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We show that r-Dehn surgery on a hyperbolic, periodic knot K with period $p > 2$ yields a hyperbolic manifold unless $p = 3$, $r = 0$ and the genus of K is one. Regarding hyperbolic, periodic knots with period 2, we show that only integral Dehn surgeries can yield toroidal manifolds.

1. Introduction.

A 3-manifold is *toroidal* if it contains an essential torus, i.e., an incompressible torus not parallel to a boundary component. A knot K in S^3 is called a *periodic knot with period* p if there is a homeomorphism $f : S^3 \to S^3$ such that $f(K) = K$, $\mathrm{Fix}(f) \cap K = \emptyset$, and $\mathrm{Fix}(f)$ is a circle. We call f a *periodic map of* K. For a knot K in a 3-manifold $M \subset S^3$ we denote by $M(K; r)$ the manifold obtained by r-Dehn surgery of M on K, where $r \in \mathbf{Q} \cup \{1/0\}$; if $M = S^3$, simply we denote $M(K; r)$ by $(K; r)$.

The hyperbolic Dehn surgery theorem of Thurston [**25**] shows that for hyperbolic knots K, $(K; r)$ is non-hyperbolic only for finitely many $r \in \mathbf{Q}$. In this paper we consider when Dehn surgery on a hyperbolic, periodic knot yields a non-hyperbolic, in particular toroidal, manifold. For example, the figure eight knot 4_1, which has period 2, has exactly 10 surgeries producing non-hyperbolic manifolds [**25**]; if $(4_1; r)$ is toroidal, then $r = 0, \pm 4$.

Theorem 1.1. *If K is a hyperbolic, periodic knot with period* 2 *and* $(K; r)$ *is toroidal, then r is an integer.*

Remark. Gordon and Luecke proved that the denominator of a toroidal surgery slope is at most two for hyperbolic knots [**11**], and furthermore if the denominator is two then the knot is strongly invertible [**12**]. Eudave-Muñoz [**5**] constructed an infinite family of strongly invertible hyperbolic knots having non-integral, toroidal surgeries. Theorem 1.1 shows that none of his knots has period 2.

Then, does a hyperbolic, periodic knot with period greater than 2 have a non-hyperbolic Dehn surgery? Our answer is "no except for a special case" (Corollary 1.4). Before giving the statement let us review what non-hyperbolic manifolds are like. Each of the following cases is an obstruction to a closed orientable manifold M being hyperbolic:

(1) M is reducible;
(2) M is a Seifert fibered manifold with a finite fundamental group;
(3) $\pi_1(M)$ has a subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

In 1981, Thurston announced the Symmetry Theorem [**26**]: If M admits an action by a finite group G such that a fixed point set of some nontrivial element of G has dimension at least one, then M has a G-invariant geometric decomposition such that G acts on each piece by isometries. The theorem implies that (1)-(3) are the only obstructions to such M being hyperbolic. Recently, the Symmetry Theorem is proved in the case when the union of fixed point sets of nontrivial elements of G is a 1-manifold by Cooper, Hodgson and Kerckhoff [**3**], and Boileau and Porti [**1**]; this case of the theorem is what we need and referred to below as the Symmetry Theorem. On the other hand, if M is irreducible, condition (3) implies ($3'$) below [**7**, Corollary 8.6].

($3'$) M is either toroidal or a Seifert fibered manifold with an infinite fundamental group.

If K is a hyperbolic, periodic knot, $(K;r)$ does not fall under case (1) by the Cabling Conjecture for symmetric knots (Hayashi and Shimokawa [**17**], Gordon and Luecke). Since the periodic map of K extends to a periodic map on $(K;r)$, the Symmetry Theorem applies to $(K;r)$. Regarding (2) and ($3'$), the authors proved that:

Theorem 1.2 ([**21**, Theorem 1.5 and Proposition 5.6])**.** *If K is a hyperbolic, periodic knot with period greater than 2, then $(K;r)$ is not Seifert fibered for any $r \in \mathbf{Q}$. (Without using the Symmetry Theorem we show that M is not a Seifert fibered manifold with an infinite fundamental group.)*

Without assuming the Symmetry Theorem, we shall prove:

Theorem 1.3. *Let K be a hyperbolic, periodic knot with period $p > 2$. Then $(K;r)$ is toroidal if and only if $p = 3, r = 0$, and the genus of K is one.*

Remark. The $(3,3,3)$ pretzel knot is an example of a genus one, hyperbolic, periodic knot with period 3.

Theorems 1.2 and 1.3 preclude the possibility of cases (2) and ($3'$). Then the Symmetry Theorem implies that:

Corollary 1.4. *Let K be a hyperbolic, periodic knot with period $p > 2$. Then $(K;r)$ is hyperbolic for any $r \in \mathbf{Q}$ except when $p = 3$, $r = 0$, and the genus of K is one.*

The if part of Theorem 1.3 is proved below. The only if part is proved in §§3, 4. Theorem 1.1 is proved in §5 by graph-theoretic arguments.

Proof of the if part of Theorem 1.3. If K has an incompressible Seifert surface of genus one, then $(K;0)$ contains a non-separating torus obtained from the

Seifert surface by attaching a meridian disk of the glued solid torus. Gabai [6] shows that such a torus is incompressible. □

2. Preliminaries.

2.1. Dehn surgery on a factor knot.

Let K be a periodic knot, and f a periodic map of K with period p. Set $C = \mathrm{Fix}(f)$, which is a trivial knot in S^3 by the positive solution to the Smith Conjecture [22]. Then f induces the p-fold cyclic covering π from S^3 to the quotient space $S^3/\langle f\rangle = S^3$ branched along the trivial knot $C_f = \pi(C)$. We denote the factor knot $\pi(K)$ by K_f. Dehn surgeries on K and K_f are related as follows.

