

*Pacific  
Journal of  
Mathematics*

POSITIVITY RESULTS FOR THE YANG–MILLS–HIGGS  
HESSIAN

GEORGE ANDROULAKIS AND STAMATIS DOSTOGLOU

Volume 194 No. 1

May 2000



## POSITIVITY RESULTS FOR THE YANG–MILLS–HIGGS HESSIAN

GEORGE ANDROULAKIS AND STAMATIS DOSTOGLOU

We study the second derivative  $Q_{c_\lambda}$  of the Yang–Mills–Higgs functional with structure group  $SU(2)$  at a spherically symmetric critical point  $c_\lambda$  when the self-interaction parameter  $\lambda$  is not close to 0. We find that if  $Q_{c_{\lambda_0}}$  is non-negative and with kernel consisting entirely of translation modes then positivity persists in a neighborhood of  $\lambda_0$ . In particular, we show that if translation modes always account for the whole kernel then  $Q_{c_\lambda}$  is always non-negative. This extends our previous results for  $\lambda$  in an neighborhood of 0.

### 1. Introduction.

The Yang–Mills–Higgs functional  $E_\lambda$  on  $\mathbb{R}^3$ , with structure group  $SU(2)$ , is the classical static version of the functional introduced by P. Higgs in [H]. The critical points of  $E_\lambda$  correspond to magnetic monopoles.

It has been known since the late 70s that spherically symmetric critical points  $c_\lambda$  of  $E_\lambda$  exist for all values of the positive parameter  $\lambda$ . The authors of this paper have recently shown that for positive  $\lambda$  in a neighborhood of 0 these critical points have non-negative Hessian  $Q_{c_\lambda}$ , i.e., they are (weakly) stable, see [AD]. The aim of this article is to investigate the stability of  $c_\lambda$  for (the more relevant for physics) large values of  $\lambda$ .

For  $\lambda = 0$  the spherically symmetric solution  $c_0$  satisfies the first order (Bogomol’nyi) equation for global minima, [JT]. This equation is unique to the  $\lambda = 0$  case and has no analogue for  $\lambda \neq 0$ . One of the crucial observations in [AD] for extending the positivity from  $\lambda = 0$  to  $\lambda \neq 0$  is that the kernel of the Hessian at 0 consists entirely of translation modes. Here we show that this is not special to  $\lambda = 0$ : For any positive  $\lambda_0$  where  $Q_{c_{\lambda_0}}$  is non-negative, if the kernel of  $Q_{c_{\lambda_0}}$  is the span of the translation modes then  $Q_{c_\lambda}$  is non-negative for  $\lambda$  in a neighborhood of  $\lambda_0$ . In the same neighborhood the dimension of the kernel may not increase. The first main result then is:

**Theorem.** *The set of  $\lambda \geq 0$  for which  $Q_{c_\lambda}$  is non-negative and has 3-dimensional kernel is open.*

In addition, since for  $\lambda$  close to  $\lambda_0$  it is shown that  $Q_{c_\lambda}$  is close to  $Q_{c_{\lambda_0}}$ , the set of  $\lambda$ 's for which  $Q_{c_\lambda} \geq 0$  is closed, yielding:

**Corollary.** *If  $\ker Q_{c_\lambda}$  is always 3-dimensional then  $Q_{c_\lambda}$  is non-negative for all  $\lambda$ .*

The proof of the main results consists of three parts. First it is shown that if  $Q_{c_{\lambda_0}}$  is non-negative then, away from its kernel, it is bounded below by a strictly positive constant. Then  $c_\lambda$  is shown to converge to  $c_{\lambda_0}$  in the configuration space as  $\lambda \rightarrow \lambda_0$ . This implies that the Hessians  $Q_{c_\lambda}$  are all defined on the same Hilbert space and that they differ by a small amount. It is then shown that the subspaces  $N_\lambda$  spanned by the translation modes of  $c_\lambda$  contain the kernel of  $Q_{c_{\lambda_0}}$  at the limit. This implies that directions orthogonal to  $N_\lambda$  are almost orthogonal to, and definitely not in, the kernel of  $Q_{c_{\lambda_0}}$ . For the last two steps in the proof the estimates of [AD] need to be extended from uniform estimates on some neighborhood of 0 to uniform estimates on any compact  $\lambda$ -interval.

In Section 2 we review the basics of the theory and state the main results. The proofs are contained in Section 3, where we focus on aspects that are genuinely different from the  $\lambda_0 = 0$  case. It is in Section 4 that we show how to improve the estimates in [AD] so that they hold on any bounded  $\lambda$ -interval.

## 2. The functional $E_\lambda$ and the symmetric solutions $c_\lambda$ .

The Yang-Mills-Higgs functional  $E_\lambda$  with self-interaction parameter  $\lambda \geq 0$  is defined by

$$(1) \quad E_\lambda(A, \Phi) = \frac{1}{2} \int_{\mathbf{R}^3} \left\{ |F_A|^2 + |d_A \Phi|^2 + \frac{\lambda}{4} (|\Phi|^2 - 1)^2 \right\} d^3x,$$

on pairs  $c = (A, \Phi)$ . Here  $A$  is a connection on the  $SU(2)$  bundle  $SU(2) \times \mathbf{R}^3$  over  $\mathbf{R}^3$  and  $\Phi$  is a section of the associated bundle  $E$  with fiber the Lie Algebra  $\mathfrak{su}(2)$ ,  $E = \mathfrak{su}(2) \times \mathbf{R}^3$ .  $F_A$  is the curvature of the connection  $A$  and  $d_A \Phi$  the covariant derivative of  $\Phi$  with respect to the connection  $A$ :

$$F_A = dA + \frac{1}{2}[A, A], \quad d_A \Phi = d\Phi + [A, \Phi].$$

All norms use the Killing inner product on  $\mathfrak{su}(2)$  and the standard metric on  $\mathbf{R}^3$ .

$E_\lambda$  is defined on the configuration space

$$\hat{\mathcal{C}} = \{(A, \Phi) : A \in L^2_{1,\text{loc}}, \Phi \in L^2_{1,\text{loc}}, E_\lambda(A, \Phi) < \infty.\}$$

Note here that  $\hat{\mathcal{C}}$  stays the same for all  $\lambda > 0$ . For the special case  $\lambda = 0$  see [AD].  $\hat{\mathcal{C}}$  is equipped with the  $L^2_{1,\text{loc}}$  topology intersected with the topology

that makes  $\|d_A\Phi\|_2$  and  $\|F_A\|_2$  continuous.  $E_\lambda$ ,  $\lambda \geq 0$  is invariant under the action of the gauge group

$$\mathcal{G} = \{g : \mathbf{R}^3 \rightarrow \text{SU}(2), \quad g \in L_{2, \text{loc}}^2\},$$

where  $g \cdot A = gAg^{-1} + gdg^{-1}$ , and  $g \cdot \Phi = g\Phi g^{-1}$ . Then  $E_\lambda$  descends to the quotient  $\mathcal{C} = \hat{\mathcal{C}}/\mathcal{G}$ .

