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EMBEDDINGS OF $\mathbf{Z}_2$-HOMOLOGY 3-SPHERES IN $\mathbf{R}^5$ UP TO REGULAR HOMOTOPY

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Let $F : M^3 \hookrightarrow \mathbf{R}^5$ be an embedding of an (oriented) $\mathbf{Z}_2$-homology 3-sphere M^3 in $\mathbf{R}^5$. Then F bounds an embedding of an oriented manifold W^4 in $\mathbf{R}^5$. It is well known that the signature $\sigma(W^4)$ of W^4 is equal to the μ-invariant of M^3 modulo 16. In this paper we prove that $\sigma(W^4)$ itself completely determines the regular homotopy class of F.

1. Introduction.

Let $\mathrm{Imm}[X, Y]$ be the set of regular homotopy classes of immersions of a manifold X in a manifold Y, and $\mathrm{Emb}[X, Y]$ denote the subset of $\mathrm{Imm}[X, Y]$ consisting of all regular homotopy classes containing an embedding. Smale [**6**] has given a 1-1 correspondence (the Smale invariant) $s : \mathrm{Imm}[S^n, \mathbf{R}^N] \to \pi_n(V_{N,n})$, where $V_{N,n}$ is the Stiefel manifold of all n-frames in $\mathbf{R}^N$. Hirsch [**2**] has generalized this to the case of immersions of an arbitrary manifold in an arbitrary manifold. These results solve the problem of the number of regular homotopy classes in terms of homotopy theory, but do not succeed in finding representatives for each class or determining which classes are represented by an embedding.

According to Hughes [**4**], $\mathrm{Imm}[S^n, \mathbf{R}^N]$ has a group structure under connected sum and the Smale invariant actually gives a group isomorphism. [**4**] gives explicit generators of $\mathrm{Imm}[S^3, \mathbf{R}^4]$ and $\mathrm{Imm}[S^3, \mathbf{R}^5]$.

Hughes-Melvin [**5**] determine which classes of $\mathrm{Imm}[S^n, \mathbf{R}^{n+2}]$ are represented by an embedding, and prove that $\mathrm{Emb}[S^n, \mathbf{R}^{n+2}]$ is isomorphic to $\mathbf{Z}$ if $n \equiv 3 \bmod 4$, and to 0 otherwise. Furthermore, [**5**] proves that the regular homotopy class of an embedding $S^n \hookrightarrow \mathbf{R}^{n+2} (n \equiv 3 \bmod 4)$ can be completely determined by the signature of its oriented "Seifert" manifold. For example, in the case $n = 3$, we have the following diagram:

$$\begin{array}{cccc} s: & \mathrm{Imm}[S^3, \mathbf{R}^5] & \xrightarrow{\approx} & \pi_3(V_{5,3}) \approx \mathbf{Z} \\ & \cup & & \cup \\ & \mathrm{Emb}[S^3, \mathbf{R}^5] & \xrightarrow{\approx} & 24\mathbf{Z} \\ & f & \longmapsto & -\frac{3}{2}\sigma(V^4) \end{array}$$

where V^4 is an oriented Seifert manifold for f.

This implies that there exist many n-knots which cannot be transformed to the standard embedding even through a smooth deformation admitting self-intersections ($n \equiv 3 \bmod 4$).

The purpose of this paper is to prove a similar statement for embeddings of $\mathbf{Z}_2$-homology 3-spheres in $\mathbf{R}^5$. More precisely we prove that the regular homotopy class of an embedding of a $\mathbf{Z}_2$-homology 3-sphere in $\mathbf{R}^5$ is completely determined by the signature of its oriented Seifert manifold.

Throughout this paper, manifolds and immersions are of class C^∞. The symbol "$\approx$" denotes an appropriate isomorphism betweeen algebraic objects; "$\sim$" and "$\sim_r$" mean respectively "homotopic" and "regularly homotopic". We often do not distinguish between an immersion f and its regular homotopy class, both of which we denote by f.

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2. Preliminaries.

We recall some results of [**9**]. Let M^n be a parallelizable n-manifold, and $f : M^n \looparrowright \mathbf{R}^N$ be an immersion. Fix a trivialization $TM \cong M^n \times \mathbf{R}^n$; we can associate to f a map $\overline{df} : M^n \to V_{N,n}$ from M^n to the Stiefel manifold $V_{N,n}$, where $V_{N,n}$ is identified with the set of all injective linear maps from $\mathbf{R}^n$ to $\mathbf{R}^N$. $\overline{df}$ is essentially the differential of f. By Hirsch's theorem [**2**], the correspondence $f \mapsto \overline{df}$ gives a bijection between $\mathrm{Imm}[M^n, \mathbf{R}^N]$ and the homotopy set $[M^n, V_{N,n}]$. Every oriented 3-manifold M^3 is parallelizable, so $\mathrm{Imm}[M^3, \mathbf{R}^5] \approx [M^3, V_{5,3}]$.

