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Let X be a compact complex manifold of dimension $n \geq 2$ and $\mathcal{E}$ an ample vector bundle of rank $r < n$ on X. As the continuation of Part I, we further study the properties of $g(X, \mathcal{E})$ that is an invariant for pairs $(X, \mathcal{E})$ and is equal to curve genus when $r = n - 1$. Main results are the classifications of $(X, \mathcal{E})$ with $g(X, \mathcal{E}) = 2$ (resp. 3) when $\mathcal{E}$ has a regular section (resp. $\mathcal{E}$ is ample and spanned) and $1 < r < n - 1$.

Introduction.

The present paper is a continuation of [**I**]. For a pair $(X, \mathcal{E})$ which consists of a compact complex manifold X of dimension $n \geq 2$ and an ample vector bundle $\mathcal{E}$ of rank $r < n$ on X, we defined in [**I**] an invariant $g(X, \mathcal{E})$ by the formula

$$2g(X, \mathcal{E}) - 2 := (K_X + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}).$$

We note that $g(X, \mathcal{E})$ is a nonnegative integer, and $g(X, \mathcal{E})$ is equal to the curve genus of $(X, \mathcal{E})$ when $r = n - 1$. As in the case of curve genus, above $(X, \mathcal{E})$ with $g(X, \mathcal{E}) \leq 1$ have been classified in [**I**]; moreover, it is shown that $g(X, \mathcal{E}) \geq q(X)$ for spanned $\mathcal{E}$ and its equality condition is given in [**I**]. ($q(X)$ is the irregularity of X.)

After we recall some preliminary results in Section 1, we consider the cases $g(X, \mathcal{E}) = 2$ and $g(X, \mathcal{E}) = 3$ when $1 < r < n-1$ in Section 2. Corresponding results for c_1-sectional genus are given in [**Fj2**] and [**BiLL**] respectively. In Section 3 we consider the cases $g(X, \mathcal{E}) = q(X) + 1$ and $g(X, \mathcal{E}) = q(X) + 2$ when $1 < r < n - 1$. Related results for c_1-sectional genus are given in [**R**]. In Section 4 we give another relation between $g(X, \mathcal{E})$ and $q(X)$, namely $g(X, \mathcal{E}) \geq 2q(X) - 1$ for $1 < r < n - 1$. When $r = 1$, this inequality is satisfied except one case. In Section 5 we show that $g(X, \mathcal{E}) \geq g(C)$ when there exists a fibration $f : X \to C$ over a curve. We also give its equality condition. Finally in Appendix we give a classification of (X, L) with $g(X, L) = q(X) + 2$ and $n = 2$ for ample and spanned line bundles L on X.

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1. Preliminaries.

We use a notation similar to that in [**I**]. For example, we denote by $H(\mathcal{E})$ the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$, the projective space bundle associated to a vector bundle $\mathcal{E}$ on a variety X. We say that a vector bundle $\mathcal{E}$ is spanned if $H(\mathcal{E})$ is spanned. A polarized manifold (X, L) is said to be a scroll over a variety W if $(X, L) \simeq (\mathbb{P}_W(\mathcal{F}), H(\mathcal{F}))$ for some ample vector bundle $\mathcal{F}$ on W. We denote by $\mathbb{F}_e$ the Hirzebruch surfaces $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ $(e > 0)$, by σ a minimal section, and by f a fiber of the ruling $\mathbb{F}_e \to \mathbb{P}^1$. Numerical equivalence is denoted by $\equiv$.

Definition 1.1. Let X be a compact complex manifold of dimension $n \geq 2$ and $\mathcal{E}$ an ample vector bundle of rank $r < n$ on X. We define a rational number $g(X, \mathcal{E})$ for the pair $(X, \mathcal{E})$ by the formula

$$2g(X, \mathcal{E}) - 2 := (K_X + (n-r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}).$$

It turns out that $g(X, \mathcal{E})$ is a nonnegative integer (see [**I**]). When $r = 1$ (resp. $r = n-1$), $g(X, \mathcal{E})$ is nothing but the sectional genus (resp. curve genus) of $(X, \mathcal{E})$.

Remark 1.2. Let $(X, \mathcal{E})$ be as above. Suppose that $(X, \mathcal{E})$ satisfies the condition

($*$) There exists a section $s \in H^0(X, \mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth submanifold of X of the expected dimension $n - r$.

Then we have $g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z)$ (see [**I**]). If $\mathcal{E}$ is spanned, then $\mathcal{E}$ satisfies ($*$) by Bertini's theorem.

The following facts are used in the subsequent sections.

Proposition 1.3. *Let X be an n-dimensional compact complex manifold and $\mathcal{E}$ an ample vector bundle of rank $r < n$ on X with the property ($*$) in (1.2). Let $\iota : Z \hookrightarrow X$ be the embedding. Then*

(1) *$H^i(\iota) : H^i(X, \mathbb{Z}) \to H^i(Z, \mathbb{Z})$ is an isomorphism for $i < n - r$.*
(2) *$H^i(\iota)$ is injective and its cokernel is torsion free for $i = n - r$.*
(3) *$\mathrm{Pic}(\iota) : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism for $n - r > 2$.*
(4) *$\mathrm{Pic}(\iota)$ is injective and its cokernel is torsion free for $n - r = 2$.*

Proof. See Theorem 1.3 in [**LM1**] and see also Theorem 1.1 in [**LM2**]. □

Proposition 1.4. *Let X be an n-dimensional compact complex manifold and $\mathcal{E}$ an ample vector bundle of rank $r \geq 2$ on X with the property ($*$).*

If $Z \simeq \mathbb{P}^{n-r} (n - r \geq 1)$, then $(X, \mathcal{E})$ is one of the following:

(P1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$;
(P2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$;
(P3) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-1)})$;
(P4) $X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ *for some vector bundle* $\mathcal{F}$ *of rank* n *on* $\mathbb{P}^1$ *and* $\mathcal{E} = \oplus_{j=1}^{n-1}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, *where* $\pi : X \to \mathbb{P}^1$ *is the bundle projection.*

If $Z \simeq \mathbb{Q}^{n-r}(n - r \geq 2)$, *then* $(X, \mathcal{E})$ *is one of the following:*

(Q1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)})$;
(Q2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus r})$;
(Q3) $X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ *and* $\mathcal{E} = \oplus_{j=1}^{n-2}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, *where* $\mathcal{F}$ *is the same as that in* (P4).

Proof. See Theorem A and Theorem B in [**LM1**]. □

Proposition 1.5. *Let* X *be a complex projective manifold of dimension* n *and let* $\mathcal{E}$ *be an ample vector bundle of rank* $n-2 \geq 2$ *on* X *satisfying* $(*)$.

(1) *If* Z *is a geometrically ruled surface over a smooth curve* B *such that* $Z \neq \mathbb{F}_0, \mathbb{F}_1$, *then* X *is a* $\mathbb{P}^{n-1}$*-bundle over* B *and* $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$ *for every fiber* F *of the bundle map* $X \to B$.
(2) *If* $Z = \mathbb{F}_0$, *then* $(X, \mathcal{E})$ *is either the type in* (1) *with* $B = \mathbb{P}^1$ *or* $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)})$ *or* $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)})$.
(3) *If* $Z = \mathbb{F}_1$, *then* $(X, \mathcal{E})$ *is either the type in* (1) *with* $B = \mathbb{P}^1$ *or possibly* $X \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$ *for some ample vector bundle* $\mathcal{F}$ *on* $\mathbb{P}^2$ *with* $c_1(\mathcal{F}) = k(n-2)+3$ *for some positive integer* k *and* $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus(n-2)}$ *for every fiber* F *of the bundle map* $X \to \mathbb{P}^2$.

