \Lambda_{\min},\ 0 \neq c, d \in \mathbb{L}.$$

We prove $(*)$ in several steps by calculation in Γ. Let $0 \neq c \in \mathbb{L}$ be fixed.

We choose notation such that the apartment (v_1, w_2, w_1, v_2) of the hermitian quadrangle Γ^D corresponds to the apartment (p, q, t, z) of Γ (this means that v_1 corresponds to the line pq, further $\langle v_1, w_2\rangle$ corresponds to q and so on). The line $\langle v_1, \lambda v_2 + w_2\rangle$ corresponds to some point on pq. The points $cw_1 + v_2$, $w_1 + v_2$ correspond to lines through z (different from pz and zt). By $\widetilde{a}$ and a, respectively, we denote the projections of q onto these lines.

We choose coordinates in Γ as follows. Let y_1 be a third point on pz. We set $y_2 := qa \cap {y_1}^\perp$, $e := zt \cap {y_2}^\perp$, $f := pq \cap e^\perp$. Then $y_2 \neq a, q$, $e \neq z, t$ and $f \neq p, q$. We choose vectors $p', q', t', z' \in V$ such that

$$\begin{array}{rclrclrcl} \pi(p) & = & \langle p'\rangle, & \pi(z) & = & \langle z'\rangle, & \pi(y_1) & = & \langle p' - z'\rangle, \\ \pi(q) & = & \langle q'\rangle, & \pi(a) & = & \langle a'\rangle, & \pi(y_2) & = & \langle a' - q'\rangle, \\ & & & \pi(t) & = & \langle t'\rangle, & \pi(e) & = & \langle t' + z'\rangle. \end{array}$$

Then there exists $0 \neq \alpha \in \mathbb{K}$ such that $\pi(f) = \langle q' - \alpha p' \rangle$. Replacing p' by $\alpha^{-1}p'$, z' by $\alpha^{-1}z'$ and t' by $\alpha^{-1}t'$, we may assume that $\pi(f) = \langle q' - p' \rangle$. Let $c_1 := qt \cap {y_1}^{\perp}$. Then $\pi(c_1) = \langle q' + t' \rangle$, using the method of Remark 6.2 with $(p, q, f; y_1, c_1, \cdot; z, t, e)$.

Set $\widetilde{y_2} := q\widetilde{a} \cap {y_1}^{\perp}$, $\widetilde{e} := zt \cap \widetilde{y_2}^{\perp}$, $\widetilde{f} := pq \cap \widetilde{e}^{\perp}$. As above, we choose vectors v'' in V spanning the images under π of $p, \dots, \widetilde{f}$. Then $\pi(y_1) = \langle p' - z' \rangle = \langle p'' - z'' \rangle$ and $\pi(c_1) = \langle q' + t' \rangle = \langle q'' + t'' \rangle$. Hence there exist $0 \neq \alpha, \beta \in \mathbb{K}$ such that $p'' = \alpha p'$, $z'' = \alpha z'$, $q'' = \beta q'$, $t'' = \beta t'$. We set $\gamma := \alpha^{-1}\beta$. Then γ is a scalar in $\mathbb{K}$ depending on $c \in \mathbb{L}$ (via $\widetilde{a}$). Note that γ is unique and well defined since one can calculate that $\pi(\widetilde{f}) = \langle \gamma q' - p' \rangle$.

We denote by r_λ the point on pq corresponding to the line $\langle v_1, \lambda v_2 + w_2 \rangle$ of the hermitian quadrangle Γ^D. Let $\mu := \mu(\lambda) \in \mathbb{K}$ with $\pi(r_\lambda) = \langle \mu p' + q' \rangle$. We name by $u_1, \dots, u_4$ the points of Γ that correspond to the lines in the hermitian quadrangle Γ^D occurring in the above calculation of $\tau_c(\langle v_1, \lambda v_2 + w_2 \rangle)$. Then

$$u_1 = z\widetilde{a} \cap {r_\lambda}^{\perp}, \;\; u_2 = qt \cap {u_1}^{\perp}, \;\; u_3 = za \cap {u_2}^{\perp}, \;\; u_4 = pq \cap {u_3}^{\perp}.$$

