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L^2 SPECTRAL DECOMPOSITION ON THE HEISENBERG GROUP ASSOCIATED TO THE ACTION OF $U(p,q)$

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Here we consider the Heisenberg group $H_n = C^n \times \Re$. $U(p,q)$, $p+q=n$, acts by automorphism on H_n by $g\cdot(z,t) = (gz,t)$.

Let $\{X_1,\dots,X_n,Y_1,\dots,Y_n,T\}$ be the standard basis of the Lie algebra of H_n and let

$$L = \sum_{j=1}^{p}\left(X_j^2+Y_j^2\right) - \sum_{j=p+1}^{n}\left(X_j^2+Y_j^2\right).$$

Via the Plancherel inversion formula, we obtain the joint spectral decomposition of $L^2(H_n)$ with respect to L and T

$$f = \sum_{k\in Z}\int_{-\infty}^{+\infty} f * S_{\lambda,k}\,|\lambda|^n\,d\lambda, \quad f\in S(H_n)$$

where each $S_{\lambda.k}$ is a tempered distribution $U(p,q)$ invariant satisfying $iTS_{\lambda,k} = \lambda S_{\lambda,k}$, $LS_{\lambda,k} = -|\lambda|(2k+p-q)S_{\lambda,k}$. We compute explicitly the distributions $S_{\lambda,k}$ and the integral $\mu_k = \int_{-\infty}^{+\infty} f * S_{\lambda,k}\,|\lambda|^n\,d\lambda$.

1. Introduction.

Let $H_n = C^n\times\Re$ with law group $(z,t)(z',t') = \left(z+z', t+t'-\frac{1}{2}\mathrm{Im}B(z,z')\right)$, where $B(z,w) = \sum_{j=1}^{p} z_j\overline{w_j} - \sum_{j=p+1}^{n} z_j\overline{w_j}$. Then H_n can be viewed as the $2n+1$ dimensional Heisenberg group. Indeed, if $n=p+q$, $Q(z,w) = -\mathrm{Im}B(z,w)$ is the standard symplectic form on $\Re^{2(p+q)}$ via the identification $\Psi : \Re^{2(p+q)} \to C^n$ given by

$$\Psi\left(x',x'',y',y''\right) = \left(x'+iy', x''-iy''\right), \quad x',y'\in\Re^p; x'',y''\in\Re^q. \tag{1.1}$$

Moreover, Ψ provides a global coordinate system (x,y,t) with $x=(x',x'')$, $y=(y',y'')$. The vector fields $X_j = -\frac{1}{2}y_j\frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}$, $Y_j = \frac{1}{2}x_j\frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}$, $j=1,\dots,n$ and $T = \frac{\partial}{\partial t}$ form a basis for the Lie algebra h_n of H_n. As usual, $\mathcal{U}(h_n)$ will denote its universal enveloping algebra, which can be identified with the algebra of left invariant differential operators on H_n.

$U(p,q) = \{g \in GL(n,\mathbb{C}) : B(gz,gw) = B(z,w)\}$ acts by automorphism on H_n by

$$g \cdot (z,t) = (gz,t), \qquad g \in U(p,q), (z,t) \in H_n. \tag{1.2}$$

It is well known that the subalgebra $\mathcal{U}(h_n)^{U(n)}$ of the elements which commute with the action of $U(n) = U(n,0)$ given by (1.2), is generated by T and the Heisenberg Laplacian $\sum\limits_{j=1}^{n}\left(X_j^2 + Y_j^2\right)$. The spherical functions associated with the Gelfand pair $(U(n), H_n)$ have been obtained independently by many authors (see e.g., [**H-R**], [**Ko**], [**St**]). Moreover in [**B-J-R**] it is developed a general calculus to provide the bounded K- spherical functions for a Gelfand pair (K, H_n), $K \subset U(n)$.

For general $p, q, p+q = n$, let

$$L = \sum_{j=1}^{p}\left(X_j^2 + Y_j^2\right) - \sum_{j=p+1}^{n}\left(X_j^2 + Y_j^2\right).$$

Then

$$L = \left(\sum_{j=1}^{p}\left(x_j^2 + y_j^2\right) - \sum_{j=p+1}^{n}\left(x_j^2 + y_j^2\right)\right)\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{p}\left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2}\right) - \sum_{j=p+1}^{n}\left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2}\right) + \frac{\partial}{\partial t}\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right). \tag{1.3}$$

It is easy to see, reasoning as in the case $p = n, q = 0$, (see Lemma 2.1 below), that the subalgebra $\mathcal{U}(h_n)^{U(p,q)}$, of the left invariant differential operators which commute with the action of $U(p,q)$ is generated by T and L. So, it is natural to ask for the joint eigendistributions of L and T and the associated decomposition of $L^2(H_n)$. In order to do this, we will use, following [**St**], the Plancherel inversion formula to decompose $f \in S(H_n)$ as

$$f = \sum_{k \in Z}\int_{-\infty}^{+\infty} f * S_{\lambda,k}\,|\lambda|^n\,d\lambda$$

where each $S_{\lambda,k}$ is a tempered and $U(p,q)$ invariant distribution satisfying $iTS_{\lambda,k} = \lambda S_{\lambda,k}$, $LS_{\lambda,k} = -|\lambda|(2k+p-q)S_{\lambda,k}$.

Next we will study the confluent hypergeometric equation in a suitable distribution space in order to obtain that, for $k \geq q$

$$\langle S_{\lambda,k}, f\rangle = c\sum_{j=0}^{n-2} c_j(\lambda)\int\limits_{\Re} e^{-i\lambda t}\delta_B^j(f(.,t))\,dt + c\int\limits_{C^n\times\Re} e^{-i\lambda t}e^{-\frac{|\lambda|}{4}B(z)}L_{k-q}^{n-1}\left(\frac{|\lambda|}{2}B(z)\right)H(|\lambda|\,B(z))\,f(z,t)\,dzdt$$

where $B(z) = B(z,z)$, H is the Heaviside function, δ_B^j are canonical distributions associated to the quadratic form B defined as in [**G-Sh**], supported on $\{z \in C^n : B(z) = 0\}$ and where L_{k-q}^{n-1} denotes, as usual, a Laguerre polynomial. The various constants $c, c_j(\lambda)$ are explicitly computed. Similar formulas are obtained if $k \leq -p$. If $-p < k < q$, $S_{\lambda,k}$ is written as a finite sum in terms of the distributions δ_B^j, $j = 1, ..., n-2$. Finally, we compute $\mu_k = \int\limits_{\Re} S_{\lambda,k} |\lambda|^n \, d\lambda$ and so the projections $\wp_k f = f * \mu_k$, $k \in Z$. In particular we recover the projections onto the kernel of $L + i(2k+p-q)T$, extending the formula given in [**M-R,2**] for $n = 2$, $p = q = 1$, to arbitrary n, p, q.

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2. Some preliminaries.

As in the case $p = n, q = 0$ we have that $\mathcal{U}(h_n)^{U(p,q)}$ is generated by T and L and the proof follows the same lines but we add it for the sake of completeness.

Lemma 2.1. *$\mathcal{U}(h_n)^{U(p,q)}$ is generated by T and L.*

Proof. Let $S(h_n)$ be the symmetric algebra generated by the set

$$\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$$

and let $\Lambda : S(h_n) \to \mathcal{U}(h_n)$ be the symmetrizer map. Since $U(p,q)$ acts on $S(h_n)$ and on $\mathcal{U}(h_n)$ by automorphism, the following diagram is commutative (see [**V**], Th. 3.3.4)

$$\begin{array}{ccc} S(h_n) & \xrightarrow{\Lambda} & \mathcal{U}(h_n) \\ \big\downarrow g & & \big\downarrow g \\ S(h_n) & \xrightarrow{\Lambda} & \mathcal{U}(h_n) \end{array} \quad , \qquad g \in U(p,q).$$

Λ is a linear isomorphism, thus Λ maps $S(h_n)^{U(p,q)}$ onto $\mathcal{U}(h_n)^{U(p,q)}$. Since the action of $U(p,q)$ preserves degree on $S(h_n)$, the lines of Theorem 3.3.8 in [**V**] say that if $\{1, u_1, \ldots, u_m\}$ is a set of generators of $S(h_n)^{U(p,q)}$, then $\{1, \Lambda(u_1), \ldots, \Lambda(u_m)\}$ generates $\mathcal{U}(h_n)^{U(p,q)}$. If $u \in S(h_n)^{U(p,q)}$ then $u = \sum P_j(X_1, \ldots, X_n, Y_1, \ldots, Y_n,) T^j$ where the sum is finite and each P_j is a polynomial $U(p,q)$ invariant. Decomposing P_j as a sum of homogeneous polynomials, the same is true for all of them. Since $SU(p,q)$ acts transitively on

$$S_1 = \left\{ (x,y) \in \Re^{2n} : \sum_{j=1}^{p} \left(x_j^2 + y_j^2\right) - \sum_{j=p+1}^{n} \left(x_j^2 + y_j^2\right) = 1 \right\}$$

each P_j must be a polynomial in $\sum_{j=1}^{p}\left(x_j^2+y_j^2\right)-\sum_{j=p+1}^{n}\left(x_j^2+y_j^2\right)$. This ends the proof. $\square$

We recall that for $\lambda\in\Re$ $\lambda\neq 0$, the Schrödinger's representation π_λ of the Heisenberg group $\Re^n\times\Re^n\times\Re$ is defined on $L^2\left(\Re^n\right)$ by

$$\pi_\lambda\left(x,y,t\right)h\left(\zeta\right)=e^{-i\left(\lambda t+sg(\lambda)\sqrt{|\lambda|}x\cdot\zeta+\frac{1}{2}\lambda x\cdot y\right)}h\left(\zeta+\sqrt{|\lambda|}y\right). \tag{2.1}$$

We denote by $E_\lambda\left(h_1,h_2\right)$ the matrix entry associated to π_λ and the vectors h_1,h_2, given by

$$E_\lambda\left(h_1,h_2\right)\left(x,y,t\right)=\left\langle\pi_\lambda\left(x,y,t\right)h_1,h_2\right\rangle.$$

We also denote by $d\pi_\lambda$ the infinitesimal representation defined on the space of C^∞ vectors for π_λ, which is, in this case, the space of the rapidly decreasing functions

$$d\pi_\lambda\left(X\right)h=\frac{d}{dt}_{|t=0}\pi_\lambda\left(\exp tX\right)h.$$

We still denote by π_λ the corresponding representation of $H_n=C^n\times\Re$ and by $E_\lambda\left(h_1,h_2\right),d\pi_\lambda$ its associated matrix entries and infinitesimal representation respectively.

It is remarked in [**St**] that

$$XE_\lambda\left(h_1,h_2\right)=E_\lambda\left(d\pi_\lambda\left(X\right)h_1,h_2\right),\quad X\in\mathcal{U}\left(h_n\right).$$

It follows that $iTE_\lambda=\lambda E_\lambda$ and that, in order to obtain matrix entries eigenfuntions of L, we must look for eigenvectors of $d\pi_\lambda\left(L\right)$ in $L^2\left(\Re^n\right)$.

