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ISOMORPHISMS OF TYPE A AFFINE HECKE ALGEBRAS AND MULTIVARIABLE ORTHOGONAL POLYNOMIALS

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We examine two isomorphisms between affine Hecke algebras of type A associated with parameters q^{-1}, t^{-1} and q, t. One of them maps the non-symmetric Macdonald polynomials $E_\eta(x;q^{-1},t^{-1})$ onto $E_\eta(x;q,t)$, while the other maps them onto non-symmetric analogues of the multivariable Al-Salam & Carlitz polynomials. Using the properties of $E_\eta(x;q^{-1},t^{-1})$, the corresponding properties of these latter polynomials can then be elucidated.

1. Introduction.

In several recent works [28]-[29], [9]-[10], eigenstates of the rational (type A) Calogero-Sutherland model have been investigated from an algebraic point of view. In particular it has been shown that the algebra governing the eigenfunctions of the *periodic* Calogero-Sutherland model (namely the type A degenerate affine Hecke algebra augmented by type A Dunkl operators) is isomorphic to its *rational* model counterpart. This enables information to be gleaned about the properties of the eigenfunctions in the rational case (the (non-)symmetric Hermite polynomials) from the corresponding periodic eigenfunctions (the (non-)symmetric Jack polynomials).

To summarize the argument, consider the type A Dunkl operators

$$d_i := \frac{\partial}{\partial x_i} + \frac{1}{\alpha}\sum_{p\neq i}\frac{1-s_{ip}}{x_i-x_p}$$

which, along with the operators representing multiplication by the variable x_i and the elementary transpositions s_{ij}, satisfy the following commutation relations

$$(1.1)\qquad [d_i,x_j] = \begin{cases} -\frac{1}{\alpha}s_{ij} & i\neq j \\ 1+\frac{1}{\alpha}\sum_{p\neq i}s_{ip} & i=j \end{cases}$$

$$d_i\,s_{ip} = s_{ip}\,d_p \qquad\qquad [d_i,s_{jp}]=0, \qquad i\neq j,p.$$

It is easily checked that the map ρ defined by

$$(1.2)\qquad \rho(x_i)=x_i-\frac{1}{2}d_i, \qquad \rho(d_i)=d_i, \qquad \rho(s_{ij})=s_{ij}$$

is an isomorphism of the algebra (1.1) [**28**].

Now, the non-symmetric Jack polynomials $E_\eta(x)$, indexed by compositions $\eta := (\eta_1, \dots, \eta_n)$ can be defined [**23**] as the unique eigenfunctions of the mutually commuting Cherednik operators

$$\xi_i := \alpha x_i d_i + \sum_{p>i} s_{ip} - n + 1 \tag{1.3}$$

with a unique expansion of the form

$$E_\eta(x) = x^\eta + \sum_{\nu<\eta} c_{\eta\nu}\, x^\nu. \tag{1.4}$$

Here, the partial order $<$ is defined on compositions by: $\nu < \eta$ iff $\nu^+ < \eta^+$ with respect to the dominance order (where ν^+ is the unique partition associated to ν etc) *or* $\nu^+ = \eta^+$, $\nu \neq \eta$ and $\sum_{i=1}^p (\eta_i - \nu_i) \geq 0$, for all $p = 1, \dots, n$. The polynomial $E_\eta(x)$ is an eigenfunction of ξ_i given by (1.3) with eigenvalue

$$\bar{\eta}_i = \alpha\eta_i - \#\{k < i | \eta_k \geq \eta_i\} - \#\{k > i | \eta_k > \eta_i\}. \tag{1.5}$$

Using the isomorphism (1.2) it follows that the polynomials [**27**, **24**, **10**]

$$E_\eta^{(H)}(x) := E_\eta(\rho(x)) \cdot 1$$

are eigenfunctions of the operators

$$h_i = \rho(\xi_i) = \xi_i - \frac{\alpha}{2} d_i^2 \tag{1.6}$$

which are precisely the eigenoperators of the non-symmetric Hermite polynomials [**2**]. The orthogonality of these latter polynomials with respect to the usual multivariable Hermite inner product then follows from the fact that the operator (1.6) is self-adjoint with respect to the inner product

$$\langle f, g\rangle := \prod_{i=1}^n \int_{-\infty}^\infty dx_i\, e^{-x_i^2} \prod_{1\leq j<k\leq n} |x_j - x_k|^{2/\alpha} f(x)\, g(x). \tag{1.7}$$

In this work, we provide a similar analysis of the Macdonald case. As such, we introduce an isomorphism of the q-analogue of the algebra (1.1), namely the subalgebra $\mathcal{S}_{q,t} := \{T_i, \omega, D_i, x_i\}$ of the algebra of endomorphisms of the polynomial ring $\mathbb{Q}(q,t)[x_1, \dots, x_n]$. Here, $\{T_i, \omega\}$ generate a subalgebra isomorphic to the (type A) affine Hecke algebra, while $\{D_i\}$ are the q-Dunkl operators introduced in [**3**, **14**]. To describe this mapping, we need to introduce some further concepts.

The generalization of the formalism of non-symmetric Jack polynomials to the Macdonald case involves replacing the Cherednik operators (1.3) by their q-analogues which can be realized as a commutative subalgebra of the affine Hecke algebra [18]. In the type A case, one can describe this using the Demazure-Lustig operators

$$T_i := t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(s_i - 1) \qquad i = 1, \dots, n-1 \tag{1.8}$$

$$T_0 := t + \frac{qtx_n - x_1}{qx_n - x_1}(s_0 - 1) \tag{1.9}$$

along with the operator

$$\omega := s_{n-1}\cdots s_2\, s_1\tau_1 = s_{n-1}\cdots s_i\tau_i s_{i-1}\cdots s_1. \tag{1.10}$$

Here τ_i is the operator which replaces x_i by qx_i, $s_i := s_{i,i+1}$ for $1 \le i \le n-1$ and $s_0 := \omega s_1 \omega^{-1}$. The affine Hecke algebra is then generated by elements T_i, $0 \le i \le n-1$ and ω, satisfying the relations

$$(T_i - t)(T_i + 1) = 0 \tag{1.11}$$

$$T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1} \tag{1.12}$$

$$T_i\, T_j = T_j\, T_i \qquad |i-j| \ge 2 \tag{1.13}$$

$$\omega\, T_i = T_{i-1}\,\omega. \tag{1.14}$$

There is a commutative subalgebra generated by elements of the form [5, 6]

$$Y_i := t^{-n+i}\, T_i \cdots T_{n-1}\,\omega\, T_1^{-1}\cdots T_{i-1}^{-1} \tag{1.15}$$

which have the following relations with the generators T_i for $1 \le i \le n-1$.

(1.16)

$$T_i\, Y_{i+1} = tY_i\, T_i^{-1}, \quad T_i\, Y_i = Y_{i+1}\, T_i + (t-1)Y_i, \quad [T_i, Y_j] = 0, \;\; j \ne i, i+1.$$

The non-symmetric Macdonald polynomials $E_\eta(x;q,t)$ are defined as the simultaneous eigenfunctions of the commuting operators Y_i with an expansion of the form (1.4). The corresponding eigenvalue is $t^{\bar\eta_i}$ with $\bar\eta_i$ given in (1.5), and $t^\alpha = q$. From now on, we drop the dependence on q and t and just write $E_\eta(x) \equiv E_\eta(x;q,t)$ when the meaning is unambiguous.

Define the following degree-raising operator

$$e_i := t^{i-1} T_i \cdots T_{n-1}\, x_n\, \omega\, T_1^{-1}\cdots T_{i-1}^{-1}. \tag{1.17}$$

Using (1.12)-(1.14) it can be shown that the operators e_i form a set of mutually commuting operators. Our first result is:

Theorem 1.1. *We have*

$$E_\eta(e_1, \dots, e_n; q^{-1}, t^{-1})\,.\,1 = \alpha_\eta(q,t)\, E_\eta(x_1, \dots, x_n; q, t)$$

where

$$\alpha_\eta(q,t) = q^{\sum_i \binom{\eta_i}{2}} t^{\sum_i (n-i)\eta_i^+ - \ell(w_\eta)} \tag{1.18}$$

with $\ell(w_\eta)$ the length of the (unique) minimal permutation sending η to η^+.