To define the space  $T_c\mathcal{C}$  of admissible infinitesimal perturbations at  $c = (A, \Phi)$  in  $\mathcal{C}$ , consider first the completion  $H_c$  of  $C_0^\infty$  sections on  $\mathbf{R}^3$  with respect to the inner product norm

$$(2) \quad \|(a, \phi)\|_c^2 = \|\nabla_A a\|_2^2 + \|\nabla_A \phi\|_2^2 + \|[\Phi, a]\|_2^2 + \|\phi\|_2^2.$$

$H_c$  contains only directions that keep  $E_\lambda$  finite, c.f. [T1], but it still contains deformations along the orbit of  $\mathcal{G}$ . This is remedied here by excluding the elements of the kernel of the (formal) adjoint of the linearization of the action

$$(3) \quad \partial_c(a, \phi) = -d_A^* a + [\Phi, \phi].$$

We define therefore

$$T_c\mathcal{C} = \{(a, \phi) \in H_c : \partial_c(a, \phi) = 0\}.$$

The variational equations for  $\lambda \geq 0$  are the Yang-Mills-Higgs equations

$$(4) \quad d_A^* F_A = [d_A \Phi, \Phi], \quad d_A^* d_A \Phi = -\frac{\lambda}{2} \Phi (|\Phi|^2 - 1).$$

For any  $c = (A, \Phi)$  in  $\mathcal{C}$ , the second derivative of the energy  $E_\lambda$  defines a bilinear form on  $T_c\mathcal{C}$

$$(5) \quad \begin{aligned} & \hat{Q}_{\lambda, c}((a_1, \phi_1), (a_2, \phi_2)) \\ &= \frac{d^2}{dsdt} \Big|_{(0,0)} E_\lambda(A + sa_1 + ta_2, \Phi + s\phi_1 + t\phi_2), \quad \lambda \geq 0, \end{aligned}$$

the Hessian of the Yang-Mills-Higgs functional. Then

$$\begin{aligned} & \hat{Q}_{\lambda, c}((a_1, \phi_1), (a_2, \phi_2)) \\ &= \langle F_A, [a_1, a_2] \rangle + \langle d_A \Phi, [a_1, \phi_2] + [a_2, \phi_1] \rangle + \langle d_A a_1, d_A a_2 \rangle \\ & \quad + \langle d_A \phi_1, d_A \phi_2 \rangle + \langle [a_1, \Phi], [a_2, \Phi] \rangle + \langle d_A \phi_1, [a_2, \Phi] \rangle \\ & \quad + \langle d_A \phi_2, [a_1, \Phi] \rangle + \frac{\lambda}{2} \int_{\mathbf{R}^3} (|\Phi|^2 - 1) \langle \phi_1, \phi_2 \rangle d^3 x \\ & \quad + \lambda \int_{\mathbf{R}^3} \langle \Phi, \phi_1 \rangle \langle \Phi, \phi_2 \rangle d^3 x \end{aligned}$$

and the corresponding quadratic form is

$$\begin{aligned} Q_{\lambda, c}(a, \phi) &= \|d_A a\|_2^2 + \|d_A \phi\|_2^2 + \|[a, \Phi]\|_2^2 \\ & \quad + \langle F_A, [a, a] \rangle + 2\langle d_A \Phi, [a, \phi] \rangle + 2\langle d_A \phi, [a, \Phi] \rangle \end{aligned}$$

$$+ \frac{\lambda}{2} \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x + \lambda \int_{\mathbf{R}^3} \langle \Phi, \phi \rangle^2 d^3x.$$

Now recall the following standard:

**Definition 2.1.** Let  $\hat{Q}$  be a bilinear form a Hilbert space  $H$ . Then  $v_0$  in  $H$  is in  $\ker \hat{Q}$  if and only if

$$\hat{Q}(v_0, v) = 0$$

for all  $v$  in  $H$ .

Throughout this paper whenever  $Q$  is the quadratic form associated to a bilinear form  $\hat{Q}$  we use the phrase “ $v_0$  is in  $\ker Q$ ” to mean “ $v_0$  is in  $\ker \hat{Q}$ ”.

The following is rewriting  $Q_{\lambda,c}$  on  $T_c\mathcal{C}$  as in [T2], page 246.

**Lemma 2.2.** For  $(a, \phi)$  in  $T_c\mathcal{C}$ ,

$$\begin{aligned} Q_{\lambda,c}(a, \phi) &= \|\nabla_A a\|_2^2 + \|\nabla_A \phi\|_2^2 + \|[a, \Phi]\|_2^2 + \|[\Phi, \phi]\|_2^2 \\ &\quad + 2\langle F_A, [a, a] \rangle + 4\langle d_A \Phi, [a, \phi] \rangle \\ &\quad + \frac{\lambda}{2} \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x + \lambda \int_{\mathbf{R}^3} \langle \Phi, \phi \rangle^2 d^3x. \end{aligned}$$

*Proof.* First separate in  $Q_{\lambda,c}$  the terms that contain  $\lambda$  from those that do not, using the obvious notation:

$$Q_{\lambda,c}(a, \phi) = Q_{0,c}(a, \phi) + \frac{\lambda}{2} \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x + \lambda \int_{\mathbf{R}^3} \langle \Phi, \phi \rangle^2 d^3x.$$

Next use the Bochner-Weitzenböck formula for 1-forms on flat spaces, see page 95 of [L], and integrate by parts to get

$$\|d_A a\|_2^2 + \|d_A^* a\|_2^2 = \|\nabla_A a\|_2^2 + \langle F_A, [a, a] \rangle.$$

Then

$$\begin{aligned} Q_{0,c}(a, \phi) &= \|\nabla_A a\|_2^2 - \|d_A^* a\|_2^2 + \|d_A \phi\|_2^2 + \|[a, \Phi]\|_2^2 \\ &\quad + 2\langle F_A, [a, a] \rangle + 2\langle d_A \Phi, [a, \phi] \rangle + 2\langle d_A \phi, [a, \Phi] \rangle. \end{aligned}$$

Therefore for  $(a, \phi)$  in  $T_c\mathcal{C}$ , where  $d_A^* a - [\Phi, \phi] = 0$  holds,

$$\begin{aligned} Q_{0,c}(a, \phi) &= \|\nabla_A a\|_2^2 + \|d_A \phi\|_2^2 + \|[a, \Phi]\|_2^2 + \|[\Phi, \phi]\|_2^2 \\ &\quad - 2\|[\Phi, \phi]\|_2^2 + 2\langle F_A, [a, a] \rangle + 2\langle d_A \Phi, [a, \phi] \rangle + 2\langle d_A \phi, [a, \Phi] \rangle. \end{aligned}$$