We now study the set $[M^3, V_{5,3}]$. Since $V_{5,3}$ is simply connected, we can make use of the results of Whitney [**8**]. Let $\pi_i = \pi_i(V_{5,3})$, then $\pi_1 = 0$, $\pi_2 \approx \pi_3 \approx \mathbf{Z}$. Therefore we must consider the secondary difference.

Identify π_2 and π_3 with $\mathbf{Z}$ in the same way as [**9**, Proof of Theorem 2]. For a map $\xi : M^3 \to V_{5,3}$ we can suppose $\xi(M^{(1)}) = p \in V_{5,3}$ because $\pi_1 = 0$, (p is a point in $V_{5,3}$ and $M^{(q)}$ denotes the q-skeleton of M). So we can consider the difference 2-cochain between ξ and the constant map to the point p. Since ξ is defined over M^3, this 2-cochain is actually a 2-cocycle. Let C^2_ξ denote its cohomology class in $H^2(M^3; \mathbf{Z})$.

Next, for two maps $\xi, \eta : M^3 \to V_{5,3}$ with $\xi|M^{(2)} \sim \eta|M^{(2)}$, denote by $\Delta^3_{\xi,\eta}$ the difference 3-cochain.

The following is an application of [**8**, Theorem 8A] to our special case of mappings of M^3 in $V_{5,3}$ (see also [**9**, proof of Theorem 2]).

Lemma 2.1 ([**8**, Theorem 8A], [**9**, Theorem 2])**.** *Two maps $\xi, \eta : M^3 \to V_{5,3}$ are homotopic if and only if*

(a) $C^2_\xi = C^2_\eta \in H^2(M^3; \mathbf{Z})$.

(b) *There is a* 1*-cocycle* X^1 *and a* 2*-cochain* Y^2 *such that* $\Delta^3_{\xi,\eta} = 4X^1 \cup C^2_\xi + \delta Y^2$.

3. Main results.

Let M^3 be a closed oriented 3-manifold. Let D^3 be the 3-disk, which from now on we will often identify with the northern hemisphere of the 3-sphere S^3. Fix an inclusion $D^3 \subset M^3$, and put $M_0 = M^3 - \operatorname{int} D^3$. Suppose $F_0 : M^3 \hookrightarrow \mathbf{R}^5$ is an embedding such that $F_0|D^3$ coincides with the northern part of the standard embedding $S^3 \subset \mathbf{R}^5$. For an immersion $f : S^3 \looparrowright \mathbf{R}^5$, we can assume $f|$(the southern hemisphere) is standard, so define the map

$$\begin{array}{cccc} \sharp_{F_0}: & \mathrm{Imm}[S^3,\mathbf{R}^5] & \longrightarrow & \mathrm{Imm}[M^3,\mathbf{R}^5] \\ & f & \longmapsto & F_0 \sharp f \end{array}$$

where $(F_0\sharp f)|M_0 = F_0|M_0$, and $(F_0\sharp f)|D^3 = f|D^3$. The normal bundle of F_0 is trivial and if F_0 is altered on D^3 its normal bundle does not change. So we can in fact define the map

$$\sharp_{F_0} : \mathrm{Imm}[S^3,\mathbf{R}^5] \longrightarrow \mathrm{Imm}[M^3,\mathbf{R}^5]_0$$

where $\mathrm{Imm}[M^3,\mathbf{R}^5]_0$ is the subset of $\mathrm{Imm}[M^3,\mathbf{R}^5]$ consisting of all regular homotopy classes of immersions with trivial normal bundle. Note that $\mathrm{Emb}[M^3,\mathbf{R}^5] \subset \mathrm{Imm}[M^3,\mathbf{R}^5]_0$.

Proposition 3.1. *If* $H^2(M^3;\mathbf{Z})$ *has no elements of even order, then*

$$\sharp_{F_0} : \mathrm{Imm}[S^3,\mathbf{R}^5] \longrightarrow \mathrm{Imm}[M^3,\mathbf{R}^5]_0$$

is bijective.