Thus we pick the orthonormal basis of $L^2\left(\Re^n\right)$ given by the Hermite functions: For $\alpha=\left(\alpha_1,\dots,\alpha_n\right)\in\left(N\cup\{0\}\right)^n$, let

$$h_\alpha\left(\zeta\right)=\left(2^{|\alpha|}\alpha!\sqrt{\pi}\right)^{-\frac{n}{2}}e^{-\frac{|\zeta|^2}{2}}\prod_{j=1}^{n}H_{\alpha_j}\left(\zeta_j\right)$$

with $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha!=\alpha_1!\dots\alpha_n!$ and where

$$H_k\left(s\right)=\left(-1\right)^k e^{s^2}\frac{d^k}{ds^k}\left(e^{-s^2}\right)$$

is the $k-th$ Hermite polynomial.

It follows from (2.1) that

$$d\pi_\lambda\left(L\right)=-\left|\lambda\right|\left(B\left(\zeta\right)-\left(\sum_{j=1}^{p}\frac{\partial^2}{\partial\zeta_j^2}-\sum_{j=p+1}^{n}\frac{\partial^2}{\partial\zeta_j^2}\right)\right)$$

where $B\left(\zeta\right)=\sum_{j=1}^{p}\zeta_j^2-\sum_{j=p+1}^{n}\zeta_j^2$.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ we set $\|\alpha\| = \sum_{j=1}^{p} \alpha_j - \sum_{j=p+1}^{n} \alpha_j$. Since $\left(\zeta_j^2 - \frac{\partial^2}{\partial \zeta_j^2}\right) h_{\alpha_j}$ $= (2\alpha_j + 1) h_{\alpha_j}$, we have that $d\pi_\lambda (L) h_\alpha = -|\lambda| (2\|\alpha\| + p - q) h_\alpha$. Thus

$$d\pi_\lambda (L) E_\lambda (h_\alpha, h_\alpha) = -|\lambda| (2\|\alpha\| + p - q) E_\lambda (h_\alpha, h_\alpha). \tag{2.2}$$

(2.2) and the Plancherel inversion formula lead us to the joint spectral resolution of iT and L.

The inversion formula asserts that, for $f \in S(H_n)$

$$f(x, y, t) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} tr (\pi_\lambda (f) \pi_\lambda (x, y, t)) |\lambda|^n d\lambda$$

where $\pi_\lambda (f) = \int_{H_n} f(x, y, t) \pi_\lambda (x, y, t)^{-1} dxdydt$. Moreover, for $f \in S(H_n)$, $(x, y, t) \in H_n$, we have that

$$\sum_\alpha \int_{-\infty}^{+\infty} |\langle \pi_\lambda (x, y, t) \pi_\lambda (f) h_\alpha, h_\alpha \rangle| |\lambda|^n d\lambda \leq M < \infty \tag{2.3}$$

with M independent of (x, y, t) (see [**R**], Th. 10.1).

Taking account of that

$$\langle \pi_\lambda (x, y, t) \pi_\lambda (f) h_\alpha, h_\alpha \rangle = (E_\lambda (h_\alpha, h_\alpha) * f)(x, y, t)$$

and that

$$E_\lambda (h_\alpha, h_\alpha) \left((x, y, t)^{-1}\right) = \overline{E_\lambda (h_\alpha, h_\alpha)(x, y, t)}$$

we have

$$\begin{aligned} f(x, y, t) &= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} \sum_\alpha \langle \pi_\lambda (x, y, t) \pi_\lambda (f) h_\alpha, h_\alpha \rangle |\lambda|^n d\lambda \\ &= \frac{1}{(2\pi)^{n+1}} \sum_\alpha \int_{-\infty}^{+\infty} (f * E_\lambda (h_\alpha, h_\alpha))(x, y, t) |\lambda|^n d\lambda \\ &= \frac{1}{(2\pi)^{n+1}} \sum_{k \in Z} \int_{-\infty}^{+\infty} \sum_{\|\alpha\| = k} (f * E_\lambda (h_\alpha, h_\alpha))(x, y, t) |\lambda|^n d\lambda. \end{aligned}$$

Lemma 2.2. *Let $\mu_k : S(H_n) \to C$ be defined by*

$$\mu_k (f) = \int_{-\infty}^{+\infty} \sum_{\|\alpha\| = k} \langle f, E_\lambda (h_\alpha, h_\alpha) \rangle |\lambda|^n d\lambda, \qquad f \in S(H_n).$$

Then $\mu_k \in S'(H_n)$.

Proof. For $k \in Z$, let H_k be the closed subspace of $L^2(\Re^n)$ generated by $\{h_\alpha : \|\alpha\| = k\}$, thus $L^2(\Re^n) = \bigoplus_{k \in Z} H_k$. Let P_k be the orthogonal projection from $L^2(\Re^n)$ onto H_k. Now, for $f \in S(H_n)$, we define $\wp_k f$ by

$$\pi_\lambda (\wp_k f) = P_k \pi_\lambda (f). \tag{2.4}$$

It follows from (2.3) that

$$\int_{-\infty}^{+\infty} \sum_{\alpha} \left| \left\langle \pi_\lambda \left(\wp_k f \right) \pi_\lambda \left(x, y, t \right) h_\alpha, h_\alpha \right\rangle \right| \left| \lambda \right|^n d\lambda < \infty$$

and so

$$\wp_k f \left(x, y, t \right) = \frac{1}{\left(2\pi \right)^{n+1}} \int_{-\infty}^{+\infty} \sum_{\|\alpha\| = k} \left(f * E_\lambda \left(h_\alpha, h_\alpha \right) \right) \left(x, y, t \right) \left| \lambda \right|^n d\lambda.$$

$\wp_k f$ commutes with left translations and by (2.4) and the Plancherel formula it extends to a bounded operator on $L^2 \left(H_n \right)$. So, there exists a unique tempered distribution, which is μ_k such that $\wp_k f = f * \mu_k$. □

We set, for $\lambda \in \Re - \{0\}$ and $f \in S \left(H_n \right)$

$$S_{\lambda, k} \left(f \right) = \sum_{\|\alpha\| = k} \left\langle f, E_\lambda \left(h_\alpha, h_\alpha \right) \right\rangle . \tag{2.5}$$

We claim that $S_{\lambda,k}$ is well defined and belongs to $S' \left(H_n \right)$. In order to see this, we consider $\overline{H_n} = H_n / N$ where $N = \{0\} \times \{0\} \times 2\pi Z$. Then $\overline{H_n} = \Re^n \times \Re^n \times S^1$, where $S^1 = \left\{ e^{i\theta} : \theta \in \Re \right\}$. Each irreducible unitary representation of $\overline{H_n}$ is unitarily equivalent to one and only one of the following representations: The representations π_m acting on $L^2 \left(\Re^n \right)$ corresponding to $\lambda = 2\pi m$, $m \in Z$ and the one dimensional representations $\sigma_{a,b} \left(x, y, t \right) = e^{i(ax+by)}$, $(a, b) \in \Re^n \times \Re^n$. For f nice enough, $\pi_m \left(f \right)$ is a Hilbert Schmidt operator. We have also $\sigma_{a,b} \left(f \right) = \int_{\Re^n \times \Re^n \times S^1} f \left(x, y, t \right) e^{-i(ax+by)} dxdydt = \widehat{f} \left(a, b, \overline{0} \right)$, where $\widehat{f}$ denotes the euclidean Fourier transform and $\overline{0}$ is the identity in N. The Plancherel identity asserts that

$$\|f\|^2_{L^2\left(\overline{H_n}\right)} = \sum_{m \neq 0} \left\| \pi_m \left(f \right) \right\|^2_{HS} \left| m \right|^n + \int_{\Re^n \times \Re^n} \left| \sigma_{a,b} \left(f \right) \right|^2 dadb.$$

Also, setting $\phi \left(a, b \right) = \sigma_{a,b} \left(f \right)$, the inversion formula is in this case

$$f \left(x, y, t \right) = \sum_{m \neq 0} tr \left(\pi_m \left(f \right) \pi_m \left(x, y, t \right)^{-1} \right) \left| m \right|^n + \widehat{\phi} \left(-x, -y \right).$$

So we can consider $L, T = \frac{\partial}{\partial \theta}$ and $\wp_k$ as above, and repeat all the arguments for $\overline{H_n}$ instead of H_n to obtain that $\nu_k \left(f \right) = \sum\limits_{m \neq 0} \left| m \right|^n \sum\limits_{\|\alpha\| = k} \left\langle f, E_m \left(h_\alpha, h_\alpha \right) \right\rangle$ defines a tempered distribution on $S \left(\Re^n \times \Re^n \times S^1 \right)$. Furthermore, the analogous of (2.3) says that the last double series converges absolutely. Now, for $\lambda \in \Re - \{0\}$, $(z, t) \in C^n \times \Re$, we can write (see, for example [Fo]), $E_\lambda \left(h_\alpha, h_\alpha \right) \left(z, t \right)$ in terms of Laguerre polynomials as

$$E_\lambda \left(h_\alpha, h_\alpha \right) \left(z, t \right) = e^{-i\lambda t} e^{-\frac{1}{4} |\lambda| |z|^2} \prod_{j=1}^{n} L^0_{\alpha_j} \left(\frac{1}{2} \left| \lambda \right| \left| z_j \right|^2 \right). \tag{2.6}$$

For $f \in S\left(\Re^{2n}\right)$, we set $\nu_{k,l}(f) = \nu_k(g_l(f))$, where $g_l(f)(z,t) = e^{ilt} f(z)$, $(z,t) \in C^n \times \Re$ and where we use the identification of C^n with $\Re^{2n}$ given by (1.1). Then $\nu_{k,l} \in S'\left(\Re^{2n}\right)$ if $l \in Z - \{0\}$. In particular, we have that the series

$$e^{-\frac{1}{4}|z|^2} \sum_{\|\alpha\|=k} \prod_{j=1}^{n} L^0_{\alpha_j}\left(\frac{1}{2}|z_j|^2\right) \tag{2.7}$$

defines an element in $S'\left(\Re^{2n}\right)$ and so $S_{1,k} \in S'(H_n)$.

We set, for $\mu \in S'(H_n)$, $\lambda \in \Re - \{0\}$

$$\langle \delta_\lambda \mu, f \rangle = |\lambda|^{-n-1} \langle \mu, \delta_{\lambda^{-1}} f \rangle \tag{2.8}$$

where $\delta_\lambda f(z,t) = f\left(\sqrt{|\lambda|} z, \lambda t\right)$.

Lemma 2.3. *$S_{\lambda,k} \in S'(H_n)$ for all $\lambda \in \Re - \{0\}, k \in Z$.*

Proof. $S_{\lambda,k} = \delta_\lambda(S_{1,k})$ and $S_{1,k} \in S'(H_n)$. □

Remark 2.4. Since the series (2.7) belongs to $S'\left(\Re^{2n}\right)$, the same dilation argument shows that the series $e^{-\frac{1}{4}|\lambda||z|^2} \sum\limits_{\|\alpha\|=k} \prod\limits_{j=1}^{n} L^0_{\alpha_j}\left(\frac{1}{2}|\lambda||z_j|^2\right)$ defines a tempered distribution $F_{\lambda,k}$ on $\Re^{2n}$ for $\lambda \in \Re - \{0\}, k \in Z$.