Now observe that on  $T_c\mathcal{C}$  we also have

$$\begin{aligned} &\langle d_A \Phi, [a, \phi] \rangle \\ &= \sum_1^3 \int_{\mathbf{R}^3} \langle (d_A)_i \Phi, [a_i, \phi] \rangle \\ &= - \sum_1^3 \int_{\mathbf{R}^3} \langle \Phi, (d_A)_i [a_i, \phi] \rangle \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{R}^3} \left\langle \Phi, \left[ -\sum_1^3 (d_A)_i a_i, \phi \right] \right\rangle - \sum_1^3 \int_{\mathbb{R}^3} \langle \Phi, [a_i, (d_A)_i \phi] \rangle \\
 &= \int_{\mathbb{R}^3} \langle \Phi, [d_A^* a, \phi] \rangle - \int_{\mathbb{R}^3} \langle \Phi, [a, d_A \phi] \rangle \\
 &= \int_{\mathbb{R}^3} \langle \Phi, [[\Phi, \phi], \phi] \rangle + \int_{\mathbb{R}^3} \langle [a, \Phi], d_A \phi \rangle \\
 &= -\|[\Phi, \phi]\|_2^2 + \langle d_A \phi, [a, \Phi] \rangle,
 \end{aligned}$$

to finally get

$$\begin{aligned}
 &Q_{0,c}(a, \phi) \\
 &= \|\nabla_A a\|_2^2 + \|d_A \phi\|_2^2 + \|[a, \Phi]\|_2^2 + \|[\Phi, \phi]\|_2^2 + 2\langle F_A, [a, a] \rangle + 4\langle d_A \Phi, [a, \phi] \rangle.
 \end{aligned}$$

□

In [tH] and [P] 't Hooft and Polyakov suggest spherically symmetric solutions for the three-dimensional Yang-Mills-Higgs equations. With respect to the standard basis  $e_a$ ,  $a = 1, 2, 3$  of  $\mathfrak{su}(2)$  their Ansatz is

$$(6) \quad A = \varepsilon_{ija} \frac{x_j}{r^2} (1 - K(r)) e_a dx_i, \quad \Phi = \frac{x_\alpha}{r} \frac{H(r)}{r} e_a,$$

with boundary conditions  $K(r) \rightarrow 0$  and  $H/r \rightarrow 1$ , as  $r \rightarrow \infty$ .

On configurations of this form,  $E_\lambda$  is

$$\begin{aligned}
 E_\lambda(H, K) = 4\pi \int_0^\infty \left\{ (K')^2 + \frac{1}{2} \left( H' - \frac{H}{r} \right)^2 + \frac{K^2 H^2}{r^2} + \frac{1}{2} \frac{(K^2 - 1)^2}{r^2} \right. \\
 \left. + \frac{\lambda}{4} \left( \frac{H^2}{r} - r \right)^2 \right\} dr.
 \end{aligned}$$

A critical point  $(K_\lambda, H_\lambda)$  of the 1-dimensional integral satisfies the variational equation

$$\left. \frac{d}{dt} \right|_{t=0} E_\lambda(H_\lambda + th, K_\lambda + tk) = 0$$

for all  $h, k$  with compact support on  $[0, \infty)$ . This yields the system of non-linear, second order, ordinary differential equations

$$K_\lambda'' = \frac{H_\lambda^2 - 1 + K_\lambda^2}{r^2} K_\lambda \quad (\text{YMH 1})$$

$$H_\lambda'' = \frac{2K_\lambda^2}{r^2} H_\lambda - 4\lambda H_\lambda \left( 1 - \frac{H_\lambda^2}{r^2} \right). \quad (\text{YMH 2})$$

It is relatively easy to produce critical points of the 1-dimensional integral by direct minimization, see [D] for example. On the other hand it is a standard fact, referred to as “the principle of symmetric criticality” in [Pa],

that due to symmetry a critical point of the 1-dimensional integral is also a critical point of  $E_\lambda$  overall.

Therefore for each  $\lambda$  there is a spherically symmetric monopole solution. Throughout this paper

$$c_\lambda = (K_\lambda, H_\lambda)$$

will denote this solution.

**Remark on notation.** The notation for the Hessian  $Q_{\lambda, c_\lambda}$  of  $E_\lambda$  over all directions in  $T_{c_\lambda}\mathcal{C}$  will be shortened to  $Q_{c_\lambda}$ . Otherwise,  $Q_{\lambda, c}$  will denote the Hessian at an arbitrary configuration  $c$  in  $\mathcal{C}$ .

There is no *a priori* reason why  $c_\lambda$  should be an overall minimum. For example, the spherically symmetric minimizer of the Skyrmin functional has (non-spherically symmetric) unstable directions, see [WB].

For  $\lambda = 0$  the the point  $c_0$  is a global minimum in the connected component of all configurations with finite  $E_0$  energy, [M].

For  $\lambda \neq 0$  and small, the main result in [AD] is:

**Theorem 2.3.** *There is  $\lambda_0 > 0$  such that  $Q_{c_\lambda}(v) \geq 0$  for all  $\lambda \leq \lambda_0$  and for all  $v$  in  $T_{c_\lambda}\mathcal{C}$ . Furthermore,*

$$\ker Q_{c_\lambda} = \left\langle \frac{\partial c_\lambda}{\partial x_i}, i = 1, 2, 3 \right\rangle.$$

The behavior of  $Q_{c_\lambda}$  for  $\lambda$  away from  $\lambda = 0$  is investigated here. For this, define

$$\Lambda = \{\lambda > 0 : Q_{c_\lambda} \geq 0\},$$

$$\Lambda' = \left\{ \lambda \geq 0 : \ker Q_{c_\lambda} = \left\langle \frac{\partial c_\lambda}{\partial x_i}, \right\rangle_{i=1,2,3} \right\}.$$

Then Theorem 2.3 states that  $\Lambda \cap \Lambda'$  is not empty and contains an interval of the form  $(0, \lambda_0)$ .

The main result here is:

**Theorem 2.4.** 1)  $\Lambda \cap \Lambda'$  is an open subset of  $(0, \infty)$ .  
 2)  $\Lambda$  is closed.  
 3)  $\Lambda$  contains the first connected component of  $\Lambda'$ .

With this, to prove that  $Q_{c_\lambda}$  is non-negative for all  $\lambda$  reduces to the following:

**Conjecture 2.5.** *For all positive  $\lambda$ ,  $\ker Q_{c_\lambda} = \left\langle \frac{\partial c_\lambda}{\partial x_i}, i = 1, 2, 3 \right\rangle$ .*



### 3. Convergence in the configuration space and Hessians.

**3.1. General Observations.** The proof of the Theorem 2.4 relies on some general observations about quadratic forms from [AD]. First, in order to describe the fact that the kernel of the Hessians changes as the solutions  $c_\lambda$  move in the configuration space  $\mathcal{C}$ , adopt the following:

**Definition.** Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Let  $V_{\lambda_0}$  be a closed subspace of  $H$  and  $V_\lambda$  be a one-parameter family of closed subspaces of  $H$ .  $V_\lambda$  contains  $V_{\lambda_0}$  at the limit as  $\lambda \rightarrow \lambda_0$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|\lambda - \lambda_0| < \delta$  implies that for any  $u$  in  $V_{\lambda_0}^\perp$  of norm 1 there exists  $v$  in  $V_\lambda^\perp$  of norm 1 such that  $\|v - u\| < \varepsilon$ .