Take an f-invariant tubular neighborhood $N(K)$ of K. We can extend $f|S^3 - \mathrm{int}N(K)$ over $(K; m/n)$ periodically. Denote by $\bar{f}$ the resulting periodic map on $(K; m/n)$; the period of $\bar{f}$ is p. We may assume that $\bar{f}$ preserves the core K^* of the reglued solid torus. Note that for any $0 < i < p$, $\mathrm{Fix}(\bar{f}^i)$ is either C or $C \cup K^*$. The projection $\pi' : (K; m/n) \to (K; m/n)/\langle \bar{f}\rangle$ is a p-fold cyclic branched covering. Then $(K; m/n)/\langle \bar{f}\rangle$ is identified with $(K_f; m/(np))$ such that $\pi'(K^*)$ is a core of the reglued solid torus in $(K_f; m/(np))$. So denote $\pi'(K^*) = K_f^*$; see Diagram 2.1.

$$\begin{array}{ccccc}
K & \longrightarrow & K_f & & \\
\cap & & \cap & & \\
S^3 & \xrightarrow{\pi} & S^3/\langle f\rangle = S^3 & & \\
\downarrow & & & \searrow & \\
(K; \frac{m}{n}) & \xrightarrow{\pi'} & (K; \frac{m}{n})/\langle \bar{f}\rangle & = & (K_f; \frac{m}{np}) \\
\cup & & \cup & & \\
K^* & \longrightarrow & K_f^* & &
\end{array}$$

Diagram 2.1. The vertical and the slanted arrows mean Dehn surgeries.

Now choosing an f-invariant tubular neighborhood $N(C)$ of C, set $V = S^3 - \mathrm{int}N(C)$ and $V_f = V/\langle f\rangle = S^3 - \mathrm{int}N(C_f)$. Just as above a Dehn surgery of V on K and that of V_f on K_f are related (Diagram 2.2).

$$\begin{array}{ccccc}
V & \xrightarrow{\pi} & V/\langle f\rangle = V_f & & \\
\downarrow & & & \searrow & \\
V(K; \frac{m}{n}) & \xrightarrow{\pi'} & V(K; \frac{m}{n})/\langle \bar{f}\rangle & = & V_f(K_f; \frac{m}{np})
\end{array}$$

Diagram 2.2. The vertical and the slanted arrows mean Dehn surgeries.

Suppose $K \subset V$ is a hyperbolic knot. Since $\pi : V - K \to V_f - K_f$ is an unbranched covering, $V_f - K_f$ is neither toroidal nor Seifert fibered. Thus K_f is hyperbolic in V_f [22]. In the next subsection, we shall show that this hypothesis is satisfied if K is hyperbolic in S^3.

2.2. Hyperbolic, periodic knots.

Proposition 2.1. *Let* $K \subset S^3$ *be a hyperbolic, periodic knot. Let* $C = \mathrm{Fix}(f)$*, where* f *is a periodic map of* K*. Then* $K \cup C$ *is a hyperbolic link in* S^3*.*

Proof. $N(K)$ and $N(C)$ denote disjoint tubular neighborhoods of K and C which are preserved by f, respectively. Set $V = S^3 - \mathrm{int}N(C)$, an unknotted solid torus. Let $\mathcal{T}$ be a characteristic family of tori for $V - \mathrm{int}N(K)$ whose union is invariant under f [**19**, Theorem 8.6]. It suffices to prove $\mathcal{T} = \emptyset$. Note that since $K \subset S^3$ is hyperbolic, any torus in $\mathcal{T}$ is compressible in $S^3 - K$; in particular, any compressing disk meets C.

Assume for a contradiction that there is a torus in $\mathcal{T}$ which separates $\partial N(K)$ and ∂V. Among such tori let T be the one closest to ∂V. Let V' be the solid torus in V such that $\partial V' = T$. Note $f(V') = V'$, and $T = \partial V'$ is compressible in $S^3 - K$. It follows that V' is unknotted in S^3. By the equivariant loop theorem [**20**] there is a meridian disk D of $S^3 - \mathrm{int}V'$ such that $f(D) = D$ or $f(D) \cap D = \emptyset$. Since $C \cap D \neq \emptyset$, we have $f(D) = D$. Hence, D meets $C = \mathrm{Fix}(f)$ in a single point. This together with the unknottedness of C in S^3 shows that C is a core of the unknotted solid torus $S^3 - \mathrm{int}V'$. A core of V' and C then form a Hopf link, so that T and ∂V bounds $T^2 \times I$. This contradicts the minimality of $\mathcal{T}$.

Hence, if $\mathcal{T} \neq \emptyset$, each torus in $\mathcal{T}$ would not separate $\partial N(K)$ and ∂V. Let T be a torus in $\mathcal{T}$ such that the manifold $E \subset V - \mathrm{int}N(K)$ bounded by T does not contain a torus in $\mathcal{T} - \{T\}$. Then for any i either $f^i(E) = E$ or $f^i(E) \cap E = \emptyset$. Set $X = S^3 - \mathrm{int}(N(K) \cup \bigcup_{i \geq 0} f^i(E))$. Since T is compressible in $S^3 - K$, T is compressible in $S^3 - \mathrm{int}(N(K) \cup E)$ and thus in X. Let D be a compressing disk for $T \subset X$ such that $f(D) = D$ or $f(D) \cap D = \emptyset$ [**20**]. Just as above, the fact $C \cap D \neq \emptyset$ implies that D meets C in a single point. Hence C winds around the knotted solid torus $S^3 - \mathrm{int}E$ geometrically once, which contradicts that C is unknotted in S^3. □

3. Proof of Theorem 1.3: Case when $(m, p) = 1$ or p.

In this section and the next, we prove the only if part of Theorem 1.3.