For $g \in U(p,q)$, let $S^g_{\lambda,k}$ be defined by $S^g_{\lambda,k}(f) = S_{\lambda,k}(f^g)$, where $f^g(z,t) = f(gz,t)$. We have

Lemma 2.5. *$S_{\lambda,k}$ is a $U(p,q)$ invariant distribution for all $\lambda \in \Re - \{0\}, k \in Z$.*

Proof. Let w be the metaplectic representation of $SU(p,q)$ on $L^2(\Re^n)$. Then, for $g \in SU(p,q)$, $(z,t) \in H_n$, we have that

$$\pi_\lambda(gz,t) = w(g)\, \pi_\lambda(z,t)\, w\left(g^{-1}\right). \tag{2.9}$$

Furthermore, $L^2(\Re^n) = \bigoplus\limits_{k \in Z} H_k$, where H_k is, as in Lemma 2.2, the closed subspace generated by $\{h_\alpha : \|\alpha\| = k\}$. It is known that (w, H_k) is $SU(p,q)$ irreducible (see 1.12, 2.7 and 2.8, Ch.VIII in [**B-W**]).

We denote by $I_k : H_k \to L^2(\Re^n)$ the inclusion map and by $P_k : L^2(\Re^n) \to H_k$ the orthogonal projection. We also set $T_{z,t} = P_k \pi_\lambda(z,t) I_k$. Then, for $f \in S(H_n)$, the operator $T = \int_{H_n} f(z,t)\, T_{z,t}\, dz dt$ is a trace class operator. Now, by (2.9)

$$\begin{aligned} \left\langle S^g_{\lambda,k}, f \right\rangle &= \sum_{\|\alpha\|=k} \int_{H_n} f(z,t) \langle \pi_\lambda(gz,t) h_\alpha, h_\alpha \rangle\, dz dt \\ &= \sum_{\|\alpha\|=k} \int_{H_n} f(z,t) \left\langle \pi_\lambda(z,t)\, w\left(g^{-1}\right) h_\alpha, w\left(g^{-1}\right) h_\alpha \right\rangle dz dt \end{aligned}$$

$$= \sum_{\beta} \langle T\theta_\beta, \theta_\beta \rangle = \langle S_{\lambda,k}, f \rangle$$

with $\theta_\beta = w\left(g^{-1}\right) h_\beta$ and where we use that $\{\theta_\beta\}_\beta$ is another orthonormal basis of H_k. Then $S_{\lambda,k}$ is $SU(p,q)$ invariant. Finally, we note also that if $g = z_0 I$, $|z_0| = 1$, I the $n \times n$ identity matrix, it is clear from (2.6) that $S^g_{\lambda,k} = S_{\lambda,k}$ and so $S_{\lambda,k}$ is a $U(p,q)$ invariant distribution. □

Remark 2.6. By the inversion Plancherel formula and Lemmas (2.2), (2.3) and (2.5) we have $f = \sum\limits_{k \in Z} \int_{-\infty}^{+\infty} f * S_{\lambda,k} |\lambda|^n \, d\lambda$, $f \in S(H_n)$.

Let $F_{\lambda,k} \in S'\left(\Re^{2n}\right)$ be the distribution defined in Remark 2.4. Since $F_{\lambda,k} \bigotimes 1 = e^{i\lambda t} S_{\lambda,k}$ we have that $F_{\lambda,k}$ is $U(p,q)$ invariant. Then

$$\sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right) F_{\lambda,k} = 0.$$

From $LS_{\lambda,k} = -|\lambda| (2k + p - q) S_{\lambda,k}$ and (1.3) we have that

$$\left(-\frac{1}{4} \lambda^2 B(z) + \Box \right) F_{\lambda,k} = -|\lambda| (2k + p - q) F_{\lambda,k} \tag{2.10}$$

where $\Box = \sum\limits_{j=1}^{p} \left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2} \right) - \sum\limits_{j=p+1}^{n} \left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2} \right)$ and $B(z) = B(z,z)$ for $z = x + iy$, $x, y \in \Re^n$.

Now, according with [T], the space of the $U(p,q)$ invariant tempered distributions can be described as the dual of the space of the functions in $C^\infty(\Re - \{0\})$ with some kind of singularity at the origin. In order to describe them, we introduce polar coordinates on $\Re^{2n}$ as follows. For $x, y \in \Re^n$ we set $\sigma = \sum\limits_{j=1}^{p} \left(x_j^2 + y_j^2 \right) - \sum\limits_{j=p+1}^{n} \left(x_j^2 + y_j^2 \right)$, $\rho = \sum\limits_{j=1}^{n} \left(x_j^2 + y_j^2 \right)$, $u = \left(\frac{\rho+\sigma}{2} \right)^{\frac{1}{2}} w_u$, $v = \left(\frac{\rho-\sigma}{2} \right)^{\frac{1}{2}} w_v$ where w_u belongs to the $2p-1$ dimensional sphere S^{2p-1} and $w_v \in S^{2q-1}$.

For $f \in S\left(\Re^{2n}\right)$ and for $\rho, \sigma \in \Re$, $\rho \geq \sigma$, $\rho \geq 0$, let

$$(Mf)(\rho,\sigma) = \int\limits_{S^{2p-1} \times S^{2q-1}} f\left(\left(\frac{\rho+\sigma}{2} \right)^{\frac{1}{2}} w_u, \left(\frac{\rho-\sigma}{2} \right)^{\frac{1}{2}} w_v \right) dw_u dw_v$$

and let, for $\tau \in \Re$,

$$(Nf)(\tau) = \int\limits_{\rho > |\tau|} (Mf)(\rho,\tau) (\rho+\tau)^{p-1} (\rho-\tau)^{q-1} \, d\rho. \tag{2.11}$$

We note that

$$\int_{\Re^{2n}} f(x)\,dx = \frac{1}{2^n}\int_{\Re} Nf(\sigma)\,d\sigma. \tag{2.12}$$

Let H be the Heaviside function, defined by $H(\tau) = 1$ if $\tau \geq 0$ and $H(\tau) = 0$ if $\tau < 0$. Let $\mathcal{H}_0$ the space of the functions $\varphi : \Re \to C$ such that $\varphi(\tau) = \varphi_1(\tau) + H(\tau)\varphi_2(\tau)\tau^{n-1}$, $\varphi_1, \varphi_2 \in D(\Re)$, where $D(\Re)$ denotes the space of the functions in $C^\infty(\Re)$ with compact support and let $\mathcal{H}$ be the space defined analogously, but where now we require $\varphi_1, \varphi_2 \in S(\Re)$.

If $\varphi \in \mathcal{H}$, then it is regular out of the origin and $\varphi \in C^{n-2}(\Re)$. Moreover, for each $m \geq n-1$, there exists $P_m(\varphi)$, polynomial of degree m, such that $\varphi - HP_m(\varphi) \in C^m(\Re)$. So, for $m \in N$, φ admits an expansion

$$\varphi(\tau) = \sum_{j=0}^{m} B_j(\varphi)\tau^j + H(\tau)\sum_{j=0}^{m} A_j(\varphi)\tau^j + o(\tau^m) \tag{2.13}$$

with $A_j(\varphi) = 0$ for $j < n-1$.

Remark 2.7. $\mathcal{H}_0$ and $\mathcal{H}$, with the topology given in [**T**], are Frechet spaces and $N : S(\Re^{2n}) \to \mathcal{H}$, $N : D(\Re^{2n}) \to \mathcal{H}_0$ are linear, continuous and surjective maps. Moreover, their adjoints $N' : \mathcal{H}' \to S'(\Re^{2n})^{U(p,q)}$, $N' : \mathcal{H}_0' \to D'(\Re^{2n})^{U(p,q)}$ are linear homeomorphisms. (see 2.1, 4.3, 5.1 and some remarks at the beginning of §7 in [**T**]). (We also remark that 5.1 in [**T**] holds for $U(p,q)$ instead of $SO(p,q)$ with the obvious changes.)

It is also proved in [**T**] that

$$N(\Box f) = D(Nf),\ f \in S(\Re^{2n}) \tag{2.14}$$

where the differential operator D is defined by

$$D = 4\left(\tau\frac{\partial^2}{\partial\tau^2} + (2-n)\frac{\partial}{\partial\tau}\right) \tag{2.15}$$

so the adjoint of D is given by $D'T = 4(\tau T'' + nT')$, $T \in \mathcal{H}'$.

We say that $T \in \mathcal{H}'$ is a solution of $D'T = 0$ if $\langle D'T, \varphi\rangle = 0$ for all $\varphi \in \mathcal{H}$. It is easy to see that $T \in \mathcal{H}'$ is a solution of

$$\frac{\lambda^2}{4}\tau T + 4(\tau T'' + nT') = -|\lambda|(2k+p-q)T \tag{2.16}$$

if and only if $N'T$ is a solution of (2.10). The same assertion is true for solutions in $\mathcal{H}_0'$.

Setting $b = -|\lambda|(2k+p-q)$, (2.16) becomes $16\tau T'' + 16nT' - (\lambda^2\tau + 4b)T = 0$. As in [**Ko**], we note that if $\beta = \pm\frac{\lambda}{4}$, $\frac{\beta}{\alpha} = -\frac{1}{2}$ and $l = \frac{4n\beta - b}{4\alpha}$ and if $w(t) = e^{\beta t}v(\alpha t)$, then w is a solution of $16\tau w'' + 16nw' -$

$\left(\lambda^2\tau + 4b\right) w = 0$ if and only if v is a solution of the confluent hypergeometric equation (C.H.E) $tv'' + (n-t)\, v' + lv = 0$.

For $T \in \mathcal{H}'$ and for $k \in Z$, $\lambda \in \Re - \{0\}$ we set

$$\left\langle T_{\lambda,k}, \varphi \right\rangle = \left\langle \delta_{\frac{|\lambda|}{2}} T, \psi_\lambda(\varphi) \right\rangle, \psi_\lambda(\varphi)(t) = e^{-\frac{|\lambda|}{4}t}\varphi(t) \tag{2.17}$$

for $k \geq 0$, where $\delta_\lambda \varphi(t) = \varphi(\lambda t)$ and $\langle \delta_\lambda T, \varphi \rangle = |\lambda|^{-1} \langle T, \delta_{\lambda^{-1}} \varphi \rangle$.

We also set

$$\left\langle T_{\lambda,k}, \varphi \right\rangle = \left\langle \delta_{-\frac{|\lambda|}{2}} T, \psi_\lambda(\varphi) \right\rangle, \psi_\lambda(\varphi)(t) = e^{\frac{|\lambda|}{4}t}\varphi(t) \tag{2.18}$$

if $k < 0$.

We note that if $k \geq 0$ then $T \in \mathcal{H}'_0$ is a solution of the C.H.E. with parameter $l = k - q$ if and only if $T_{\lambda,k}$ is a solution in $\mathcal{H}'_0$ of (2.16). If $k < 0$ then $T \in \mathcal{H}'_0$ solves the C.H.E. with parameter $l = -k - p$ if and only if $T_{\lambda,k}$ solves (2.16).

Our aim is to find all the solutions in $\mathcal{H}'$ of (2.16). We note that if S is such a solution, then $S = T_{\lambda,k}$ for some solution $T \in \mathcal{H}'_0$ of the C.H.E. with parameter $l = k - q$ if $k \geq 0$ and $l = -k - p$ if $k < 0$. This leads us to determine all the solutions in $\mathcal{H}'_0$ of C.H.E. with parameter $l \geq -n+1$ such that the corresponding $T_{\lambda,k} \in \mathcal{H}'$.