The symmetric Al-Salam & Carlitz (ASC) polynomials were examined in [**1**] as q-analogues of multivariable Hermite polynomials. There are two families of ASC polynomials, denoted $U_\lambda^{(a)}(x;q,t)$ and $V_\lambda^{(a)}(x;q,t)$, which are simply related by

$$V_\lambda^{(a)}(x;q^{-1},t^{-1}) = U_\lambda^{(a)}(x;q,t). \tag{1.19}$$

The polynomials $V_\lambda^{(a)}$ can be defined as the unique polynomials of the form

$$V_\lambda^{(a)}(x;q,t) = P_\lambda(x;q,t) + \sum_{\mu<\lambda} b_{\lambda\mu} P_\mu(x;q,t)$$

which are orthogonal with respect to the inner product

$$\langle f,g\rangle^{(V)} := \int_{[1,\infty]^n} f(x)g(x)\, d_q\mu^{(V)}(x), \tag{1.20}$$

$$d_q\mu^{(V)}(x) := \Delta_q^{(k)}(x)\prod_{l=1}^n w_V(x_l;q)d_qx_l.$$

Here, $P_\lambda(x;q,t)$ denotes the symmetric Macdonald polynomial [**19**] and we use the notation for q-integrals

$$\int_1^\infty f(x)d_qx := (1-q)\sum_{n=0}^\infty f(q^{-n})q^{-n} \tag{1.21}$$

while

$$\begin{aligned} w_V(x;q) &= \frac{(q;q)_\infty(\frac{1}{a};q)_\infty(qa;q)_\infty}{(x;q)'_\infty(\frac{x}{a};q)_\infty} \\ \Delta_q^{(k)}(x_1,\dots,x_n) &:= \prod_{p=-(k-1)}^{k}\ \prod_{1\le i<j\le n}(x_i-q^px_j), \end{aligned} \tag{1.22}$$

where the dash in $(x;q)'_\infty$ denotes that any vanishing factor is to be deleted, and it is assumed $a<0$. Moreover, in (1.20) and in what follows, we assume $t=q^k$, where k is a positive integer.

The polynomials $U_\lambda^{(a)}$ are orthogonal with respect to the inner product

$$\langle f|g\rangle^{(U)} := \int_{[a,1]^n} f(x)g(x)\, d_q\mu^{(U)}(x), \tag{1.23}$$

$$d_q\mu^{(U)}(x) := \Delta_q^{(k)}(x)\prod_{l=1}^n w_U(x_l;q)d_qx_l$$

where $\Delta_q^{(k)}$ is given by (1.22) and

$$w_U^{(a)}(x;q) := \frac{(qx;q)_\infty(\frac{qx}{a};q)_\infty}{(q;q)_\infty(a;q)_\infty(\frac{q}{a};q)_\infty} \tag{1.24}$$

$$\int_a^1 f(x)\, d_q x := (1-q)\left(\sum_{n=0}^{\infty} f(q^n)q^n - a\sum_{n=0}^{\infty} f(aq^n)q^n\right), \quad (a<0). \tag{1.25}$$

This can be regarded as a consequence of (1.19), and the formulas

$$\frac{1}{1-q}\int_a^1 w_U^{(a)}(x;q)f(x)\, d_q x\Big|_{q\mapsto q^{-1}} = \frac{1}{1-q}\int_1^{\infty} w_V^{(a)}(x;q)f(x)\, d_q x \tag{1.26}$$

$$\Delta_{q^{-1}}^{(k)}(x) = q^{-kn(n-1)}\Delta_q^{(k)}(x^R) \tag{1.27}$$

where $x^R = (x_n, x_{n-1}, \ldots, x_1)$. The formula (1.26) is established in [**1**, eq. (2.23)], while (1.27) follows immediately from the definition (1.22).

Non-symmetric analogues of the ASC polynomials can be introduced in the following manner: Consider the following q-analogues of the type A Dunkl operators [**8**] examined in [**3**],

$$D_i := x_i^{-1}\left(1 - t^{n-1}\, T_i^{-1}\cdots T_{n-1}^{-1}\,\omega\, T_1^{-1}\cdots T_{i-1}^{-1}\right) \tag{1.28}$$

and let

$$E_i := D_i + (1+a^{-1})t^{n-1}\, Y_i - a^{-1}\, e_i. \tag{1.29}$$

The operators E_i mutually commute, and our second main result is that:

Theorem 1.2. *The polynomials*

$$E_\eta^{(V)}(x;q,t) = \frac{(-a)^{|\eta|}}{\alpha_\eta(q,t)}\, E_\eta(E;q^{-1},t^{-1})\,\cdot\, 1 \tag{1.30}$$

where $\alpha_\eta(q,t)$ is given by (1.18) are the unique polynomials with an expansion of the form

$$E_\eta^{(V)}(x;q,t) = E_\eta(x;q,t) + \sum_{|\nu|<|\eta|} c_{\eta\nu}\, E_\nu(x;q,t)$$

which are orthogonal with respect to the inner product (1.20). Furthermore, these polynomials are simultaneous eigenfunctions of the commuting family of eigenoperators

$$h_i = Y_i + (1+a)t^{1-n}D_i + at^{2-2n}D_iY_i^{-1}D_i \tag{1.31}$$

with eigenvalue $t^{\bar\eta_i}$.

An immediate consequence of Thm. 1.2, (1.19), and (1.26), (1.27) is:

Corollary 1.3. *The polynomials*

$$E_\eta^{(U)}(x;q,t) := E_\eta^{(V)}(x^R;q^{-1},t^{-1}) \tag{1.32}$$

are the unique polynomials with an expansion of the form

$$E_\eta^{(U)}(x;q,t) = E_\eta(x^R;q^{-1},t^{-1}) + \sum_{|\nu|<|\eta|} d_{\eta\nu}\, E_\nu(x^R;q^{-1},t^{-1})$$

which are orthogonal with respect to the inner product (1.23). *These polynomials are simultaneous eigenfunctions of the operators* $\hat{h}_i$*, where* $\hat{h}_i$ *denotes the operator* (1.31) *modified by the involution* $\hat{}$*, which is defined by the mappings* $q \mapsto q^{-1}$*,* $t \mapsto t^{-1}$ *and* $x_i \mapsto x_{n+1-i}$.

In Section 2, we examine the various properties of non-symmetric Macdonald polynomials used in subsequent calculations, including raising and lowering operators, and introduce a non-symmetric analogue of Kaneko's kernel [**11**]. We finish the section with a proof of Thm. 1.1. An isomorphism between Hecke algebras is introduced in Section 3, facilitating a proof of Thm. 1.2. Various properties of these non-symmetric ASC polynomials are then described including their normalization and a generating function. We conclude by clarifying their relationship to the non-symmetric analogues of the shifted Macdonald polynomials.

2. Non-symmetric Macdonald polynomials.

In this section we gather together some (old and new) results concerning non-symmetric Macdonald polynomials $E_\eta(x)$ in preparation of the proof of Thm. 1.1, as well as the forthcoming section on the non-symmetric ASC polynomials.

For future reference we note that the operators T_i and ω defined by (1.8) and (1.10) have the properties

$$
\begin{aligned}
T_i^{-1}\, x_{i+1} &= t^{-1} x_i\, T_i & T_i^{-1}\, x_i &= x_{i+1}\, T_i^{-1} + (t^{-1}-1)x_i \\
T_i\, x_i &= t x_{i+1}\, T_i^{-1} & T_i\, x_{i+1} &= x_i\, T_i + (t-1)x_{i+1} \\
\omega\, x_1 &= q x_n \omega & \omega\, x_{i+1} &= x_i\, \omega
\end{aligned} \tag{2.1}
$$

valid for $1 \le i \le n-1$. Also note the following action of T_i on monomials

$$
T_i\, x_i^a x_{i+1}^b = \begin{cases} (1-t)x_i^{a-1}x_{i+1}^{b+1} + \cdots + (1-t)x_i^{b+1}x_{i+1}^{a-1} + x_i^b x_{i+1}^a & a > b \\ t x_i^a x_{i+1}^a & a = b \\ (t-1)x_i^a x_{i+1}^b + \cdots + (t-1)x_i^{b-1}x_{i+1}^{a+1} + t x_i^b x_{i+1}^a & a < b. \end{cases} \tag{2.2}
$$

There exists a variant of the q-Dunkl operator (1.28) which is relevant to the forthcoming discussion. With $\hat{}$ denoting the involution defined in the statement of Corollary 1.3, this operator is defined as

$$
\begin{aligned}
\mathcal{D}_i &:= -q\hat{D}_{n+1-i} \\
&= -q x_i^{-1}\Big(1 - t^{-n+1} T_{i-1}\cdots T_1 \omega^{-1} T_{n-1}\cdots T_i\Big) \\
&= q t^{-2n+i+1} D_i Y_i^{-1} T_i \cdots T_{n-1} T_{n-1} \cdots T_i.
\end{aligned} \tag{2.3}
$$

In obtaining the first equality in (2.3), the facts that

$$
\hat{T}_i = T_{n-i}^{-1} \quad \text{and} \quad \hat{\omega} = \omega^{-1} \tag{2.4}
$$

have been used in applying the operation ˆ to (1.28), while the second equality can be verified by substituting for Y_i^{-1} using (1.15) and for D_i using (1.28) and comparing with the first equality.

