Pacific Journal of Mathematics

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Volume 194 No. 1 May 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 194, No. 1, 2000

CHARACTER ANALOGUES OF DEDEKIND SUMS AND TRANSFORMATIONS OF ANALYTIC EISENSTEIN SERIES

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In this paper we give a transformation formula for an analytic generalized Eisenstein series. In the same way that the ordinary Dedekind sums arise in the transformation formula for the logarithm of the Dedekind eta-function, similar sums arise in the formula for the following generalized Eisenstein series.

1. Introduction.

We put $e(z) = e^{2\pi iz}$ and let $r := (r_1, r_2)$ and $h := (h_1, h_2)$. Throughout this paper χ denotes a primitive character of modulus k. We extend the definition of χ by setting $\chi(x) = 0$, if $x \notin \mathbb{Z}$. For $\sigma = \mathrm{Re}(s) > 2$ and $\mathrm{Im}(z) > 0$, define

$$G(z,s;\chi;r,h) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\chi(m)\overline{\chi}(n)e((mh_1+nh_2)/k)}{((m+r_1)z+n+r_2)^s}, \tag{1.1}$$

where the dash $'$ means that the possible pair $m = -r_1$, $n = -r_2$ is excluded from the summation. Unless otherwise stated, we use the branch of the argument defined by $-\pi \le \arg z < \pi$. We use the modular transformation $V(z) = Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$ and $ad-bc = 1$. As applications of the main theorem we derive a reciprocity law for the new analogue of the Dedekind sum and give proofs of several series relations.

The definition of the analytic Eisenstein series in (1.1) and the methods presented in the sequel are suggested by [**2**] and [**4**].

2. The function $G(z,s;\chi;r,h)$.

We begin with a study of the function $G(z,s;\chi;r,h)$. From its definition we see that

$$(2.1)\qquad G(z,s;\chi;r,h) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\chi(m)\overline{\chi}(n)e((mh_1+nh_2)/k)}{((m+r_1)z+n+r_2)^s}$$

$$= \chi(-r_1)e(-r_1h_1/k)\sum_{\substack{n=-\infty\\ n\neq -r_1}}^{\infty}\frac{\overline{\chi}(n)e(nh_2/k)}{(n+r_2)^s}$$
$$+\left(\sum_{m<-r_1}\sum_{n=-\infty}^{\infty}+\sum_{m>-r_1}\sum_{n=-\infty}^{\infty}\right)\frac{\chi(m)\overline{\chi}(n)e((mh_1+nh_2)/k)}{((m+r_1)z+n+r_2)^s}$$

$$:=\Sigma_1+\Sigma_2+\Sigma_3.$$

We separate Σ_1 into two sums and rewrite it as

$$\Sigma_1 = \chi(-r_1)e(-r_1h_1/k) \cdot \left\{\psi(s;\overline{\chi},r_2,h_2)+\overline{\chi}(-1)e(s/2)\psi(s;\overline{\chi},-r_2,-h_2)\right\}, \tag{2.2}$$

where

$$\psi(s,\chi,\alpha,\beta)=\sum_{n>-\alpha}\frac{e(n\beta/k)\chi(n)}{(n+\alpha)^s}.$$

The Gauss sum $G(z,\chi)$ is defined by

$$G(z,\chi)=\sum_{h=1}^{k-1}\chi(h)e(hz/k).$$

We set $G(1,\chi)=G(\chi)$. It is well-known that if χ is a primitive character and n is an integer, then [1, p. 168]

$$G(n,\chi)=\overline{\chi}(n)G(\chi). \tag{2.3}$$

In [3], Berndt gives a character analogue of the Lipschitz Summation Formula. We use it in the following form.

Theorem 1. *Let χ be a primitive character of modulus k. For* Re $z>0$, *α real and $\sigma>1$ we have*

$$\sum_{n=-\infty}^{\infty}\chi(n)e(-n\alpha/k)(z+n)^{-s}$$
$$=\frac{G(\chi)(-2\pi i/k)^s}{\Gamma(s)}\sum_{n>-\alpha}\overline{\chi}(n)e(z(n+\alpha)/k)(n+\alpha)^{s-1}.$$