3. About the confluent hypergeometric equation.

As in [**Sz**], if m, β are non negative integers, we denote by $\{L^\beta_m\}$, the Laguerre polynomials. Then $L^\beta_m(x)$ is defined as the only polynomial solution of

$$tv'' + (\beta + 1 - t)v' + mv = 0$$

and normalized by the condition

$$\int_0^\infty e^{-x} x^\beta L^\beta_m(x) L^\beta_{m'}(x)\, dx = \Gamma(\beta+1)\binom{m+\beta}{m}\delta_{m,m'}. \tag{3.1}$$

We have that

$$L^0_m(t) = \sum_{j=0}^m \binom{m}{j} (-1)^j \frac{x^j}{j!} \tag{3.2}$$

and that $\frac{d}{dt} L^\beta_m = -L^{\beta+1}_{m-1}$.

Let D_l be the differential operator on $\mathcal{H}$ given by

$$D_l\varphi(\tau) = \tau\varphi'' + (2-n)\varphi' + \tau\varphi' + (l+1)\varphi. \tag{3.3}$$

Then its adjoint D'_l is $D'_l T = tT'' + (n-t)T' + lT$. We recall that $A_j(\varphi) = 0$ for $\varphi \in \mathcal{H}$, $j \leq n-2$. It is easy to see that if φ admits an asymptotic development

$$\sum_{j \geq 0} B_j(\varphi)\, \tau^j + H \sum_{j \geq 0} A_j(\varphi)\, \tau^j$$

then the expansion around $\tau = 0$ of $D_l\varphi$ is

$$\sum_{j\geq 0}\left[(l+1+j)B_j(\varphi)+(j+1)(j+2-n)B_{j+1}(\varphi)\right]\tau^j + H\sum_{j\geq 0}\left[(l+1+j)A_j(\varphi)+(j+1)(j+2-n)A_{j+1}(\varphi)\right]\tau^j. \tag{3.4}$$

With the natural restrictions on f, integration by parts gives

$$\int_a^b f(t)\left(D_l\varphi\right)(t)\,dt = \int_a^b \left(D_l' f\right)(t)\varphi(t)dt + R(b,\varphi) - R(a,\varphi) \tag{3.5}$$

where $-\infty \leq a < b \leq +\infty$ and

$$R(b,\varphi) = (1-n+b)f(b)\varphi(b) + bf(b)\varphi'(b) - bf'(b)\varphi(b). \tag{3.6}$$

Proposition 3.1. *For $l \geq 0$, $T = \left(L^0_{l+n-1}H\right)^{(n-1)}$ is a solution in $\mathcal{H}_0'$ of $D_l'T = 0$.*

Proof. Let $c_{j,l} = \left(L^0_{l+n-1}\right)^{(n-2-j)}(0)\,,\ 0 \leq j \leq n-2$. Then a computation shows that

$$T = \left(L^0_{l+n-1}\right)^{(n-1)}H + \sum_{j=0}^{n-2} c_{j,l}\delta^{(j)}$$

and so $T \in \mathcal{H}'$ since every $\varphi \in \mathcal{H}$ is in $C^{n-2}(\Re)$. Also

$$\begin{aligned}\left\langle D_l'T,\varphi\right\rangle &= \left\langle T, D_l\varphi\right\rangle \\ &= \int_0^\infty \left(L^0_{l+n-1}\right)^{(n-1)}(t)\left(D_l\varphi\right)(t)\,dt + \left\langle \sum_{j=0}^{n-2} c_{j,l}\delta^{(j)}, D_l\varphi\right\rangle.\end{aligned}$$

By (3.4), (3.5) and (3.6) we have

$$\int_0^\infty \left(L^0_{l+n-1}\right)^{(n-1)}(t)\left(D_l\varphi\right)(t)\,dt = (n-1)\left(L^0_{l+n-1}\right)^{(n-1)}(0)\,B_0(\varphi)$$

and by (3.4)

$$\begin{aligned}&\left\langle \sum_{j=0}^{n-2} c_{j,l}\delta^{(j)}, D_l\varphi\right\rangle \\ &= \sum_{j=0}^{n-2} c_{j,l}(-1)^j j!B_j\left(D_l\varphi\right) \\ &= \sum_{j=0}^{n-2} c_{j,l}(-1)^j j!\left((l+1+j)B_j\varphi + (j+1)(j+2-n)B_{j+1}(\varphi)\right) \\ &= \sum_{j=0}^{n-2} d_{j,l}B_j(\varphi)\end{aligned}$$

where $d_{0,l} = (l+1)\, c_{0,l}$ and $d_{j,l} = (-1)^j\, j!\, ((l+1+j)\, c_{j,l} + (n-j-1)\, c_{j-1,l})$ if $1 \leq j \leq n-2$. Since $c_{j,l} = (-1)^{n-j} \binom{l+n-1}{n-j-2}$ the lemma follows. $\square$

Now, it is proved in [T] that if $S \in \mathcal{H}'$ and $\operatorname{supp}(S) = \{0\}$ then there exists $m_1, m_2 \in N \cup \{0\}$ $\alpha_0, \ldots \alpha_{m_1}, \alpha'_0, \ldots \alpha'_{m_2} \in C$ such that

$$S(\varphi) = \sum_{j=0}^{m_1} \alpha_j B_j(\varphi) + \sum_{j=0}^{m_2} \alpha'_j A_j(\varphi), \quad \varphi \in \mathcal{H}.$$

We will need the following:

Lemma 3.2. *Assume $l \geq -n+1$. If $S \in \mathcal{H}'$,* supp$S = \{0\}$ *and if*

$$D'_l S = c_{n-1} B_{n-1} + d_{n-1} A_{n-1} + \sum_{j=0}^{n-2} c_j B_j$$

with $c_0, \ldots, c_{n-1}, d_{n-1} \in C$, then $c_{n-1} = d_{n-1} = 0$.

Proof. We write $S = \sum_{j=0}^{m_1} \alpha_j B_j + \sum_{j=0}^{m_2} \alpha'_j A_j$. Suppose $c_{n-1} \neq 0$. By (3.4) the coefficient of $B_j(\varphi)$ in the expansion of $D_l(\varphi)$ is $(l+1+j)\, \alpha_j$ $+ j\,(j+1-n)\, \alpha_{j-1}$ and so $c_{n-1} = (l+n)\, \alpha_{n-1}$ and $\alpha_j = -\frac{j(j+1-n)}{l+1+j} \alpha_{j-1}$ for $j \geq -l$. Then $\alpha_j \neq 0$ if $j \geq n$. Contradiction. Analogously $d_{n-1} \neq 0$ would imply $\alpha'_j \neq 0$ for $j \geq n$. $\square$

If $l \geq 0$, a solution of the C.H.E. is the function $f_1(t) = L_l^{n-1}(t)$. Another solution $f_2 \in C^2((-\infty, 0))$ of the C.H.E., linearly independent with f_1, is obtained setting $f_2(t) = c(t) f_1(t)$ where $c(t)$ satisfy

$$t f_1(t) c''(t) + \left[2t f'_1(t) + (n-t) f_1(t)\right] c'(t) = 0.$$

Then for $t < 0$,

$$f_2(t) = f_1(t) \int_{-\infty}^{t} f_1(s)^{-2} s^{-n} e^s ds \tag{3.7}$$

is well defined since the zeros of the Laguerre's polynomials are in $(0, +\infty)$.

Also

$$\left.\begin{array}{c} f_2(t) \underset{t\to-\infty}{=} o\left(e^t\right), \\[2ex] f'_2(t) \underset{t\to-\infty}{=} o\left(e^t\right), \\[2ex] f_2(t) \backsim -\frac{1}{f_1(0)(n-1)} t^{-n+1} \quad \text{as} \quad t \to 0. \end{array}\right\} \tag{3.8}$$

Lemma 3.3. *Let for* $\varphi \in \mathcal{H}$,

$$\langle Pf(f_2), \varphi\rangle = \lim_{\epsilon\to 0^+} \int_{-\infty}^{-\epsilon} f_2(t)\left(\varphi(t) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} t^j\right) dt.$$

Then $Pf(f_2) \in \mathcal{H}'$ *and* $D_l' Pf(f_2) = -\frac{1}{f_1(0)} B_{n-1}(\varphi)$.

Proof. $Pf(f_2) \in \mathcal{H}'$ by Lemma 3.3 in [**T**]. On the other hand, from (3.4) it follows that if $\psi(t) = \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} t^j$ then $D_l\psi = \sum_{j=0}^{n-2} \frac{(D_l\varphi)^{(j)}(0)}{j!} t^j$. Thus

$$\begin{aligned}\left\langle D_l' Pf(f_2), \varphi\right\rangle &= \left\langle Pf(f_2), D_l'\varphi\right\rangle \\ &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{-\epsilon} f_2(t)\left((D_l\varphi)(t) - \sum_{j=0}^{n-2} \frac{(D_l\varphi)^{(j)}(0)}{j!} t^j\right) dt \\ &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{-\epsilon} f_2(t) D_l(\varphi - \psi)(t)\, dt = \lim_{\epsilon\to 0^+} R(-\epsilon, \varphi_1)\end{aligned}$$

where $\varphi_1 = \varphi - \psi$ and $R(-\epsilon, \varphi_1)$ is given by (3.6).

As by (3.8)

$$\lim_{s\to 0^-} (1-n+s) f_2(s) \varphi_1(s) = (1-n)\frac{1}{f_1(0)(1-n)} B_{n-1}(\varphi),$$

$$\lim_{s\to 0^-} s f_2(s) \varphi_1'(s) = \frac{1}{f_1(0)} \lim_{s\to 0^-} \frac{s^{-n+2}}{1-n}\left((n-1) B_{n-1} s^{n-2} + ...\right)$$
$$= -\frac{1}{f_1(0)} B_{n-1}$$

and

$$\lim_{s\to 0^-} s f_2'(s) \varphi_1(s) = \frac{1}{f_1(0)} B_{n-1}$$

the lemma follows. □

Proposition 3.4. *Let* T *be in* $\mathcal{H}_0'$. *Suppose that either* $k \geq q$ *or* $k \leq -p$ *and* $\lambda \in \Re - \{0\}$, *let* $T_{\lambda,k}$ *be defined as in* (2.17) *and* (2.18). *If* $T_{\lambda,k}$ *is a tempered solution* (*i.e.,* $T_{\lambda,k} \in \mathcal{H}'$) *of* (2.16) *then* T *is a multiple of* $\left(L_{l+n-1}^0 H\right)^{(n-1)}$ *where* $l = k - q$ *if* $k \geq q$ *and* $l = -k - p$ *if* $k \leq -p$.

Proof. We know that there exists a basis of the solution space in $C^2(0, +\infty)$ given by $f_1(t)$ and a certain function $g(t)$ where $g(t) \backsim e^t$ as $t \to +\infty$ [**Se**]. In particular when we write T restricted to $(0, +\infty)$, as a linear combination $af_1 + bg$, the condition $T_{\lambda,k} \in \mathcal{H}'$ implies $b = 0$.