Now, slightly abusing notation, let  $Q_{\lambda_0}$  be a quadratic form and  $Q_\lambda$  be a one-parameter family of quadratic forms on  $H$ . The following describes the steps for the proof of Theorem 2.4 in this general setting:

**Proposition 3.1.** *Let  $Q_{\lambda_0}$  be a quadratic form and  $Q_\lambda$  be a one-parameter family of quadratic forms defined on a Hilbert space  $H$  and assume that:*

- 1)  $Q_{\lambda_0}$  is uniformly continuous on the unit sphere of  $H$ .
- 2)  $\alpha := \inf\{Q_{\lambda_0}(v) : v \perp \ker Q_{\lambda_0}, \|v\| = 1\} \geq 0$ .
- 3)  $\sup_{\|v\|=1} |Q_{\lambda_0}(v) - Q_\lambda(v)| \rightarrow 0$  as  $\lambda \rightarrow \lambda_0$ .
- 4) There are subspaces  $N_\lambda$  of  $\ker Q_\lambda$  such that  $N_\lambda$  contains  $\ker Q_{\lambda_0}$  at the limit as  $\lambda \rightarrow \lambda_0$ .

Then there exists  $\varepsilon > 0$  such that whenever  $|\lambda - \lambda_0| < \varepsilon$  then

- 1)  $\inf\{Q_\lambda(v) : v \perp N_\lambda, \|v\| = 1\} > \frac{\alpha}{3} \geq 0$ ,
- 2)  $N_\lambda = \ker Q_\lambda$ .

*Proof.* If  $v$  is in  $N_\lambda^\perp$  and of norm 1 then, for  $\lambda$  sufficiently close to  $\lambda_0$ , there is  $v'$  of norm 1 in  $\ker Q_{c_{\lambda_0}}^\perp$  close to  $v$ . Therefore  $Q_{c_\lambda}(v) \approx Q_{c_{\lambda_0}}(v) \approx Q_{c_{\lambda_0}}(v') > \alpha/3$ .  $\square$

**3.2. Reduction to a single Hilbert space.** Before Proposition 3.1 can take over, one has to establish that as  $\lambda \rightarrow \lambda_0$  the spaces  $T_{c_\lambda}\mathcal{C}$  are isomorphic and therefore all Hessians  $Q_{c_\lambda}$  are defined on the same space.

**Lemma 3.2.** *For any  $c$  and  $c'$  in  $\mathcal{C}$  with  $c - c'$  in  $T_c\mathcal{C}$  the following holds:*

$$\| \|v\|_{c'} - \|v\|_c \| \leq M_c \|v\|_c \|c - c'\|_c.$$

*In particular, for  $\|c - c'\|_c$  sufficiently small the identity on  $C_0^\infty$  induces an isomorphism between  $T_c\mathcal{C}$  and  $T_{c'}\mathcal{C}$ .*

*Proof.* The proof for the norm  $\|\cdot\|_c$  as defined here, is the same as the proof of Proposition B6.2 of [T1].  $\square$

That the conditions of this Lemma are satisfied for spherically symmetric solutions is proved in the following:

**Proposition 3.3.**  $\|c_\lambda - c_{\lambda_0}\|_{c_{\lambda_0}} \rightarrow 0$ , as  $\lambda \rightarrow \lambda_0 > 0$ .

*Proof.* It is a matter of straightforward calculation to show that this follows by the fact that the following norms over  $[0, \infty)$  go to 0 as  $\lambda \rightarrow \lambda_0$ :

$$(7) \quad \left\| \frac{1}{r}(H_\lambda - H_{\lambda_0}) \right\|_2, \quad \|H'_\lambda - H'_{\lambda_0}\|_2, \quad \|(H_\lambda - H_{\lambda_0})\|_2.$$

$$(8) \quad \|K_\lambda - K_{\lambda_0}\|_2, \quad \left\| \frac{1}{r}(K_\lambda - K_{\lambda_0}) \right\|_2, \quad \|K'_\lambda - K'_{\lambda_0}\|_2.$$

For these estimates work as follows:

**Step 1. Estimates for  $H_\lambda$  and  $K_\lambda$ .** First obtain uniform in  $\lambda$  pointwise bounds on the fields on  $[0, r_0)$ , for  $r_0$  sufficiently small, see (23) and (24) in Section 4.

Then obtain uniform in  $\lambda$  pointwise exponential decay estimates on the fields on  $[r_1, \infty)$  for  $r_1$  sufficiently large, see Proposition 4.2 in Section 4.

For the intervals of the form  $[r_0, r_1]$ , Proposition 7.1 of [AD] shows that  $\|F_{A_\lambda}\|_2 + \|d_{A_\lambda\Phi_\lambda}\|_2$  is a non-decreasing function of  $\lambda$ , therefore bounded on bounded intervals. As an elementary case of Uhlenbeck's compactness, this suffices for uniform convergence of the fields on bounded domains, see Proposition 7.4 of [AD].

**Step 2. Estimates for  $H''_\lambda$  and  $K''_\lambda$ .** These follow from the estimates on  $H_\lambda$  and  $K_\lambda$  after using equations (YMH1) and (YMH2) that do not involve first derivatives, see Proposition 4.1 below.

**Step 3. Estimates for  $H'_\lambda$  and  $K'_\lambda$ .** For these, obtain uniform in  $\lambda$  pointwise decay estimates on the first derivatives on  $[r_1, \infty)$ , c.f. Propositions 8.2, 9.4 and 9.5 of [AD]. On  $[0, r_1)$  use Step 2, the Poincaré inequality and the fact that  $H'_\lambda(0) = K'_\lambda(0) = 0$ .

The details follow from the arguments in [AD], after observing that the pointwise estimates there on the fields  $H_\lambda$ ,  $K_\lambda$  and their first derivatives are valid for  $\lambda$  in any specified bounded interval; see Section 4 below.  $\square$

**Remark.**  $\|H_\lambda - H_{\lambda_0}\|_2 \rightarrow 0$  does not hold for  $\lambda_0 = 0$  over  $[0, \infty)$  but only over compact intervals. This reflects the fact that  $H_0$  decays in power law whereas  $H_\lambda$  decays exponentially for all  $\lambda > 0$ , see Proposition 4.2.

The remaining subsections of this section show that the conditions of Proposition 3.1 hold for the Hessians  $Q_{c_\lambda}$ .