Let K be a hyperbolic, periodic knot, and f a periodic map of K with period $p > 2$. We use the notation in §2.1 in what follows.

Assume that $(K; m/n)$ is toroidal. Note that $(K; m/n)$ is irreducible and not Seifert fibered (Theorem 1.2). By the equivariant torus decomposition theorem [**19**], $(K; m/n)$ contains an incompressible torus T such that for any i, $\bar{f}^i(T) = T$ or $\bar{f}^i(T) \cap T = \emptyset$. By rechoosing T, if necessary, the $\langle \bar{f} \rangle$-equivariant torus T meets $C \cup K^*$ transversely, and $N(C)$ and $N(K^*)$ in (possibly empty) meridian disks. Note $T \cap K^* \neq \emptyset$.

The proof is divided into three cases: (1) $(m, p) = 1$, (2) $(m, p) = p$, (3) $1 < (m, p) < p$, where (m, p) is the greatest common divisor of m and p.

The first two cases are dealt with in this section. Cases 1 and 3 will lead to contradictions.

Case 1. $(m,p)=1$; then $\mathrm{Fix}(\bar{f}^i)=C$ for $0<i<p$.

Claim 3.1. $T\cap C=\emptyset$.

Proof. Assume that T intersects C in $k(>0)$ points. Then $f(T)=T$. Moreover, since f fixes C pointwise, $\bar{f}$ preserves the orientation of T, and thus $T/\langle\bar{f}\rangle$ is an orientable surface. The assumption $(m,p)=1$ implies $m\neq 0$, and then any closed orientable surface in $(K;m/n)$ is separating. Thus k is even. The projection $\pi':T\to T/\langle\bar{f}\rangle=\pi'(T)$ is a p-fold cyclic branched covering along k branch points of index p. The Riemann-Hurewitz formula gives

$$0=\chi(T)=p\left(\chi(\pi'(T))-k\left(1-\frac{1}{p}\right)\right). \tag{1}$$

It follows $\chi(\pi'(T))>0$. Since $\pi'(T)$ is a closed, orientable surface, $\chi(\pi'(T))$ must be 2. Hence, $2=k(1-1/p)$. We then obtain $(p-1)(k-2)=2$. The solution sets in positive integers are $(k,p)=(3,3),(4,2)$. The former contradicts the fact that k is even; the latter does the assumption $p>2$. □

By Claim 3.1 $\pi':T\to\pi'(T)$ is an unbranched covering, thus $\pi'(T)$ is a Klein bottle or a torus. Hence, the $m/(np)$-surgery of the solid torus V_f on K_f contains a Klein bottle or a torus. Note that K_f is a hyperbolic knot in V_f, for K is hyperbolic in V (Proposition 2.1). Then, if $\pi'(T)$ is a Klein bottle, by [**11**] $|np|=1$. This contradicts $p>1$. It follows that $\pi'(T)$ is a torus. The fact that $\pi':V(K;m/n)\to V_f(K_f;m/(np))$ is an unbranched covering implies that $\pi'(T)$ is an essential torus in $V_f(K_f;m/(np))$. For hyperbolic knots in S^3, Gordon and Luecke [**11**] proved that the denominator of a toroidal surgery slope is at most 2. As pointed out in [**13**], their proof works also for hyperbolic knots in a solid torus. Hence $|np|\leq 2$, a contradiction.

Case 2. $(m,p)=p$; then $\mathrm{Fix}(\bar{f}^i)=C\cup K^*$ for $1<i<p$.

Let $k=|T\cap(C\cup K^*)|$. The projection $\pi':T\to T/\langle\bar{f}\rangle$ is a p-fold cyclic branched covering along k branch points of index p. As in Case 1 we obtain Equation (1), and the relevant solution set is $(k,p)=(3,3)$. This implies that T intersects C or K^* in an odd number of points, so T is a non-separating incompressible torus in $(K;m/n)$. By considering the first homology group we see $m=0$. [**6**, Corollary 8.3] shows if $(K;0)$ contains such a torus, then the genus of K is one as desired.

4. Proof of Theorem 1.3: Case when $1 < (m,p) < p$.

In this case, $\mathrm{Fix}(\bar f) = C$, $\bar f|K^*$ has period $p/(m,p)$, and $\bar f|S^3 - \mathrm{int}N(K^* \cup C)$ has period p. Note that $(K; m/n)$ and $(K_f; m/(np))$ do not contain non-separating closed surfaces because $m \neq 0$. Set $n_1 = |T \cap C|$ and $n_2 = |T \cap K^*|$. Then n_i are even numbers, and $n_2 > 0$.

Subcase 1. $T \cap C \neq \emptyset$.

Then $\bar f^{\frac{p}{(m,p)}}|T$ has $n_1 + n_2$ fixed points. This implies that $\pi' : T \to T/\langle \bar f\rangle = \pi'(T)$ is a p-fold cyclic branched covering along n_1 branch points of index p and $n_2(m,p)/p$ branch points of index (m,p). Note $n_2(m,p)/p = |\pi'(T) \cap K_f^*|$.