We now consider $S = T - a\left(L^0_{l+n-1}H\right)^{(n-1)}$. Then supp$S \subset (-\infty, 0]$, $D'_l S = 0$ and the corresponding $S_{\lambda,k} \in \mathcal{H}'$.

Writing S restricted to $(-\infty, 0)$ as a linear combination $\alpha f_1 + \beta f_2$ we obtain that $\alpha = 0$. Thus $S - \beta Pf\,(f_2)$ has support at $t = 0$ and by Lemma 3.3

$$D'_l\left(S - \beta Pf\,(f_2)\right) = -\beta \frac{1}{f_1(0)} B_{n-1}.$$

If $\beta \neq 0$, this contradicts Lemma 3.2. Thus supp$S = \{0\}$. But, from (3.4), it is easy to see that there is not nontrivial solution S supported at the origin of $D'_l S = 0$ if $l \geq 0$. So $S = 0$ and the proof is complete. □

To state a similar result for $-p < k < q$ we will need some facts about the equation

$$tv'' + (n-t)\,v' - lv, \quad l = 1, \dots, n-1. \tag{3.9}$$

Lemma 3.5. *For $l = 1, \dots, n-1$ there exists a polynomial P_{l-1} of degree $l-1$ with $P_{l-1}(0) = 1$ such that for all open interval $I \subset \Re - \{0\}$ (not necessarily finite) two linearly independent solutions in $C^2(I)$ are given by $g_1(t) = t^{1-n}P_{l-1}(t)\,e^t$ and $g_2(t) = t^{1-n}T_{n-2}\left(P_{l-1}(t)\,e^t\right)$ where $T_{n-2}(g)$ denotes the Taylor polynomial of degree $n-2$ around the origin for the function g.*

Proof. Following the notation of [**Se**], we can write every solution of (3.9) belonging to $C^2(I)$ as $\alpha.{}_1F_1(l, n, t) + \beta t^{1-n}.{}_1F_1(1+l-n, 2-n, t)$ where

$$ {}_1F_1(a, c, t) = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j} \frac{t^j}{j!} \tag{3.10}$$

and $(a)_j = a(a+1)\dots(a+j-1)$.

By (3.10) ${}_1F_1(1+l-n, 2-n, t) = \sum\limits_{j=0}^{\infty} p_{l-1}(j)\frac{t^j}{j!}$ where $p_{l-1}(j) = \sum\limits_{k=0}^{l-1} a_k j^k$ for some $a_1, \dots, a_{k-1} \in \Re$ and $a_0 = 1$. Induction on k shows that $\sum\limits_{j=0}^{\infty} j^k \frac{t^j}{j!} = q_k(t)\,e^t$ with q_k a polynomial of degree k such that $q_k(0) = 0$ for $k > 0$. So $g_1(t) = t^{1-n}.{}_1F_1(1+l-n, 2-n, t)$ is a solution of the desired form.

Also

$$\begin{aligned}
&{}_1F_1(l, n, t)\\
&= \sum_{j=0}^{\infty} \frac{(l)_j}{(n)_j}\frac{t^j}{j!} = \frac{(n-1)!}{(l-1)!}\sum_{j=0}^{\infty}\frac{(j+1)\dots(j+l-1)}{(n+j-1)!}t^j\\
&= \frac{(n-1)!}{(l-1)!}\sum_{j=0}^{\infty}\frac{(j+(n-1)+(2-n))\dots((j+n-1)+(l-n))}{(n+j-1)!}t^j
\end{aligned}$$

$$= \frac{(n-1)!}{(l-1)!} \frac{1}{t^{n-1}} \sum_{j=n-1}^{\infty} (j+2-n) \dots (j+l-n) \frac{t^j}{j!}$$

$$= \frac{(n-1)!}{(l-1)!} (2-n) \dots (l-n)$$

$$\cdot \frac{1}{t^{n-1}} \left({}_1F_1 (1+l-n, 2-n, t) - T_{n-2} \left({}_1F_1 (1+l-n, 2-n, t) \right) \right).$$

So we can take $g_2(t) = t^{1-n} T_{n-2} \left({}_1F_1 (1+l-n, 2-n, t) \right).$ □

Lemma 3.6. *For $\varphi \in \mathcal{H}$, let $Pf^-(g_1)$ and $Pf^+(g_2)$ be defined by*

$$\left\langle Pf^-(g_1), \varphi \right\rangle = \lim_{\epsilon \to 0^+} \int_{-\infty}^{-\epsilon} g_1(t) \left(\varphi(t) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} t^j \right) dt,$$

$$\left\langle Pf^+(g_2), \varphi \right\rangle = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} g_2(t) \left(\varphi(t) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} t^j \right) dt$$

$$+ \int_1^{\infty} g_2(t) \varphi(t) \, dt.$$

Then $Pf^-(g_1)$ and $Pf^+(g_2)$ belong to $\mathcal{H}'$ and they satisfy:

(i) $D_l'(Pf^-(g_1)) = (n-1) B_{n-1}$,

(ii) $D_l'(Pf^+(g_2)) = -(n-1)(B_{n-1} + A_{n-1}) + \sum_{j=0}^{n-2} \beta_j B_j$ *for some constants $\beta_1, \dots, \beta_{n-2}$.*

Proof. The proof follows similar lines those of Lemma 3.3, but now, to prove (i) we take account of that $P_{l-1}(0) = 1$ where P_{l-1} is as in Lemma 3.5. For (ii) we observe that if $\varphi \in \mathcal{H}$ and if $\psi(t) = \sum_{j=0}^{n-2} B_j(\varphi) t^j$, we have $R(1, \varphi - \psi) - R(1, \varphi) = -(2-n)\psi(1) - \psi'(1) f_2(1) + f_2'(1)\psi(1)$.

The constants β_j are determined by $f_2(1)$ and $f_2'(1)$. □

Lemma 3.7. *For each $l = -1, -2, \dots, -n+1$, the space of the solutions $T \in \mathcal{H}_0'$ which are supported at the origin of the equation $D_l' T = 0$ is one dimensional.*

Proof. For such a T we write $T = \sum_{j=0}^{m_1} \alpha_j B_j + \sum_{j=n-1}^{m_2} \alpha_j' A_j$. From $\langle T, D_l \varphi \rangle = 0$ and (3.4) we obtain that $\alpha_j (l+1+j) + \alpha_{j-1}(j+1-n) = 0$ for all j. If $j = n-1$, this implies that $\alpha_{n-1}(l+n) = 0$ and so $\alpha_j = 0$ for all $j \geq n-1$. The same argument says that $\alpha_j' = 0$, $j \geq n-1$ and thus $T = \sum_{j=0}^{n-2} \alpha_j B_j$. Let

$j_0 = -l - 1$. Then $\alpha_{j_0 - 1} = 0$. . Since

$$\alpha_j = -\frac{j+1-n}{l+1+j}\alpha_{j-1} \tag{3.11}$$

for $j \neq j_0$ we have $\alpha_0 = \alpha_1 = \cdots = \alpha_{j_0-1} = 0$. So T is completely determined by α_{j_0}. On the other hand, it is clear that for each α_{j_0} we obtain in this way a solution supported at $\{0\}$. ☐

Remark 3.8. Let l, T be as in Lemma 3.7. If we write $T = \sum_{j=0}^{n-2} \gamma_{j,l}\delta^{(j)}$ instead of $\sum_{j=0}^{n-2} \alpha_j B_j$, by (3.11) we see that $\{\gamma_{j,l}\}$ satisfy

$$(l+1+j)\,\gamma_{j,l} + (n-j-1)\,\gamma_{j-1,l} = 0$$

for $0 \leq j \leq n-2$. But this is also the recurrence relation for the successive derivatives at the origin of the polynomial L^0_{l+n-1}, so we can choose a nontrivial solution as $T_0 = \sum_{j=0}^{n-2} \gamma_{j,l}\delta^{(j)}$ with $\gamma_{j,l} = \left(L^0_{l+n-1}\right)^{(n-j-2)}(0)$, $0 \leq j \leq n-2$. Now, a computation shows that $T_0 = \left(L^0_{l+n-1}H\right)^{(n-1)}$.

Proposition 3.9. *Let T be in $\mathcal{H}'_0$. Suppose $-p < k < q$, $\lambda \in \Re - \{0\}$, let $T_{\lambda,k}$ be defined as in* (2.17) *and* (2.18). *If $T_{\lambda,k}$ is a tempered solution (i.e., $T_{\lambda,k} \in \mathcal{H}'$) of* (2.16) *then T is a multiple of the distribution T_0 defined in Remark* 3.8.

Proof. We argue as in Proposition 3.4. Suppose $0 \leq k < q$. So $T_{\lambda,k}$ is given by (2.17). Now, $T_{\lambda,k} \in \mathcal{H}'$ implies that T restricted to $(0, +\infty)$ agrees with αg_2 and T restricted to $(-\infty.0)$ agrees with βg_1, for some $\alpha, \beta \in C$ and where g_1, g_2 are defined as in Lemma 3.5. So $S = T - \beta Pf^-(g_1) - \alpha Pf^+(g_2)$ has support at the origin and, by Lemma 3.6, it satisfies $D'_l(S) = -\beta(n-1)B_{n-1} + \alpha(n-1)(B_{n-1} + A_{n-1}) + \sum_{j=0}^{n-2} \beta_j B_j$. But, by Lemma 3.2 $\alpha = \beta = 0$ and so T has support at the origin and the lemma follows from Lemma 3.7. The case $-p < k < 0$ is analogous. ☐

4. Determination of $S_{\lambda,k}$ and $\wp_k$.

In this section we compute explicitly the distributions $S_{\lambda,k}$ and μ_k. Taking account of Remark 3.8 and Proposition 3.1, we consider the particular distribution T given by $T = \left(L^0_{l+n-1}H\right)^{(n-1)}$ where $l = k - q$ if $k \geq 0$ and $l = -k - p$ if $k < 0$. Let $F_{\lambda,k} \in S'\left(\Re^{2n}\right)$ be defined as in Remark 2.4. Since $F_{\lambda,k} \in S'(H_n)^{U(p,q)}$ and satisfies (2.10), the considerations in Remark 2.7 and Propositions 3.4 and 3.9 imply that $F_{\lambda,k} = c_{\lambda,k}N'(T_{\lambda,k})$ for some $c_{\lambda,k} \in C$. In order to compute $c_{\lambda,k}$ we apply both distributions to the function

(4.1)

$$f_\lambda(z) = f_\lambda(z_1, \dots z_n) = e^{-\frac{|\lambda|}{4}|z|^2} \sum_{\substack{\beta_1 + \cdots + \beta_n = |k|, \\ \beta_1 \geq 0, \dots \beta_n \geq 0}} \prod_{j=1}^{n} L^0_{\beta_1}\left(\frac{1}{2}|\lambda||z_j|^2\right).$$