### 3.3. Uniform continuity of $Q_{\lambda,c}$ on the unit sphere.

**Lemma 3.4.** *For any  $\lambda$ , and for any  $c = (A, \Phi)$  in  $\mathcal{C}$  with  $\Phi$  bounded the Hessian  $Q_{\lambda,c} : (T_c\mathcal{C}, \|\cdot\|_c) \rightarrow \mathbf{R}$  is uniformly continuous on the unit sphere.*

*Proof.* It suffices to show that  $\hat{Q}_{\lambda,c}(v, w) \leq K\|v\|_c \cdot \|w\|_c$  for some constant  $K$  and all  $v$  and  $w$ . To see that this holds separately for each term of  $\hat{Q}_{\lambda,c}$ , use the fact that  $\Phi$  is bounded and the inequalities

$$(9) \quad |\langle F_A, [a, a] \rangle| \leq \|F_A\|_2 \| [a, a] \|_2 \leq C \|F_A\|_2 \| (a, \phi) \|_c^2,$$

$$(10) \quad |\langle d_A \Phi, [a, \phi] \rangle| \leq \|d_A \Phi\|_2 \| [a, \phi] \|_2 \leq C \|d_A \Phi\|_2 \| (a, \phi) \|_c^2,$$

c.f. [T1]. □

**Remark.** It is standard to show using maximum principle that for any critical point  $c = (A, \Phi)$  of  $E_\lambda$ , and in particular for the spherically symmetric solutions  $c_\lambda$ , the Higgs field  $\Phi$  satisfies  $|\Phi|(x) < 1$ .

### 3.4. When $Q_{\lambda,c}$ is non-negative.

**Theorem 3.5.** *For any  $c$  in  $\mathcal{C}$  the following hold:*

- 1) *If  $\Phi$  is bounded then  $Q_{\lambda,c} : (T_c \mathcal{C}, \|\cdot\|_c) \rightarrow \mathbf{R}$  is continuously differentiable.*
- 2) *There is  $\varepsilon > 0$  such that  $Q_{\lambda,c} - \varepsilon \|\cdot\|_c^2$  is weakly lower semi-continuous in  $(T_c \mathcal{C}, \|\cdot\|_c)$ .*

*Proof.* 1) Since  $\hat{Q}_{\lambda,c}$  is continuous, this follows from the fact that  $2\hat{Q}_{\lambda,c}(v, \cdot)$  is the differential of  $Q_{\lambda,c}$  at  $v$ .

2) First use Lemma 2.2 to rewrite the Hessian as

$$\begin{aligned} Q_{\lambda,c}(a, \phi) &= \|\nabla_A a\|_2^2 + \|d_A \phi\|_2^2 + \|[a, \Phi]\|_2^2 + \|[\Phi, \phi]\|_2^2 \\ &\quad + 2\langle F_A, [a, a] \rangle + 4\langle d_A \Phi, [a, \phi] \rangle \\ &\quad + \frac{\lambda}{2} \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x + \lambda \int_{\mathbf{R}^3} \langle \Phi, \phi \rangle^2 d^3x. \end{aligned}$$

Then for  $\varepsilon \leq \min(\lambda, 1)$  we have:

$$\begin{aligned} &Q_{\lambda,c}(a, \phi) - \varepsilon \|(a, \phi)\|_c^2 \\ &= (1 - \varepsilon)\|\nabla_A a\|_2^2 + (1 - \varepsilon)\|d_A \phi\|_2^2 \\ &\quad + (1 - \varepsilon)\|[a, \Phi]\|_2^2 + (1 - \varepsilon)\|[\phi, \Phi]\|_2^2 + (\lambda - \varepsilon) \int_{\mathbf{R}^3} \langle \Phi, \phi \rangle^2 d^3x \\ &\quad + 2\langle F_A, [a, a] \rangle + 4\langle d_A \Phi, [a, \phi] \rangle + \frac{\lambda}{2} \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x \\ &\quad + \varepsilon\|[\phi, \Phi]\|_2^2 + \varepsilon \int_{\mathbf{R}^3} \langle \Phi, \phi \rangle^2 d^3x - \varepsilon\|\phi\|_2^2. \end{aligned}$$

Each term of the first two lines is weakly lower semi-continuous with respect to the  $\|\cdot\|_c$ -norm: For the first term, for example, note that if  $(a_n, \phi_n)$  converges weakly to  $(a, \phi)$  in  $T_c \mathcal{C}$  then

$$\|\nabla_A a\|_2 \leq \|(a, \phi)\|_c \leq \liminf_{n \rightarrow \infty} \|(a_n, \phi_n)\|_c$$

since  $\|\cdot\|_c$  is a norm and hence weakly lower semicontinuous.

The weak continuity of the terms in the third line follows as in Section VI of [T2] from the fact that  $F_A$ ,  $d_A\Phi$  and  $\|\Phi\|^2 - 1$  belong to  $L^2$  (and therefore Property (\*) of [T2] is satisfied).

The terms in the fourth line are also weakly continuous since they regroup as

$$(11) \quad \varepsilon \int_{\mathbf{R}^3} (|\Phi|^2 - 1)|\phi|^2 d^3x.$$

□

The following is the main application of the weak lower semi-continuity offered by Theorem 3.5.

**Proposition 3.6.** *For any  $\lambda$ , if  $Q_{\lambda,c}(v) \geq 0$  for all  $v$  in  $T_c\mathcal{C}$  then*

$$(12) \quad \inf\{Q_{\lambda,c}(v) : v \perp_c \ker Q_{\lambda,c}, \|v\|_c = 1\} \not\geq 0.$$

*Proof.* Let  $v_n$  be a sequence in  $\ker Q_{\lambda,c}^\perp$  with  $\|v_n\|_c = 1$  and  $Q_{\lambda,c}(v_n) \rightarrow 0$ . By considering a subsequence, assume that  $v_n$  is weakly convergent and let  $v \in \ker Q_{\lambda,c}^\perp$  be its weak limit. Since  $Q_{\lambda,c}$  is weakly lower semicontinuous (by Theorem 3.5) and non-negative, obtain that  $Q_{\lambda,c}(v) = 0$ . Moreover, since there is an  $\varepsilon > 0$  such that  $Q_{\lambda,c}(\cdot) - \varepsilon\|\cdot\|_c^2$  is weakly lower semicontinuous, it follows that

$$-\varepsilon\|v\|_c^2 = Q_{\lambda,c}(v) - \varepsilon\|v\|_c^2 \leq \liminf_n (Q_{\lambda,c}(v_n) - \varepsilon\|v_n\|_c^2) = -\varepsilon$$

which implies that  $\|v\|_c \geq 1$  and thus  $\|v\|_c = 1$  (therefore  $(v_n)$  converges to  $v$  in the Hilbert space norm  $\|\cdot\|_c$ ). Hence

$$(13) \quad Q_{\lambda,c}(v) = \min\{Q_{\lambda,c}(u) : u \perp_c \ker Q_{\lambda,c}, \|u\|_c = 1\}.$$

Then by the differentiability and the non-negativity of  $Q_{\lambda,c}$  it is easy to see using a Lagrange multiplier that for each  $w \in \ker Q_{\lambda,c}^\perp$

$$\hat{Q}_{\lambda,c}(v, w) = Q_{\lambda,c}(v)\langle v, w \rangle_c$$

which gives that  $v$  is in  $\ker Q_{\lambda,c}$ , a contradiction to the fact that  $v \in \ker Q_{\lambda,c}^\perp$  of norm 1. □