Proof. See [**LM3**]. □

Proposition 1.6. *Let* X *be a complex projective manifold of dimension* n *and let* $\mathcal{E}$ *be an ample vector bundle of rank* $r \geq 2$ *on* X. *If* $g(X, \det \mathcal{E}) = 2$, *then* $n = 2$ *and* $(X, \mathcal{E})$ *is one of the following:*

(1) X *is the Jacobian variety of a smooth curve* B *of genus* 2 *and* $\mathcal{E} \simeq \mathcal{E}_r(B, o) \otimes N$ *for some* $N \in \operatorname{Pic} X$ *with* $N \equiv 0$, *where* $\mathcal{E}_r(B, o)$ *is the Jacobian bundle for some point* o *on* B;
(2) $X \simeq \mathbb{P}_B(\mathcal{F})$ *for some stable vector bundle* $\mathcal{F}$ *of rank* 2 *on an elliptic curve* B *with* $c_1(\mathcal{F}) = 1$. *There is an exact sequence*

$$0 \to \mathcal{O}_X[2H(\mathcal{F}) + \rho^*G] \to \mathcal{E} \to \mathcal{O}_X[H(\mathcal{F}) + \rho^*T] \to 0,$$

where $G, T \in \operatorname{Pic} B$ *and* ρ *is the projection* $X \to B$. *We have* $(\deg G, \deg T) = (-2, 1)$ *or* $(-1, 0)$;
($2^\sharp$) $X, \mathcal{F}, B$ *and* ρ *are as in* (2) *and* $\mathcal{E} \simeq \rho^*\mathcal{G} \otimes H(\mathcal{F})$ *for some stable vector bundle* $\mathcal{G}$ *of rank* 3 *on* B *with* $c_1(\mathcal{G}) = -1$;
(3) $X \simeq \mathbb{P}_B(\mathcal{F})$ *and* $\mathcal{E} \simeq \rho^*\mathcal{G} \otimes H(\mathcal{F})$ *for some semistable vector bundles* $\mathcal{F}$ *and* $\mathcal{G}$ *of rank* 2 *on an elliptic curve* B *with* $(c_1(\mathcal{F}), c_1(\mathcal{G})) = (1, 0)$ *or* $(0, 1)$;

(4) $-K_X$ *is ample,* $K_X^2 = 1$ *and* $\det \mathcal{E} = -2K_X$*. We have* $\mathcal{E} \simeq [-K_X]^{\oplus 2}$*, or* $c_2(\mathcal{E}) = 3$ *and* $r = 2$*;*
(5_0) $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *and* $\mathcal{E} \simeq \mathcal{O}(1,1) \oplus \mathcal{O}(1,2)$*;*
(5_1) X *is the blowing-up of* $\mathbb{P}^2$ *at a point and* $\mathcal{E} \simeq [2L - E]^{\oplus 2}$*, where* L *is the pull-back of* $\mathcal{O}_{\mathbb{P}^2}(1)$ *and* E *is the exceptional curve.*

Proof. See (2.25) Theorem in [**Fj2**]. □

Proposition 1.7. *Let* X *be a complex projective manifold of dimension* n *and let* $\mathcal{E}$ *be an ample and spanned vector bundle of rank* $r \geq 2$ *on* X*. If* $g(X, \det \mathcal{E}) = 3$*, then* $n = 2$ *and* $(X, \mathcal{E})$ *is one of the following:*
(1a) $X = \mathbb{P}^2$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$*;*
(1b) $X = \mathbb{P}^2$*, and either* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ *or* $\mathcal{E} = T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$*;*
(1c) $X = \mathbb{P}^2$, $\operatorname{rank} \mathcal{E} = 2$ *and* $\det \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(4)$*;*
(2a) $X = \mathbb{F}_0$*, and either* $\mathcal{E} = [\sigma + f] \oplus [\sigma + 3f]$ *or* $\mathcal{E} = [\sigma + 2f]^{\oplus 2}$*;*
(2b) $X = \mathbb{F}_1$, $\mathcal{E} = [\sigma + 2f] \oplus [\sigma + 3f]$*;*
(2c) $X = \mathbb{F}_2$, $\mathcal{E} = [\sigma + 3f]^{\oplus 2}$*;*
(3) X *is a Del Pezzo surface with* $K_X^2 = 2$ *and either* $\mathcal{E} = [-K_X]^{\oplus 2}$*, or* $\mathcal{E} = \psi^*(\mathcal{Q}|_Y)$*, where* ψ *is a birational morphism from* X *to a surface* Y *of bidegree* $(4,4)$ *in the Grassmannian of lines of* $\mathbb{P}^3$*, and* $\mathcal{Q}$ *is the universal rank* 2 *quotient bundle;*
(4) $X = \mathbb{P}(\mathcal{F})$*, where* $\mathcal{F}$ *is a rank* 2 *vector bundle on an elliptic curve* B *with* $c_1(\mathcal{F}) = 1$ *and* $\mathcal{E} = H(\mathcal{F}) \otimes \rho^*\mathcal{G}$*, where* $\rho : X \to B$ *is the bundle projection and* $\mathcal{G}$ *is any rank* 2 *vector bundle on* B *defined by a nonsplitting exact sequence* $0 \to \mathcal{O}_B \to \mathcal{G} \to \mathcal{O}_B(x) \to 0$*, where* $x \in B$*.*

Proof. See (1.10) Theorem in [**BiLL**]. □

2. The cases $g(X, \mathcal{E}) = 2$ and $g(X, \mathcal{E}) = 3$.

Theorem 2.1. *Let* X *be a compact complex manifold of dimension* n *and* $\mathcal{E}$ *an ample vector bundle of rank* r *on* X *with* $1 < r < n - 1$ *and the property* $(*)$ *in* (1.2)*. If* $g(X, \mathcal{E}) = 2$*, then* $(X, \mathcal{E})$ *is one of the following:*
(i) *There exists an ample line bundle* A *on* X *such that* (X, A) *is a Del Pezzo* 4*-fold of degree* 1 *and* $\mathcal{E} = A^{\oplus 2}$ *(see also* (2.2.1)*);*
(ii) $X \simeq \mathbb{P}_B(\mathcal{F})$ *and* $\mathcal{E} = H(\mathcal{F}) \otimes \pi^*\mathcal{G}$*, where* $\mathcal{F}$ *and* $\mathcal{G}$ *are vector bundles on an elliptic curve* B *such that* $\operatorname{rank} \mathcal{F} = 4$, $\operatorname{rank} \mathcal{G} = 2$, $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 1$*, and* $\pi : X \to B$ *is the bundle projection;*
(iii) $X \simeq \mathbb{P}_B(\mathcal{F})$ *and* $\mathcal{E} = H(\mathcal{F}) \otimes \pi^*\mathcal{G}$*, where* $\mathcal{F}$ *and* $\mathcal{G}$ *are vector bundles on an elliptic curve* B *such that* $\operatorname{rank} \mathcal{F} = 5$, $\operatorname{rank} \mathcal{G} = 3$, $3c_1(\mathcal{F}) + 5c_1(\mathcal{G}) = 1$*, and* $\pi : X \to B$ *is the bundle projection.*

Proof. Suppose that $g(X, \mathcal{E}) = 2$. Since $\mathcal{E}$ satisfies $(*)$, there exists a nonzero section $s \in H^0(X, \mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth submanifold of X of dimension $n - r$ and $2 = g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z)$. From (1.6) we see that $n-r=2$ and $(Z,\mathcal{E}_Z)$ is one of the cases in (1.6). We make a case by case analysis in the following.

(2.1.1) If $(Z,\mathcal{E}_Z)$ is in case (1.6;1), then $K_Z=\mathcal{O}_Z$. We have $K_X+\det\mathcal{E}=\mathcal{O}_X$ since $[K_X+\det\mathcal{E}]_Z=K_Z$ and $\mathrm{Pic}(\iota):\mathrm{Pic}(X)\to\mathrm{Pic}(Z)$ is injective by (1.3). We get also that $H^1(\iota):H^1(X,\mathbb{Z})\to H^1(Z,\mathbb{Z})$ is an isomorphism by (1.3), but this is impossible since X is a Fano manifold and Z is an abelian surface.

(2.1.2) If $(Z,\mathcal{E}_Z)$ is in case $(1.6;5_0)$, we have $r=2$ and $n=4$. By (1.4), $(X,\mathcal{E})$ is one of the cases (Q1),(Q2) and (Q3). We easily see that $g(X,\mathcal{E})\neq 2$ in cases (Q1) and (Q2). In case (Q3), we can write $\mathcal{F}=\oplus_{i=1}^4\mathcal{O}_{\mathbb{P}^1}(a_i)$. Since $\mathcal{E}$ is ample, $H(\mathcal{F})+\pi^*\mathcal{O}_{\mathbb{P}^1}(b_j)$ is ample and so is $\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^1}(b_j)$. Hence we get $a_i+b_j>0$ for every i and j. Then it follows that

$$\begin{aligned}2=2g(X,\mathcal{E})-2&=(K_X+2c_1(\mathcal{E}))c_1(\mathcal{E})c_2(\mathcal{E})\\&=2\left(-2+\sum_{i=1}^4 a_i+2(b_1+b_2)\right)\geq 4,\end{aligned}$$

a contradiction.