Since the D_i commute, it follows from the definition of $\mathcal{D}_i$ that the $\{\mathcal{D}_i\}$ also form a commuting set. Moreover, using (2.4), one can check that the operators $\mathcal{D}_i$ possess the same relations with the generators T_i, ω as do the D_i, namely

$$\begin{aligned} &T_i\mathcal{D}_{i+1} = t\mathcal{D}_iT_i^{-1}, \quad T_i\mathcal{D}_i = \mathcal{D}_{i+1}T_i + (t-1)\mathcal{D}_i, && 1 \le i \le n-1 \\ &[T_i, \mathcal{D}_j] = 0, && j \ne i, i+1 \\ &\mathcal{D}_n\,\omega = q\,\omega\,\mathcal{D}_1, \qquad \mathcal{D}_i\,\omega = \omega\,\mathcal{D}_{i+1} && 1 \le i \le n-1. \end{aligned} \tag{2.5}$$

To conclude the preliminaries, we follow Sahi [**25**] and introduce the generalized arm and leg (co-)lengths for a node $s \in \eta$ via

(2.6)
$$\begin{aligned} &a(s) = \eta_i - j \quad l(s) = \#\{k > i | j \le \eta_k \le \eta_i\} \;+\; \#\{k < i | j \le \eta_k + 1 \le \eta_i\} \\ &a'(s) = j - 1 \quad l'(s) = \#\{k > i | \eta_k > \eta_i\} \;+\; \#\{k < i | \eta_k \ge \eta_i\} \end{aligned}$$

and define the quantities

$$\begin{aligned} d_\eta(q,t) &:= \prod_{s\in\eta}\left(1 - q^{a(s)+1}t^{l(s)+1}\right) && l(\eta) := \sum_{s\in\eta} l(s) \\ d'_\eta(q,t) &:= \prod_{s\in\eta}\left(1 - q^{a(s)+1}t^{l(s)}\right) && l'(\eta) := \sum_{s\in\eta} l'(s) \\ e_\eta(q,t) &:= \prod_{s\in\eta}\left(1 - q^{a'(s)+1}t^{n-l'(s)}\right) && a(\eta) := \sum_{s\in\eta} a(s). \end{aligned} \tag{2.7}$$

The statistics $l(\eta)$, $l'(\eta)$ and $a(\eta)$ generalize the quantity

$$b(\lambda) := \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2} \tag{2.8}$$

from partitions to compositions. From [**25**] these quantities have the following properties

Lemma 2.1. *Let $\Phi\eta := (\eta_2, \ldots, \eta_n, \eta_1 + 1)$. We have*

$$\begin{aligned} \frac{d_{\Phi\eta}(q,t)}{d_\eta(q,t)} &= \frac{e_{\Phi\eta}(q,t)}{e_\eta(q,t)} = 1 - qt^{n+\bar\eta_1}, \quad \frac{d'_{\Phi\eta}(q,t)}{d'_\eta(q,t)} = 1 - qt^{n-1+\bar\eta_1}, \\ &\qquad\qquad\qquad\qquad e_{s_i\eta}(q,t) = e_\eta(q,t), \\ \frac{d_{s_i\eta}(q,t)}{d_\eta(q,t)} &= \frac{1 - t^{\delta_{i,\eta}+1}}{1 - t^{\delta_{i,\eta}}}, \qquad \frac{d'_{s_i\eta}(q,t)}{d'_\eta(q,t)} = \frac{1 - t^{\delta_{i,\eta}}}{1 - t^{\delta_{i,\eta}-1}} \\ &\qquad\qquad\qquad\qquad \text{for } \eta_i > \eta_{i+1}, \quad \delta_{i,\eta} := \bar\eta_i - \bar\eta_{i+1} \\ a(\Phi\eta) &= \eta_1 + a(\eta), \qquad l(\Phi\eta) = l(\eta) + \#\{k > 1 | \eta_k \le \eta_1\} \end{aligned}$$

$$\begin{aligned} l'(\Phi\eta) &= l'(\eta) + n - 1 - \#\{k > 1 | \eta_k \le \eta_1\} \\ a(s_i\eta) &= a(\eta) \qquad l'(s_i\eta) = l'(\eta) \quad l(s_i\eta) = l(\eta) + 1 \quad \text{for } \eta_i > \eta_{i+1}. \end{aligned}$$

A consequence of the first two equations in the final line is that

$$l'(\eta) = l'(\eta^+) = b(\eta^+), \qquad a(\eta) = a(\eta^+) = b((\eta^+)') \tag{2.9}$$

where $(\eta^+)'$ denotes the partition conjugate to η^+.

2.1. Raising Operators and Lowering Operators.

There are two distinct raising operators which have a very simple action on non-symmetric Macdonald polynomials. Define [13, 3]

$$\begin{aligned} \Phi_1 &:= x_n\, \omega, \\ \Phi_2 &:= x_n\, T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1}. \end{aligned} \tag{2.10}$$

A direct calculation reveals that for $i = 1, 2$

$$\begin{aligned} Y_n\, \Phi_i &= q \Phi_i\, Y_1 \\ Y_j\, \Phi_i &= \Phi_i\, Y_{j+1} \qquad\qquad 1 \le j \le n-1 \end{aligned}$$

whence $\Phi_i E_\eta$ is a constant multiple of $E_{\Phi\eta}$, where $\Phi\eta := (\eta_2, \ldots, \eta_n, \eta_1 + 1)$. This constant is determined by looking at the coefficient of $x^{\Phi\eta}$ with the result that

$$\begin{aligned} \Phi_1\, E_\eta &= q^{\eta_1}\, E_{\Phi\eta}, \\ \Phi_2\, E_\eta &= t^{-\#\{i|\eta_i \le \eta_1\}}\, E_{\Phi\eta}. \end{aligned}$$

Remark. These operators are simply related via $\Phi_1 = t^{n-1}\Phi_2\, Y_1$. Of course any function of the operators Y_i multiplied by Φ_1 will be a raising operator for the non-symmetric Macdonald polynomials but these two are in some sense the simplest.

In a similar manner, one can use the q-Dunkl operators (1.28) to construct lowering operators as follows,

$$\begin{aligned} \Psi_1 &:= \omega^{-1}\, D_n, \\ \Psi_2 &:= T_1 T_2 \cdots T_{n-1}\, D_n. \end{aligned} \tag{2.11}$$

Ψ_2 was introduced previously in [3]. These operators intertwine with the Cherednik operators as

$$\begin{aligned} Y_1\, \Psi_i &= q^{-1} \Psi_i\, Y_n \\ Y_j\, \Psi_i &= \Psi_i\, Y_{j-1} \qquad\qquad 2 \le j \le n \end{aligned}$$

and it is seen that

$$\begin{aligned} \Psi_1\, E_\eta &= q^{-\eta_n + 1}(1 - t^{n-1+\bar\eta_n})\, E_{\Psi\eta}, \\ \Psi_2\, E_\eta &= t^{\#\{i|\eta_i < \eta_n\}}(1 - t^{n-1+\bar\eta_n}) E_{\Psi\eta} \end{aligned}$$

where $\Psi\eta := (\eta_n - 1, \eta_1, \ldots, \eta_{n-1})$.

2.2. Kernel.

Let $\widetilde{\ }$ denote the involution on the ring of polynomials with coefficients in $\mathbb{C}(q,t)$, which acts on the the coefficients by sending $q \mapsto q^{-1}$, $t \mapsto t^{-1}$, and extend it to act on operators in the obvious way. Define the kernel

$$\mathcal{K}_A(x;y;q,t) = \sum_\eta q^{a(\eta)} t^{(n-1)|\eta|-l'(\eta)} \frac{d_\eta}{d'_\eta e_\eta} E_\eta(x) \widetilde{E}_\eta(y). \tag{2.12}$$

It follows from (2.7) that this kernel is related to the previously introduced kernel [3]

$$K_A(x;y;q,t) = \sum_\eta \frac{d_\eta}{d'_\eta e_\eta} E_\eta(x) \widetilde{E}_\eta(y) \tag{2.13}$$

((2.13) was denoted by $\mathcal{K}_A$ in [3], but for the present purpose it is desirable to use this notation for (2.12)) by means of

$$\widetilde{\mathcal{K}}_A(x;y;q,t) = K_A(-qy;x;q,t). \tag{2.14}$$

The kernel $\mathcal{K}_A(x;y;q,t)$ satisfies the following properties:

Theorem 2.2.