### 3.5. Uniform convergence on the sphere.

**Lemma 3.7.** *For  $c_i = (A_i, \Phi_i)$  and  $c = (A, \Phi)$  in  $\mathcal{C}$  assume that  $\|c_i - c\|_c \rightarrow 0$ . If  $\lambda_i$  converge to  $\lambda$ ,  $|\Phi_i - \Phi|$  tends to zero uniformly on  $\mathbf{R}^3$  and  $|\Phi|$  is bounded on  $\mathbf{R}^3$ , then*

$$(14) \quad \sup_{\|v\|_c=1} |Q_{\lambda_i, c_i}(v) - Q_{\lambda, c}(v)| \rightarrow 0.$$

*Proof.* For the part of the Hessian with no  $\lambda$  coefficient, see statement (5) of Proposition A.4.3. of [T3]. For the remaining terms note that

$$\begin{aligned} & \left| \lambda_i \left( \frac{1}{2} \int (|\Phi_i|^2 - 1) |\phi|^2 d^3x + \int \langle \Phi_i, \phi \rangle^2 d^3x \right) \right. \\ & \quad \left. - \lambda \left( \frac{1}{2} \int (|\Phi|^2 - 1) |\phi|^2 d^3x + \int \langle \Phi, \phi \rangle^2 d^3x \right) \right| \\ & \leq |\lambda_i - \lambda| \frac{1}{2} \int (|\Phi|^2 - 1) |\phi|^2 d^3x + |\lambda_i| \frac{1}{2} \int ||\Phi_i|^2 - |\Phi|^2| |\phi|^2 d^3x \\ & \quad + |\lambda_i - \lambda| \int \langle \Phi, \phi \rangle^2 d^3x + |\lambda_i| \int |\langle \Phi_i, \phi \rangle^2 - \langle \Phi, \phi \rangle^2| d^3x. \end{aligned}$$

The first and third term in the last expression obviously tend to zero. The second term tends to zero since  $\Phi_i$  tends to  $\Phi$  uniformly on  $\mathbf{R}^3$  and  $\|\phi\|_2 \leq 1$ . The fourth term tends to zero since

$$(15) \quad \int |\langle \Phi_i, \phi \rangle^2 - \langle \Phi, \phi \rangle^2| d^3x \leq \int |\Phi_i - \Phi| |\phi|^2 |\Phi_i + \Phi| d^3x$$

and  $|\Phi_i + \Phi|$  is bounded on  $\mathbf{R}^3$ .  $\square$

**Lemma 3.8.** *For  $\lambda_0 \geq 0$ ,  $\Phi_\lambda$  tends to  $\Phi_{\lambda_0}$  uniformly on  $\mathbf{R}^3$  as  $\lambda \rightarrow \lambda_0$ .*

*Proof.* First note that in terms of  $H_\lambda$  it is enough to show that as  $\lambda \rightarrow \lambda_0$

$$(16) \quad \left\| \frac{H_\lambda}{r} - \frac{H_{\lambda_0}}{r} \right\|_\infty \rightarrow 0.$$

For this use Corollary 4.3 for large  $r$ , estimate (24) as it appears in the proof of Proposition 4.1 for small  $r$ , and the uniform convergence on compact intervals (as in Step 1, Proposition 3.3) in between.  $\square$

**3.6.  $\lambda$ -subspaces containing the kernel of  $Q_{c_{\lambda_0}}$  at the limit.** For  $\lambda \geq 0$  consider the subspace of  $T_{c_\lambda} \mathcal{C}$  spanned by the translation modes, i.e., the partial derivatives of  $c_\lambda$ :

$$(17) \quad N_\lambda = \left\langle \frac{\partial c_\lambda}{\partial x_i}, i = 1, 2, 3 \right\rangle.$$

As a result of a straight-forward calculation

$$(18) \quad \begin{aligned} & \frac{d^2}{dsdt} \Big|_{t=0, s=0} E_\lambda(c(x + te_i) + sv(x + te_i)) \\ & = \hat{Q}_{\lambda, c} \left( \frac{\partial c}{\partial x_i}, v \right) + \nabla(E_\lambda)_c \left( \frac{\partial v}{\partial x_i} \right) \end{aligned}$$

for any  $v$  in  $T_c \mathcal{C}$ . Since  $E_\lambda$  is translation invariant

$$(19) \quad \frac{d}{dt} E_\lambda(c(x + te_i) + sv(x + te_i)) = 0.$$

In addition the first variation of  $E_\lambda$  vanishes at  $c_\lambda$ , therefore

$$(20) \quad Q_{c_\lambda} \left( \frac{\partial c_\lambda}{\partial x_i}, v \right) = 0,$$

which shows that  $N_\lambda$  is a subspace  $\ker Q_{c_\lambda}$ .

**Proposition 3.9.**  $N_\lambda$  contain  $S_{\lambda_0}$  at the limit in  $(T_{c_{\lambda_0}}\mathcal{C}, \|\cdot\|_{c_{\lambda_0}})$  as  $\lambda \rightarrow \lambda_0 \neq 0$ .

*Proof.* As a matter of a straightforward calculation using that  $|\Phi_{\lambda_0}|(x) < 1$ , and that  $|A_{\lambda_0}|$  is uniformly bounded by [AD],

$$(21) \quad \left\| \frac{\partial c_\lambda}{\partial x_a} - \frac{\partial c_{\lambda_0}}{\partial x_a} \right\|_{c_{\lambda_0}}^2 \rightarrow 0$$

if, in addition to the estimates in the proof of 3.3, the following norms over  $[0, \infty)$  tend to 0 as  $\lambda \rightarrow \lambda_0$ :

$$(22) \quad \left\| \frac{1}{r^2} (H_\lambda - H_{\lambda_0}) \right\|_2, \quad \|H''_\lambda - H''_{\lambda_0}\|_2, \\ \left\| \frac{1}{r} (H'_\lambda - H'_{\lambda_0}) \right\|_2, \quad \|K''_\lambda - K''_{\lambda_0}\|_2.$$

Observe that this involves estimates on the fields, their first and their second derivatives. These follow again as in [AD] using the fact that the pointwise estimates on the fields and their derivatives hold for  $\lambda$  on any specified bounded interval, see Section 4.  $\square$

*Proof or Theorem 2.4.* Fix  $\lambda_0$  in  $\Lambda \cap \Lambda_0$ . Then subsections 3.2 to 3.6 show that  $Q_{c_{\lambda_0}}$  satisfies the conditions 1) to 4) respectively of Proposition 3.1, and that for  $\lambda$  close to  $\lambda_0$  all the Hessians  $Q_{c_\lambda}$  are defined on the same space as  $Q_{c_{\lambda_0}}$ . Then Proposition 3.1 applies to give part 1) of Theorem 2.4.

The second part of Theorem 2.4 follows immediately from Lemma 3.7, which shows that the complement of  $\Lambda$  is open.