(2.1.3) If $(Z,\mathcal{E}_Z)$ is in case $(1.6;5_1)$, we have $r=2$ and $n=4$. Since $Z=\mathbb{F}_1$, we see that $(X,\mathcal{E})$ is in case (1.5;3). If $(X,\mathcal{E})$ is the type (1.5;1) with $B=\mathbb{P}^1$, then we come to a contradiction by the argument of (2.1.2). Hence we have $X\simeq\mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$ for some ample vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ with $c_1(\mathcal{F})=2k+3$ $(k>0)$, and $\mathcal{E}_F=\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ for every fiber F of the bundle map $\pi:X\to\mathbb{P}^2$. We set $H:=H(\mathcal{F})$; we can write $\mathcal{E}=H\otimes\pi^*\mathcal{G}$ for some vector bundle $\mathcal{G}$ of rank 2 on $\mathbb{P}^2$. Since $\mathcal{E}_Z=[2L-E]^{\oplus 2}$, we can write $H_Z=aL-E$ $(2\leq a\in\mathbb{Z})$. Then we get $\mathcal{G}=\mathcal{O}_{\mathbb{P}^2}(2-a)^{\oplus 2}$, hence $\mathcal{E}=[H+\pi^*\mathcal{O}_{\mathbb{P}^2}(2-a)]^{\oplus 2}$ by $(\pi|_Z)^*\mathcal{G}=\mathcal{E}_Z\otimes[-H_Z]=[(2-a)L]^{\oplus 2}$. Since $\mathcal{E}$ is ample, $H+\pi^*\mathcal{O}_{\mathbb{P}^2}(a)$ is ample and so is $\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^2}(a)$. Then we get $c_1(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^2}(2-a))\geq 3$, hence $2k-3a+6\geq 0$. We note that

$$3=(2L-E)^2=c_2(\mathcal{E}_Z)=c_2(\mathcal{E})^2=s_2(\mathcal{F})+4c_1(\mathcal{F})\cdot(2-a)+6(2-a)^2.$$

On the other hand, we have

$$a^2-1=(aL-E)^2=H_Z^2=H^2\cdot c_2(\mathcal{E})=s_2(\mathcal{F})+2c_1(\mathcal{F})\cdot(2-a)+(2-a)^2.$$

From these two equalities we get $(2-a)(2k-3a+7)=0$. Since $2k-3a+6\geq 0$, we have $a=2$ and then $c_2(\mathcal{F})=3$ and $\mathcal{E}=H^{\oplus 2}$. It follows that

$$2=2g(X,\mathcal{E})-2=(K_X+2c_1(\mathcal{E}))c_1(\mathcal{E})c_2(\mathcal{E})=2s_2(\mathcal{F})+4k\geq 10,$$

a contradiction.

(2.1.4) If $(Z,\mathcal{E}_Z)$ is in case (1.6;4), then $r=2$ and $n=4$. We have $2K_X+3\det\mathcal{E}=\mathcal{O}_X$ since, by adjunction, $[2K_X+3\det\mathcal{E}]_Z=2K_Z+\det\mathcal{E}_Z=\mathcal{O}_Z$ and the restriction map $\mathrm{Pic}(X)\to\mathrm{Pic}(Z)$ is injective. By setting $A:=K_X+2\det\mathcal{E}$, we get $\det\mathcal{E}=2A$ and $K_X+3A=\mathcal{O}_X$, hence (X,A) is a

Del Pezzo 4-fold. Then we set $\mathcal{E}' := \mathcal{E} \oplus A$; we get $K_X + \det \mathcal{E}' = \mathcal{O}_X$ and $\mathcal{E}' \simeq A^{\oplus 3}$ by using Proposition 7.4 in [**PSW**]. It follows that $\mathcal{E} \simeq A^{\oplus 2}$ and

$$2 = 2g(X, \mathcal{E}) - 2 = (K_X + 2c_1(\mathcal{E}))c_1(\mathcal{E})c_2(\mathcal{E}) = 2A^4,$$

hence $A^4 = 1$. Thus we obtain that $(X, \mathcal{E})$ is the case (i) of our theorem.

(2.1.5) If $(Z, \mathcal{E}_Z)$ is in case (1.6;2), then $r = 2$ and $n = 4$. Since Z is a geometrically ruled surface over an elliptic curve B, by (1.5), X is a $\mathbb{P}^3$-bundle over B and $\mathcal{E}|_{\widetilde{F}} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ for every fiber $\widetilde{F}$ of the ruling $\pi : X \to B$. On the other hand, we have $\mathcal{E}_Z|_F = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ for every fiber F of the ruling $\rho : Z \to B$. This is a contradiction since $\pi|_Z = \rho$. If $(Z, \mathcal{E}_Z)$ is in case (1.6;$2^\sharp$) or (1.6;3), by using (1.5), we obtain that $(X, \mathcal{E})$ is the case (ii) or (iii) of our theorem respectively. This completes the proof. □

Remark 2.2. We make some comments on (2.1).

(2.2.1) In case (2.1; i), Del Pezzo 4-folds of degree 1 have been classified in [**Fj1**], Part III. In particular, they are weighted hypersurfaces of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, 1, 1, 1)$.

(2.2.2) We give an example of $(X, \mathcal{E})$ in case (2.1; ii) in the following. Let L_1 and L_2 be line bundles on an elliptic curve B such that $\deg L_1 = \deg L_2$ and $L_1 \neq L_2$ in $\operatorname{Pic} B$. Let $\mathcal{F}$ be an indecomposable vector bundle of rank 4 on B with $c_1(\mathcal{F}) = 1 - 2\deg L_1 - 2\deg L_2$. We set $X := \mathbb{P}_B(\mathcal{F})$, $\mathcal{G} := L_1 \oplus L_2$, and $\mathcal{E} := H(\mathcal{F}) \otimes \pi^*\mathcal{G} = \oplus_{i=1}^2 [H(\mathcal{F}) + \pi^* L_i]$, where $\pi : X \to B$ is the bundle projection. Since $c_1(\mathcal{F} \otimes L_i) = 1$, $\mathcal{F} \otimes L_i$ is ample and $h^0(B, \mathcal{F} \otimes L_i) = 1$. Then there exists an exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{F} \otimes L_i \to Q_i \to 0,$$

where Q_i is a locally free sheaf of rank 3 on B. Since Q_i is ample and $c_1(Q_i) = 1$, we see that Q_i is indecomposable. We set $D_i := \mathbb{P}_B(Q_i)$ and $Z := D_1 \cap D_2$. Since $c_1(Q_2 \otimes [L_1 - L_2]) = 1$, there exists an exact sequence

$$0 \to \mathcal{O}_B \to Q_2 \otimes [L_1 - L_2] \to Q \to 0,$$

where Q is a locally free sheaf of rank 2 on B. Then we have $Z = \mathbb{P}_B(Q)$ in $|H(Q_2) + (\pi|_{D_2})^*(L_1 - L_2)|$. Thus we see that $(X, \mathcal{E})$ satisfies the condition $(*)$ and $(X, \mathcal{E})$ is an example of (2.1; ii).

(2.2.3) The authors have no example for case (2.1; iii). We note that without the condition $(*)$ we have examples for the case. Indeed, we can take semistable vector bundles $\mathcal{F}$ and $\mathcal{G}$ on an elliptic curve B with the property that $\operatorname{rank} \mathcal{F} = 5$, $\operatorname{rank} \mathcal{G} = 3$, and $3c_1(\mathcal{F}) + 5c_1(\mathcal{G}) = 1$. Let $\pi : \mathbb{P}(\mathcal{F}) \to B$ and $\pi' : \mathbb{P}(\mathcal{G}) \to B$ be the bundle projections. Then $5H(\mathcal{F}) - \pi^* \det \mathcal{F}$ is nef on $\mathbb{P}(\mathcal{F})$ and $3H(\mathcal{G}) - (\pi')^* \det \mathcal{G}$ is nef on $\mathbb{P}(\mathcal{G})$ by Theorem 3.1 in [**Mi**]. We set $\mathcal{E} := H(\mathcal{F}) \otimes \pi^*\mathcal{G}$ and let $p : \mathbb{P}(\mathcal{E}) \to B$ be the composition of the projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{F})$ and π. Then $15H(\mathcal{E}) - F$ is nef on $\mathbb{P}(\mathcal{E})$ for a fiber F of p, hence $\mathcal{E}$ is ample. But it is uncertain that such $\mathcal{E}$ satisfies $(*)$.

(2.2.4) We see that every vector bundle $\mathcal{E}$ appeared in (2.1) is not spanned. Indeed, it is clear for case (2.1; i). For cases (2.1; ii) and (2.1; iii), we use the following:

Lemma 2.2.5. *Let $\mathcal{F}$ be a vector bundle of rank r on an elliptic curve. Then there exists a line sub-bundle L of $\mathcal{F}$ such that $\deg L \geq [c_1(\mathcal{F})/r]$, where $[c_1(\mathcal{F})/r]$ is the largest integer that is not greater than $c_1(\mathcal{F})/r$.*

This is a consequence of the Mukai-Sakai theorem [**MuS**], hence proof is omitted.