In Σ_2, we replace m by $-m$ and n by $-n$. So

$$\Sigma_2=\sum_{m>r_1}\sum_{n=-\infty}^{\infty}\frac{\chi(m)\overline{\chi}(n)e((-mh_1-nh_2)/k)}{((-m+r_1)z-n+r_2)^s} \tag{2.4}$$
$$=e(s/2)\sum_{m>r_1}e(-mh_1/k)\chi(m)\sum_{n=-\infty}^{\infty}\frac{\overline{\chi}(n)e(-nh_2/k)}{((m-r_1)z+n-r_2)^s}.$$

We apply Theorem 1 to the inner sum in (2.4) with χ replaced by $\overline{\chi}$, $\alpha = h_2$ and z replaced by $(m-r_1)z - r_2$ to see that

$$\Sigma_2 = e(s/2) \sum_{m>r_1} e(-mh_1/k)\chi(m) \frac{G(\overline{\chi})(-2\pi i/k)^s}{\Gamma(s)} \cdot \sum_{n>-h_2} \chi(n) e\big(((m-r_1)z - r_2)(n+h_2)/k\big)(n+h_2)^{s-1} = \frac{G(\overline{\chi})(-2\pi i/k)^s e(s/2)}{\Gamma(s)} A(z,s;\chi;-r,-h), \tag{2.5}$$

where

$$A(z,s;\chi;r,h) = \sum_{m>-r_1} e(mh_1/k)\chi(m) \cdot \sum_{n>h_2} \chi(n) e\big(((m+r_1)z + r_2)(n-h_2)/k\big)(n-h_2)^{s-1}.$$

Similarly we deduce that

$$\Sigma_3 = \frac{G(\overline{\chi})(-2\pi i/k)^s}{\Gamma(s)} A(z,s;\chi;r,h). \tag{2.6}$$

From (2.1), (2.2), (2.5) and (2.6) we conclude that

$$G(z,s;\chi;r,h) = \frac{G(\overline{\chi})(-2\pi i/k)^s}{\Gamma(s)} \big(A(z,s;\chi;r,h) + e(s/2)A(z,s;\chi;-r,-h)\big) + \chi(-r_1)e(-r_1h_1/k)\big(\psi(s;\overline{\chi},r_2,h_2) + \overline{\chi}(-1)e(s/2)\psi(s;\overline{\chi},-r_2,-h_2)\big). \tag{2.7}$$

We are interested in transformations of the function $A(z,s;\chi;r,h)$. But the derivation of the formulas is more manageable and straightforward if we use the function $G(z,s;\chi;r,h)$ and then the relationship in (2.7).

3. A Transformation Formula for $G(z,s;\chi;r,h)$.

In this section we present a transformation formula for the function $G(z,s;\chi;r,h)$. We need the following lemma [7].

Lemma 1. *Let A, B, C and D be real with A and B not both zero and $C > 0$. Then, for* $\operatorname{Im}(z) > 0$,

$$\arg\left((Az+B)/(Cz+D)\right) = \arg(Az+B) - \arg(Cz+D) + 2\pi k,$$

where k is independent of z and

$$k = \begin{cases} 1, & \text{if } A \le 0 \text{ and } AD - BC > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2. *Let $Q = \{z = x + iy : x > -d/c, y > 0\}$. Define $R = (R_1, R_2)$ where $R_1 = ar_1 + cr_2$ and $R_2 = br_1 + dr_2$, for arbitrary real numbers r_1 and r_2. Set $H = (H_1, H_2)$ where $H_1 = dh_1 - bh_2$ and $H_2 = -ch_1 + ah_2$, for arbitrary real numbers h_1 and h_2. Let $\rho = \{R_2\}c - \{R_1\}d$, where $\{x\} = x - [x]$ is the fractional part of x. Let $G(z, s; \chi; r, h)$ be defined by (1.1). Suppose first that $a \equiv d \equiv 0 \pmod k$. Then for $z \in Q$ and all s,*

$$\begin{aligned} &(cz+d)^{-s}\Gamma(s)G(Vz, s; \chi; r, h) \\ &= \overline{\chi}(b)\chi(c)\Bigg\{\Gamma(s)G(z, s; \overline{\chi}; R, H) \\ &\quad - 2i\Gamma(s)\sin(\pi s)\overline{\chi}(R_1)e(-R_1H_1/k)\psi(s; \chi; -R_2, -H_2) \\ &\quad + e\left(-\frac{s}{2}\right)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\Bigg(-H_1(c\mu - ck + j + [R_1])/k \\ &\quad - H_2\left(k + d\mu - dk + [R_2] + \left[\frac{dj+\rho}{c}\right] - \nu\right)\Big/k\Bigg) \\ &\qquad \cdot \overline{\chi}(c\mu + j + [R_1])\chi\left([R_2] + \left[\frac{dj+\rho}{c}\right] - \nu\right) f(z, s; R, H)\Bigg\}, \end{aligned}$$