Claim 4.1. $n_1 = n_2 = 2$.

Proof. The Riemann-Hurewitz formula to the covering above gives:

$$(2)\quad 0 = \chi(T) = p\left(\chi(\pi'(T)) - n_1\left(1 - \frac{1}{p}\right) - \frac{n_2(m,p)}{p}\left(1 - \frac{1}{(m,p)}\right)\right).$$

As in the proof of Claim 3.1, we obtain $\chi(\pi'(T)) = 2$. It follows:

$$(3)\qquad 2 = n_1\left(1 - \frac{1}{p}\right) + \frac{n_2(m,p)}{p}\left(1 - \frac{1}{(m,p)}\right).$$

The right hand side of (3) is greater than $n_1/2$, therefore $4 > n_1$. Since $n_1(>0)$ is even, $n_1 = 2$ as claimed.

Multiplying (3) by p and substituting $n_1 = 2$, we obtain $2p = 2p - 2 + n_2(m,p) - n_2$. Thus $2 + n_2 = n_2(m,p) \geq 2n_2$. It follows that the even number n_2 must be 2. □

Since $(m,p) < p$, we have $n_2(m,p)/p < n_2 = 2$. Hence $n_2(m,p)/p = 1$. This implies that the 2-sphere $\pi'(T)$ in $(K_f; m/(np))$ meets K_f^* in a single point, a contradiction.

Subcase 2. $T \cap C = \emptyset$.

Then the closed surface $\pi'(T)$ is contained in $V_f(K_f; m/(np))$.

Claim 4.2. (1) $\pi'(T)$ is a 2-sphere.
(2) K_f^* meets $\pi'(T)$ in 4 points.

Proof. Let i be the least positive integer such that $\bar f^i(T) = T$; then $\pi' : T \to T/\langle \bar f^i\rangle = \pi'(T)$ is a p/i-fold cyclic branched covering along $in_2(m,p)/p$ branch points of index (m,p). For simplicity set $k = in_2(m,p)/p$. We then have:

$$(4)\qquad 0 = \chi(T) = \frac{p}{i}\left(\chi(\pi'(T)) - k\left(1 - \frac{1}{(m,p)}\right)\right).$$

This shows $\chi(\pi'(T)) > 0$, so $\pi'(T)$ is $\mathbf{R}P^2$ or S^2. If the orientable manifold $V_f(K_f; m/(np))$ contains $\mathbf{R}P^2$, it has a $\mathbf{R}P^3$ factor in its prime decomposition. This is absurd because no surgery on a hyperbolic knot in a solid torus yields a reducible manifold [**23**]. Therefore $\pi'(T)$ is a 2-sphere as claimed.

Letting $\chi(\pi'(T)) = 2$ in (4), we obtain $2 = k(1-1/(m,p))$. The right hand side is smaller than k and greater than or equal to $k/2$, so that $2 < k \leq 4$. Since $k = |\pi'(T) \cap K_f^*|$ is even, it must be 4. □

Claim 4.3. The 2-sphere $\pi'(T)$ in $V_f(K_f; m/(np))$ gives an essential tangle decomposition (defined below) of K_f^*.

Definition. Let K be a knot in a 3-manifold M. A separating 2-sphere $\widehat{S} \subset M$ gives an *essential tangle decomposition* of K if $\widehat{S}$ meets K in 4 points and $S = \widehat{S} - \mathrm{int}N(K)$ is incompressible in $M - \mathrm{int}N(K)$. Note that such an S is boundary-incompressible in $M - \mathrm{int}N(K)$.

Proof. By Claim 4.2 it suffices to see $S = \pi'(T) - \mathrm{int}N(K_f^*)$ is incompressible in $V_f(K_f; m/(np)) - \mathrm{int}N(K_f^*)$. Assume for a contradiction that S has a compressing disk D. Under the unbranched cyclic covering $\pi' : V(K; m/n) - \mathrm{int}N(K^*) \to V_f(K_f; m/(np)) - \mathrm{int}N(K_f^*)$, ${\pi'}^{-1}(D)$ consists of disks. Since T is incompressible in $V(K; m/n)$, each component of ${\pi'}^{-1}(\partial D) \cap T$ bounds a unique disk in T which meets K^*. Let Δ be an innermost one among such disks. Recall $\bar{f}|K^*$ has period $p/(m,p)$. Then $g = \bar{f}^{\frac{p}{(m,p)}}$ preserves Δ, and thus $g(\partial\Delta) = \partial\Delta$. This contradicts that $\bar{f}$ permutes the p components of ${\pi'}^{-1}(D)$ cyclically. □

The following proposition is essentially proved in Wu [**27**, Theorem 4.4]. We say that a Dehn surgery on a knot K is *integral* if the surgery slope on $\partial N(K)$ meets a meridian of K in a single point.

Proposition 4.4. *Let K be a knot in an irreducible 3-manifold M. Suppose that $K \subset M$ admits an essential tangle decomposition. Then $M - \mathrm{int}N(K)$ contains an incompressible, closed orientable surface of genus 1 or 2 which remains incompressible after any non-integral, nontrivial surgery on $K \subset M$.*

In our setting, $M = V_f(K_f; m/(np))$ is irreducible by [**23**], and $K_f^* \subset M$ admits an essential tangle decomposition. The solid torus $V_f = V_f(K_f; 1/0)$ contains no incompressible closed surface. But the 1/0-slope of $K_f \subset V_f$ does not meet a meridian of $K_f^* \subset M$ in a single point by $\begin{vmatrix} 1 & m/(m,np) \\ 0 & np/(m,np) \end{vmatrix} = np/(m,np) = np/(m,p) \neq \pm 1$. This contradicts Proposition 4.4. Hence, Subcase 2 does not occur (Theorem 1.3). □

Proof of Proposition 4.4. Let $\widehat{S}$ be a 2-sphere giving an essential tangle decomposition of $K \subset M$, and set $S = \widehat{S} - \mathrm{int}N(K)$. Let B be a 3-ball in M bounded by $\widehat{S}$, and let $B \cap K = t_1 \cup t_2$, two arcs properly embedded in B.