Suppose that $\mathcal{E}$ is spanned in case (2.1; ii). Applying the lemma to $\mathcal{F}^\vee$ and $\mathcal{G}^\vee$, we get quotient line bundles L_1 and L_2 of $\mathcal{F}$ and $\mathcal{G}$ respectively, with the property that $\deg L_1 \leq -[-c_1(\mathcal{F})/4]$ and $\deg L_2 \leq -[-c_1(\mathcal{G})/2]$. The surjection $\mathcal{F} \to L_1$ gives a section $C := \mathbb{P}(L_1)$ of the projection $\pi : \mathbb{P}_B(\mathcal{F}) \to B$. Since $H(\mathcal{F})|_C = (\pi|_C)^* L_1$, we see that $(\pi|_C)^*(L_1 \otimes L_2)$ is a quotient line bundle of $\mathcal{E}_C$, hence $L_1 \otimes L_2$ is ample and spanned. From $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 1$ we get $\deg L_1 + \deg L_2 \leq -[(2c_1(\mathcal{G}) - 1)/4] - [-c_1(\mathcal{G})/2] = 1$; this leads to a contradiction since B is an elliptic curve. Similarly we can show that $\mathcal{E}$ is not spanned in case (2.1; iii).

Theorem 2.3. *Let X be a compact complex manifold of dimension n and $\mathcal{E}$ an ample and spanned vector bundle of rank r on X with $1 < r < n-1$. If $g(X, \mathcal{E}) = 3$, then $(X, \mathcal{E})$ is one of the following:*

(i) $(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)^{\oplus 4})$;
(ii) $(\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,1)^{\oplus 2})$;
(iii) *There exists a double covering $f : X \to \mathbb{P}^4$ with branch locus $B \in |\mathcal{O}_{\mathbb{P}^4}(4)|$ and $\mathcal{E} = f^*\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}$.*

Proof. Suppose that $g(X, \mathcal{E}) = 3$. We argue as in the proof of (2.1). Since $\mathcal{E}$ is spanned, there exists a nonzero section $s \in H^0(X, \mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth submanifold of X of dimension $n-r$ and $3 = g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z)$. From (1.7) we see that $n - r = 2$ and $(Z, \mathcal{E}_Z)$ is one of the cases in (1.7).

(2.3.1) If $(Z, \mathcal{E}_Z)$ is in case (1a), (1b), or (1c) of (1.7), then $Z = \mathbb{P}^2$ and $(X, \mathcal{E})$ is the case (P1) of (1.4) since $n - r = 2$. We obtain that $(X, \mathcal{E})$ is the case (i) of our theorem by $g(X, \mathcal{E}) = 3$.

(2.3.2) If $(Z, \mathcal{E}_Z)$ is in case (3) of (1.7), then $r = 2$ and $n = 4$. By setting $A := K_X + 2\det \mathcal{E}$, we infer that (X, A) is a Del Pezzo manifold and $\mathcal{E} = A^{\oplus 2}$ from the same argument as that in (2.1.4). Then we find that $A^4 = 2$ since $g(X, \mathcal{E}) = 3$. Hence we obtain that $(X, \mathcal{E})$ is the case (iii) of our theorem by [**Fj1**], Part I.

(2.3.3) If $(Z, \mathcal{E}_Z)$ is in case (2a), (2b), (2c), or (4) of (1.7), then $r = 2$ and $n = 4$. Since Z is a geometrically ruled surface, by (1.5), $(X, \mathcal{E})$ is one of the following:

(R1) $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2))$;
(R2) $(\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})$;
(R3) X is a $\mathbb{P}^3$-bundle over a smooth curve B and $\mathcal{E}_{\widetilde{F}} = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ for every fiber $\widetilde{F}$ of the bundle map $\pi : X \to B$;
(R4) $Z = \mathbb{F}_1$, $X \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$ for some ample vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = 2k+3$ $(k > 0)$, and $\mathcal{E}_{\widetilde{F}} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ for every fiber $\widetilde{F}$ of the bundle map $\pi : X \to \mathbb{P}^2$.

Cases (R1) and (R2) are ruled out by $g(X, \mathcal{E}) = 3$. Case (R4) comes from (2b) of (1.7), hence $\pi|_Z$ is the blowing-up $\mathbb{F}_1 \to \mathbb{P}^2$ and $\mathcal{E}_Z = [\sigma + 2f] \oplus [\sigma + 3f]$. We can write $\mathcal{E} = H(\mathcal{F}) \otimes \pi^*\mathcal{G}$ for some vector bundle $\mathcal{G}$ of rank 2 on $\mathbb{P}^2$ and $H(\mathcal{F})_Z = a\sigma + bf$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} 2\sigma + 5f = \det \mathcal{E}_Z &= 2H(\mathcal{F})_Z + (\pi|_Z)^* \det \mathcal{G} \\ &= (2a + c_1(\mathcal{G}))\sigma + (2b + c_1(\mathcal{G}))f, \end{aligned}$$

hence $2a - 2b = -3$, a contradiction. In case (R3), we have $X \simeq \mathbb{P}_B(\mathcal{F})$ and $\mathcal{E} = H(\mathcal{F}) \otimes \pi^*\mathcal{G}$ for some vector bundles $\mathcal{F}$ and $\mathcal{G}$ on B such that $\operatorname{rank} \mathcal{F} = 4$ and $\operatorname{rank} \mathcal{G} = 2$. Then

$$\begin{aligned} 4 = 2g(X, \mathcal{E}) - 2 &= (K_X + 2c_1(\mathcal{E}))c_1(\mathcal{E})c_2(\mathcal{E}) \\ &= 2(2g(B) - 2 + c_1(\mathcal{F}) + 2c_1(\mathcal{G})), \end{aligned}$$

where $g(B)$ is the genus of B. Since $\mathcal{E}$ is ample, we find that $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) > 0$ from $(\det \mathcal{E})^4 > 0$. It follows that $g(B) \leq 1$. In case $g(B) = 0$, we have $B \simeq \mathbb{P}^1$ and $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 4$. Then we can write $\mathcal{F} = \sum_{i=1}^4 \mathcal{O}(a_i)$ and $\mathcal{G} = \sum_{j=1}^2 \mathcal{O}(b_j)$. By the same argument as that in (2.1.2), we infer that $a_i + b_j = 1$ for every i and j. It follows that $a_1 = \cdots = a_4$ and $b_1 = b_2$, hence $\mathbb{P}_B(\mathcal{F}) \simeq \mathbb{P}^1 \times \mathbb{P}^3$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,1)^{\oplus 2}$, which is the case (ii) of our theorem. In case $g(B) = 1$, we have $c_1(\mathcal{F}) + 2c_1(\mathcal{G}) = 2$. Then we get a contradiction by the same argument as that in (2.2.4). We have thus completed the proof. □

3. The cases $g(X, \mathcal{E}) = q(X) + 1$ and $g(X, \mathcal{E}) = q(X) + 2$.

Theorem 3.1. *Let X be a compact complex manifold of dimension n and let $\mathcal{E}$ be an ample and spanned vector bundle of rank r with $1 < r < n - 1$. Then $g(X, \mathcal{E}) = q(X) + 1$ if and only if $(X, \mathcal{E})$ is one of the following:*

(1) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2})$;
(2) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 3})$;
(3) $(\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})$.

Proof. First we note that if $(X, \mathcal{E})$ is one of the cases (1),(2) and (3) of our theorem, then we easily see that $g(X, \mathcal{E}) = 1 = q(X) + 1$. Suppose that $g(X, \mathcal{E}) = q(X) + 1$ on the contrary. Let Z be a smooth submanifold of X with $\dim Z = n - r$ defined as the zero locus of some $s \in H^0(X, \mathcal{E})$.

Then $g(X,\mathcal{E}) = g(Z, \det \mathcal{E}_Z)$. We put $A := \det \mathcal{E}_Z$; then A is ample and spanned. If $n-r \geq 3$, we take general members $D_1, \ldots, D_{n-r-2} \in |A|$ with the property that $S := D_1 \cap \cdots \cap D_{n-r-2}$ is a smooth surface. If $n-r=2$, we set $S = Z$. Hence there exists a polarized surface (S, A_S) such that $g(Z,A) = g(S,A_S)$. We get $q(X) = q(Z) = q(S)$ by using (1.3). Thus we get $g(S,A_S) = q(S)+1$.