where

$$f(z, s; R, H) = \int_C u^{s-1} \frac{e^{-u(c\mu + j - \{R_1\})(cz+d)/c}}{e^{-ku(cz+d)} - e(H_1c + H_2d)} \frac{e^{u(\nu + \{(dj+\rho)/c\})}}{e^{ku} - e(-H_2)} du,$$

and

$$\psi(s, \chi, \alpha, \beta) = \sum_{n > -\alpha} \frac{e(n\beta/k)\chi(n)}{(n+\alpha)^s}.$$

We choose the branch of u^s with $0 < \arg u < 2\pi$. Here C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of $(\exp(-ku(cz+d)) - e(H_1c + H_2d))(\exp(ku) - e(-H_2))$ lying “within” the loop, and then returning to $+\infty$ in the lower half-plane.

Secondly, if $b \equiv c \equiv 0 \pmod k$, we have for $z \in Q$ and all s,

$$\begin{aligned} &(cz+d)^{-s}\Gamma(s)G(Vz, s; \chi; r, h) \\ &= \overline{\chi}(a)\chi(d)\Bigg\{\Gamma(s)G(z, s; \chi; R, H) \end{aligned}$$

$$- 2i\Gamma(s)\sin(\pi s)\chi(R_1)e(-R_1H_1/k)\psi(s;\overline{\chi};-R_2,-H_2)$$
$$+ e\left(-\frac{s}{2}\right)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\Big(-H_1(c\mu - ck + j + [R_1])/k$$
$$- H_2\left(k + d\mu - dk + [R_2] + \left[\frac{dj+\rho}{c}\right] - \nu\right)\Big/k\Big)$$
$$\cdot\, \chi(j+[R_1])\overline{\chi}\left([R_2] + \left[\frac{dj+\rho}{c}\right] + d\mu - \nu\right) f(z,s;R,H)\Bigg\}.$$

Proof. Let $M = ma+nc$ and $N = mb+nd$. Then, since $a \equiv d \equiv 0 \pmod{k}$,

$$\begin{aligned} G(Vz,s;\chi;r,h) &= \sum_{M,N=-\infty}^{\infty}{}' \chi(Md-Nc)\overline{\chi}(Na-Mb)e((MH_1+NH_2)/k) \\ &\qquad \cdot \left\{\frac{(M+R_1)z+N+R_2}{cz+d}\right\}^{-s} \\ &= \chi(c)\overline{\chi}(b)\sum_{m,n=-\infty}^{\infty}{}' \chi(n)\overline{\chi}(m)e((H_1m+H_2n)/k) \\ &\qquad \cdot \left\{\frac{(m+R_1)z+n+R_2}{cz+d}\right\}^{-s}. \end{aligned} \tag{3.1}$$

Similarly, if $b \equiv c \equiv 0 \pmod{k}$,

$$G(Vz,s;\chi;r,h) = \chi(d)\overline{\chi}(a)\sum_{m,n=-\infty}^{\infty}{}' \chi(m)\overline{\chi}(n)e((H_1m+H_2n)/k) \cdot \left\{\frac{(m+R_1)z+n+R_2}{cz+d}\right\}^{-s}.$$

For the rest of the proof we assume that $a \equiv d \equiv 0 \pmod{k}$. The proof for the other case is completely analogous.

Now by Lemma 1,

$$\left\{\frac{(m+R_1)z+n+R_2}{cz+d}\right\}^{-s} = \frac{((m+R_1)z+n+R_2)^{-s}}{(cz+d)^{-s}}e(-ls), \tag{3.2}$$

where $l = 1$, if $m+R_1 \le 0$ and $d(m+R_1) - c(n+R_2) > 0$, and $l = 0$ otherwise. Then, from (3.1) and (3.2) we deduce that