(a) $\quad (T_i^{\pm 1})^{(x)}\, \mathcal{K}_A(x;y;q,t) = \left(\widetilde{T_i^{\mp 1}}\right)^{(y)} \mathcal{K}_A(x;y;q,t)$

(b) $\quad \Psi_1^{(x)}\, \mathcal{K}_A(x;y;q,t) = \widetilde{\Phi_2}^{(y)}\, \mathcal{K}_A(x;y;q,t)$

(c) $\quad \mathcal{D}_i^{(x)}\, \mathcal{K}_A(x;y;q,t) = y_i\, \mathcal{K}_A(x;y;q,t).$

Proof. The proof of this result follows the same line of reason as in [3, Thm. 5.2], using the facts that

$$x_i = t^{-n+i}\, \widetilde{T_i^{-1}} \cdots \widetilde{T_{n-1}^{-1}}\, \widetilde{\Phi_2}\, \widetilde{T_1} \cdots \widetilde{T_{i-1}} \tag{2.15}$$

$$\mathcal{D}_i = t^{-n+i} T_{i-1}^{-1} \cdots T_1^{-1}\, \Psi_1\, T_{n-1} \cdots T_i. \tag{2.16}$$

□

We recall from [3] that the analogue of property (c) for the kernel $K_A(x;y;q,t)$ is

$$D_i^{(x)} K_A(x;y;q,t) = y_i K_A(x;y;q,t). \tag{2.17}$$

A feature of both property (c) and (2.17) is that the q-Dunkl operator $\mathcal{D}_i$ (resp. D_i) act on the left set of variables *only*. However, by applying the operation $\widetilde{\ }$ and using (2.14), we can form similar identities where they act on the right set of variables, namely:

Corollary 2.3.

$$(\widetilde{\mathcal{D}_i})^{(x)}\, K_A(z;x;q,t) = -q^{-1} z_i\, K_A(z;x;q,t) \tag{2.18}$$

$$(\widetilde{D_i})^{(y)}\, \mathcal{K}_A(x;y;q,t) = -q x_i\, \mathcal{K}_A(x;y;q,t). \tag{2.19}$$

2.3. First isomorphism.

Returning to the proof of Thm. 1.1, we claim that it follows from the subsequent

Proposition 2.4. *Let $\mathcal{R}_{q,t}$ be the subalgebra of the algebra of endomorphism on the polynomial ring $\mathbb{Q}(q,t)[x_1,\dots,x_n]$ generated by the elements $\{T_i,\omega,x_i\}$ with relations given by (1.11)-(1.14), (2.1) The map $\phi:\mathcal{R}_{q^{-1},t^{-1}} \longrightarrow \mathcal{R}_{q,t}$ defined by*

$$\phi(\widetilde{\omega^{-1}}) = T_1\cdots T_{n-1}\,\omega\,T_1^{-1}\cdots T_{n-1}^{-1}, \quad \phi(x_i)=e_i, \quad \phi(\widetilde{T_i^{\pm1}}) = T_i^{\mp1}. \tag{2.20}$$

is an algebra isomorphism.

Proof. Note that a simple consequence of the definition of ϕ given above, is the relation

$$\phi(\widetilde{Y_i^{-1}}) = Y_i. \tag{2.21}$$

The proof that ϕ is indeed an isomorphism follows by a standard calculation. □

Proof of Thm. 1.1. We know that $E_\eta(x;q^{-1},t^{-1})$ is an eigenfunction of $\widetilde{Y_i^{-1}}$. This is shown by utilizing the relations amongst the operators $\{\tilde{Y}_i^{-1}, \tilde{T}_i^{\pm1}, x_i\}$ to move the operators $\widetilde{Y_i^{-1}}$ through the terms in $E_\eta(x;q^{-1},t^{-1})$, until one obtains $\widetilde{Y_i^{-1}}\cdot 1 = t^{-n+1}\cdot 1$. By adopting this viewpoint in the eigenvalue equation (considered as an identity in $\mathcal{R}_{q^{-1},t^{-1}}$)

$$\widetilde{Y_i^{-1}}\,E_\eta(x;q^{-1},t^{-1})\,\cdot\,1 = t^{\bar{\eta}_i}\,E_\eta(x;q^{-1},t^{-1})\,\cdot\,1$$

and applying the map ϕ to both sides it then follows from (2.20), (2.21) that $E_\eta(e;q^{-1},t^{-1})\cdot 1$ is an eigenfunction of $\phi(\widetilde{Y_i^{-1}})=Y_i$, with leading order term x^η and hence must be proportional to $E_\eta(x;q,t)$.

To determine the proportionality constant $\alpha_\eta(q,t)$ say, it follows from the action of T_i given by (2.2) that

$$e_1^{\eta_1}e_2^{\eta_2}\cdots e_n^{\eta_n}\,\cdot\,1 = q^{f(\eta)}\,t^{g(\eta)}\,x^\eta + \sum_{\nu<\eta} b_{\eta\nu}x^\nu$$

where $f(\eta)=\sum_i \binom{\eta_i}{2}$ and

$$\begin{aligned} g(q) \;&=\; \sum_{i=1}^{n}(\eta_i-1)(i-1) + \sum_{i=0}^{\eta_{n-1}}\chi(\eta_n\le i) + \sum_{i=0}^{\eta_{n-2}}\chi(\eta_n\le i)+\chi(\eta_{n-1}\le i) \\ &\quad + \cdots + \sum_{i=0}^{\eta_1}\chi(\eta_n\le i)+\cdots+\chi(\eta_2\le i) \end{aligned} \tag{2.22}$$

where $\chi(P)=1$ if P is true, and zero otherwise. The simplification $g(q)=\sum_i(n-i)\eta_i^+ - \ell(w_\eta)$ then follows from the above expression by induction on $\ell(w_\eta)$. □

3. Al-Salam & Carlitz polynomials.

The isomorphism ϕ introduced in the previous section can be generalized to another isomorphism ψ_a such that $\psi_a(x_i)$ includes not just degree-raising parts, but degree-preserving and lowering parts as well. It will turn out that this isomorphism is precisely what is needed to obtain non-symmetric analogues of the Al-Salam&Carlitz polynomials in the same way as was done for the Hermite case.

As previously mentioned, the symmetric ASC polynomials $V_\lambda^{(a)}$ can be defined via their orthogonality with respect to the inner product (1.20). We remark that under this inner product we have the important result that the adjoint operators of $T_i^{\pm 1}$, ω are given by

$$(T_i^{\pm 1})^* = T_i^{\pm 1}, \qquad (\omega^{-1})^* = \frac{t^{n-1}}{aq}\,\omega\,(x_1-q)(x_1-aq). \tag{3.1}$$

The ASC polynomials $V_\lambda^{(a)}$ can equivalently be defined by means of the generating function [1]

$$\prod_{i=1}^{n}\frac{1}{\rho_a(t^{-(n-1)}x_i;q)}\,{}_0\psi_0(x;y;q,t) = \sum_\lambda \frac{(-1)^{|\lambda|}q^{b(\lambda')}V_\lambda^{(a)}(y;q,t)P_\lambda(x;q,t)}{d'_\lambda(q,t)P_\lambda(1,t,\ldots,t^{n-1};q,t)}.$$

Here, $\rho_a(x) := (x;q)_\infty(ax;q)_\infty$, $b(\lambda)$ is defined by (2.8) and

(3.2)

$$P_\lambda(1,t,\ldots,t^{n-1};q,t) = t^{l(\lambda)}\prod_{s\in\lambda}\frac{(1-q^{a'(s)}t^{n-l'(s)})}{(1-q^{a(s)}t^{l(s)+1})}$$

$${}_0\psi_0(x;y;q,t) := \sum_\lambda \frac{(-1)^{|\lambda|}q^{b(\lambda)}}{d'_\lambda(q,t)P_\lambda(1,t,\ldots,t^{n-1};q,t)}P_\lambda(x;q,t)P_\lambda(y;q,t).$$

This latter kernel was previously introduced by Kaneko [11] in connection with hypergeometric solutions of systems of q-difference equations.