We show that $h^0(K_S) = 0$. Indeed, it is obvious if $\kappa(S) = -\infty$, where $\kappa(S)$ is the Kodaira dimension of S. When $\kappa(S) \geq 0$, by Riemann-Roch Theorem and Vanishing Theorem, we get

$$h^0(K_S + A_S) - h^0(K_S) = g(S,A_S) - q(S) = 1.$$

If $h^0(K_S) > 0$, then

$$h^0(K_S + A_S) \geq h^0(K_S) + h^0(A_S) - 1.$$

But this is impossible since $h^0(A_S) \geq 3$. Hence $h^0(K_S) = 0$. Thus we get $g(S,A_S) \geq 2q(S)$ by Lemma 1.4 in [**Ma1**] since (S,A_S) is not a scroll over a smooth curve. Then $q(S) \leq 1$ and $g(X,\mathcal{E}) \leq 2$ by the above argument. So we obtain that $(X,\mathcal{E})$ is the case (1),(2), or (3) of our theorem by using (2.1), (2.2.4) and [**I**]. □

Remark 3.2. Let L be an ample and spanned line bundle on a compact complex manifold X of dimension $n \geq 2$. When $n \geq 3$, we have $g(X,L) = q(X)+1$ if and only if (X,L) is a Del Pezzo manifold (see [**Fk3**]). When $n=2$, we have $g(X,L) = q(X)+1$ if and only if (X,L) is a Del Pezzo surface (i.e., $L = -K_X$) or $X \simeq \mathbb{P}_B(\mathcal{F})$ and $L \equiv 2H(\mathcal{F})$ for some ample vector bundle $\mathcal{F}$ of rank 2 on an elliptic curve B with $c_1(\mathcal{F}) = 1$. We can prove this by the argument in (3.1) and Theorem 3.1 in [**LP**].

Proposition 3.3. *Let X be a compact complex manifold of dimension n and let $\mathcal{E}$ be an ample and spanned vector bundle of rank r with $1 < r < n-1$. Then we have $g(X,\mathcal{E}) \neq q(X)+2$.*

Proof. The following argument is similar to the proof of (3.1). Suppose that $g(X,\mathcal{E}) = q(X)+2$. Let Z be a smooth submanifold of X with $\dim Z = n-r$ defined as the zero locus of some $s \in H^0(X,\mathcal{E})$. Then $g(X,\mathcal{E}) = g(Z, \det\mathcal{E}_Z)$ and $\det\mathcal{E}_Z$ is ample and spanned. As in the proof of (3.1), we get a smooth surface S such that $g(Z,\det\mathcal{E}_Z) = g(S,\det\mathcal{E}_S)$. We have $q(X) = q(Z) = q(S)$, thus we get $g(S,\det\mathcal{E}_S) = q(S)+2$. Then we find that $q(S) \leq 1$ by Theorem 3.4 in [**R**]. It follows that $g(X,\mathcal{E}) \leq 3$ and we infer that $(X,\mathcal{E})$ does not exist from (2.1), (2.2.4) and (2.3). This completes the proof. □

Remark 3.4. We see that the pairs $(X,\mathcal{E})$ in (2.3) satisfy $g(X,\mathcal{E}) = q(X)+3$. In Appendix we give a classification of polarized surfaces (X,L) such that $g(X,L) = q(X)+2$ and L is spanned.

4. Another Lower bound for $g(X, \mathcal{E})$.

Proposition 4.1. *Let L be an ample and spanned line bundle on a compact complex manifold X with $\dim X = n \geq 2$. Then $g(X, L) \geq 2q(X) - 1$ unless (X, L) is a scroll over a smooth curve B of genus $g(B) \geq 2$.*

Proof. Since L is ample and spanned, if $n \geq 3$, we can take general members $D_1, \dots, D_{n-2} \in |L|$ such that $S := D_1 \cap \cdots \cap D_{n-2}$ is a smooth surface. If $n = 2$, we set $S = X$. Then we get $g(X, L) = g(S, L_S)$ and $q(X) = q(S)$.

If $\kappa(S) \geq 0$, then $g(X, L) = g(S, L_S) \geq 2q(S) - 1 = 2q(X) - 1$ by Corollary 3.2 in [**Fk1**].

If $\kappa(S) = -\infty$ and (S, L_S) is not a scroll over a smooth curve, then $g(X, L) = g(S, L_S) \geq 2q(S) = 2q(X)$ by Lemma 1.4 in [**Ma1**].

If $\kappa(S) = -\infty$ and (S, L_S) is a scroll over a smooth curve, then $g(X, L) = g(S, L_S) = q(S) = q(X)$. Hence we get $g(X, L) \geq 2q(X) - 1$ if $q(S) \leq 1$. So we may assume that $q(S) \geq 2$. Then we obtain that (X, L) is a scroll over a smooth curve B of genus $g(B) \geq 2$ by using Theorem 3 in [**Bă**]. □

Theorem 4.2. *Let X be a compact complex manifold with $\dim X = n$ and let $\mathcal{E}$ be an ample and spanned vector bundle of rank r with $1 < r < n - 1$. Then $g(X, \mathcal{E}) \geq 2q(X) - 1$.*

Proof. Let Z be the zero locus of some $s \in H^0(X, \mathcal{E})$ such that Z is a smooth submanifold of X with $\dim Z = n - r$. Then $g(X, \mathcal{E}) = g(Z, \det \mathcal{E}_Z)$ and $q(X) = q(Z)$. We put $A := \det \mathcal{E}_Z$; then A is ample and spanned. Since (Z, A) is not a scroll, by (4.1), we obtain that $g(X, \mathcal{E}) = g(Z, A) \geq 2q(Z) - 1 = 2q(X) - 1$. □

5. The case of a fiber space over a curve.

Definition 5.1. Here we say that a quartet $(f, X, C, \mathcal{E})$ is a *generalized polarized fiber space over a curve* if:

(1) X and C are compact complex manifolds with $1 = \dim C < \dim X = n$,
(2) $f : X \to C$ is a surjective morphism with connected fibers, and
(3) $\mathcal{E}$ is an ample vector bundle of rank r on X.

Theorem 5.2. *Let $(f, X, C, \mathcal{E})$ be a generalized polarized fiber space over a curve with $r \leq n - 1$. Then $g(X, \mathcal{E}) \geq g(C)$.*

Proof. First we remark that the following equality holds:

$$(5.2.1) \qquad g(X, \mathcal{E}) = g(C) + \frac{1}{2}(K_{X/C} + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) + (g(C) - 1)(c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E})F - 1),$$

where $K_{X/C} := K_X - f^*(K_C)$ and F is a general fiber of f.

If $g(C) = 0$, then Theorem 5.2 is true by [**I**]. So we may assume that $g(C) \geq 1$.

(I) The case in which $K_{X/C} + (n-r)c_1(\mathcal{E})$ is f-nef.

Then there exists a surjective map

$$f^* \circ f_*(\mathcal{O}(m(K_{X/C} + (n-r)c_1(\mathcal{E})))) \to \mathcal{O}(m(K_{X/C} + (n-r)c_1(\mathcal{E})))$$

for any large m by base point free theorem.

By Theorem A in Appendix in [**Fk2**], $f_*(\mathcal{O}(m(K_{X/C} + (n-r)c_1(\mathcal{E}))))$ is semipositive. Hence $K_{X/C} + (n-r)c_1(\mathcal{E})$ is nef. So we get

$$(K_{X/C} + (n-r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 0.$$

Hence we obtain $g(X, \mathcal{E}) \geq g(C)$ because of (5.2.1) and $c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E})F \geq 1$.

(II) The case in which $K_{X/C} + (n-r)c_1(\mathcal{E})$ is not f-nef.

Then $K_X + (n-r)c_1(\mathcal{E})$ is not nef. So by Mori Theory, there exists an extremal rational curve l such that $(K_X + (n-r)c_1(\mathcal{E}))l < 0$. Hence

$$n + 1 \geq -K_X l > (n-r)c_1(\mathcal{E})l \geq (n-r)r \geq n-1.$$

If $(n-r)r = n$, then $(n, r) = (4, 2)$.

If $(n-r)r = n-1$, then $r = 1$ or $r = n-1$.

(II-1) The case where $(n, r) = (4, 2)$.

Then $-K_X l = 5 = n+1$. So we have $\operatorname{Pic} X \cong \mathbb{Z}$ by [**W**]. But this is impossible because X has a nontrivial fibration.

(II-2) The case in which $r = 1$.

Then Theorem 5.2 is true by Theorem 1.2.1 in [**Fk2**].

(II-3) The case in which $r = n-1$.