(3.3)

$$(cz+d)^{-s}G(Vz,s;\chi;r,h)$$

$$= \chi(c)\overline{\chi}(b)\left\{ \left(e(-s) \sum_{\substack{m \le -R_1 \\ d(m+R_1) > c(n+R_2)}} + \sum_{\substack{m,n \\ \text{otherwise}}}' \right) \frac{e((H_1 m + H_2 n)/k)\chi(n)\overline{\chi}(m)}{((m+R_1)z+n+R_2)^s} \right\}$$

$$= \chi(c)\overline{\chi}(b)\left\{ G(z,s;\overline{\chi};R,H) + (e(-s)-1)g(z,s;\chi;R,H) \right\},$$

where

$$g(z,s;\chi;R,H) = \sum_{\substack{m \le -R_1 \\ d(m+R_1) > c(n+R_2)}} \frac{e((H_1 m + H_2 n)/k)\chi(n)\overline{\chi}(m)}{((m+R_1)z+n+R_2)^s}.$$

We now replace n by $-n$ and m by $-m$ in $g(z,s;\chi;R,H)$ and find that

(3.4) $g(z,s;\chi;R,H)$

$$= \sum_{\substack{m \ge R_1 \\ d(R_1-m) > c(R_2-n)}} \frac{e((-H_1 m - H_2 n)/k)\chi(n)\overline{\chi}(m)}{((R_1-m)z-n+R_2)^s}$$

$$= e(s/2)\Bigg(\overline{\chi}(R_1)e(-R_1H_1/k) \sum_{n > R_2} \frac{e(-H_2 n/k)\chi(n)}{(n-R_2)^s}$$

$$+ \sum_{\substack{m > R_1 \\ n > d(m-R_1)/c + R_2}} \frac{e((-H_1 m - H_2 n)/k)\chi(n)\overline{\chi}(m)}{((m-R_1)z+n-R_2)^s} \Bigg)$$

$$= e(s/2)\left\{ \overline{\chi}(R_1)e(-R_1H_1/k)\psi(s;\chi,-R_2,-H_2) + h(z,s;\chi;R,H) \right\},$$

where

$$h(z,s;\chi;R,H) = \sum_{\substack{m > R_1 \\ n > d(m-R_1)/c + R_2}} \frac{e((-H_1 m - H_2 n)/k)\chi(n)\overline{\chi}(m)}{((m-R_1)z+n-R_2)^s}.$$

We use Euler's integral representation of $\Gamma(s)$ to find that for $z \in Q$ and $\sigma > 2$,

$$\Gamma(s)h(z,s;\chi;R,H)$$

$$= \int_0^\infty u^{s-1}e^{-u} \sum_{\substack{m > R_1 \\ n > d(m-R_1)/c + R_2}} \frac{e((-H_1 m - H_2 n)/k)\chi(n)\overline{\chi}(m)}{((m-R_1)z+n-R_2)^s}$$

$$= \sum_{\substack{m > R_1 \\ n > d(m-R_1)/c + R_2}} e((-H_1 m - H_2 n)/k)\chi(n)\overline{\chi}(m)$$

$$\cdot \int_0^\infty u^{s-1}e^{-u} \left((m-R_1)z+n-R_2\right)^{-s} du$$

$$= \sum_{\substack{m>R_1 \\ n>d(m-R_1)/c+R_2}} e((-H_1m - H_2n)/k)\chi(n)\overline{\chi}(m) \cdot \int_0^\infty u^{s-1}e^{-u((m-R_1)z+n-R_2)}du.$$

Now let $m = m'c + j + [R_1] + 1$; thus $0 \le m' < \infty$ and $0 \le j \le c-1$. And let $n = n' + [R_2 + d(m-R_1)/c] + 1$; thus $0 \le n' < \infty$. We conclude that

$$\begin{aligned}
&\Gamma(s)h(z,s;\chi;R,H) \\
&= \sum_{j=0}^{c-1}\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty} e\big(-H_1(m'c+j+[R_1]+1)/k \\
&\quad - H_2(n' + \big[R_2 + d(m'c+j+1-\{R_1\})/c\big]+1)/k\big) \\
&\quad\cdot \chi(n' + [R_2 + d(m'c+j+1-\{R_1\})/c]+1)\overline{\chi}(m'c+j+[R_1]+1) \\
&\quad\cdot \int_0^\infty u^{s-1}e^{-zu(m'c+j+1-\{R_1\})-u(n'+[R_2+d(m'c+j+1-\{R_1\})/c]+1-R_2)}du.
\end{aligned}$$