Similarly the ASC polynomials $U_\kappa^{(a)}$ can be defined by the generating function [1]

(3.3)

$$\rho_a(x_1;q)\cdots\rho_a(x_n;q)\,{}_0\mathcal{F}_0(x;y;q,t) = \sum_\kappa \frac{t^{b(\kappa)}U_\kappa^{(a)}(y;q,t)P_\kappa(x;q,t)}{d'_\kappa(q,t)P_\kappa(1,t,\ldots,t^{n-1};q,t)}$$

where the hypergeometric function ${}_0\mathcal{F}_0$ is defined by

$${}_0\mathcal{F}_0(x;y;q,t) := \sum_\kappa \frac{t^{b(\kappa)}}{d'_\kappa(q,t)P(1,t,\ldots,t^{n-1};q,t)}P_\kappa(x;q,t)P_\kappa(y;q,t). \tag{3.4}$$

3.1. Second isomorphism.

Consider the involution $\hat{}$ on polynomials and operators defined in the statement of Corollary 1.3. The operator E_i introduced in (1.29) has its origins in this involution, namely,

$$E_i := (\hat{D}_{n+1-i})^* := \left(-\frac{1}{q}\mathcal{D}_i\right)^* . \tag{3.5}$$

The form (1.29) follows from (3.5) by making use of the adjoint formulae (3.1). The relations between the operators E_i and the operators $\{D_i, T_i, \omega\}$, can be derived using (3.5). Thus, for example, application of the adjoint operation * to the relations involving $\mathcal{D}_i$, T_i gives, in place of the first relation in (2.5),

$$T_i^{-1}\, E_i\, T_i^{-1} = t^{-1} E_{i+1}. \tag{3.6}$$

Now consider the following mapping $\psi_a : \{\widetilde{\omega}^{-1}, \widetilde{T}_i, x_i, \widetilde{\mathcal{D}}_i\} \longrightarrow \{\omega, T_i, x_i, d_i\}$ where each set of operators defines a certain algebra of endomorphisms on the ring $\mathbb{Q}(q,t)[x_1, \ldots, x_n]$, defined by

$$\begin{aligned} (3.7)\quad \psi_a(x_i) &= E_i, \\ \psi_a(\widetilde{\omega}^{-1}) &= T_1 \cdots T_{n-1} \left(Y_n + (1+a)t^{1-n} D_n + a t^{2-2n} D_n Y_n D_n\right), \\ \psi_a(\widetilde{T}_i^{-1}) &= T_i, \\ \psi_a(\widetilde{\mathcal{D}}_i) &= -a t^{n+1-2i} T_{i-1} \cdots T_1 T_1 \cdots T_{i-1}\, E_i^*\, T_i^{-1} \cdots T_{n-1}^{-1} T_{n-1}^{-1} \cdots T_i^{-1}. \end{aligned}$$

Then Theorem 1.2 will follow from:

Proposition 3.1. *The map ψ_a is an algebra isomorphism.*

Proof. The proof of this result consists of checking that the operators $\psi_a(u)$ given in (3.7) satisfy the same relations as the original operators u, given by (1.11)-(1.14), (2.1) and (2.5), (after application of the involution $\widetilde{\ }$). For example, the first formula in (2.1), after application of the involution $\widetilde{\ }$, reads

$$\tilde{T}_i^{-1} x_{i+1} = t x_i \tilde{T}_i.$$

Now applying the mapping ψ_a gives

$$T_i E_{i+1} = t E_i T_i^{-1}.$$

But this is equivalent to (3.6) so the algebra is indeed preserved. The calculations involved in checking the other relations are typically more involved; however they are similar to those undertaken in [**3**], and so for brevity will be omitted. □

As with the relationship between Prop. 2.4 and the proof of Thm. 1.1 we are in a position to complete the:

Proof of Thm. 1.2. From Thm. 1.1, and the definition (1.29) of the operators E_i it follows that $E_\eta^{(V)}$ has leading term $E_\eta(x;q,t)$. In addition, it follows from (3.7) that

$$\psi_a(\widetilde{Y_i^{-1}}) = Y_i + (1+a)t^{1-n}D_i + at^{2-2n}D_i\,Y_i^{-1}\,D_i \tag{3.8}$$

and from Prop. 3.1, that these are eigenoperators for the non-symmetric ASC polynomials defined by (1.30). The corresponding eigenvalue is simply $t^{\bar\eta_i}$. By writing these operators out explicitly, it is seen that they are self-adjoint w.r.t. the inner product (1.20). Hence by standard arguments, the polynomials (1.30) are orthogonal w.r.t. (1.20). ☐

3.2. Normalization.

The images of the raising and lowering operators (2.10), (2.11) (after application of $\widetilde{\ }$) under the map ψ_a are guaranteed, by virtue of Prop. 3.1, to be raising and lowering operators for the polynomials $E_\eta^{(V)}(x)$.

In particular, using (2.16) and (3.7) we see that

$$\psi_a(\widetilde{\Psi_1}) = aq^{-1}t^{1-n}\,\Psi_1$$

so that Ψ_1 remains a raising operator for the polynomials $E_\eta^{(V)}$. By examination of the leading terms, we must have

$$\Psi_1\,E_\eta^{(V)} = q^{\eta_n+1}\frac{d'_\eta}{d'_{\Psi\eta}}\,E_{\Psi\eta}^{(V)}. \tag{3.9}$$

Also, use of (2.15) and (3.7) gives

$$\psi_a(\widetilde{\Phi_2}) = -q^{-1}\,\Psi_1^*$$

so that Ψ_1^* is a raising operator for $E_\eta^{(V)}$. Indeed,

$$\Psi_1^*\,E_\eta^{(V)} = a^{-1}t^{n-1}q^{\eta_1+1}\,E_{\Phi\eta}^{(V)}. \tag{3.10}$$

By an argument similar to that used in [4, Prop. 3.6] it follows from (3.9) and (3.10) that

$$\left\langle E_{\Phi\eta}^{(V)}, E_{\Phi\eta}^{(V)}\right\rangle^{(V)} = at^{1-n}q^{-2\eta_1-1}\frac{d'_{\Phi\eta}}{d'_\eta}\left\langle E_\eta^{(V)}, E_\eta^{(V)}\right\rangle^{(V)}. \tag{3.11}$$

Also, we have

$$\left\langle E_{s_i\eta}^{(V)}, E_{s_i\eta}^{(V)}\right\rangle^{(V)} = \frac{(1-t^{\delta_{i\eta}-1})(1-t^{\delta_{i\eta}+1})}{t(1-t^{\delta_{i\eta}})^2}\left\langle E_\eta^{(V)}, E_\eta^{(V)}\right\rangle^{(V)}. \tag{3.12}$$

The solution of the recurrence relations (3.11), (3.12) gives:

Proposition 3.2.

(3.13)

$$\mathcal{N}_\eta^{(V)} := \left\langle E_\eta^{(V)}, E_\eta^{(V)}\right\rangle^{(V)} = \left(aq^{-1}t^{2-2n}\right)^{|\eta|}\,q^{-2a(\eta)}t^{l(\eta)+l'(\eta)}\frac{d'_\eta\,e_\eta}{d_\eta}\,\mathcal{N}_0^{(V)}$$

where for $t = q^k$, [1]

$$\mathcal{N}_0^{(V)} = (1-q)^n a^{kn(n-1)/2} t^{-2k\binom{n}{3} - k\binom{n}{2}} \prod_{l=1}^{n} \frac{(q;q)_{kl}}{(q;q)_k}.$$

By using the formulas (1.32), (1.26) and (1.27) we see that the norm $\mathcal{N}_\eta^{(U)}$ of the non-symmetric ASC polynomials $E_\eta^{(U)}$ with respect to the inner product (1.23) is given by simply replacing q, t by q^{-1}, t^{-1} in (3.13). Use of (2.7) then gives:

Corollary 3.3.

$$\mathcal{N}_\eta^{(U)} := \left\langle E_\eta^{(U)}, E_\eta^{(U)} \right\rangle^{(U)} = \left(at^{n-1}\right)^{|\eta|} q^{a(\eta)} t^{-l(\eta)} \frac{d'_\eta\, e_\eta}{d_\eta} \mathcal{N}_0^{(U)} \tag{3.14}$$

where for $t = q^k$, [1]

$$\mathcal{N}_0^{(U)} = (1-q)^n (-a)^{kn(n-1)/2} t^{k\binom{n}{3} - \frac{k-1}{2}\binom{n}{2}} \prod_{l=1}^{n} \frac{(q;q)_{kl}}{(q;q)_k}.$$

3.3. Generating function.

The raising operator expression (1.30) facilitates the derivation of the generating function for the non-symmetric ASC polynomials. Also required will be the q-symmetrization of (2.12).