With β, γ from above, we obtain

$$(**) \qquad \begin{array}{ll} \pi(u_1) = \langle \mu z' + \beta^{-1} a'' \rangle, & \text{using } (r_\lambda, p, q; \cdot, y_1, \widetilde{y_2}; u_1, z, \widetilde{a}), \\ \pi(u_2) = \langle \mu\gamma t' - q' \rangle, & \text{using } (u_1, \widetilde{a}, z; \cdot, \widetilde{y_2}, \widetilde{e}; u_2, q, t), \\ \pi(u_3) = \langle \mu\gamma z' + a' \rangle, & \text{using } (u_2, q, t; \cdot, y_2, e; u_3, a, z), \\ \pi(u_4) = \langle \mu\gamma p' + q' \rangle, & \text{using } (u_3, z, a; \cdot, y_1, y_2; u_4, p, q). \end{array}$$

This yields that there is a scalar $\mu \in \mathbb{K}$ depending on $\lambda \in \Lambda_{\min}$ and a scalar $\gamma \in \mathbb{K}$ depending on $0 \neq c \in \mathbb{L}$, such that the mapping $\langle v_1, \lambda v_2 + w_2 \rangle \mapsto \tau_c(\langle v_1, \lambda v_2 + w_2 \rangle)$ in the hermitian quadrangle Γ^D reads in Γ as $\langle \mu p' + q' \rangle \mapsto \langle \mu\gamma p' + q' \rangle$. Let $0 \neq d \in \mathbb{L}$ and denote by δ the corresponding scalar in $\mathbb{K}$. To prove $(*)$, we have to show that the following equation holds in Γ:

$$\langle (\mu\gamma)\delta p' + q' \rangle = \langle (\mu\delta)\gamma p' + q' \rangle \qquad \text{for all } \mu = \mu(\lambda),\, \lambda \in \Lambda_{\min}.$$

We see that $\gamma\delta = \delta\gamma$ as follows. If we let vary λ in $\Lambda_{\min} \setminus \{0\}$ in the line $\langle v_1, \lambda v_2 + w_2 \rangle$ of the hermitian quadrangle Γ^D, then the corresponding point r_λ in Γ reaches all points on pq, different from p, q. Hence there exists $\lambda_0 \in \Lambda_{\min}$ with $r_{\lambda_0} = f$. Since $\pi(f) = \langle q' - p' \rangle$, we have $\mu(\lambda_0) = -1$.

The calculation in $(**)$ with $\mu = -1$ and γ and δ, respectively, shows that there are points r, g on pq and qt, respectively, with $\pi(r) = \langle -\gamma p' + q' \rangle$ and $\pi(g) = \langle -\delta t' - q' \rangle$. The projection s of r onto zt satisfies $\pi(s) = \langle t' + \gamma z' \rangle$ (use $(r, p, q; \cdot, y_1, c_1; s, z, t)$). The projection h of g onto pz satisfies $\pi(h) = \langle p' - \delta z' \rangle$ (use $(f, p, q; \cdot, h, g; e, z, t)$). The calculation of the remaining point j in the grid $(r, p, q; j, h, g; s, z, t)$ yields $\gamma\delta = \delta\gamma$.

Hence $(*)$ holds, and we obtain Lemma 6.3. This finishes the determination of Γ in the case where W is 4-dimensional. $\square$

The final step in this section is to show that in the situation of Lemma 6.3 the vector space W is necessarily 4-dimensional.

Lemma 6.4. *Let Γ be a generalized quadrangle weakly embedded of degree 2 in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$. Assume that Γ^D is a hermitian quadrangle with natural embedding in the projective space $\mathbf{PG}(W)$, where W is a vector space over the skew field $\mathbb{L}$ (see Notation 6.1).*

Then the vector space W is 4-dimensional. Hence the possibilities for Γ and Γ^D are determined by Lemma 6.3.

Proof. We identify Γ with Γ^π in this proof. Since Γ is weakly embedded of degree 2 in $\mathbf{PG}(V)$, all lines of Γ are regular (see Lemma 4.2). Hence Γ^D has regular points. By the discussion in Van Maldeghem [**21**], Table 5.1, we may conclude that $W = H_1 \perp H_2 \perp \mathrm{Rad}(W, f)$, where H_1, H_2 are hyperbolic lines and f is the underlying $(\sigma, -1)$-hermitian form.

We set $\Delta = \Gamma^D$. Let Δ' be the full subquadrangle of Δ obtained by intersecting the natural embedding of Δ with $\mathbf{PG}(H_1 \perp H_2)$. Our aim is to show that $\Delta' = \Delta$, then $W = H_1 \perp H_2$ is a 4-dimensional vector space.