If $n = 2$, then $r = 1$ and so we may assume that $n \geq 3$. Since X has a nontrivial fibration, $(X, \mathcal{E})$ is the following type by [**YZ**]: There exists a surjective morphism $\pi : X \to B$ such that any fiber of π is $\mathbb{P}^{n-1}$ and $\mathcal{E}|_{F_\pi} \cong \mathcal{O}(1)^{\oplus n-1}$, where B is a smooth curve and F_π is a fiber of π.

Since any fiber of π is $\mathbb{P}^{n-1}$, there exists a morphism $\delta : B \to C$ such that $f = \delta \circ \pi$. Because f has connected fibers, δ is an isomorphism. In particular, $g(B) = g(C)$. On the other hand, by [**Ma2**], $g(X, \mathcal{E}) = g(B)$. Hence $g(X, \mathcal{E}) = g(C)$. This completes the proof of Theorem 5.2. $\square$

Theorem 5.3. *Let $(f, X, C, \mathcal{E})$ be a generalized polarized fiber space over a curve with $2 \leq r \leq n-1$. If $g(X, \mathcal{E}) = g(C)$, then $r = n-1$, any fiber F of f is isomorphic to $\mathbb{P}^{n-1}$ and $\mathcal{E}|_F \cong \oplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.*

Proof. (I) The case in which $g(C) \leq 1$.

Then $g(X, \mathcal{E}) = g(C) \leq 1$, and by the classification results of [**I**] and [**Ma2**], we get the following: X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ or a smooth elliptic curve and $\mathcal{E}|_{F_\pi} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n-1}$, where F_π is a fiber of its bundle map

$\pi : X \to B$ and B is $\mathbb{P}^1$ or a smooth elliptic curve. Since any fiber of π is $\mathbb{P}^{n-1}$, there exists a morphism $\delta : B \to C$ such that $f = \delta \circ \pi$. Because f has connected fibers, δ is an isomorphism. Therefore we get the assertion.

(II) The case in which $g(C) \geq 2$.

(II-1) $n - r \geq 2$ case.

If $K_{X/C} + (n - r - 1)c_1(\mathcal{E})$ is f-nef, then by the same argument as in the proof of Theorem 5.2 we get

$$(K_{X/C} + (n - r - 1)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 0$$

and

$$(K_{X/C} + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 1.$$

Hence we obtain that $g(X, \mathcal{E}) > g(C)$ by (5.2.1). So we may assume that $K_{X/C} + (n - r - 1)c_1(\mathcal{E})$ is not f-nef. Then by Mori Theory, there exists an extremal rational curve l such that $(K_X + (n - r - 1)c_1(\mathcal{E}))l < 0$. Hence we get

$$n + 1 \geq -K_X l > (n - r - 1)c_1(\mathcal{E})l \geq (n - r - 1)r \geq n - 2.$$

If $(n - r - 1)r = n$, then $-K_X l = n + 1$ and $\operatorname{Pic} X \cong \mathbb{Z}$ by [**W**]. But this is impossible.

If $(n - r - 1)r = n - 1$, then $n = 5$ and $r = 2$.

Here we prove the following Lemma.

Lemma 5.4. *Let $(f, X, C, \mathcal{E})$ be a generalized polarized fiber space over a curve with $2 \leq r \leq n-1$ and $g(C) \geq 1$. If $\kappa(K_F + xc_1(\mathcal{E}_F)) \geq 0$ for a rational number x with $x < n-r$ and a general fiber F of f, then $g(X, \mathcal{E}) \geq g(C)+1$.*

Proof. By assumption, there exists a natural number N such that

$$f_*(\mathcal{O}(N(K_{X/C} + xc_1(\mathcal{E})))) \neq 0.$$

By Remark 1.3.2 in [**Fk2**], $N(K_{X/C} + xc_1(\mathcal{E}))$ is pseudo effective. Therefore

$$(K_{X/C} + xc_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 0$$

and we get

$$(K_{X/C} + (n - r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) \geq 1.$$

Since $g(C) \geq 1$, we get that $g(X, \mathcal{E}) \geq g(C) + 1$ by (5.2.1). □

We continue the proof of Theorem 5.3. If $K_F + xc_1(\mathcal{E}_F)$ is nef for a rational number x with $x < 3$, then we can prove that $g(X, \mathcal{E}) > g(C)$ by Lemma 5.4.

Assume that $K_F + xc_1(\mathcal{E}_F)$ is not nef for a rational number x with $x < 3$. Then there exists an extremal rational curve l on F such that $n \geq -K_F l > xc_1(\mathcal{E}_F)l \geq rx$. Since $n = 5$ and $r = 2$, we have $x < 5/2$. Therefore there exists a rational number $y < 3$ such that $K_F + yc_1(\mathcal{E}_F)$ is nef, and we get $g(X, \mathcal{E}) > g(C)$.

If $(n-r-1)r = n-2$, then $r = n-2$ by assumption. Assume that $K_F + xc_1(\mathcal{E}_F)$ is not nef for a rational number x with $x < 2$. Then we get $n > rx$ by the same argument as above. Since $r = n-2$, we get $x < n/(n-2) = 1 + 2/(n-2)$. By assumption, we get $n \geq 4$. So we have $x < 2$. Therefore there exists a rational number $y < 2$ such that $K_F + yc_1(\mathcal{E}_F)$ is nef. Hence we have $g(X,\mathcal{E}) > g(C)$.

(II-2) $n-r=1$ case.

First we assume that $K_F + c_1(\mathcal{E}_F)$ is nef for a general fiber F of f. If $K_F + c_1(\mathcal{E}_F)$ is ample, then there exists a rational number $t > 0$ such that $\kappa(K_F + (1-t)c_1(\mathcal{E}_F)) \geq 0$ by Kodaira's Lemma. So we get that $g(X,\mathcal{E}) > g(C)$ by the same argument as above. Assume that $K_F + c_1(\mathcal{E}_F)$ is not ample. Since $\dim F = \operatorname{rank} \mathcal{E}_F$, by [**Fj3**], we get that $(F, \mathcal{E}_F)$ is one of the following:

(A) $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n-2})$,
(B) $(\mathbb{P}^{n-1}, T_{\mathbb{P}^{n-1}})$,
(C) $(\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus n-1})$,
(D) F is a $\mathbb{P}^{n-2}$-bundle over a smooth curve B and $\mathcal{E}_{F_\pi} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}$ for every fiber F_π of the projection $\pi : F \to B$.

If $(F, \mathcal{E}_F)$ is one of the type (A), (B), or (C), then $h^0(K_F + c_1(\mathcal{E}_F)) > 0$ by easy calculation. Here we prove the following Lemma.

Lemma 5.5. *Let $(f, X, C, \mathcal{E})$ be a generalized polarized fiber space over a curve with $2 \leq r \leq n-1$. If $h^0(K_F + c_1(\mathcal{E}_F)) > 0$ for a general fiber F of f, then $(K_{X/C} + c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) > 0$.*

Proof. By hypothesis, $f_*\mathcal{O}(K_{X/C} + c_1(\mathcal{E})) \neq 0$. By Theorem 2.4 and Corollary 2.5 in [**EV**], we get that $f_*\mathcal{O}(K_{X/C} + c_1(\mathcal{E}))$ is ample. By the proof of Lemma 1.4.1 in [**Fk2**], we get that $m(K_{X/C} + c_1(\mathcal{E})) - f^*A$ is an effective divisor for a large number m and an ample divisor A on C. Hence we obtain $(K_{X/C} + c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}) > 0$. □

By Lemma 5.5, we get that $g(X,\mathcal{E}) > g(C)$ if $(F, \mathcal{E}_F)$ is one of the type (A), (B), or (C).

Assume that $(F, \mathcal{E}_F)$ is the type (D). Then there exist vector bundles $\mathcal{F}$ and $\mathcal{G}$ on B with $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{G} = n-1$ such that $\mathcal{E}_F \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$, where $H(\mathcal{F})$ is the tautological line bundle of $\mathbb{P}(\mathcal{F})$. Then $K_F + c_1(\mathcal{E}_F) = \pi^*(K_B + \det \mathcal{F} + \det \mathcal{G})$. Since $K_F + c_1(\mathcal{E}_F)$ is nef, we get $(K_{X/C} + c_1(\mathcal{E}))c_r(\mathcal{E}) \geq 0$ by the proof of Lemma 5.4. We have $g(X,\mathcal{E}) = g(C)$, then

$c_r(\mathcal{E})F = 1$ by (5.2.1). Since $1 = c_r(\mathcal{E}_F) = c_1(\mathcal{F}) + c_1(\mathcal{G})$, we obtain that

$$\begin{aligned} &h^0(K_B + \det\mathcal{F} + \det\mathcal{G}) \\ &\geq 1 - g(B) + \deg(K_B + \det\mathcal{F} + \det\mathcal{G}) \\ &= g(B) - 1 + c_1(\mathcal{F}) + c_1(\mathcal{G}) \\ &= g(B). \end{aligned}$$

Because $K_F + c_1(\mathcal{E}_F)$ is nef, we obtain that $\deg(K_B + \det\mathcal{F} + \det\mathcal{G}) \geq 0$. Hence $g(B) \geq 1$. Therefore $h^0(K_F + c_1(\mathcal{E}_F)) \geq 1$. By Lemma 5.5 we obtain that $g(X, \mathcal{E}) > g(C)$ and this is a contradiction.