Proposition 3.4. *Let* [18] $U^+ = \sum_\sigma T_\sigma$ *where* $T_\sigma := T_{i_1} \cdots T_{i_p}$ *for a reduced word decomposition* $\sigma = s_{i_1} \cdots s_{i_p}$. *We have*

$$(U^+)^{(x)}\, \mathcal{K}_A(x;y;q,t) = [n]_t!\, {}_0\psi_0(x; -t^{n-1}y; q,t) \tag{3.15}$$

where ${}_0\psi_0$ *is defined by* (3.2).

Proof. We remark that this is the analogue of the result [3, Prop. 5.4]

$$(U^+)^{(x)}\, K_A(x;y;q,t) = [n]_t!\, {}_0F_0(x;y;q,t). \tag{3.16}$$

In fact in our proof of (3.15) we will use the formula

$$U^+ E_\eta(x) = [n]_t! t^{l(\eta)} \frac{e_\eta}{P_\lambda(t^\delta) d_\eta} P_\lambda(x), \qquad \lambda = \eta^+ \tag{3.17}$$

which was deduced [3, eqs. (5.8)&(5.18)] as a corollary of (3.16). Thus we apply U^+ to (2.12) and use (3.17) to compute its action. Simplifying the result using the first equation in (2.9) and the formula [18]

$$P_\lambda(y) = \sum_{\eta:\eta^+=\lambda} \frac{d'_\lambda}{d'_\eta} E_\eta(y), \tag{3.18}$$

the result then follows. □

Consider now the generating function

$$F_1(y;z) = \sum_{\nu} A_\nu E^{(V)}_\nu(y)\, \widetilde{E}_\nu(z)$$

where

$$A_\nu = (a/q)^{|\nu|}\, \frac{\mathcal{N}^{(V)}_0}{\alpha_\nu(q,t)\mathcal{N}^{(V)}_\nu} = q^{a(\nu)} t^{(n-1)|\nu|-l'(\nu)}\, \frac{d_\nu}{d'_\nu e_\nu}. \tag{3.19}$$

Here we have used the fact that $l(\eta) = l(\eta^+) + \ell(w_\eta)$ to rewrite $\alpha_\eta(q,t)$ as defined by (1.18) as

$$\alpha_\eta(q,t) = q^{a(\eta)}\, t^{(n-1)|\eta|-l(\eta)}.$$

Clearly

$$\Big\langle F_1(y;z), E^{(V)}_\eta(y)\Big\rangle^{(V)}_y = (a/q)^{|\eta|}\, \frac{\mathcal{N}^{(V)}_0}{\alpha_\eta(q,t)}\, \widetilde{E}_\eta(z).$$

Next note the integration formula

$$\begin{aligned}
\langle \mathcal{K}_A(y;z), 1\rangle^{(V)}_y &= \frac{1}{[n]_t!}\, \big\langle U^+_y\, \mathcal{K}_A(y;z), 1\big\rangle^{(V)}_y \\
&= \big\langle {}_0\psi_0(y; -t^{n-1}z), 1\big\rangle^{(V)}_y = \mathcal{N}^{(V)}_0 \prod_{i=1}^n \rho_a(-z_i) \qquad (3.20)
\end{aligned}$$

which follows from the symmetrization formula (3.15), the fact that U^+_y is self adjoint w.r.t. $\langle\, ,\,\rangle^{(V)}_y$ and an integral formula for the kernel ${}_0\psi_0(y;z)$ given in [**1**, Prop 4.8], and consider the generating function

$$F_2(y;z) = \prod_{i=1}^n \frac{1}{\rho_a(-z_i)}\, \mathcal{K}(y;z).$$

We have

$$\begin{aligned}
\Big\langle F_2(y;z), E^{(V)}_\eta(y)\Big\rangle^{(V)}_y &= \frac{(-a)^{|\eta|}}{\alpha_\eta(q,t)} \prod_i \frac{1}{\rho_a(-z_i)}\, \Big\langle \mathcal{K}(y;z), \widetilde{E}_\eta(E^{(y)})\Big\rangle^{(V)}_y \\
&= \frac{(a/q)^{|\eta|}}{\alpha_\eta(q,t)} \prod_i \frac{1}{\rho_a(-z_i)}\, \Big\langle \widetilde{E}_\eta(\mathcal{D}^{(y)})\, \mathcal{K}(y;z), 1\Big\rangle^{(V)}_y \\
&= \frac{(a/q)^{|\eta|}}{\alpha_\eta(q,t)} \prod_i \frac{1}{\rho_a(-z_i)} \widetilde{E}_\eta(z)\, \langle \mathcal{K}(y;z), 1\rangle^{(V)}_y \\
&= (a/q)^{|\eta|} \frac{\mathcal{N}^{(V)}_0}{\alpha_\eta(q,t)} \widetilde{E}_\eta(z).
\end{aligned}$$

In the above chain of equalities, we have used (1.30), (3.5), the kernel property Thm. 2.2 (c) and (3.20) respectively. The non-symmetric ASC polynomials $E^{(V)}_\eta(y)$ are a complete basis for polynomials in y and hence from above we have $F_1 = F_2$. That is, we have the generating function for non-symmetric ASC polynomials $E_\nu^{(V)}$.

Proposition 3.5. *With A_ν given by (3.19)*

$$\prod_{i=1}^{n} \frac{1}{\rho_a(-z_i)}\, \mathcal{K}_A(y;z) = \sum_\nu A_\nu E_\nu^{(V)}(y)\, \widetilde{E}_\nu(z). \tag{3.21}$$

We remark that this generating function could also be derived in a manner similar to that used in the symmetric case [**1**], namely by applying the operator $(\widetilde{Y_i^{-1}})^{(z)}$ to both sides of (3.21) and deducing that $E_\eta^{(V)}(y)$ is an eigenfunction of

$$h_i = \psi_a(\widetilde{Y_i^{-1}}) = Y_i\, T_{i-1} \cdots T_1\, (1+\mathcal{D}_1)\, (1+a\mathcal{D}_1)\, T_1^{-1} \cdots T_{i-1}^{-1} \tag{3.22}$$

with leading term $E_\eta(y)$ (some manipulation using (2.5) and (2.3) casts this into the form given in (1.31)). Note also that by applying the operation $\hat{}$ with the respect to the y-variables in (3.21) and using the formula (2.14) as well as

$$\frac{1}{\rho_a(x;q)}\bigg|_{q\mapsto q^{-1}} = \rho_a(qx;q),$$

(see e.g. [**1**]) we deduce the generating function formula for the polynomials $E_\nu^{(U)}$.

Corollary 3.6.

$$\prod_{i=1}^{n} \rho_a(z_i)\, K_A(z;y^R;q,t) = \sum_\nu \frac{d_\nu}{d'_\nu e_\nu} E_\nu^{(U)}(y) E_\nu(z). \tag{3.23}$$

The generating function formulas in turn imply a further class of operator formulas relating the ASC polynomials and the non-symmetric Jack polynomials (c.f. [**1**, eqs. (3.9)&(3.10)]).

Corollary 3.7. *We have*

$$E_\eta^{(V)}(y) \;=\; \prod_{i=1}^{n} \frac{1}{\rho_a(-\mathcal{D}_i^{(y)})}\, E_\eta(y) \tag{3.24}$$

$$E_\eta^{(U)}(y) \;=\; \prod_{i=1}^{n} \rho_a\Big(-q\widetilde{\mathcal{D}_i^{(y)}}\Big)\, \widetilde{E}_\eta(y^R). \tag{3.25}$$

Proof. The first identity follows from (3.21) by using Thm. 2.2 (c) and comparing coefficients of $\widetilde{E}_\eta(z)$, while the second identity follows similarly from (3.23) and (2.18). □

As further applications of the generating functions we will present some evaluation formulas for $E_\eta^{(V)}$ at the special points $t^{\bar{\delta}-n+1}$ and $at^{\bar{\delta}-n+1}$, where $t^{\bar{\delta}} := (1,t,t^2,\ldots,t^{n-1})$.