By (3.1) we have that, if $k \geq 0$

$$\langle F_{\lambda,k},\, f_\lambda\rangle = 2^n\pi^n |\lambda|^{-n} \sum_{\substack{\beta_1 + \cdots + \beta_p = |k|, \\ \beta_1 \geq 0, \dots \beta_p \geq 0}} 1 = 2^n\pi^n |\lambda|^{-n} \binom{p+k-1}{p-1} \tag{4.2}$$

and if $k < 0$

(4.3)

$$\langle F_{\lambda,k},\, f_\lambda\rangle = 2^n\pi^n |\lambda|^{-n} \sum_{\substack{\beta_1 + \cdots + \beta_q = |k|, \\ \beta_1 \geq 0, \dots \beta_q \geq 0}} 1 = 2^n\pi^n |\lambda|^{-n} \binom{q-k-1}{q-1}.$$

On the other hand, by well known properties of the Laguerre polynomials,

$$f_\lambda(z) = e^{-\frac{|\lambda|}{4}|z|^2} L^{n-1}_{|k|}\left(\frac{1}{2}|\lambda||z|^2\right). \tag{4.4}$$

So, for $t \geq 0$, and taking account of that the volume of the n dimensional sphere is $2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$, we have

(4.5)

$$\begin{aligned} & N f_\lambda\left(2|\lambda|^{-1} t\right) \\ & = \frac{4\pi^{p+q}}{(p-1)!\,(q-1)!} \int\limits_{2|\lambda|^{-1}t}^{\infty} e^{-\frac{|\lambda|}{4}\rho} L^{n-1}_{|k|}\left(\frac{|\lambda|\rho}{2}\right) \\ & \qquad \cdot \left(\rho + 2|\lambda|^{-1} t\right)^{p-1} \left(\rho - 2|\lambda|^{-1} t\right)^{q-1} d\rho \\ & = \frac{4\pi^{p+q}}{(p-1)!\,(q-1)!} 2^{n-1} |\lambda|^{-(n-1)} \int_t^\infty e^{-\frac{s}{2}} L^{n-1}_{|k|}(s)\,(s+t)^{p-1}(s-t)^{q-1}\, ds. \end{aligned}$$

Now,

$$\langle F_{\lambda,k}, f_\lambda\rangle = c_{\lambda,k}\left\langle N'(T_{\lambda,k}), f_\lambda\right\rangle = c_{\lambda,k}\langle T_{\lambda,k}, N(f_\lambda)\rangle.$$

From (4.5), the definition of $T_{\lambda,k}$ and (4.2) we obtain that $c_{\lambda,k}$ is independent of λ. In order to compute $c_{\lambda,k}$ we consider first the case $k \geq 0$. By (2.17)

$$\langle T_{\lambda,k}, N(f_\lambda)\rangle = \left\langle 2|\lambda|^{-1} \delta_{\frac{|\lambda|}{2}} T, t \to e^{-\frac{|\lambda|}{4}t} N(f_\lambda)(t)\right\rangle$$

$$= 2\left|\lambda\right|^{-1}\left\langle T, t \rightarrow e^{-\frac{t}{2}} N\left(f_{\lambda}\right)\left(2\left|\lambda\right|^{-1} t\right)\right\rangle$$

thus, by (4.5), we need to evaluate $T\left(\psi_0\right)$ where $T = \left(L^0_{k-q+n-1} H\right)^{(n-1)}$ and $\psi_0\left(t\right) = e^{-\frac{t}{2}}\varphi_0\left(t\right)$ with

$$\varphi_0\left(t\right) = e^{-\frac{t}{2}} \int_0^{\infty} e^{-\frac{\rho}{2}} L_k^{n-1}\left(\rho + t\right)\left(\rho + 2t\right)^{p-1} \rho^{q-1} d\rho.$$

Since $k-q+n-1 = k+p-1$ and $L_k^{n-1}\left(\rho+t\right)\left(\rho+2t\right)^{p-1}$ is a polynomial in t of degree $k+p-1$ we can use the Leibnitz formula for the derivatives of a product, the fact that every polynomial can be written as a linear combination of the Laguerre polynomials and the orthogonality relations (3.1) to obtain that

$$\begin{aligned} &T\left(\psi_0\right) \\ &= \left(-1\right)^{n-1} \int_0^{\infty} L^0_{k+p-1}\left(t\right) \int_0^{\infty} e^{-\frac{\rho}{2}} \rho^{q-1} e^{-t} L_k^{n-1}\left(\rho+t\right)\left(\rho+2t\right)^{p-1} d\rho dt. \end{aligned}$$

Since $L_k^{n-1}\left(\rho+t\right) = \sum\limits_{m+j=k} L_m^{n-2}\left(\rho\right) L_j^0\left(t\right)$, we repeat the same argument to obtain that

$$\begin{aligned} &T\left(\psi_0\right) \\ &= 2^{p-1}\left(-1\right)^{n-1} \int_0^{\infty} L^0_{k+p-1}\left(t\right)\left[\int_0^{\infty} e^{-\frac{\rho}{2}} \rho^{q-1} L_0^{n-2}\left(0\right) d\rho\right] e^{-t} L_k^0\left(t\right) t^{p-1} dt \\ &= \left(-1\right)^{n-1} 2^{p-1}\left(-1\right)^q 2^q \left(q-1\right)! \int_0^{\infty} e^{-t} L^0_{k+p-1}\left(t\right) \frac{\left(-1\right)^k}{k!} t^{k+p-1} dt \\ &= \left(-1\right)^{n+q-1} 2^{n-1}\left(q-1\right)! \frac{\left(-1\right)^k}{k!}\left(-1\right)^{k+p-1}\left(k+p-1\right)! \end{aligned}$$

where we have used (3.1) and (3.2).

Finally, by (4.2), we find that

$$2^n \pi^n \frac{\left(p+k-1\right)!}{k!\left(p-1\right)!} = c_{\lambda,k} 2^n \frac{4\pi^n}{\left(p-1\right)!\left(q-1\right)!} 2^{n-1} \frac{\left(k+p-1\right)!}{k!}\left(q-1\right)!$$

and so

$$c_{\lambda,k} = \frac{1}{2^{n+1}}.$$

If $k < 0$, we can repeat the above computation, using (2.18) instead of (2.17) and replacing $L^0_{k-q+n-1}$ by $L^0_{-k-p+n-1}$. In this case we also find $c_{\lambda,k} = \frac{1}{2^{n+1}}$.

Theorem 4.1. *If* $k \geq q$, $\lambda \in \Re - \{0\}$, $f \in S\left(C^n\right)$, *then*

$$\left\langle F_{\lambda,k}, f\right\rangle = \frac{1}{2} \int\limits_{B(z)\geq 0} e^{-\frac{|\lambda|}{4} B(z)} L_{k-q}^{n-1}\left(\frac{|\lambda|}{2} B\left(z\right)\right) f\left(z\right) dz$$

$$+\frac{1}{2^n}\sum_{l=0}^{n-2}4^l\left|\lambda\right|^{-(l+1)}\sum_{j=l}^{n-2}\frac{1}{2^j}\binom{j}{l}(-1)^{n-j}\binom{n+k-q-1}{k-q+j+1}\left\langle\delta_B^l,f\right\rangle$$

where $\delta_B^l=N'\left(\delta^{(l)}\right).$

Proof.

$$\begin{aligned}\left\langle F_{\lambda,k},f\right\rangle&=\frac{1}{2^{n+1}}\left\langle N'T_{\lambda,k},f\right\rangle=\frac{1}{2^{n+1}}\left\langle T_{\lambda,k},Nf\right\rangle\\&=\frac{1}{2^{n+1}}\left\langle T,t\to 2\left|\lambda\right|^{-1}e^{-\frac{t}{2}}Nf\left(2\left|\lambda\right|^{-1}t\right)\right\rangle.\end{aligned}$$

Now, as at the beginning of the proof of Proposition 3.1,

$$T=L_{k-q}^{n-1}H+\sum_{j=0}^{n-2}\left(L_{k-q+n-1}^{0}\right)^{(n-2-j)}(0)\,\delta^{(j)}.$$

But

$$\begin{aligned}&2\left|\lambda\right|^{-1}\int_0^\infty L_{k-q}^{n-1}(t)\,e^{-\frac{t}{2}}Nf\left(2\left|\lambda\right|^{-1}t\right)dt\\&=\int_0^\infty L_{k-q}^{n-1}\left(\frac{\left|\lambda\right|t}{2}\right)e^{-\frac{\left|\lambda\right|t}{4}}Nf(t)\,dt\\&=2^n\int_{B(z)\geq 0}e^{-\frac{\left|\lambda\right|}{4}B(z)}L_{k-q}^{n-1}\left(\frac{\left|\lambda\right|}{2}B(z)\right)f(z)\,dz\end{aligned}$$

where the last equality follows from (2.12) applied to the function

$$F(z)=L_{k-q}^{n-1}\left(\frac{\left|\lambda\right|B(z)}{2}\right)e^{-\frac{\left|\lambda\right|B(z)}{4}}f(z).$$

On the other hand, a computation shows that

$$\begin{aligned}&\left\langle\sum_{j=0}^{n-2}\left(L_{k-q+n-1}^{0}\right)^{(n-2-j)}(0)\,\delta^{(j)},t\to 2\left|\lambda\right|^{-1}e^{-\frac{t}{2}}Nf\left(2\left|\lambda\right|^{-1}t\right)\right\rangle\\&=2\sum_{l=0}^{n-2}4^l\left|\lambda\right|^{-(l+1)}\sum_{j=l}^{n-2}\binom{j}{l}\left(L_{k-q+n-1}^{0}\right)^{(n-2-j)}(0)\,\frac{1}{2^j}\left\langle\delta_B^l,f\right\rangle\end{aligned}$$

and the theorem follows. □

Remark 4.2. Theorem 4.1 remains true for $k\leq -p$, with the obvious changes in the proof, if we replace L_{k-q}^{n-1} by L_{-k-p}^{n-1}, $\binom{n+k-q-1}{k-q+j+1}$ by $\binom{n-k-p-1}{-k-p+j+1}$

and the integration region $\{z : B(z) \geq 0\}$ by $\{z : B(z) \leq 0\}$. It is also immediate to see that if $-p < k < q$, $\lambda \in \Re - \{0\}$, $f \in S(C^n)$, then

$$\langle F_{\lambda,k}, f\rangle = \frac{1}{2^n}\sum_{l=0}^{n-2} 4^l |\lambda|^{-(l+1)} \sum_{j=l}^{n-2} \frac{1}{2^j}\binom{j}{l}\gamma_{j,k}\left\langle \delta_B^l, f\right\rangle$$

with $\gamma_{j,l}$ as in Remark 3.8, i.e.,

$$\gamma_{j,k} = \left(L^0_{k-q+n-1}\right)^{(n-j-2)}(0) = (-1)^{n-j}\binom{n+k-q-1}{n-j-2}$$

for $q-k-1 \leq j \leq n-2$ and $\gamma_{j,k} = 0$ if $j < q-k-1$ and where δ_B^l is as in Theorem 4.1.

Remark 4.3. We have computed the distributions $F_{\lambda,k}$ and the constant $c_{\lambda,k}$, and so also $S_{\lambda,k} = e^{-i\lambda t}F_{\lambda,k}$.