Now for the third part of Theorem 2.4 argue as follows: by Theorem 2.3, both the first connected component  $\Lambda_0$  of  $\Lambda$  and the first connected component  $\Lambda'_0$  of  $\Lambda'$  are intervals starting from 0. If  $\Lambda'_0$  is not contained in  $\Lambda$  then  $\Lambda_0$  is a proper subset of  $\Lambda'_0$ . Let  $\lambda_0$  be the supremum of  $\Lambda_0$ , which by the second part of the theorem belongs to  $\Lambda_0$ , i.e.,  $\lambda_0$  is in the intersection of  $\Lambda$  and  $\Lambda'$ . This contradicts the first part of the theorem. The proof of theorem 2.4 is now complete.

#### 4. Estimates.

This section will substantiate the claim that the estimates of [AD] for  $\lambda$  in a neighborhood of 0 can be extended to estimates on any bounded  $\lambda$ -interval.

Proposition 4.1 is typical of the  $L^2$ -norm estimates required. Proposition 4.2 is typical of the uniform in  $\lambda$  and pointwise in  $r$  estimates required.

**Proposition 4.1.**  $\|H''_\lambda - H''_{\lambda_0}\|_{L^2[0,\infty)} \rightarrow 0$ , as  $\lambda \rightarrow \lambda_0 \neq 0$ .

*Proof.* For this, first note that it follows as in Section 10 of [AD] that there is a constant  $C$  such that for all  $\lambda < \lambda_0 + 1$

$$(23) \quad |K_\lambda(r)| \leq C$$

for all  $r$  and

$$(24) \quad \left| \frac{H_\lambda(r)}{r^2} \right| \leq C$$

for  $r$  in  $[0, 1]$ . Then

$$\begin{aligned} & \|H''_\lambda - H''_{\lambda_0}\|_{L^2[0,\infty)} \\ &= \left\| \frac{2K_\lambda^2 H_\lambda}{r^2} - \frac{2K_{\lambda_0}^2 H_{\lambda_0}}{r^2} - 4\lambda H_\lambda \left(1 - \frac{H_\lambda^2}{r^2}\right) - 4\lambda_0 H_{\lambda_0} \left(1 - \frac{H_{\lambda_0}^2}{r^2}\right) \right\|_{L^2[0,\infty)} \\ &\leq \left\| \frac{2K_\lambda^2 H_\lambda}{r^2} - \frac{2K_{\lambda_0}^2 H_{\lambda_0}}{r^2} \right\|_{L^2[0,\infty)} \\ &\quad + \left\| 4\lambda H_\lambda \left(1 - \frac{H_\lambda^2}{r^2}\right) - 4\lambda_0 H_{\lambda_0} \left(1 - \frac{H_{\lambda_0}^2}{r^2}\right) \right\|_{L^2[0,\infty)}. \end{aligned}$$

For the first term in this sum find appropriately small  $r_0$  and large  $r_1$  such that:

$$\begin{aligned} & \left\| \frac{2K_\lambda^2 H_\lambda}{r^2} - \frac{2K_{\lambda_0}^2 H_{\lambda_0}}{r^2} \right\|_{L^2[0,\infty)} \\ &\leq \left\| \frac{K_\lambda^2 H_\lambda}{r^2} \right\|_{L^2[0,r_0]} + \left\| \frac{K_{\lambda_0}^2 H_{\lambda_0}}{r^2} \right\|_{L^2[0,r_0]} \\ &\quad + \left\| \frac{K_\lambda^2 (H_\lambda - H_{\lambda_0})}{r^2} \right\|_{L^2[r_0,r_1]} + \left\| \frac{(K_\lambda^2 - K_{\lambda_0}^2) H_{\lambda_0}}{r^2} \right\|_{L^2[r_0,r_1]} \\ &\quad + 2 \left\| \alpha \frac{e^{-r/2}}{r} \right\|_{L^2[r_1,\infty)} \\ &\leq C^3 M \varepsilon + C^3 \varepsilon \\ &\quad + \frac{C^2}{r_0^2} \|H_\lambda - H_{\lambda_0}\|_{L^2[r_0,r_1]} + \frac{r_1}{r_0^2} \|K_\lambda^2 - K_{\lambda_0}^2\|_{L^2[r_0,r_1]} \\ &\quad + 2\varepsilon. \end{aligned}$$

As already remarked in Step 1. of Proposition 3.3 above,  $E_0$  is an increasing function of  $\lambda$ , and therefore bounded on a bounded interval, and that this is enough to give uniform convergence of  $H_\lambda$  to  $H_0$  and of  $K_\lambda$  to  $K_0$  on bounded domains. Therefore the last expression becomes smaller than  $5\varepsilon$  if  $|\lambda - \lambda_0|$  is small enough, by the uniform convergence of  $K_\lambda$  to  $K_{\lambda_0}$  and  $H_{\lambda_0}$  to  $H_0$  on the interval  $[r_0, r_1]$ .

For the second term and for a choice of  $0 < r_0 < r_1$ ,

$$\begin{aligned}
& \left\| 4\lambda H_\lambda \left( 1 - \frac{H_\lambda^2}{r^2} \right) - 4\lambda_0 H_{\lambda_0} \left( 1 - \frac{H_{\lambda_0}^2}{r^2} \right) \right\|_{L^2[0, \infty)} \\
& \leq \left\| 4\lambda H_\lambda \left( 1 - \frac{H_\lambda^2}{r^2} \right) \right\|_{L^2[0, r_0]} + \left\| 4\lambda_0 H_{\lambda_0} \left( 1 - \frac{H_{\lambda_0}^2}{r^2} \right) \right\|_{L^2[0, r_0]} \\
& \quad + 4 \left\| (\lambda - \lambda_0) H_\lambda \left( 1 - \frac{H_\lambda^2}{r^2} \right) \right\|_{L^2[r_0, r_1]} \\
& \quad + 4 \left\| \lambda_0 (H_\lambda - H_{\lambda_0}) \left( 1 - \frac{H_\lambda^2}{r^2} \right) \right\|_{L^2[r_0, r_1]} \\
& \quad + 4 \left\| \lambda_0 \frac{H_{\lambda_0}}{r^2} (H_{\lambda_0}^2 - H_\lambda^2) \right\|_{L^2[r_0, r_1]} \\
& \quad + 4 \left\| \lambda_0 (H_\lambda - H_{\lambda_0}) \left( 1 - \frac{H_\lambda^2}{r^2} \right) \right\|_{L^2[r_1, \infty)} \\
& \quad + 4 \left\| \lambda_0 \frac{H_{\lambda_0}}{r} \left( \left( 1 - \frac{H_\lambda}{r} \right) \left( 1 - \frac{H_{\lambda_0}}{r} \right) \right) (H_{\lambda_0} + H_\lambda) \right\|_{L^2[r_1, \infty)}.
\end{aligned}$$