Proof. Let ν_F be the normal bundle of an immersion $F : M^3 \looparrowright \mathbf{R}^5$. Since there is the bundle map

$$\begin{array}{ccc} \nu_F & \longrightarrow & V_{5,5} \\ \downarrow & & \downarrow \\ M^3 & \xrightarrow{\overline{dF}} & V_{5,3} \end{array}$$

and since the Euler class of the S^1-bundle $V_{5,5} \to V_{5,3}$ is equal to $2\Sigma^2$ for a generator $\Sigma^2 \in H^2(V_{5,3};\mathbf{Z}) \approx \mathbf{Z}$, we have

ν_F is trivial,
$\Leftrightarrow$ the normal Euler class of F (denoted by χ_F) is zero,
$\Leftrightarrow \overline{dF}^*(2\chi_F) = 2\overline{dF}^*(\chi_F) = 0$,
$\Leftrightarrow 2C^2_{\overline{dF}} = 0$,
$\Leftrightarrow C^2_{\overline{dF}} = 0$.

Therefore, $\mathrm{Imm}[M^3_0,\mathbf{R}^5]_0 \approx H^3(M_0;\mathbf{Z}) = 0$ by [**9**, Theorem 2]. This means that $\sharp_{F_0}$ is surjective from the covering homotopy property for immersion spaces (see [**7**]).

We next prove the injectivity. For two immersions $f, g : S^3 \looparrowright \mathbf{R}^5$, by Lemma 2.1,

$$\begin{aligned}&F_0\sharp f \sim_r F_0\sharp g,\\ &\Leftrightarrow \overline{d(F_0\sharp f)} \sim \overline{d(F_0\sharp g)},\\ &\Leftrightarrow \Delta^3_{\overline{d(F_0\sharp f)},\overline{d(F_0\sharp g)}} \text{ is a coboundary.}\end{aligned}$$

If we consider D^3 as a 3-cell, $\Delta^3_{\overline{d(F_0\sharp f)},\overline{d(F_0\sharp g)}}$ is a 3-cochain which assigns $s(f) - s(g) \in \pi_3(V_{5,3})$ to D^3, and $0 \in \pi_3(V_{5,3})$ to other 3-cells by definition. So clearly

$$\begin{aligned}&\Delta^3_{\overline{d(F_0\sharp f)},\overline{d(F_0\sharp g)}} \text{ is a coboundary,}\\ &\Leftrightarrow s(f) = s(g) \in \pi_3(V_{5,3}),\\ &\Leftrightarrow f \sim_r g : S^3 \to \mathbf{R}^5.\end{aligned}$$

This completes the proof. □

Remark 3.2. We already know, by a result of Wu ([**9**, Theorem 2]), that for a closed oriented 3-manifold M^3, $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \sqcup \cdots \sqcup \mathbf{Z}$, i.e., the disjoint union of as many copies of $\mathbf{Z}$ as the number of elements $c \in H^2(M^3; \mathbf{Z})$ with $2c = 0$. In particular, in our case where $H^2(M^3; \mathbf{Z})$ has no elements of even order, this implies that $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathbf{Z}$.

We now investigate $\sharp_{F_0}$ restricted to $\mathrm{Emb}[M^3, \mathbf{R}^5]$. We want to show that $\sharp_{F_0}$ gives a bijection between $\mathrm{Emb}[S^3, \mathbf{R}^5]$ and $\mathrm{Emb}[M^3, \mathbf{R}^5]$.

Theorem 3.3. *If* $H^1(M^3; \mathbf{Z}_2) = 0$, *then*

$$\sharp_{F_0} : \mathrm{Emb}[S^3, \mathbf{R}^5] \longrightarrow \mathrm{Emb}[M^3, \mathbf{R}^5]$$

is bijective.

Furthermore, under the identification $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \overset{\text{Prop. 3.1}}{\approx} \mathrm{Imm}[S^3, \mathbf{R}^5] \overset{\text{Smale inv.}}{\approx} \mathbf{Z}$,

$$\begin{array}{ccc}\mathrm{Emb}[M^3, \mathbf{R}^5] & \approx & 24\mathbf{Z}\\ F & \longmapsto & \frac{3}{2}(\sigma(W^4_F) - \sigma(W^4_{F_0}))\end{array}$$

where W^4_F *stands for an oriented Seifert manifold for* F, *and* $\sigma(W^4_F)$ *is its signature.*

Proof. Extend the embedding $F_0 : M^3 \hookrightarrow \mathbf{R}^5$ to an embedding $\overline{F_0} : W^4_{F_0} \hookrightarrow \mathbf{R}^5$. Take a suitable neighbourhood of M^3 in $W^4_{F_0}$ diffeomorphic to $M^3 \times [0, 1)$, and further extend $\overline{F_0}$ to an embedding (denoted again by $\overline{F_0}$)

$$\overline{F_0} : W^4_{F_0} \underset{M\times\{0\}}{\cup} M^3 \times (-1, 0] \hookrightarrow \mathbf{R}^5.$$

Let $F : M^3 \hookrightarrow \mathbf{R}^5$ be an embedding, and extend F to

$$\overline{F} : W^4_F \underset{M\times\{0\}}{\cup} M^3 \times (-1, 0] \hookrightarrow \mathbf{R}^5.$$

in the same way as above.