Let $\Gamma' = (\Delta')^D$. Then Γ' is an ideal subquadrangle of Γ. Further, Γ' is weakly embedded in $\mathbf{PG}(V)$. To prove this, let p be a point of Γ'. Let ξ_p be spanned by the set of points of Γ collinear with p in Γ and let ξ'_p be spanned by the set of points of Γ' collinear with p in Γ'. Then ξ'_p is contained in ξ_p, which is a hyperplane of $\mathbf{PG}(V)$. If a point x of Γ is in ξ_p, then x is collinear with p in Γ. Since Γ' is an ideal subquadrangle of Γ, the line on p and x is a line of Γ' and x is collinear with p in Γ'. Hence $\xi'_p = \xi_p$. For q in Γ', q not collinear with p in Γ, this yields $\mathbf{PG}(V) = \langle \xi_p, q\rangle = \langle \xi'_p, q\rangle$ and the point set of Γ' generates $\mathbf{PG}(V)$.

Hence we may apply Lemma 6.3 for the weak embedding $\pi : \Gamma' \to \mathbf{PG}(V)$. We obtain that $\Lambda_{\min}$ is a field and $\mathbb{L}$ is quadratic or quaternion over $\Lambda_{\min}$. Now let Δ'' be a symplectic subquadrangle of Δ', obtained by restricting scalars to $\Lambda_{\min}$. Then Δ'' is an ideal subquadrangle of Δ'. Hence $\Gamma'' = (\Delta'')^D$ is a full subquadrangle of Γ' and Γ'' is weakly embedded in $\mathbf{PG}(V')$, where the latter is spanned by the set of points of Γ''. (Let p be a point of Γ''. If $\mathbf{PG}(V')$ is contained in ξ''_p, notation as above, then every point of Γ'' is collinear in Γ with p. This is a contradiction, since Γ'' contains ordinary quadrangles.)

Now $\mathbf{PG}(V')$ meets Γ in a full subquadrangle Γ''', which is weakly embedded in $\mathbf{PG}(V')$. (Let p be a point of Γ'''. With notation as above, $\mathbf{PG}(V')$ is not contained in ξ'''_p. If p is in Γ'', then $\xi''_p = \xi'''_p$ is a hyperplane of $\mathbf{PG}(V')$.) This implies that Γ'' is an ideal subquadrangle of Γ'''. But all points of Γ'' (a dual symplectic, hence mixed quadrangle) are regular, hence Γ''' has also regular points. Since all lines of Γ''' are regular (this even holds for Γ), Γ''' is a mixed quadrangle.

Now the symplectic quadrangle Δ'' is a full subquadrangle of the dual of Γ''', with the latter a mixed quadrangle. That implies $\Gamma''' = \Gamma''$ (indeed, with the notation of the next section, any full subquadrangle of the mixed quadrangle $Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ can be written as $Q(\mathbb{L}', \mathbb{L}_0^2; \Lambda', \Lambda_0^2)$, with $\mathbb{L}_0^2 \subseteq \Lambda' \subseteq \mathbb{L}' \subseteq \Lambda_0 \subseteq \mathbb{L}_0$ and $\mathbb{L}_0 \subseteq \mathbb{L}$ and $\Lambda_0 \subseteq \Lambda$, by Tits [**17**]; the latter is a symplectic quadrangle only if $\Lambda_0^2 = \mathbb{L}_0^2 = \mathbb{L}' = \Lambda'$, implying $\Lambda^2 = \mathbb{L}^2 = \mathbb{L}'$).

So Γ'' is a full subquadrangle of Γ. Since it is also a full subquadrangle of Γ', we deduce that Γ' is a full subquadrangle of Γ. But it is also an ideal subquadrangle. Consequently, $\Gamma' = \Gamma$ (cp. Van Maldeghem [**21**], (1.8.2)). Passing to the dual, we obtain $\Delta' = \Delta$. $\square$

7. Weakly embedded mixed quadrangles of degree 2.

Let Γ be a mixed quadrangle weakly embedded of degree 2 in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$. In this section, we deduce from Steinbach & Van Maldeghem [**9**, (6.1.3)] that there exists a commutative subfield $\mathbb{F}$ of $\mathbb{K}$ and a quadric of Witt index 2 over this subfield, such that Γ^π is part of that quadric (except for the universal weak embedding of $W(2)$).