Next we assume that $K_F + c_1(\mathcal{E}_F)$ is not nef. Then $K_X + c_1(\mathcal{E})$ is not nef and the same argument as in the proof of Theorem 5.2, case (II-3), shows that $(f, X, C, \mathcal{E})$ is as required. This completes the proof of Theorem 5.3. □

Remark 5.6. Let $(f, X, C, \mathcal{E})$ be as in Theorem 5.2. Suppose that $g(X, \mathcal{E}) = g(C)$ and $r = 1$. Then by Theorem 1.4.2 and Proposition 1.4.3 in [**Fk2**], $(f, X, C, \mathcal{E})$ is a scroll (in the sense of [**Fk2**], §0) unless $n = 2$ and $(f, X, C, \mathcal{E}) \cong (\pi, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1, L)$, where π is one projection such that $LF_\pi \geq 2$ for a fiber F_π of π. By the other projection ρ, however, $(\rho, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1, L)$ becomes a scroll.

Appendix.

Proposition A. *Let (X, L) be a quasi-polarized surface (i.e., L is a nef and big line bundle on a smooth surface X) such that $\kappa(X) = 2$ and $h^0(L) \geq 2$. Then $K_X L \geq 2q(X) - 2$. If equality holds and (X, L) is L-minimal (i.e., $LE > 0$ for any (-1)-curve E on X), then (X, L) is the following:*

$X \cong F \times C$ and $L \equiv C + 2F$, where F and C are smooth curves with $g(F) = 2$ and $g(C) \geq 2$.

Proof. See [**Fk4**]. □

Proposition B. *Let (X, L) be a polarized surface with $\kappa(X) = 0$ or 1. Assume that L is spanned. Then $g(L) := g(X, L) \geq 2q(X)$. Furthermore if $g(L) = 2q(X)$, then (X, L) is one of the following:*

(1) *(X, L) is a polarized abelian surface with $L^2 = 6$ such that $(X, L) \not\cong (E_1 \times E_2, p_1^*(D_1) + p_2^*(D_2))$, where E_i is a smooth elliptic curve, p_i is the i-th projection, and $D_i \in \operatorname{Pic}(E_i)$ for $i = 1, 2$ with $\deg D_1 = 1$ and $\deg D_2 = 3$.*

(2) *X is a one point blowing up of S, and $L = \mu^* A - 2E$, where S is an abelian surface, A is an ample line bundle with $A^2 = 8$, $\mu : X \to S$ is its blowing up, and E is a (-1)-curve of μ.*

(3) *$\kappa(X) = 1$, $L^2 = 4$, $q(X) = 3$, X has a locally trivial elliptic fibration $f : X \to C$, and $LF = 3$ for a fiber F of f, where C is a smooth curve with $g(C) = 2$.*

Proof. See [**Fk5**]. □

Theorem. *Let X be a smooth projective surface and let L be an ample and spanned line bundle on X. If $g(L) = q(X) + 2$, then (X, L) is one of the following:*

(1) *(X, L) is a relatively minimal conic bundle over a smooth curve B of genus two (i.e., X is a $\mathbb{P}^1$-bundle over B and $L_F = \mathcal{O}_{\mathbb{P}^1}(2)$ for every fiber F of the ruling).*

(2) *X is a $\mathbb{P}^1$-bundle X_0 blown-up at s $(0 \leq s \leq 4)$ points $p_1, \dots, p_s$ on distinct fibers and $L = \pi^* L_0 - E_1 - \cdots - E_s$, where $\pi : X \to X_0$ is the blowing up, $E_i = \pi^{-1}(p_i)$, X_0 is an elliptic $\mathbb{P}^1$-bundle of invariant $e \leq 0$, and $L_0 \equiv 2\sigma + (e+2)f$ (σ is a minimal section with $\sigma^2 = -e$ and f is a fiber).*

(3) *X is an $\mathbb{F}_e$ $(e \leq 2)$ blown-up at s $(0 \leq s \leq 9)$ points $p_1, \dots, p_s$ on distinct fibers and $L = \pi^* L_0 - E_1 - \cdots - E_s$, where $\pi : X \to \mathbb{F}_e$ is the blowing up, $E_i = \pi^{-1}(p_i)$, and $L_0 \equiv 2\sigma + (e+3)f$.*

(4) *X is a Del Pezzo surface of degree one and there exists a double covering $\pi : X \to \mathcal{Q} \subset \mathbb{P}^3$ of a quadric cone $\mathcal{Q}$ branched at the vertex and along the transverse intersection of $\mathcal{Q}$ with a cubic surface and $L = \pi^*(\mathcal{O}_{\mathcal{Q}}(1))$.*

(5) *(X, L) is a polarized abelian surface with $L^2 = 6$ such that $(X, L) \ncong (E_1 \times E_2, p_1^*(D_1) + p_2^*(D_2))$, where E_i is a smooth elliptic curve, p_i is the i-th projection, and $D_i \in \mathrm{Pic}(E_i)$ for $i = 1, 2$ with* $\deg D_1 = 1$ *and* $\deg D_2 = 3$.

(6) *X is a blowing up of an abelian surface S at one point p and $L = \pi^* A - 2E$, where $\pi : X \to S$ is the blowing up, $E = \pi^{-1}(p)$, and A is an ample line bundle on S with $A^2 = 8$.*

(7) *X is a $K3$ surface which is a double covering of $\mathbb{P}^2$ branched along a smooth curve of degree six and L is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$.*

Proof. (I) The case in which $\kappa(X) = 0$ or 1.

Then by Proposition B, we get that $g(L) \geq 2q(X)$. So we obtain $q(X) \leq 2$ by assumption.

(I-1) If $q(X) = 2$, then $g(L) = q(X) + 2 = 2q(X)$ and by Proposition B we get the type (5) and (6) in Theorem.

(I-2) If $q(X) \leq 1$, then $g(L) \leq 3$ and $L^2 \leq 4$ by $K_X L \geq 0$.

(I-2-1) If $L^2 = 4$, then $\kappa(X) = 0$ and X is minimal since $K_X L = 0$. So by Kodaira vanishing Theorem and Riemann-Roch Theorem, we get the equality: $h^0(L) = L^2/2 + \chi(\mathcal{O}_X) = 2 + \chi(\mathcal{O}_X)$. Because L is ample and spanned, we obtain $h^0(L) \geq 3$ and $\chi(\mathcal{O}_X) \geq 1$. But then $q(X) = 0$ by the classification theory of surfaces and this is impossible.

(I-2-2) If $L^2 = 3$, then $g(L) = 3$, $K_X L = 1$, and $q(X) = 1$. We have $h^0(L) \geq 3$ since L is ample spanned.

If $h^0(L) \geq 4$, then $g(L) > \Delta(L)$ and $L^2 \geq 2\Delta(L) + 1$, where $\Delta(L) := 2 + L^2 - h^0(L)$ is the Δ-genus of L. But then $q(X) = 0$ (see e.g. (I.3.5) in [**Fj4**]).

If $h^0(L) = 3$, then there is a triple covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Let $\mathcal{E}$ be a vector bundle of rank two on $\mathbb{P}^2$ such that $\pi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$. By Lemma 3.2 in [**Be**], we get the following two equalities:

(i) $\chi(\mathcal{O}_X) = (1/2)g(L)(g(L) + 1) + 2 - c_2$,
(ii) $K_X^2 = 2g(L)^2 - 4g(L) + 11 - 3c_2$,

where $c_2 := c_2(\mathcal{E})$. Since $g(L) = 3$, we get that $3\chi(\mathcal{O}_X) - K_X^2 = 7$ by the above equalities.

If $\kappa(X) = 0$, then $K_X^2 = -1$ because $K_X L = 1$. So we get $\chi(\mathcal{O}_X) = 2$. But by the classification theory of surfaces, this is impossible because $q(X) = 1$.