Proposition 3.8. *We have*

$$E_\eta^{(V)}(t^{\bar\delta-n+1}) = (-a)^{|\eta|}q^{-a(\eta)}t^{l'(\eta)-(n-1)|\eta|}E_\eta(t^{\bar\delta}) \tag{3.26}$$

$$E_\eta^{(V)}(at^{\bar\delta-n+1}) = (-1)^{|\eta|}q^{-a(\eta)}t^{l'(\eta)-(n-1)|\eta|}E_\eta(t^{\bar\delta}) \tag{3.27}$$

where

$$E_\eta(t^{\bar\delta}) = t^{l(\eta)}\frac{e_\eta}{d_\eta}. \tag{3.28}$$

Proof. The formula (3.28) is a special case of a result of Cherednik [**7**] (see also [**20**]). For the derivation of (3.26) and (3.27) we follow the strategy of the proof of the analogous result in the symmetric case [**1**, Prop. 4.3]. First, note from the definition (1.8) that in general

$$T_i f(t^{\bar\delta}) = tf(t^{\bar\delta}),$$

and so

$$\left(U^+ f\right)(t^{\bar\delta}) = (U^+ 1)f(t^{\bar\delta}) = [n]_t!f(t^{\bar\delta}).$$

Use of this latter formula in (3.15) with $y = t^{\bar\delta}$ gives

$$\mathcal{K}_A(t^{\bar\delta}; z; q, t) = {}_0\psi_0(t^{\bar\delta}; -t^{n-1}z; q, t) = \prod_{i=1}^{n}(-t^{n-1}z_i; q)_\infty, \tag{3.29}$$

and similarly, from (3.16)

$$K_A(t^{\bar\delta}; z; q, t) = {}_0F_0(t^{\bar\delta}; z; q, t) = \frac{1}{\prod_{i=1}^n (z_i; q)_\infty}, \tag{3.30}$$

where the final equalities in (3.29) and (3.30) are known results [**17**, **12**]. Now set $y = t^{\bar\delta-n+1}$ in the generating function (3.15). Use of (3.29) with z replaced by $t^{-n+1}z$, and then use of (3.30) allows the l.h.s. of the resulting expression to be written

$$\frac{1}{\prod_{i=1}^n(-az_i; q)_\infty} = K_A(t^{\bar\delta}; -az; q, t) = \sum_\eta \frac{(-a)^{|\eta|}d_\eta}{d'_\eta e_\eta}E_\eta(t^{\bar\delta})\widetilde{E}_\eta(z).$$

Equating with $\widetilde{E}_\eta(z)$ on the r.h.s. of the resulting expression gives (3.26). The formula (3.27) follows similarly, by substituting $y = at^{\bar\delta-n+1}$ in (3.15). □

3.4. Relationship to the symmetric ASC polynomials.

The non-symmetric ASC polynomials are related to the corresponding symmetric ASC polynomials in an analogous way to the relationship (3.17) between the non-symmetric and symmetric Macdonald polynomials.

Proposition 3.9. *Let*

$$a_\eta(q, t) = [n]_t!t^{\ell(\eta)}\frac{e_\eta}{P_{\eta^+}(t^{\bar\delta})d_\eta}.$$

We have

$$U^+E_\eta^{(V)}(y) \;=\; a_\eta(q,t)V_{\eta^+}^{(a)}(y;q,t) \tag{3.31}$$

$$U^+E_\eta^{(U)}(y) \;=\; a_\eta(q,t)U_{\eta^+}^{(a)}(y;q,t). \tag{3.32}$$

Proof. Consider the action of the U^+ operator on (3.24) and (3.25). From the first three equations of (2.5) one can check that T_i commutes with any symmetric function of the $\mathcal{D}_i$. Thus the action of U^+ can be commuted to act to the right of $\prod_i \rho_a(-\frac{1}{q}\tilde{\mathcal{D}}_i)$ and $1/\prod_i \rho_a(-\mathcal{D}_i)$. Use of (3.17) then gives

$$\begin{aligned} U^+E_\eta^{(V)}(y) &= a_\eta(q,t)\frac{1}{\prod_i \rho_a(-\mathcal{D}_i)}P_{\eta^+}(y) = a_\eta(q,t)\frac{1}{\prod_i \rho_a(q\widetilde{D}_i)}P_{\eta^+}(y) \\ U^+E_\eta^{(U)}(y) &= a_\eta(q,t)\prod_i \rho_a(-q\widetilde{\mathcal{D}}_i)\,P_{\eta^+}(y) = a_\eta(q,t)\prod_i \rho_a(D_i)\,P_{\eta^+}(y), \end{aligned}$$

where in obtaining the first equality in the second formula we have used the fact that $\widetilde{P}_\eta(y^R) = P_\eta(y)$, while the second equalities in both formulas make use of (2.3) and the fact that P_{η^+} is a symmetric function. But the resulting operator formulas are precisely representations obtained in [**1**, Eq. (3.9)&(3.10)] for the symmetric ASC polynomials. □

We can also relate the eigenoperators h_i for the non-symmetric ASC polynomials $E_\eta^{(V)}$ to the eigenoperator [**1**, Eq. (3.28)]

$$\begin{aligned} \mathcal{H} = t^{1-n}\sum_{i=1}^n Y_i^{-1} - (1+a)\sum_{i=1}^n t^{1-i}D_i\,Y_i^{-1} + a\sum_{i=1}^n t^{1-i}D_i^2\,Y_i^{-1} \\ + a(1-t^{-1})\sum_{1\le i<j\le n} t^{1-i}D_j\,D_i\,Y_i^{-1} \end{aligned} \tag{3.33}$$

for the symmetric ASC polynomials $U_\lambda^{(a)}$.

Proposition 3.10. *Let h_i be given by (1.31) and $\mathcal{H}$ by (3.33). When acting on symmetric functions*

$$\sum_{i=1}^n h_i = t^{1-n}\widetilde{\mathcal{H}}.$$

Proof. From Theorem 1.2, by summing over i in (1.31) we have

$$\sum_{i=1}^n h_i\,E^{(V)}(x;q,t) = t^{1-n}e(\eta^+)E^{(V)}(x;q,t),$$

where $e(\eta^+) = \sum_{i=1}^n t^{\bar{\eta}_i} = \sum_{i=1}^n q^{\eta_i^+}t^{n-i}$. We would next like to apply the operator U^+ to both sides of this eigenvalue equation. For this purpose we require the fact that T_i commutes with $\sum_{i=1}^n h_i$ (this follows from (1.16), and the fact that these same equations apply with the Y_i replaced by D_i). Thus, making use of (3.31), this operation gives

$$\sum_{i=1}^{n} h_i\, V^{(a)}_{\eta^+}(x;q,t) = t^{1-n} e(\eta^+) V^{(a)}_{\eta^+}(x;q,t).$$

But from [1] we know that this same eigenvalue equation applies with $\sum_{i=1}^{n} h_i$ replaced by $t^{1-n}\tilde{\mathcal{H}}$. The result now follows from the fact that $\{V^{(a)}_{\eta^+}\}$ are a basis for symmetric functions.

We remark that an alternative proof is to establish directly that when acting on symmetric functions

$$\sum_{i=1}^{n} \tilde{Y}_i^{-1} = \sum_{i=1}^{n} Y_i \tag{3.34}$$

$$-\sum_{i=1}^{n} t^{-1+i}\tilde{D}_i\tilde{Y}_i^{-1} = \sum_{i=1}^{n} D_i \tag{3.35}$$

$$\begin{aligned}\sum_{i=1}^{n} t^{-1+i}\tilde{D}_i^2\tilde{Y}_i^{-1} + (1-t)\sum_{1\le i<j\le n} t^{-1+i}\tilde{D}_j\tilde{D}_i\tilde{Y}_i^{-1}&\\ = t^{1-n}\sum_{i=1}^{n} D_iY_i^{-1}D_i.&\end{aligned} \tag{3.36}$$

□

3.5. Non-symmetric shifted Macdonald polynomials.

In [1] it was observed that the symmetric ASC polynomials $V^{(a)}_{\lambda}(x)$ coincide (up to a factor and change of variables) with the shifted Macdonald polynomials when $a=0$. We show now that this behaviour carries over to the non-symmetric case.