7.1. Definition of mixed quadrangles. First, we recall the definition of a mixed quadrangle (introduced by Tits [**17**]), cp. Steinbach & Van Maldeghem [**9**, (6.1.1)]. Let $\mathbb{L}$ be a (commutative) field of characteristic 2 and let

$$\mathbb{L}^2 \subseteq \Lambda' \subseteq \mathbb{L}' \subseteq \Lambda \subseteq \mathbb{L},$$

where $\mathbb{L}'$ is a subfield of $\mathbb{L}$, Λ is a subspace of $\mathbb{L}$ considered as vector space over $\mathbb{L}'$ and Λ' is a subspace of $\mathbb{L}'$ considered as vector space over $\mathbb{L}^2$. We suppose that $\mathbb{L}$ and $\mathbb{L}'$ are generated as rings by Λ and Λ', respectively .

A mixed quadrangle is a certain subquadrangle of the symplectic quadrangle $W(\mathbb{L}')$. Passing from the symplectic quadrangle to an isomorphic orthogonal quadrangle, a mixed quadrangle is a subquadrangle of the orthogonal quadrangle $Q(W, q)$, associated to the vector space $W := \Lambda \times (\mathbb{L}')^4$ with usual scalar multiplication and the (nondegenerate) quadratic form $q : W \to \mathbb{L}'$ defined by $q((x_0; x_1, x_2, x_3, x_4)) = {x_0}^2 + x_1x_2 + x_3x_4$ for $x_0 \in \mathbb{L}$, $x_i \in \mathbb{L}'$ $(i = 1, \ldots, 4)$.

The mixed quadrangle $Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ consists of the points of $Q(W, q)$ spanned by vectors of the form

$$(0; 1, 0, 0, 0), \ (0; a, 0, 1, 0), \ (k; b, 0, k^2, 1), \ (l; l^2 + aa', 1, a', a),$$

where $a, b, a' \in \Lambda'$, $k, l \in \Lambda$.

Lemma 7.2. *Assume that the mixed quadrangle $\Gamma := Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ with $\mathbb{L} \neq \mathbf{GF}(2)$ is weakly embedded of degree 2 in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$.*

Then the weak embedding is induced by an injective semi-linear mapping. In other words, there exists a commutative subfield $\mathbb{F}$ *of* $\mathbb{K}$*, a subspace* V_0 *of* V *(over* $\mathbb{F}$*) and a quadratic form* $Q : V_0 \to \mathbb{F}$ *of Witt index* 2*, such that for each point* x *of* Γ *there is a point* $\mathbb{F}x'$ *of the associated quadric with* $\pi(x) = \mathbb{K}x'$*.*

Proof. Recall that $W = \Lambda \times (\mathbb{L}')^4$ and that W is endowed with the quadratic form $q : W \to \mathbb{L}'$. By Steinbach & Van Maldeghem [**9**, (6.1.3)] there exists an embedding $\alpha : \mathbb{L}' \to \mathbb{K}$ and a semi-linear mapping $\varphi : W \to V$ (with respect to α) such that $\ker\varphi \subseteq \Lambda$ (the kernel of the symplectic form associated to q) and $\pi(\mathbb{L}'w) = \mathbb{K}(\varphi(w))$ for all points $\mathbb{L}'w$ of Γ.

Since the degree of π is 2, we have $\varphi(k) \neq 0$, for all $0 \neq k \in \Lambda$. Namely, otherwise $\pi(\mathbb{L}'(k; 0, 0, k^2, 1))$ is a third point on the secant line spanned by $\pi(\mathbb{L}'(0; 0, 0, 1, 0))$ and $\pi(\mathbb{L}'(0; 0, 0, 0, 1))$, a contradiction. This yields that the semi-linear mapping φ is injective.

The lemma is now obvious with $\mathbb{F} := \alpha(\mathbb{L}')$, $V_0 := \varphi(W)$ and the quadratic form $Q : V_0 \to \mathbb{F}$ of Witt index 2 defined by $Q(\varphi(w)) := \alpha(q(w))$, for $w \in W$. □

8. Appendix.

In this appendix, we present the list of all weakly embedded generalized quadrangles, by putting together the results of the present paper and the results in Steinbach & Van Maldeghem [**9**]. We also mention a corollary on quadrangles Γ for which both Γ and its dual are weakly embeddable in some projective space. Finally, we show that certain finitely generated free generalized quadrangles admit lax embeddings in some finite dimensional projective space.