If $E = B - \mathrm{int}N(t_1 \cup t_2)$ contains an incompressible torus F, then it is incompressible in $M - \mathrm{int}N(K)$. Assume for a contradiction that F compresses after a non-integral, nontrivial surgery on $K \subset M$. We can apply [**4**, Theorem 2.4.4] after cutting $M - \mathrm{int}N(K)$ along F. Then we obtain an annulus $A \subset M - \mathrm{int}N(K)$ such that ∂A consists of an essential loop on F and a longitude of $\partial N(K)$. Isotop A so as to meet S transversely and to minimize $|A \cap S|$. Each component of $A \cap S$ is an arc whose ends are in the longitude. An outermost disk of the components of $A - S$ is then a boundary-compressing disk for S. This contradicts the definition of an essential tangle decomposition. Hence, we may assume that E does not contain an incompressible torus.

Claim 4.5 (Hayashi [**14**])**.** $M - \mathrm{int}N(K)$ contains an incompressible, closed orientable surface F of genus 2 which has a compressing disk in M intersecting K in a single point.

By applying [**4**, Lemma 2.5.3] or the arguments in [**24**] to $M - \mathrm{int}N(K)$ cut along F, it follows that F remains incompressible after any non-integral, nontrivial surgery on $K \subset M$. This completes the proof of Proposition 4.4. □

Proof of Claim 4.5. The arcs t_i $(i = 1, 2)$ are attached to $\widehat{S}$ such that $t_i \cap \widehat{S} = \partial t_i$. First surger $\widehat{S}$ along a 1-handle $N(t_i)$ attached to $\widehat{S}$; we obtain a torus meeting K in 2 points. Then surger the torus along a 1-handle $N(\overline{K - t_j})$ where $i \neq j$. Let F_i $(i = 1, 2)$ be the resulting closed surface of genus 2; see Figure 4.1. A cocore D of the 1-handle $N(\overline{K - t_j})$ is a compressing disk for $F_i \subset M$ meeting K in a single point, as desired.

The closed surface F_i splits $M - \mathrm{int}N(K)$ into two components. Let X be the one containing $\partial N(K)$, and Y the other. To prove the claim it suffices to see that either F_1 or F_2 is incompressible in both X and Y. If F_i compresses in X, the intersection of the compressing disk and D can be eliminated by a cut and paste argument, so $F_i - \partial D$ is compressible in $X - D$. This implies that S surgered along t_i is compressible in $E = B - \mathrm{int}N(t_1 \cup t_2)$. However, [**27**, Lemma 2.2] shows that for the atoroidal nontrivial tangle $(B, t_1 \cup t_2)$, S surgered along t_i is incompressible in E for $i = 1$ or 2. Hence, either F_1 or F_2 is incompressible in X.

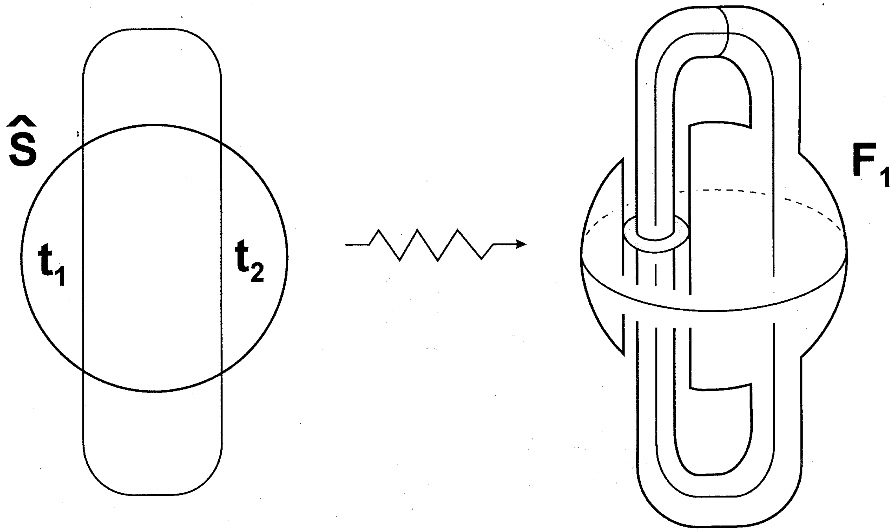

Figure 4.1.

Assume for a contradiction that there is a compressing disk Δ for $F_i \subset Y$, where $i = 1$ or 2. Let $A \subset Y$ be an annulus such that ∂A consists of meridians of the 1-handles $N(t_i)$ and $N(\overline{K - t_j})$ (the shaded annulus in Figure 4.1). By isotopy we may assume that Δ meets A transversely in arcs. Let Δ_0 be the closure of an outermost component in $\Delta - A$. If $\partial\Delta_0 \cap A$ is an arc connecting distinct components of ∂A, then S is boundary-compressible in $M - \mathrm{int}N(K)$, a contradiction. If $\partial\Delta_0 \cap A$ is an arc connecting the same component of ∂A, then $F_i - \partial A$ is compressible in $Y - A$. This implies that S is compressible in $M - \mathrm{int}N(K)$, a contradiction. □

5. Proof of Theorem 1.1.

Although Boyer and Zhang [**2**] showed the theorem when $(K; m/n)$ is Seifert fibered, we proceed without assuming their result.