Take a neighbourhood M_0' of M_0 in M^3. Since $M_0' \times (-1,1)$ is parallelizable,

$$\mathrm{Imm}[M_0' \times (-1,1), \mathbf{R}^5] \approx [M_0' \times (-1,1), V_{5,4}] \approx [M_0, SO(5)].$$

And it follows by obstruction theory that $\mathrm{Imm}[M_0' \times (-1,1), \mathbf{R}^5] \approx [M_0, SO(5)]$ consists of a unique element, because $\pi_2(SO(5)) = 0$, $H^3(M_0; \pi_3(SO(5))) = 0$, and $H^1(M_0; \pi_1(SO(5))) \approx H^1(M^3; \mathbf{Z}_2) = 0$. Therefore we can alter $\overline{F}$ by a regular homotopy (we use again the letter $\overline{F}$ to represent the resulting immersion) so that

$$\overline{F}|(M_0' \times (-1,1))(x,t) = \overline{F_0}|(M_0' \times (-1,1))(x,-t), \quad (x,t) \in M_0' \times (-1,1).$$

Consider the manifold $V_F^4 = W_{F_0}^4 \underset{M_0 \times \{0\}}{\cup} W_F^4$ (the orientation of V_F^4 is taken to be in accord with the one of $W_{F_0}^4$), whose boundary is S^3. Using $\overline{F}$ and $\overline{F_0}$, construct a map from V_F^4 to $\mathbf{R}^5$. This map is an immersion except on $S^2 = \partial M_0 \subset \partial V_F^4$. Pushing a neighbourhood of S^2 into V_F^4, we have an immersion G of the whole V_F^4 in $\mathbf{R}^5$ (Figure 1).

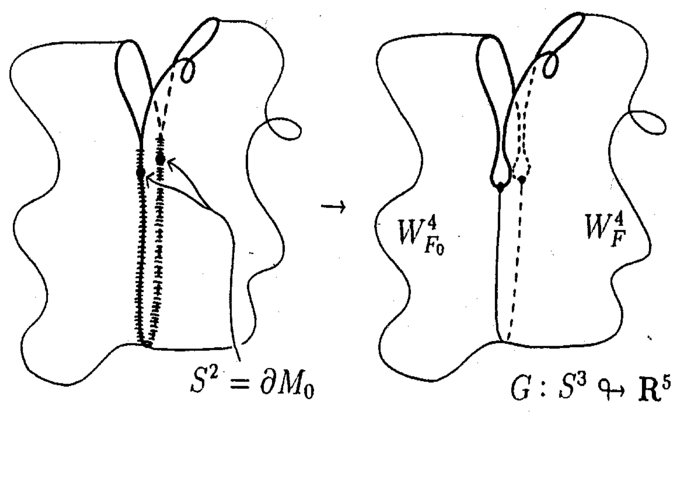

Figure 1.

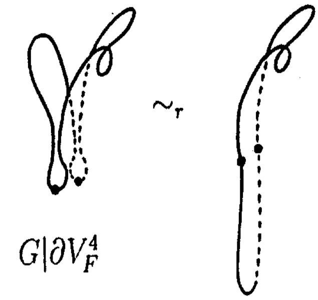

Figure 2.

Now clearly $F \sim_r F_0 \sharp (G|\partial V^4_F) : M^3 \looparrowright \mathbf{R}^5$ (Figure 2). By Proposition 3.1, the regular homotopy class of F depends only on the regular homotopy class of $G|\partial V^4_F : S^3 \looparrowright \mathbf{R}^5$. Since [**5**, Proof of Theorem and Corollary 2] actually proves that if an immersion $f : S^3 \looparrowright \mathbf{R}^5$ bounds an immersion of an oriented 4-manifold V^4 then $s(f)$ is equal to $-\frac{3}{2}\sigma(V^4)$, we can see

$$s\left(G|\partial V^4_F\right) = -\frac{3}{2}\sigma(V^4_F) \in 24\mathbf{Z},$$

and $G|\partial V^4_F \in \mathrm{Emb}[S^3, \mathbf{R}^5]$. Thus, the map $\sharp_{F_0}$ gives a bijection from $\mathrm{Emb}[S^3, \mathbf{R}^5]$ to $\mathrm{Emb}[M^3, \mathbf{R}^5]$. Therefore, identifying $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \overset{\text{Prop. 3.1}}{\approx} \mathrm{Imm}[S^3, \mathbf{R}^5] \overset{\text{Smale inv.}}{\approx} \mathbf{Z}$, $F \in \mathrm{Emb}[M^3, \mathbf{R}^5]$ corresponds to $\frac{3}{2}\sigma(V^4_F) = -\frac{3}{2}(\sigma(W^4_{F_0}) - \sigma(W^4_F))$ by Novikov additivity. This completes the proof. $\square$