Recall from Subsection 2.2 that for any skew field $\mathbb{L}$ with involutory anti-automorphism σ and $\epsilon = \pm 1$, we have $\Lambda_{\min} := \{c - \epsilon c^\sigma \mid c \in \mathbb{L}\} \subseteq \Lambda_{\max} := \{c \in \mathbb{L} \mid \epsilon c^\sigma = c\}$.

8.1. Classification of generalized quadrangles weakly embedded in projective space. Let π be a weak embedding of a thick generalized quadrangle Γ in the projective space $\mathbf{PG}(V)$, where V is a vector space over the skew field $\mathbb{K}$ (not necessarily finite-dimensional). Then Γ is a Moufang quadrangle. Up to isomorphism, we have the following cases:

(1) Γ is a classical quadrangle arising as the geometry of points and lines of $\mathbf{PG}(W)$, W a (left) vector space over the skew field $\mathbb{L}$, where one of the following nondegenerate forms of Witt index 2 vanishes:
   (a) a pseudo-quadratic form $q : W \to \mathbb{L}/\Lambda_{\min}$ (with associated (σ, ϵ)-hermitian form f),
   (b) a (σ, ϵ)-hermitian form $f : W \times W \to \mathbb{L}$ with $\Lambda_{\min} = \Lambda_{\max}$.

   Furthermore, there exists a semi-linear mapping $\varphi : W \to V$ (with respect to an embedding $\alpha : \mathbb{L} \to \mathbb{K}$) with $\ker\varphi \subseteq \mathrm{Rad}(W, f)$ such that

$\pi(\mathbb{L}w) = \mathbb{K}\varphi(w)$ for all points $\mathbb{L}w$ of Γ (i.e., the weak embedding π is induced by a semi-linear mapping).

(2) There exists a quaternion skew field $\mathbb{L}$ with standard anti-automorphism σ and center Z such that Γ is isomorphic to the hermitian quadrangle arising from the (left) vector space $W = \mathbb{L}^4$ endowed with the $(\sigma, -1)$-quadratic form $q : W \to \mathbb{L}/Z$ defined by $q(x_1, x_2, x_3, x_4) = x_1 x_3^\sigma + x_2 x_4^\sigma + Z$ for $x_1, x_2, x_3, x_4 \in \mathbb{L}$ (i.e., Γ is a quaternion quadrangle). Furthermore, the composition of some (nontrivial) automorphism of Γ and the weak embedding π is induced by an injective semi-linear mapping.

(3) Γ is a so-called special subquadrangle of a quaternion quadrangle Γ' and the weak embedding $\pi : \Gamma \to \mathbf{PG}(V)$ may be extended to a weak embedding $\pi' : \Gamma' \to \mathbf{PG}(V)$ (for which Case (1) or (2) applies).

(4) There exists a nonperfect commutative field $\mathbb{L}$ of characteristic 2 with

$$\mathbb{L}^2 \subseteq \Lambda' \subseteq \mathbb{L}' \subseteq \Lambda \subseteq \mathbb{L},$$

where $\mathbb{L}'$ is a subfield of $\mathbb{L}$, Λ is an $\mathbb{L}'$-subspace of $\mathbb{L}$ which generates $\mathbb{L}$ as a ring and Λ' is an $\mathbb{L}^2$-subspace of $\mathbb{L}'$ which generates $\mathbb{L}'$ as a ring, such that Γ is the mixed subquadrangle $Q(\mathbb{L}', \mathbb{L}^2; \Lambda', \Lambda^2)$ of the orthogonal quadrangle arising from the vector space $W = \Lambda \times (\mathbb{L}')^4$ over $\mathbb{L}'$ endowed with the quadratic form $q : W \to \mathbb{L}'$ defined by $q(x_0; (x_1, x_2, x_3, x_4)) := x_0^2 + x_1x_2 + x_3x_4$ for $x_0 \in \Lambda$ and $x_1, x_2, x_3, x_4 \in \mathbb{L}'$. Furthermore, the weak embedding π is induced by a semi-linear mapping $\varphi : W \to V$ (with respect to an embedding $\alpha : \mathbb{L}' \to \mathbb{K}$) with $\ker\varphi \subseteq \Lambda$.