If $\kappa(X) = 1$, then X is minimal and $K_X^2 = 0$ because $K_X L = 1$. But then $3\chi(\mathcal{O}_X) = 7$ and this is impossible.

(I-2-3) If $L^2 = 2$, then $K_X L = 0$ or 2. Since $\kappa(X) \geq 0$, we get that $\Delta(L) \geq 1$ and $h^0(L) = 3$. Then there exists a double covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. We remark that $K_X = \varphi_{|L|}^*(K_{\mathbb{P}^2} + D)$ for some $D \in \mathrm{Pic}(\mathbb{P}^2)$. Since $\kappa(X) = 0$ or 1, we get that $\kappa(X) = 0$ and so X is minimal. In particular $K_X = \mathcal{O}_X$. Therefore $K_X L = 0$ and $g(L) = 2$. Since $h^0(L) = L^2/2 + \chi(\mathcal{O}_X) = 1 + \chi(\mathcal{O}_X)$, we get $\chi(\mathcal{O}_X) = 2$. Hence X is a K3 surface by the Classification theory of surfaces. This is the type (7) in Theorem.

(II) The case in which $\kappa(X) = 2$.

Then by Corollary 3.2 in [**Fk1**], we get $g(L) \geq 2q(X) - 1$. So we obtain $q(X) \leq 3$ and $g(L) \leq 5$ by assumption. Furthermore $L^2 \leq 3$ by Proposition A because L is spanned. (We remark that L is L-minimal if L is ample.)

If $h^0(L) \geq 4$, then $g(L) > 1 \geq \Delta(L)$ and $L^2 \geq 2\Delta(L) + 1$. On the other hand, since $\kappa(X) \geq 0$, we obtain that $\Delta(L) = 1$ and $L^2 = 3$. So we get $q(X) = 0$ and $g(L) \geq 3$ and this is impossible. Therefore $h^0(L) = 3$.

If $L^2 = 3$, then there exists a triple covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. In this case, by the same argument as above, we get

$$2(K_X^2 - 3\chi(\mathcal{O}_X)) = (g(L) - 1)(g(L) - 10).$$

Since $3 \leq g(L) \leq 5$, we get the following:

(α) $(g(L), q(X), K_X L, K_X^2 - 3\chi(\mathcal{O}_X)) = (5, 3, 5, -10)$,
(β) $(g(L), q(X), K_X L, K_X^2 - 3\chi(\mathcal{O}_X)) = (4, 2, 3, -9)$,
(γ) $(g(L), q(X), K_X L, K_X^2 - 3\chi(\mathcal{O}_X)) = (3, 1, 1, -7)$.

Claim. *The above three cases cannot occur.*

Proof. (II-1) The case (γ).

In this case X is minimal because $K_X L = 1$. But then this is impossible by Hodge index Theorem.

(II-2) The case (β).

If X is minimal, then $K_X^2 \geq 2q(X) = 4$ by Théorème 6.1 in [**D**]. On the other hand, $K_X^2 \leq 3$ by Hodge index Theorem and this is a contradiction.

So we get that X is not minimal. Let $\mu := \mu_r \circ \cdots \circ \mu_1 : X := X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r =: X'$ be an admissible minimalization of X and let $m = (m_r, \dots, m_1)$ be the weight sequence of this minimalization (see (II.14.4) in [**Fj4**]). We remark that $m_r \geq \cdots \geq m_1$.

If $m_1 = 1$, then $g(L_1) = q(X_1) + 1$ and $h^0(L_1) \geq 2$, where $L_1 := (\mu_1)_*(L)$ in the sense of cycle theory. But then this is impossible by Proposition A because $2 = K_X L > K_{X_1} L_1$. So we get $m_1 \geq 2$. Then $L_1^2 \geq 7$ and $K_{X_1} L_1 \leq 1$. Hence X_1 is minimal and this is a contradiction by Hodge index Theorem.

(II-3) The case (α).

If X is minimal, then $\chi(\mathcal{O}_X) \geq 4$ because $3\chi(\mathcal{O}_X) = K_X^2 + 10$. Furthermore $p_g(X) \geq 6$ since $q(X) = 3$. Hence $K_X^2 \geq 2p_g(X) \geq 12$ by Théorème 6.1 in [**D**]. But this is impossible by Hodge index Theorem. So we get that X is not minimal. By the same argument as in the case (II-2) we get a contradiction. □

We continue the proof of Theorem.

If $L^2 = 2$, then there exists a double covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Let $\mathcal{O}_{\mathbb{P}^2}(a)$ be a line bundle on $\mathbb{P}^2$ such that $B \in |\mathcal{O}_{\mathbb{P}^2}(2a)|$, where B is the branch locus. Then $(\varphi_{|L|})_*(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-a)$. Hence

$$h^1(\mathcal{O}_X) = h^1((\varphi_{|L|})_*(\mathcal{O}_X)) = h^1(\mathcal{O}_{\mathbb{P}^2}) + h^1(\mathcal{O}_{\mathbb{P}^2}(-a)) = 0.$$

So we get $g(L) = 2$. But since $K_X L > 0$ and $L^2 = 2$, this is impossible.

(III) The case in which $\kappa(X) = -\infty$.

Since (X, L) is not a scroll over a smooth curve, we get $g(L) \geq 2q(X)$ by Lemma 1.4 in [**Ma1**]. So $q(X) \leq 2$.

(III-1) The case in which $q(X) = 2$.

In this case, $g(L) = q(X) + 2 = 2q(X)$. Since $K_X + L$ is nef, we get

$$\begin{aligned} 0 \leq (K_X + L)^2 &= (K_X)^2 + 2(K_X + L)L - L^2 \\ &\leq 8(1 - q(X)) + 4(g(L) - 1) - L^2 \\ &= 4(g(L) - 2q(X) + 1) - L^2. \end{aligned}$$

Hence $L^2 \leq 4$ in this case.

If $L^2 = 4$, then X is relatively minimal and $(K_X + L)^2 = 0$, that is, (X, L) is a relatively minimal conic bundle over a smooth curve. This is the type (1) in Theorem.

If $L^2 \leq 3$ and $h^0(L) \geq 4$, then we get a contradiction as in (I-2-2). So we may assume that $L^2 \leq 3$ and $h^0(L) = 3$.

If $L^2 = 3$, then $K_X L = 3$ and there is a triple covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Since $\chi(\mathcal{O}_X) = -1$, we get that $K_X^2 = -12$ by Lemma 3.2 in [**Be**]. Here we calculate $(K_X + L)^2$;

$$(K_X + L)^2 = K_X^2 + 2K_X L + L^2 = -12 + 6 + 3 < 0.$$

But this is a contradiction because $K_X + L$ is nef.

If $L^2 = 2$, then there is a double covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. But then $q(X) = 0$ and this is a contradiction.

(III-2) The case in which $q(X) = 1$.

Then $g(L) = 3$. Here we use the classification of polarized surfaces with sectional genus three by [**LL**].

Claim. *The case in which $L^2 = 3$ cannot occur.*

Proof. If $L^2 = 3$ and $h^0(L) \geq 4$, then $g(L) > 1 \geq \Delta(L)$ and $L^2 \geq 2\Delta(L)+1$. But this is impossible because $q(X) = 1$. So we may assume that $h^0(L) = 3$. Then there is a triple covering $\varphi_{|L|} : X \to \mathbb{P}^2$ which is defined by $|L|$. Since $\chi(\mathcal{O}_X) = 0$, we get $K_X^2 = -7$ by Lemma 3.2 in [**Be**]. But in the table II of [**LL**], the case in which $L^2 = 3$ cannot occur. □

Next we prove that the following case cannot occur (see (2.6) in [**LL**]):

X is an elliptic $\mathbb{P}^1$-bundle $X_\sharp$ of invariant $e = 0$, blown up at a single point p not lying on a curve $D \in |m\sigma|$, $m \leq 2$ and $L = \eta^*[4\sigma + (2e+1)f] \otimes [E]^{-2}$. (Here we use the same notations as in [**LL**].)

Let σ' be the strict transform of σ under η. Since

$$0 < L\sigma' = (4\sigma + f)\sigma - 2E\sigma' = 1 - 2E\sigma',$$

we see that $E\sigma' = 0$ and $L\sigma' = 1$. It follows that $\sigma \cong \sigma' \cong \mathbb{P}^1$ since L is spanned. This is a contradiction.

By the above argument, we obtain the type (2) in Theorem by the classification of polarized surfaces with sectional genus three (see [**LL**]).

(III-3) The case in which $q(X) = 0$.

Then $g(L) = 2$. So by Theorem 3.1 in [**LP**] we get the type (3) and (4) in Theorem. □

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