Let K be a hyperbolic, periodic knot, and f a periodic map of K with period 2. Assume that $(K; m/n)$ is toroidal. If m is even, then [**11**] implies that $|n| = 1$ as desired. In the following we assume that m is odd.

Lemma 5.1. *There is an incompressible torus T in $(K; m/n)$ meeting $C = \mathrm{Fix}(f) = \mathrm{Fix}(\bar{f})$ transversely such that $\bar{f}(T) = T$ or $\bar{f}(T) \cap T = \emptyset$.*

Proof. The lemma follows from the equivariant torus theorem for involutions [**18**, Corollary 4.6] unless $(K; m/n)$ is a Seifert fibered manifold over S^2 with four exceptional fibers. If $(K; m/n)$ is such a Seifert fibered manifold, first choose an $\bar{f}$-invariant Seifert fibration $p : (K; m/n) \to B = S^2$ [**19**] (see also [**21**, Lemma 5.4]). By [**21**, Proposition 5.1] C cannot be a fiber of $(K; m/n)$;

then $\bar{f}$ preserves each fiber meeting C but reverses the orientation of it. It follows that $\bar{f}$ induces an orientation reversing involution, φ, of B which fixes each point on $p(C)$. Then, φ is a reflection about the embedded circle $p(C)$. Let l be a φ-invariant circle in B which meets $p(C)$ transversely and encloses two cone points in each side (Figure 5.1). Then $p^{-1}(l)$ is an $\bar{f}$-invariant incompressible torus meeting C transversely. □

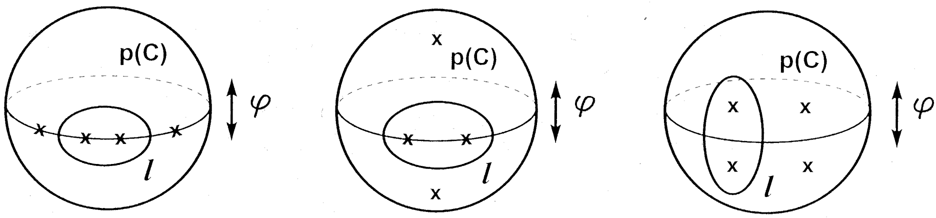

Figure 5.1.

Let T be the torus in Lemma 5.1.

Case 1. $T \cap C = \emptyset$.

The argument in the paragraph just after the proof of Claim 3.1 shows that $|n| = 1$.

Case 2. $T \cap C \neq \emptyset$.

Lemma 5.2. *If* $|n| \geq 2$*, then* $(K_f; m/(2n))$ *has two lens space summands.*

Theorem 1.1 readily follows from this lemma. If $|n| \geq 2$, then by Lemma 5.2 the non-integral surgery $(K_f; m/(2n))$ would be reducible, contradicting [**9**, Theorem 1] (Theorem 1.1). □

The rest of this section is devoted to proving Lemma 5.2 by graph-theoretic technique. The arguments are variants of those in Hayashi and Motegi [**16**, §4].

From the argument in the proof of Claim 3.1, $T/\langle \bar{f} \rangle \cong S^2$ and T meets C in four points. Consider the unbranched covering $\pi' : V(K; m/n) \to V_f(K_f; m/(2n))$. We set $S = \pi'(T - \mathrm{int}N(C))$, a 2-sphere with four open disks removed; S is properly embedded in $V_f(K_f; m/(2n))$ with components of ∂S preferred longitudes of $V_f(\subset S^3)$. Since T is separating in $(K; m/n)$, $T/\langle \bar{f} \rangle$ is separating in $(K_f; m/(2n))$ and hence S separates $V_f(K_f; m/(2n))$.

Claim 5.3. S is essential in $V_f(K_f; m/(2n))$.

Proof. (Cf. the proof of Claim 4.3.) If D is a compressing disk of S in $V_f(K_f; m/(2n))$, $\pi'^{-1}(D)$ consists of two compressing disks of $T - \mathrm{int}N(C)$ in $V(K; m/n)$. However T is incompressible, so each component of $\pi'^{-1}(\partial D)$

bounds a disk in T which meets C. Since $C = \text{Fix}(\bar{f})$, each such disk is preserved by $\bar{f}$. This contradicts that $\bar{f}$ exchanges the components of $\pi'^{-1}(D)$. □

In the following we write $M = V_f - \text{int}N(K_f)$, which is hyperbolic (Proposition 2.1).

Isotoping S so as to minimize $q_S = |S \cap K_f^*|$, we obtain an essential (i.e., incompressible and boundary-incompressible) planar surface $P_S = S \cap M$ in M. Since q_S is even and $(K; m/n) - \text{int}N(K^*)$ is atoroidal, we have $q_S \geq 2$. Let D be a meridian disk of V_f such that $q_D = |D \cap K_f|$ is minimal. Then we have an essential planar surface $P_D = D \cap M$ in M. Since K has period 2, the linking number $lk(C_f, K_f) = lk(C, K)$ is odd, so q_D is odd. If $q_D = 1$, then K is a trivial knot or a composite knot, contradicting the hyperbolicity of K. Thus $q_D \geq 3$.