(5) Γ is the unique generalized quadrangle of order $(2, 2)$ (symplectic quadrangle $W(2)$ over $\mathbf{GF}(2)$) and the weak embedding is the universal weak embedding in the sense of Thas & Van Maldeghem [**12**], with $\mathrm{char}\mathbb{K} \neq 2$.

A special subquadrangle of a quaternion quadrangle is exactly a quadrangle arising from the standard embedding of the dual of the quaternion quadrangle as orthogonal quadrangle by intersecting with a suitable hyperplane, see Steinbach & Van Maldeghem [**9**, (7.2.1)].

If in Case (1) the radical of the corresponding (σ, ϵ)-hermitian form has (vector) dimension at most 1 (which happens for instance whenever $\mathrm{char}\mathbb{K} \neq 2$), then an $\alpha(\mathbb{L})$-basis of $\varphi(W)$ is a $\mathbb{K}$-basis of V, see Lemma 5.4.

For further details on weak embeddings of degree 2 or degree > 2, we refer to the Main Result of the present paper and the one by Steinbach & Van Maldeghem [**9**].

As a corollary of Theorem 8.1, we obtain a list of all generalized quadrangles, Γ say, such that Γ and the dual generalized quadrangle Γ^D are weakly embeddable. Compare Tits [**15**, (10.10)] for the case of full embeddings in polarities.

Proposition 8.2. *Let Γ be a thick generalized quadrangle. If Γ and the dual generalized quadrangle Γ^D are weakly embeddable, then, up to duality, Γ is one of the following:*

(1) *Γ is an orthogonal quadrangle and the weak embedding of Γ^D is in a projective 3-space. Further, Γ has a standard embedding in a d-dimensional projective space with*
   (a) *$d = 4$ (Γ^D is a symplectic quadrangle),*
   (b) *$d = 5$ (Γ^D is a hermitian quadrangle),*
   (c) *$d = 6$ (Γ^D is a special subquadrangle of some quaternion quadrangle),*
   (d) *$d = 7$ (Γ^D is a quaternion quadrangle).*

(2) *Γ is any mixed quadrangle (possibly $W(2)$ with universal weak embedding).*

For the free construction of a finitely generated generalized quadrangle, see Tits [**18**, (4.4)]. It is also contained in Van Maldeghem [**21**, (1.3.13)]. We end by proving the following result.

Proposition 8.3. *Let Γ be a freely constructed generalized quadrangle generated by the finite geometry Γ_0. If Γ_0 can be laxly embedded in some projective space $\mathbf{PG}(V)$, with V a vector space over any infinite skew field, then Γ can be laxly embedded in $\mathbf{PG}(V)$.*

Proof. One step of the free construction is as follows: For each point-line pair (x, L) with $d(x, L) = 5$, introduce a 'new' point y and a 'new' line M with $x\ I\ M\ I\ y\ I\ L$ (i.e., $d(x, L) = 3$ in the new geometry).

To prove the proposition, one has to show that, if x and L are a point and a line, respectively, of some laxly embedded finite point-line geometry Γ', then we can find a point y of $\mathbf{PG}(V)$ on L off Γ' such that the line xy of $\mathbf{PG}(V)$ is not a line of Γ'. By the finiteness of Γ', this is clear. □

Of course, one can extend in the obvious way the previous proposition to quadrangles Γ generated by n points and lines, with n some infinite cardinal number, and projective spaces $\mathbf{PG}(V)$ over any field with m elements, m a cardinal number with $m > n$. Indeed, in the free construction process, in each step, no more than $2n^2 = n$ new elements are introduced.

It is now clear that one can produce laxly embedded non-Moufang generalized quadrangles. As generating structure Γ_0 one can for instance choose a usual pentagon, or a finite generalized hexagon or octagon laxly embedded in the standard way in some projective space over some finite field and then extend the field to an infinite field to obtain an embedding in $\mathbf{PG}(V)$, with V a vector space over an infinite field. In general, 'free' quadrangles have zero probability of being Moufang quadrangles. In our examples, Γ_0 does not contain an ordinary quadrangle, and consequently one can easily see that the corresponding free generalized quadrangle Γ does not contain a (3×3)-grid, or a dual such grid. This implies that Γ cannot admit any central or axial elation, and hence cannot be a Moufang quadrangle.

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