Next, we compute μ_k. We first assume $k \geq q$. Taking account of Theorem 4.1. We recall that for $f = f(z,t) \in S'(H_n)$

$$\langle \mu_k, f\rangle = \int_{-\infty}^{\infty} \left\langle e^{-i\lambda t}F_{\lambda,k}, f\right\rangle |\lambda|^n \, d\lambda.$$

By Theorem 4.1 $|\lambda|^n e^{-i\lambda t}\langle F_{\lambda,k}, f(.,t)\rangle = J_1(f)(\lambda,t) + J_2(f)(\lambda,t)$, $t \in \Re$, where

$$J_1(f)(\lambda,t) = \frac{1}{2}|\lambda|^n e^{-i\lambda t}\int_{B(z)\geq 0} e^{-\frac{|\lambda|}{4}B(z)}L^{n-1}_{k-q}\left(\frac{|\lambda|}{2}B(z)\right)f(z,t)\,dz$$

and

$$\begin{aligned} &J_2(f)(\lambda,t) \\ &= \frac{1}{2^n}e^{-i\lambda t}\sum_{l=0}^{n-2} 4^l |\lambda|^{n-(l+1)}\sum_{j=l}^{n-2}\frac{1}{2^j}\binom{j}{l}\left(L^0_{k-q+n-1}\right)^{(n-j-2)}(0)\left\langle \delta_B^l, f(.,t)\right\rangle. \end{aligned}$$

So, by well known properties of the Fourier transform on $S'(\Re)$,

$$\begin{aligned} &\int_{\Re}\left(\int_{\Re} J_2(f)(\lambda,t)\,dt\right)d\lambda \\ &= \frac{1}{2^n}\sum_{l=0}^{n-2} 4^l (-i)^{n-l-1}\sum_{j=l}^{n-2}\frac{1}{2^j}\binom{j}{l}\gamma_{j,k}\left\langle \nu_l, \frac{\partial^{n-l-1}f}{\partial t^{n-l-1}}\right\rangle \end{aligned} \tag{4.6}$$

where $\nu_l = \delta_B^l \otimes pv\left(\frac{1}{t}\right)$ if $n-l-1$ is odd and $\nu_l = \delta_B^l \otimes \delta$ if $n-l-1$ is even. Let $I_1(f) = \int_{\Re}\left(\int_{\Re} J_1(f)(\lambda,t)\,dt\right)d\lambda$. The properties of the Fourier transform in $S'(\Re)$ imply that

$$(4.7)\quad I_1(f) = \int\limits_{\Re} \left\langle e^{-\lambda it} e^{-\frac{|\lambda|}{4}B(z)} L_{k-q}^{n-1}\left(\frac{|\lambda|}{2}B(z)\right) H(B(z)), f \right\rangle |\lambda|^n \, d\lambda$$
$$= i \int\limits_{\Re} \left\langle e^{-\lambda it} e^{-\frac{|\lambda|}{4}B(z)} L_{k-q}^{n-1}\left(\frac{|\lambda|}{2}B(z)\right) H(B(z)), h \right\rangle |\lambda|^{n-1} \, d\lambda$$

where $h(z,t) = \frac{\partial\left(pv\left(\frac{1}{t} * f\right)\right)}{\partial t}(z,t)$.

Now, following [**St**], we will compute (4.7).

Lemma 4.4. *For $f \in S(C^n \times \Re)$ there exists $\int\limits_{C^n \times \Re} \frac{H(B(z))}{B(z)+it} f(z,t)\, dzdt$ and*

$$\lim_{\epsilon \to 0} \int\limits_{C^n \times \Re} \frac{H(B(z))}{B(z)+\epsilon+it} f(z,t)\, dzdt = \int\limits_{C^n \times \Re} \frac{H(B(z))}{B(z)+it} f(z,t)\, dzdt.$$

Proof. We write

$$\frac{1}{B(z)+\epsilon+it} = P(t, B(z)+\epsilon) - iQ(t, B(z)+\epsilon)$$

where $P(t,s) = \frac{s}{s^2+t^2}$, $Q(t,s) = \frac{t}{s^2+t^2}$, $t,s \in \Re$. Thus, for $s \in \Re$ $\|P(.,s)\|_{L^1(\Re)} = \pi$. So

$$\int\limits_{\Re} |P(t, B(z)+\epsilon) f(z,t)|\, dt \leq \pi \|f(z,.)\|_{L^\infty(\Re)}, \quad z \in C^n.$$

Also, for $B(z) \neq 0$, we have

$$\lim_{\epsilon \to 0} \left(P(., B(z)+\epsilon) * f(z,.)\right)(0) = \left(P(., B(z)) * f(z,.)\right)(0).$$

Since $\sup\limits_{t\in\Re} |f(z,t)| \in L^1(C^n)$, the dominated convergence theorem implies that $P(t, B(z)) f(z,t) \in L^1(C^n \times \Re)$ and

$$\lim_{\epsilon \to 0} \int\limits_{C^n \times \Re} P(t, B(z)+\epsilon) H(B(z)) f(z,t)\, dzdt$$
$$= \int\limits_{C^n \times \Re} P(t, B(z)) H(B(z)) f(z,t)\, dzdt.$$

On the other hand, let $G_\epsilon(z) = \int\limits_{\Re} Q(t, B(z)+\epsilon) f(z,t)\, dt$. So

$$G_\epsilon(z) = \int\limits_{|t|<1} Q(t, B(z)+\epsilon)\left[f(z,t) - f(z,0)\right] dt$$

$$+\int\limits_{|t|\geq 1} Q\left(t, B\left(z\right)+\epsilon\right) f\left(z,t\right) dt.$$

Now, for $|t|<1$

$$\left|\frac{f\left(z,t\right)-f\left(z,0\right)}{t}\right| = \left|\frac{\partial f}{\partial t}\left(z,\zeta\left(z,t\right)\right)\right| \leq \sup_{|u|<1}\left|\frac{\partial f}{\partial t}\left(z,u\right)\right|.$$

Also

$$\sup_{|t|<1}\left|tQ\left(t,B\left(z\right)+\epsilon\right)\right| \leq 1, \quad \sup_{|t|\geq 1}\left|Q\left(t,B\left(z\right)+\epsilon\right)\right| \leq 1.$$

Thus $|G_\epsilon(z)| \leq \sup\limits_{|u|<1}\left|\frac{\partial f}{\partial t}(z,u)\right| + \|f(z,.)\|_{L^1(\Re-[-1,1])}$. So, as above, we can use the dominated convergence theorem to obtain that $Q(t,B(z))H(B(z))f(z,t) \in L^1(C^n \times \Re)$ and

$$\lim_{\epsilon\to 0}\int\limits_{C^n\times\Re} Q\left(t,B\left(z\right)+\epsilon\right) H\left(B\left(z\right)\right) f\left(z,t\right) dzdt$$
$$= \int\limits_{C^n\times\Re} Q\left(t,B\left(z\right)\right) H\left(B\left(z\right)\right) f\left(z,t\right) dzdt.$$

□

Following [**St**], we use the generatrix identity for the Laguerre polynomials

(4.8) $$\sum_{s=0}^{\infty} L_s^{n-1}\left(t\right) r^s = \left(1-r\right)^{-n} e^{-\frac{r}{1-r}t}$$

to obtain, for $\epsilon>0$

(4.9) $$\int_0^\infty e^{-\epsilon\lambda} e^{-i\lambda t} e^{-\frac{\lambda}{4}B(z)} L_{k-q}^{n-1}\left(\frac{\lambda}{2}B\left(z\right)\right) H\left(B\left(z\right)\right)\lambda^{n-1} d\lambda$$
$$= \alpha_k \frac{\left[B\left(z\right)-4\epsilon-4it\right]^{k-q}}{\left[B\left(z\right)+4\epsilon+4it\right]^{k+p}} H\left(B\left(z\right)\right)$$

where

(4.10) $$\alpha_\kappa = 4^n\left(n-1\right)!\binom{p+k-1}{k-q}\left(-1\right)^{k-q}.$$

Indeed, by (4.8), we can write, for $|r|<1$, $B(z)\geq 0, t\in\Re$, $\epsilon>0$

$$\sum_{k=q}^{\infty} r^{k-q}\int_0^\infty e^{-\epsilon\lambda} e^{-i\lambda t} e^{-\frac{\lambda}{4}B(z)} L_{k-q}^{n-1}\left(\frac{\lambda}{2}B\left(z\right)\right)\lambda^{n-1} d\lambda$$
$$= \sum_{s=0}^{\infty} r^s\int_0^\infty e^{-\epsilon\lambda} e^{-i\lambda t} e^{-\frac{\lambda}{4}B(z)} L_s^{n-1}\left(\frac{\lambda}{2}B\left(z\right)\right)\lambda^{n-1} d\lambda$$

$$= (1-r)^{-n}\int_0^\infty \exp\left(-\lambda\left(\frac{B(z)(1+r)+4(\epsilon+it)(1-r)}{4(1-r)}\right)\right)\lambda^{n-1}d\lambda$$
$$= \frac{4^n(n-1)!}{[B(z)+4\epsilon+4it+r(B(z)-4\epsilon-4it)]^n}.$$

Now, we compare the Taylor developments to obtain (4.9).

Write

$$\frac{B(z)-it}{B(z)+it} = \frac{2B(z)}{B(z)+it} - 1.$$

Now, letting $\epsilon \to 0^+$, and taking account of Lemma 4.4, we have

$$\int_0^\infty \left\langle e^{-i\lambda t}e^{-\frac{\lambda}{4}B(z)}L_{k-q}^{n-1}\left(\frac{\lambda}{2}B(z)\right)H(B(z))\lambda^{n-1}, f\right\rangle d\lambda = \alpha_k \lim_{\epsilon\to 0}\left\langle \frac{[B(z)-4\epsilon-4it]^{k-q}}{[B(z)+4\epsilon+4it]^{k+p}}H(B(z)), f\right\rangle. \tag{4.11}$$

Now, this limit is

$$\alpha_k \lim_{\epsilon\to 0}\left\langle \left[\frac{2B(z)}{B(z)+4\epsilon+4it}-1\right]^{k-q}\frac{H(B(z))}{[B(z)+4\epsilon+4it]^n}, f\right\rangle$$
$$= \alpha_k \lim_{\epsilon\to 0}\sum_{l=0}^{k-q}\binom{k-q}{l}(-1)^{k-q-l}2^l\left\langle \frac{B(z)^l H(B(z))}{[B(z)+4\epsilon+4it]^{n+l}}, f\right\rangle$$
$$= \alpha_k \sum_{l=0}^{k-q}\binom{k-q}{l}(-1)^{k-q-l}\frac{2^l(-4i)^{n+l-1}}{(n+l-1)!}\left\langle \frac{B(z)^l H(B(z))}{B(z)+4it}, \frac{\partial^{n+l-1}f}{\partial t^{n+l-1}}\right\rangle.$$

So

$$\int_0^\infty \left\langle e^{-i\lambda t}e^{-\frac{\lambda}{4}B(z)}L_{k-q}^{n-1}\left(\frac{\lambda}{2}B(z)\right)H(B(z))\lambda^{n-1}, f\right\rangle d\lambda = \alpha_k\sum_{l=0}^{k-q}\beta_{k,l}\left\langle \frac{B(z)^l H(B(z))}{B(z)+4it}, \frac{\partial^{n+l-1}f}{\partial t^{n+l-1}}\right\rangle \tag{4.12}$$

where

$$\beta_{k,l} = \binom{k-q}{l}(-1)^{k-q-l}\frac{2^l(-4i)^{n+l-1}}{(n+l-1)!}. \tag{4.13}$$