We define graphs in D and S as in [**4**] and introduce the concepts of (great) x-edge cycles and $[x, x+1]$-Scharlemann cycles as in [**16**]. By an isotopy we may assume that ∂P_D and ∂P_S intersect in minimum number of points, and $P_D \cap P_S$ consists of loops and arcs which are essential in both P_D and P_S. We define Γ_D to be the graph in D such that its (fat) vertices are the disks $D \cap N(K_f)$ and its edges are the arc components e of $P_D \cap P_S$ with at least one endpoint of e in a fat vertex. Similarly, we define the graph Γ_S in S. An edge with one endpoint in ∂D or ∂S is a *boundary edge*.

Number the fat vertices of Γ_D (resp. Γ_S) $1, 2, \ldots, q_D$ (resp. $1, 2, \ldots, q_S$) in the order of appearence on K_f (resp. K_f^*). We next define a *sign* of a vertex of Γ_D to be the sign of the corresponding intersection point of K_f with D with respect to some chosen orientations of D, K_f and M. Similarly, give a sign to each vertex of Γ_S. An edge of Γ_α ($\alpha = D, S$) joining vertices of Γ_α with the same sign is a *positive edge*, and an edge joining the opposite signs is a *negative edge*.

Let p be some edge's endpoint at a fat vertex of Γ_D labelled x. Then p is in the boundary of some fat vertex of Γ_S labelled y (say). We label the edge-endpoint at the fat vertex x with y. Around each fat vertex of Γ_D the edge-endpoint labels occur in order $1, 2, \ldots, q_S, \ldots, 1, 2, \ldots, q_S$ repeated $2|n|$ times; the ordering is, without loss of generality, anticlockwise (resp. clockwise) at a positive (resp. negative) vertex. Label edge-endpoints at fat vertices of Γ_S, similarly. An edge with label x at one endpoint is an *x-edge*.

For a subgraph σ of Γ_D (resp. Γ_S), we call components of $D - \sigma$ (resp. $S - \sigma$) *faces* of σ. For a face P of a subgraph $\sigma \subset \Gamma_\alpha$ ($\alpha = D$ or S), ∂P denotes the subgraph of σ which consists of vertices and edges of σ meeting the closure of P in α. A subgraph σ of Γ_α is an *x-edge cycle* if its edges are positive x-edges, and there is a disk face P of σ such that $\sigma = \partial P$. Furthermore, if all the vertices of Γ_α in P have the same sign as the vertices of σ, then σ is a *great x-edge cycle*. A *Scharlemann cycle* is an x-edge cycle for some label x which bounds a disk face of Γ_α. In our setting Γ_α does not contain a Scharlemann cycle with only one edge. Note that a Scharlemann/x-edge cycle σ is not necessarily a "cycle", i.e., σ with its vertices regarded as points may not be homeomorphic to a circle; see Figure 5.2. The above definition of a Scharlemann cycle is a mild extension of the definition by Gordon and Luecke [4], but the same as in Gordon [8]. We orient a Scharlemann cycle $\sigma \subset \Gamma_\alpha$ anticlockwise (resp. clockwise) if the sign of the vertices of σ is positive (resp. negative). Then, if an edge of σ has a label x at its tail, then its head has the label $x+1$ (mod q_α). (Cf. Figure 5.2.) We say that σ is a *Scharlemann cycle for the interval* $[x, x+1]$, or simply $[x, x+1]$-*Scharlemann cycle.*

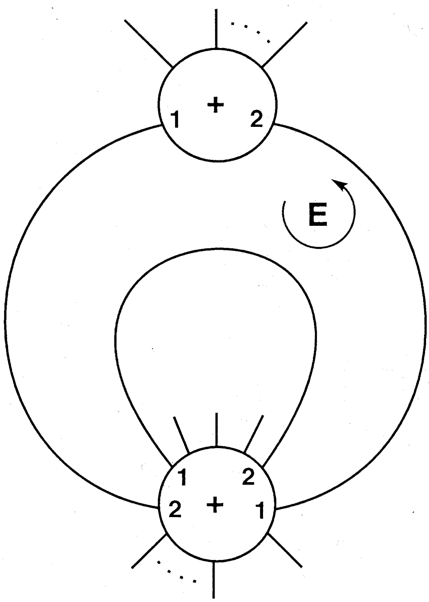

Figure 5.2.

Lemma 5.4. *The graph Γ_S does not contain a Scharlemann cycle.*

Proof. If Γ_S contains a Scharlemann cycle, then by [8, Theorem 4.1] we have a lens space summand in the solid torus V_f. This is a contradiction. □

Lemma 5.5. *If the graph Γ_D contains Scharlemann cycles for distinct intervals, then $(K_f; m/(2n))$ has at least two lens space summands.*

Proof. Let σ_i be $[x_i, y_i]$-Scharlemann cycles for $i = 1, 2$ such that $[x_1, y_1] \neq [x_2, y_2]$. Let $E_i \subset D$ be the disk face of σ_i. Then E_i is disjoint from the separating 2-sphere $\widehat{S} = T/\langle \bar{f} \rangle$ in $(K_f; m/(2n))$. There are three posibilities:

$\{x_1, y_1\} \cap \{x_2, y_2\}$ is empty, consists of one element, or two elements. (The last case occurs only when $q_S = 2$, $[x_1, y_1] = [1, 2]$ and $[x_2, y_2] = [2, 1]$.) Except for the first case, E_1 and E_2 are contained in the opposite sides of $\widehat{S}$, thus $(K_f; m/(2n))$ contains two disjoint punctured lens spaces by [**8**, Theorem 4.1].