Obviously,  $r_0$  can be chosen small enough to make the first two terms arbitrarily small. Then  $r_1$  can be chosen large enough to make the last two terms arbitrarily small (for the last term use triangle inequality of the norm and Proposition 4.2; for the anti-penultimate term use Corollary 4.3 and Proposition 4.2). The rest of the terms tend to zero as  $\lambda \rightarrow \lambda_0$  by uniform convergence on compact intervals.  $\square$

**Proposition 4.2.** *For all  $\lambda_0 \geq 0$  there exist  $\alpha > 0$ ,  $r_0 > 0$  such that for all  $\lambda \in [\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$  and for all  $r \geq r_0$*

$$(25) \quad \left| 1 - \frac{H_\lambda(r)}{r} \right| \leq \alpha e^{-\min(2\sqrt{\lambda}, 1)r}.$$



*Proof.* Let  $u_\lambda(r) = 1 - \frac{H_\lambda(r)}{r}$ . Differentiate twice to obtain

$$(26) \quad u_\lambda'' + \frac{2}{r}u_\lambda' = -\frac{H_\lambda''}{r}.$$

Replacing  $H_\lambda''$  from (YMH-2) yields

$$(27) \quad u_\lambda'' + \frac{2}{r}u_\lambda' - \frac{4\lambda H_\lambda}{r} \left(1 + \frac{H_\lambda}{r}\right) u_\lambda = -\frac{2K_\lambda^2 H_\lambda}{r^3}.$$

Now let

$$s(r) = \alpha e^{-\min(2\sqrt{\lambda}, 1)r}$$

to be the test function. The aim is to show that there exist  $\alpha \geq 1$  and  $r_0 > 0$  such that

$$|u_\lambda(r)| \leq s(r),$$

for all  $r \geq r_0$  and for all  $\lambda$  in an appropriate range. Since

$$\left|1 - \frac{H_\lambda(r)}{r}\right| \leq \sqrt{\frac{C}{r}}$$

and

$$|K_\lambda(r)| \leq \alpha e^{-r/2},$$

(see [AD]), for  $\lambda \in [\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$

$$\begin{aligned} & (s \pm u_\lambda)'' + \frac{2}{r}(s \pm u_\lambda)' - \frac{4\lambda H_\lambda}{r} \left(1 + \frac{H_\lambda}{r}\right) (s \pm u_\lambda) \\ &= \mp \frac{2K_\lambda^2 H_\lambda}{r^3} + \alpha \min(4\lambda, 1) e^{-\min(2\sqrt{\lambda}, 1)r} - \frac{2}{r} \alpha \min(2\sqrt{\lambda}, 1) e^{-\min(2\sqrt{\lambda}, 1)r} \\ &\quad - \frac{4\lambda H_\lambda}{r} \left(1 + \frac{H_\lambda}{r}\right) \alpha e^{-\min(2\sqrt{\lambda}, 1)r} \\ &\leq \frac{2e^{-r}}{r^2} + \alpha e^{-\min(2\sqrt{\lambda}, 1)r} \\ &\quad - 4\lambda \left(1 - \sqrt{\frac{C}{r}}\right) \left(1 + 1 - \sqrt{\frac{C}{r}}\right) \alpha e^{-\min(2\sqrt{\lambda}, 1)r} \\ &\leq \frac{2e^{-r}}{r^2} + \alpha e^{-\min(2\sqrt{\lambda}, 1)r} \\ &\quad - 8\lambda \alpha e^{-\min(2\sqrt{\lambda}, 1)r} + 12(\lambda_0 + 1) \sqrt{\frac{C}{r}} \alpha e^{-\min(2\sqrt{\lambda}, 1)r}. \end{aligned}$$

Note that the term  $-8\lambda \alpha e^{-\min(2\sqrt{\lambda}, 1)r}$  is negative and it eventually makes the above expression negative as well since  $|\lambda - \lambda_0| \leq 1$ , i.e., there exists

$r_0 > 0$  such that the last expression is negative for all  $r \geq r_0$  and for all  $\lambda \in [\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$ . Since

$$u_\lambda(r_0) \leq 1 + \sqrt{\frac{C}{r_0}}$$

for all  $\lambda$  in the range  $[\lambda_0 - 1, \lambda_0 + 1] \cap [0, \infty)$ , choose  $\alpha \geq 1$  such that

$$s_\lambda(r_0) \pm u_\lambda(r_0) > 0$$

for all such  $\lambda$ 's. □

**Corollary 4.3.** *For every  $\lambda_0 > 0$  there exists  $M > 0$  such that for  $\lambda$  in  $[0, \lambda_0]$  the following holds for  $r \geq 0$*

$$(28) \quad |H_\lambda(r) - H_{\lambda_0}(r)| \leq M.$$

## References

- [AD] G. Androulakis and S. Dostoglou, *On the stability of monopole solutions*, Nonlinearity, **11**(3) (1998), 377-408.
- [D] S. Dostoglou, *Remarks on the Existence of Solutions of the Yang-Mills-Higgs Equations with Arbitrary Higgs Potential*, Preprint available at <http://www.math.missouri.edu/people/sdostoglou.html>
- [H] P. Higgs, *Spontaneous symmetry breakdown without massless boson*, Phys. Rev., **145** (1966), 1156-1163.
- [tH] G. 't Hooft, *Magnetic monopoles in unified gauge theories*, Nucl. Phys. B, **79** (1976), 276.
- [JT] A. Jaffe and C.H. Taubes, *Vortices and Monopoles*, Birkhauser, 1980.
- [L] H.B. Lawson, *The theory of gauge fields in four dimensions*, CBMS Regional Conference Series in Mathematics, **58**; American Mathematical Society, Providence, R.I., 1985.
- [M] D. Maison, *Uniqueness of the Prasad-Sommerfield monopole solution*, Nucl. Phys. B, **182** (1981), 144-150.
- [Pa] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys., **69**(1) (1979), 19-30.
- [P] A.M. Polyakov, JETP Letters, **20**, 1974, 194.
- [T1] C.H. Taubes, *Monopoles and maps from  $S^2$  to  $S^2$ ; the topology of the configuration space*, Commun. Math. Phys., **95** (1984), 345-391.
- [T2] ———, *Stability in Yang-Mills Theories*, Commun. Math. Phys., **91** (1983), 235-263.
- [T3] ———, *Existence of a non-minimal solutions to the  $SU(2)$  Yang-Mills-Higgs equations*, Part I, Comm. Math. Phys., **86** (1982), 257-298.

[WB] A. Wirzba and H. Bang, *The mode spectrum and stability analysis of skyrmions on the 3-sphere*, Nuclear Phys. A, **515(4)** (1990), 571- 598.

Received June 17, 1998 and revised July 20, 1999.

UNIVERSITY OF MISSOURI-COLUMBIA  
COLUMBIA, MO 65211  
*E-mail address:* giorgis@math.missouri.edu

UNIVERSITY OF MISSOURI-COLUMBIA  
COLUMBIA, MO 65211  
*E-mail address:* stamatis@euler.math.missouri.edu