Remark 3.4. We actually proved here that if an immersion $F : M^3 \looparrowright \mathbf{R}^5$ bounds an immersion of an oriented 4-manifold W^4_F then F corresponds to $\frac{3}{2}(\sigma(W^4_F) - \sigma(W^4_{F_0}))$ under the above identification $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathbf{Z}$.

Remark 3.5. Suppose M^3 is a $\mathbf{Z}_2$-homology sphere. By Theorem 3.3, we can choose F_0 so that $\sigma(W^4_{F_0}) = \mu(M^3)'$, where $\mu(M^3)'$ is the integer in $\{0, 1, \cdots, 15\}$ representing the μ-invariant $\mu(M^3) \in \mathbf{Z}/16\mathbf{Z}$. Let $S : \mathrm{Imm}[M^3, \mathbf{R}^5]_0 \to \mathbf{Z}$ denote the previous identification through this F_0, $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathrm{Imm}[S^3, \mathbf{R}^5] \approx \mathbf{Z}$. Then Theorem 3.3 implies that $S(F) = \frac{3}{2}(\sigma(W^4_F) - \mu(M^3)') \in 24\mathbf{Z}$ if $F \in \mathrm{Emb}[M^3, \mathbf{R}^5]$.

4. Realizing h-cobordisms in $\mathbf{R}^5$.

In this section, we study the following problem. Suppose M_1, M_2 are two $\mathbf{Z}_2$-homology 3-spheres which are mutually h-cobordant and let $S_i : \mathrm{Imm}[M_i, \mathbf{R}^5]_0 \to \mathbf{Z}$ $(i = 1, 2)$ denote the bijections as in Remark 3.5. Is it possible to relate S_1 to S_2?

Let M_1, M_2 be as above, and V be an h-cobordism between M_1 and M_2. Let $F_i : M_i \hookrightarrow \mathbf{R}^5$ be embeddings and W_i be oriented Seifert manifolds for them ($i = 1, 2$). Abstractly each M_i bounds a simply connected spin 4-manifold W_i' of signature $\sigma(W_i') = \sigma(W_i)$ (taking a connected sum with some copies of the $\pm K3$-surface if necessary) ($i = 1, 2$)(see [**3**]). Consider the closed manifold

$$Y = W_1' \underset{M_1}{\cup} V \underset{M_2}{\cup} W_2'.$$

Y is a simply connected spin 4-manifold of signature $\pm(\sigma(W_1') - \sigma(W_2'))$, since $W_1' \underset{M_1}{\cup} V$ is homotopy equivalent to W_1' and since each M_i admits a unique spin structure. By Cochran [**1**], Y can embed in $\mathbf{R}^5$ if $\sigma(W_1') = \sigma(W_2')$. Clearly this embedding restricted to each M_i is regularly homotopic to F_i (i=1,2), using Theorem 3.3.

Conversely, suppose $H : V \hookrightarrow \mathbf{R}^5$ is an embedding. H can extend to an immersion of $W_1 \cup V$ in $\mathbf{R}^5$ for a Seifert manifold W_1 for $H|M_1$, if the trivialization of the normal bundle of $H|M_1$ (for the construction of W_1) is suitably chosen. This, together with Theorem 3.3, implies that $S_1(H|M_1) = S_2(H|M_2) \in \mathbf{Z}$ because $\sigma(W_1) = \sigma(W_1 \cup V)$.

Thus, we have:

Proposition 4.1. *Let M_i, S_i $(i = 1, 2)$ and V be as above. For embeddings $F_i : M_i \hookrightarrow \mathbf{R}^5$ $(i = 1, 2)$, $S_1(F_1) = S_2(F_2) \in \mathbf{Z}$ if and only if there is an embedding $H : V \hookrightarrow \mathbf{R}^5$ with $H|M_i \sim_r F_i$ $(i = 1, 2)$ (or equivalently, there is an immersion $H : V \looparrowright \mathbf{R}^5$ with $H|M_i = F_i$ $(i = 1, 2)$).*

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