Assume the first case happens. We consider the subgraph $\widehat{\sigma}_i$ of Γ_S consisting of two vertices of labels x_i, y_i and the edges of σ_i. Then $\widehat{S} - \widehat{\sigma}_1$ consists of open disks; $\widehat{\sigma}_2$ is contained in one of such disks because $\widehat{\sigma}_1 \cap \widehat{\sigma}_2 = \emptyset$. Hence we can choose disjoint disks D_1 and D_2 so that $\widehat{\sigma}_i$ lies in D_i. Thus there are two disjoint punctured lens spaces in $(K_f; m/(2n))$. □

Remark. Since m is assumed to be odd, $H_1((K_f; m/(2n)))$ has odd order. This implies that each Scharlemann cycle in Γ_D has an odd number of edges.

Lemma 5.6 ([**15**, Proposition 5.1])**.** *If Γ_α contains a great x-edge cycle σ, then the disk face of σ contains a Scharlemann cycle.*

Claim 5.7. Γ_S contains at most $q_D(q_S + 2)/2$ positive edges.

Proof. First we show that Γ_S contains at most $q_S + 2$ positive x-edges for every label x. Let x be an arbitrary label of fat vertices of Γ_D. Let Λ be the subgraph of Γ_S consisting of all positive x-edges and all vertices of Γ_S. (The graph Λ may have an isolated vertex.) Note that if Λ has a disk face, its boundary is a great x-edge cycle of Γ_S. Let f_d be the number of disk faces of Λ. Applying Euler's formula to the graph Λ on S, we have $q_S - k + \Sigma\chi(\text{face}) = \chi(S) = -2$, where k is the number of edges of Λ. Thus if $k \geq q_S + 3$, then $f_d \geq \Sigma\chi(\text{face}) \geq 1$, so that Γ_S contains a great x-edge cycle. Hence, Γ_S contains a Scharlemann cycle by Lemma 5.6. This contradicts Lemma 5.4.

Assume for a contradiction that Γ_S contains more than $q_D(q_S + 2)/2$ positive edges. Then the number of their endpoints is more than $q_D(q_S+2)$. By the parity rule [**4**, §2.5] every positive edge has distinct labels at its two endpoints. Since there are q_D kinds edge-endpoint labels in Γ_S, there are more than $q_S + 2$ positive x-edges for some label x. This contradicts what we show above. □

Claim 5.8. If Γ_D has at least $(q_D-1)q_S$ positive edges, then Γ_D has Scharlemann cycles for distinct intervals.

Proof. We first show that Γ_D contains at least $q_D - 1$ Scharlmann cycles by the arguments in the proof of Claim 5.7 or [**16**, Lemmas 4.5, 4.6]. In fact, using the arguments in the second paragraph of the proof of Claim 5.7, we see that Γ_D has at least $2(q_D-1)$ positive x-edges for some label x. Then, as in the first paragraph of the proof, apply Euler's formula to the graph Λ on D consisting of all vertices of Γ_D and all positive x-edges of Γ_D. It follows that the Euler number of the faces of Λ is at least $\chi(D)-q_D+2(q_D-1) = q_D-1$.

This implies that Γ_D contains at least $q_D - 1$ great x-edge cycles bounding mutually disjoint disk faces. The claimed result then follows from Lemma 5.6.

Following the proof of [**10**, Theorem 2.3] ([**16**, Lemma 4.4]), we find Scharlemann cycles for distinct intervals. Assume for a contradiction that Γ_D contains Scharlemann cycles only for the interval (say) $[x, x+1]$. Let k be the number of Scharlemann cycles in Γ_D. As in Figure 8 in [**16**], we form a dual graph $\Lambda \subset D$ for Scharlemann cycles. First take one dual (fat) vertex in the disk face of each Scharlemann cycle in Γ_D, and then draw edges from each dual vertex to the vertices of the corresponding Scharlemann cycle. The vertices of Λ consist of q_D vertices of Γ_D and k dual vertices; the edges of Λ consist of the edges defined above. We apply Euler's formula to the graph Λ in D. The number of the vertices is $q_D + k$; the number of the edges is at least $3k$ by Remark after the proof of Lemma 5.5. It follows that the Euler number of the faces of Λ is at least $\chi(D) - (q_D + k) + 3k = 2k + 1 - q_D \geq q_D - 1 > 0$. This implies that there is a disk face of Λ, which contains a great x-edge cycle of Γ_D as shown in [**16**, Figure 9] and thus a Scharlemann cycle (Lemma 5.6). Hence, Γ_D contains more than k Scharlemann cycles, a contradiction. □

Proof of Lemma 5.2. Since each component of ∂S is a longitude of V_f, the graph Γ_S has at most four boundary edges. Each vertex of Γ_S has $|2n|q_D (\geq 4q_D)$ edge-endpoints; Γ_S has at most $q_D(q_S + 2)/2$ positive edges (Claim 5.7). Thus, the number of endpoints of the negative edges of Γ_S is at least

$$\begin{aligned}
&4q_D q_S - 4 - q_D(q_S + 2) \\
&= 3q_D q_S - 2q_D - 4 \\
&= 2(q_D - 1)q_S + (q_S - 2)q_D + 2q_S - 4.
\end{aligned}$$

Since $q_S \geq 2$, this number is greater than or equal to $2(q_D - 1)q_S$. By the parity rule Γ_D then has at least $(q_D - 1)q_S$ positive edges. Hence Γ_D contains Scharlemann cycles for distinct intervals (Claim 5.8). Lemma 5.2 now follows from Lemma 5.5. □

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