From (4.11) a change of variable gives

$$\int_{-\infty}^0 \left\langle e^{-i\lambda t}e^{-\frac{|\lambda|}{4}B(z)}L_{k-q}^{n-1}\left(\frac{|\lambda|}{2}B(z)\right)H(B(z))|\lambda|^{n-1}, f\right\rangle d\lambda = \alpha_k\sum_{l=0}^{k-q}\overline{\beta}_{k,l}\left\langle \frac{B(z)^l H(B(z))}{B(z)-4it}, \frac{\partial^{n+l-1}f}{\partial t^{n+l-1}}\right\rangle \tag{4.14}$$

where, by (4.13), $\overline{\beta}_{k,l} = (-1)^{n+l-1}\,\beta_{k,l}$. So we have:

Theorem 4.5. *For $k \geq q$ and $0 \leq l \leq k-q$, let $\alpha_k, \beta_{k,l}$ defined by (4.10) and (4.13) respectively. Then we have $\mu_k(f) = I_1(f) + I_2(f)$ where*

$$I_1(f)$$
$$= \frac{i\alpha_k}{2}\sum_{l=0}^{k-q}\beta_{k,l}\left\langle \left(\frac{B(z)^l\,H(B(z))}{B(z)+4it} + (-1)^{n+l-1}\,\frac{B(z)^l\,H(B(z))}{B(z)-4it}\right), \frac{\partial^{n+l}\left(pv\left(\frac{1}{t}*f\right)\right)}{\partial t^{n+l}}\right\rangle$$

and

$$I_2(f)$$
$$= \frac{1}{2^n}\sum_{l=0}^{n-2}4^l\sum_{j=l}^{n-2}(-i)^{n-l-1}\,\frac{1}{2^j}\binom{j}{l}\left(L^0_{k-q+n-1}\right)^{(n-j-2)}(0)\left\langle \nu_l, \frac{\partial^{n-l-1}f}{\partial t^{n-l-1}}\right\rangle$$

where $\nu_l = \delta_B^l \otimes pv\left(\frac{1}{t}\right)$ if $n-l-1$ is odd and $\nu_l = \delta_B^l \otimes \delta$ if $n-l-1$ is even.

Proof. It follows from (4.12), (4.14), (4.7) and (4.6). □

Remark 4.6. If $k \leq -p$, Theorem 4.5 remains true if we replace $k-q$ by $-k-p$ and $H(B(z))$ by $H(-B(z))$ with the same proof, using (2.18) instead of (2.17). If $-p < k < q$ the same arguments give us $\mu_k(f) = I_2(f)$, with

$$I_2(f) = \frac{1}{2^n}\sum_{l=0}^{n-2}4^l\sum_{j=l}^{n-2}\frac{1}{2^j}\binom{j}{l}\gamma_{j,k}\left\langle \nu_l, \frac{\partial^{n-l-1}f}{\partial t^{n-l-1}}\right\rangle$$

where $\gamma_{j,k}$ is defined as in Remark 3.8.

Remark 4.7. Let $A = \{(z,t) \in C^n \times \Re : B(z) = 0\}$. If $f \in S(H_n)$ and $\operatorname{supp}(f) \cap A = \emptyset$ thus $\operatorname{supp}\left(\frac{\partial}{\partial t}\left(p.v.\left(\frac{1}{t}\right)*f\right)\right) \cap A = \emptyset$, then from (4.7) and (4.11) and taking account of that $I_2(f) = 0$, we have

$$\begin{aligned}
\mu_k(f) &= I_1(f)\\
&= i\alpha_k \lim_{\epsilon\to 0}\left\langle \frac{[B(z)-4\epsilon-4it]^{k-q}}{[B(z)+4\epsilon+4it]^{k+p}}H(B(z)), \frac{\partial}{\partial t}\left(p.v.\left(\frac{1}{t}\right)*f\right)\right\rangle\\
&= i\alpha_k\left\langle \frac{[B(z)-4it]^{k-q}}{[B(z)+4it]^{k+p}}H(B(z)), \frac{\partial}{\partial t}\left(p.v.\left(\frac{1}{t}\right)*f\right)\right\rangle\\
&= i\alpha_k\left\langle -\frac{\partial}{\partial t}\left(\frac{[B(z)-4it]^{k-q}}{[B(z)+4it]^{k+p}}\right)H(B(z)), p.v.\left(\frac{1}{t}\right)*f\right\rangle.
\end{aligned}$$

This is an analogous expression to those obtained in [**St**], p. 362.

Remark 4.8. For $\epsilon = \pm 1$, $k \in Z$, we set $R_{k,\epsilon} = \{\epsilon\rho, \rho(2k+p-q) : \rho > 0\}$. The rays $R_{k,\epsilon}$ are closely related to the study of the kernels of the operators $L - i\alpha T$, $\alpha \in C$. In order to describe $\ker(L - i\alpha T)$, with $\alpha \in$

$2Z$ for n even and $\ker(L - i\alpha T)$, with $\alpha \in 1 + 2Z$ for n odd, we define $\wp_k^+, \wp_k^- : L^2(H_n) \to L^2(H_n)$ via the Plancherel inversion formula requiring that for $\lambda \in \Re - \{0\}$, $\pi_\lambda \wp_k^+ = \chi_{(0,\infty)}(\lambda) P_k \pi_\lambda$ and $\pi_\lambda \wp_k^- = \chi_{(-\infty,0)}(\lambda) P_k \pi_\lambda$, where P_k is define as at the beginning of the proof of Lemma 2.2. Thus $\wp_k^+, \wp_k^-$ are orthogonal projections over certain subspaces of $L^2(H_n)$. As in Lemma 2.2 we have $\wp_k^+ f = \int_0^{+\infty} f * S_{\lambda,k} |\lambda|^n d\lambda$, $f \in S(H_n)$ (and the analogous formula for $\wp_k^-$). If m has the same parity than n, we define $k_1(m) = -\frac{1}{2}(m + p - q)$ and $k_2(m) = \frac{1}{2}(m - p + q)$. Thus $k_1(m), k_2(m) \in Z$. We observe that $R\left(\wp_{k_1(m)}^+\right) \subset \ker(L - imT) \cap L^2(H_n)$, where $\ker(L - imT) = \{S \in S'(H_n) : (L - imT) S = 0\}$. In order to see this inclusion, we proceed as follows. As in Lemma 2.2 we construct $\mu_{k_1(m)}^{\pm} \in S'(H_n)$ such that $\wp_{k_1(m)}^{\pm} f = f * \mu_{k_1(m)}^{\pm}$. As there, we have $\left\langle \mu_{k_1(m)}^+, \varphi \right\rangle = \int_0^{+\infty} \left\langle S_{\lambda.k_1(m)}, \varphi \right\rangle |\lambda|^n d\lambda$, $\varphi \in S'(H_n)$. Then

$$\begin{aligned}
&\left\langle (L - imT)\left(\mu_{k_1(m)}^+\right), \varphi \right\rangle \\
&= \left\langle \mu_{k_1(m)}^+, (L + imT)(\varphi) \right\rangle \\
&= \int_0^{+\infty} \left\langle S_{\lambda.k_1(m)}, (L + imT)(\varphi) \right\rangle |\lambda|^n d\lambda \\
&= \int_0^{+\infty} \left\langle (L - imT) S_{\lambda.k_1(m)}, \varphi \right\rangle |\lambda|^n d\lambda = 0.
\end{aligned}$$

Now, since L, T commute with left translations $(L - imT)\left(f * \mu_{k_1(m)}^+\right) = f * \left((L - imT)\mu_{k_1(m)}^+\right) = 0$. So $R\left(\wp_{k_1(m)}^+\right) \subset \ker(L - imT) \cap L^2(H_n)$. Similarly, $R\left(\wp_{k_2(m)}^-\right) \subset \ker(L - imT) \cap L^2(H_n)$. So $R\left(\wp_{k_1(m)}^+\right) \oplus R\left(\wp_{k_2(m)}^-\right) \subset \ker(L - imT) \cap L^2(H_n)$. On the other hand, Plancherel theorem implies that $R\left(\wp_k^{\pm}\right) \perp R\left(\wp_s^{\pm}\right)$ if $k \neq s$ and $R\left(\wp_k^+\right) \perp R\left(\wp_k^-\right)$,$k \in \dot{Z}$. We know also that, as operator on $L^2(H_n)$, iLT^{-1} has a closed and self-adjoint extension (see [**M-R,1**], Th. 7.4) that we still denote by iLT^{-1}. We have $\ker(L - i\alpha T) \cap L^2(H_n) = \ker\left(LT^{-1} - i\alpha\right)$, $\alpha \in C$ (see [**M-R,2**], Proposition 1.4). Since iLT^{-1} is a self adjoint operator, we have $\ker\left(LT^{-1} - im\right) \perp \ker\left(LT^{-1} - i\widetilde{m}\right)$ for $m \neq \widetilde{m}$. Now, $L^2(H_n) = \bigoplus_{k \in Z} R(\wp_k)$. Thus we have the direct orthogonal sum

$$L^2(H_n) = \bigoplus_{m \in Z} \left(R\left(\wp_{k_1(m)}^+\right) \bigoplus R\left(\wp_{k_2(m)}^-\right)\right).$$

Then we conclude that

$$\ker(L - imT) \cap L^2(H_n) = R\left(\wp_{k_1(m)}^+\right) \bigoplus R\left(\wp_{k_2(m)}^-\right)$$

and that if n is even then $\ker(L - i\alpha T) \cap L^2(H_n) = 0$ if $\alpha \notin 2Z$ and that if n is odd then $\ker(L - i\alpha T) \cap L^2(H_n) = 0$ if $\alpha \notin 1 + 2Z$.

The projectors $\wp_k^{\pm}$, $k \in Z$ can be computed proceeding as in the determination of $\wp_k$. As in Lemma 2.2 we construct $\mu_k^{\pm} \in S'(H_n)$ such that $\wp_k^{\pm} f = f * \mu_k^{\pm}$, and then, with the same arguments used for μ_k, we decompose $\mu_k^{+}(f) = I_1^{+}(f) + I_2^{+}(f)$, where

$$I_1^{+}(f) = \int_0^{\infty} \left\langle e^{-\lambda it} e^{-\frac{\lambda}{4} B(z)} L_{k-q}^{n-1}\left(\frac{\lambda}{2} B(z)\right) H(B(z)), f \right\rangle \lambda^n d\lambda$$

and

$$I_2^{+}(f) = \int\limits_{\Re}\int\limits_{\Re} \frac{1}{2^n} e^{-i\lambda t} H(\lambda) \sum_{l=0}^{n-2} 4^l \lambda^{-(l+1)}$$

$$\cdot \sum_{j=l}^{n-2} \frac{(-1)^{n-j}}{2^j} \binom{j}{l} \binom{n+l-1}{l+j+1} \left\langle \delta_B^l, f(.,t) \right\rangle dt d\lambda$$

thus, using the properties of the Fourier transform and taking account of that $\widehat{H} = \delta - ip.v.\left(\frac{1}{t}\right)$ we can obtain explicit formulas for μ_k^{+} of similar type those given for μ_k. Since $\mu_k^{-} = \mu_k - \mu_k^{+}$ we obtain also an explicit description for μ_k^{-}.

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