Following Knop [14], Knop and Sahi [15] and Sahi [26], the non-symmetric shifted Macdonald polynomials $G_\eta(z)$ are defined, in the notation of [14], as the unique polynomial with expansion

$$G_\eta(z;q,t) = \widetilde{E}_\eta(z) + \sum_{|\nu|<|\eta|} b_{\eta\nu}\widetilde{E}_\nu(z)$$

which vanishes at the points $z=t^{\bar{\xi}}$ for all compositions $\xi\neq\eta$ such that $|\xi|\le|\eta|$. Here $t^{\bar{\xi}}$ is given by (1.5). Equivalently [13, 22] they can be defined as eigenfunctions of the "inhomogeneous" Cherednik operators

$$\Xi_i = \widetilde{Y}_i + \widetilde{D}_i$$

where the operators are defined with the variables z_i. For such polynomials, Knop [14] defined a raising operator $\Phi_K = (z_n - t^{1-n})\omega^{-1}$ with a simple action on $G_\eta(z;q,t)$. It is easily seen that

$$\lim_{a\to 0}\frac{-a}{q}\Psi_1^* = \widetilde{\Phi}_K\Big|_{z_i=t^{n-1}x_i}, \qquad \lim_{a\to 0} h_i = \widetilde{\Xi}_i\Big|_{z_i=t^{n-1}x_i}$$

which immediately implies the sought relationship between G_η and $E_\eta^{(V)}$.

Proposition 3.11.

$$E_\eta^{(V)}(x;q,t)\Big|_{a=0} = t^{-(n-1)|\eta|}G_\eta(t^{n-1}x;q^{-1},t^{-1}) \tag{3.37}$$

or equivalently

$$E_\eta^{(U)}(x;q,t)\Big|_{a=0} = t^{(n-1)|\eta|}G_\eta(t^{1-n}x;q,t). \tag{3.38}$$

One immediate application of (3.37) is the evaluation of $G_\eta(0;q,t)$, which follows from (3.27). This is a special case of a result of Sahi [**26**, Th. 1.1], in which an evaluation formula is given for $G_\eta(\alpha t^{\bar\delta};q,t)$, for a general scalar α. In fact use of (3.37) also allows this more general evaluation formula to be deduced.

Proposition 3.12. *With* $(\alpha)_\lambda^{(q,t)} := \prod_{s\in\lambda}(t^{l'(s)} - q^{a'(s)}\alpha)$ *we have*

$$G_\eta(t^{-\bar\delta}\alpha;q,t) = \alpha^{|\eta|}(1/\alpha)_{\eta^+}^{(q,t)}t^{-(n-1)|\eta|}\frac{e_\eta}{d_\eta}.$$

Proof. Choosing $a=0$ and $y=t^{n-1-\bar\delta}\alpha$ in (3.23), and using (3.30) and (3.38), we see that

$$\sum_\eta \alpha^{-|\eta|}t^{(n-1)|\eta|}\frac{d_\eta}{d'_\eta e_\eta}G_\eta(t^{-\bar\delta}\alpha;q,t)E_\eta(z) = \prod_{i=1}^n\frac{(z_i/\alpha;q)_\infty}{(z_i;q)_\infty}.$$

But we know that [**17**, **12**]

$$\prod_{i=1}^n\frac{(z_i/\alpha;q)_\infty}{(z_i;q)_\infty} = \sum_\lambda\frac{(1/\alpha)_\lambda^{(q,t)}}{d'_\lambda}P_\lambda(z;q,t) = \sum_\eta\frac{(1/\alpha)_{\eta^+}^{(q,t)}}{d'_\eta}E_\eta(z).$$

The result follows by equating coefficients of $E_\eta(z)$. □

3.6. q-binomial coefficients.

Sahi [**26**] uses the polynomials G_η to introduce non-symmetric q-binomial coefficients $\begin{bmatrix}\eta\\ \nu\end{bmatrix}_{q,t}$ according to

$$\begin{bmatrix}\eta\\ \nu\end{bmatrix}_{q,t} := \frac{G_\nu(t^{\bar\eta})}{G_\nu(t^{\bar\nu})} \tag{3.39}$$

($\bar\eta_i$ is defined by (1.5)). Our generating function characterization of the ASC polynomials, and thus by Proposition 3.11 of the polynomials G_η, makes it natural to extend Lassalle's [**16**] definition of the symmetric q-binomial coefficients to the non-symmetric case by defining the non-symmetric q-binomial coefficients $\binom{\eta}{\nu}_{q,t}$ according to the generating function formula

$$\widetilde{E}_\nu(x)\prod_{i=1}^{n}\frac{1}{(x_i;q)_\infty}=\sum_\eta\binom{\eta}{\nu}_{q,t}t^{l(\eta)-l(\nu)}\frac{d'_\nu}{d'_\eta}\widetilde{E}_\eta(x). \tag{3.40}$$

We can then use the generating function (3.15) to relate these binomial coefficients to the polynomials G_η.

Proposition 3.13. *With $\binom{\eta}{\nu}_{q,t}$ defined by (3.40), we have*

$$\frac{G_\eta(x)}{G_\eta(0)}=\sum_\nu\binom{\eta}{\nu}_{q^{-1},t^{-1}}\frac{\widetilde{E}_\nu(x)}{G_\nu(0)}. \tag{3.41}$$

Proof. Multiply both sides of (3.40) by $q^{a(\nu)}t^{(n-1)|\nu|-l'(\nu)}\frac{d_\nu}{d'_\nu e_\nu}E_\nu(y)$ and sum over ν, rewriting the l.h.s. according to (3.15). Now equate coefficients of $\widetilde{E}_\nu(x)$ on both sides. The result then follows upon using (3.28) and (3.37). ☐

Since (3.41) is a formula satisfied by the non-symmetric q-binomial coefficients of Sahi [**26**, Cor. 1.3], and this formula suffices to implicitly define these coefficients, we have that

$$\binom{\eta}{\nu}_{q,t}=\begin{bmatrix}\eta\\ \nu\end{bmatrix}_{q,t}. \tag{3.42}$$

Finally, let us present some formulas relating the coefficients $\binom{\eta}{\nu}_{q,t}$ to their symmetric counterparts $\binom{\kappa}{\mu}_{q,t}$, which can be characterized by either of the formulas [**16**, **21**]

$$P_\mu(x;q,t)\prod_{i=1}^{n}\frac{1}{(x_i;q)_\infty} = \sum_\lambda\binom{\lambda}{\mu}_{q,t}t^{b(\lambda)-b(\mu)}\frac{d'_\mu}{d'_\lambda}P_\lambda(x;q,t), \tag{3.43}$$

$$\frac{P^*_\lambda(y;q^{-1},t^{-1})}{P^*_\lambda(0;q^{-1},t^{-1})} = \sum_\mu\binom{\lambda}{\mu}_{q,t}\frac{P_\mu(yt^{\bar{\delta}};q,t)}{P^*_\lambda(0;q^{-1},t^{-1})}. \tag{3.44}$$

Here P^*_λ is the shifted Macdonald polynomial, which is related to the symmetric ASC polynomial $V^{(0)}_\lambda$ by [**1**, Prop. 4.4]

$$P^*_\lambda(yt^{-\bar{\delta}+n-1};q^{-1},t^{-1})=t^{(n-1)|\lambda|}V^{(0)}_\lambda(y;q,t). \tag{3.45}$$

Proposition 3.14. *With $\eta^+=\kappa$, $\nu^+=\mu$,*

$$\sum_{\nu:\nu^+=\mu}\binom{\eta}{\nu}_{q,t} = \binom{\kappa}{\mu}_{q,t}, \tag{3.46}$$

$$\frac{d'_\kappa}{d'_\mu}\frac{P_\kappa(t^{\bar\delta})}{P_\mu(t^{\bar\delta})}\frac{d'_\nu}{E_\nu(t^{\bar\delta})}\sum_{\eta:\eta^+=\kappa}\binom{\eta}{\nu}_{q,t}\frac{E_\eta(t^{\bar\delta})}{d'_\eta} = \binom{\kappa}{\mu}_{q,t}. \tag{3.47}$$

Proof. The proof follows the strategy given in [1] for the proof of the corresponding results in the $q = t^\alpha$, $q \to 1$ limit (binomial coefficients associated with non-symmetric Jack polynomials). For (3.46) we apply the U^+ operator to (3.41), making use of (3.17) and (3.31). Use of the fact that

$$\frac{a_\nu}{E_\nu^{(V)}(0)} = \frac{[n]_t!}{V_{\eta^+}^{(0)}(0;q,t)}$$

and (3.45) then gives

$$\frac{P^*_\lambda(xt^{-\bar\delta};q^{-1},t^{-1})}{P^*_\lambda(0;q^{-1},t^{-1})} = \sum_\nu \binom{\eta}{\nu}_{q,t}\frac{P_{\nu^+}(x;q,t)}{P^*_{\nu^+}(0;q^{-1},t^{-1})}.$$

Comparison with (3.44) implies (3.46) The identity (3.47) follows similarly, by applying U^+ to (3.40) and comparing with (3.43). □

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