For the next calculation, we replace j by $j-1$, m' by $mk+\mu$, and n' by $nk+\nu$ where $0 \le \mu,\nu \le k-1$. From the interchange of summation and integration and the fact that $d \equiv 0 \pmod k$, we see that

$$\begin{aligned}
&\Gamma(s)h(z,s;\chi;R,H) \\
&= \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big(-H_1(\mu c+j+[R_1])/k \\
&\quad - H_2(\nu + d\mu + [R_2 + d(j-\{R_1\})/c]+1)/k\big) \\
&\quad\cdot \chi(\nu + [R_2 + d(j-\{R_1\})/c]+1)\overline{\chi}(c\mu+j+[R_1]) \\
&\quad\cdot \int_0^\infty u^{s-1}e^{-zu(c\mu+j-\{R_1\})-u(\nu+d\mu+[R_2+d(j-\{R_1\})/c]+1-R_2)} \\
&\quad\cdot \sum_{m,n=0}^{\infty} e^{-2\pi i(H_1mc+H_2md+H_2n)-mkczu-mkdu-nku}du \\
&= \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big(-H_1(\mu c+j+[R_1])/k \\
&\quad - H_2(\nu + d\mu + [R_2 + d(j-\{R_1\})/c]+1)/k\big) \\
&\quad\cdot \chi(\nu + [R_2 + d(j-\{R_1\})/c]+1)\overline{\chi}(c\mu+j+[R_1]) \\
&\quad\cdot \int_0^\infty u^{s-1}\frac{e^{-zu(c\mu+j-\{R_1\})-d\mu u}}{1-e(-H_1c-H_2d)e^{-ku(cz+d)}}\frac{e^{u(-\nu-1+R_2-[R_2+d(j-\{R_1\})/c])}}{1-e(-H_2)e^{-ku}}du
\end{aligned}$$

$$= -\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big(-H_1(\mu c - kc + j + [R_1])/k$$
$$- H_2(k - kd + d\mu + [R_2 + d(j - \{R_1\})/c] - \nu)/k\big)$$
$$\cdot \chi([R_2 + d(j - \{R_1\})/c] - \nu)\overline{\chi}(c\mu + j + [R_1])$$
$$\cdot \int_0^\infty u^{s-1} \frac{e^{-zu(c\mu + j - \{R_1\} + d\mu/z)}}{e^{-ku(cz+d)} - e(H_1 c + H_2 d)} \frac{e^{u(\nu + R_2 - [R_2 + d(j - \{R_1\})/c])}}{e^{ku} - e(-H_2)} du.$$

We recall that $\rho = \{R_2\}c - \{R_1\}d$ and we add and subtract the quantity $u(dj + \rho)/c$ in the exponent of the numerator in one of the factors of the integrand to find that

$$\Gamma(s)h(z, s; \chi; R, H)$$
$$= -\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big(-H_1(\mu c - kc + j + [R_1])/k$$
$$- H_2(k - kd + d\mu + [R_2] + [(dj + \rho)/c] - \nu)/k\big)$$
$$\cdot \chi([R_2] + [(dj + \rho)/c] - \nu)\overline{\chi}(c\mu + j + [R_1])$$
$$\cdot \int_0^\infty u^{s-1} \frac{e^{-u(c\mu + j - \{R_1\})(cz+d)/c}}{e^{-ku(cz+d)} - e(H_1 c + H_2 d)} \frac{e^{u(\nu + \{(dj+\rho)/c\})}}{e^{ku} - e(-H_2)} du.$$

In the next step we use the method outlined in [9] to convert the integral over $(0, \infty)$ into a loop integral. Thus

$$\Gamma(s)h(z, s; \chi; R, H) \tag{3.5}$$
$$= -\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big(-H_1(\mu c - kc + j + [R_1])/k$$
$$- H_2(k - kd + d\mu + [R_2] + [(dj + \rho)/c] - \nu)/k\big)$$
$$\cdot \chi([R_2] + [(dj + \rho)/c] - \nu)\overline{\chi}(c\mu + j + [R_1])$$
$$\cdot \frac{1}{e(s) - 1} f(z, s; R, H),$$

where

$$f(z, s; R, H) = \int_C u^{s-1} \frac{e^{-u(c\mu + j - \{R_1\})(cz+d)/c}}{e^{-ku(cz+d)} - e(H_1 c + H_2 d)} \frac{e^{u(\nu + \{(dj+\rho)/c\})}}{e^{ku} - e(-H_2)} du.$$

Now from (3.3) and (3.4) we deduce that

$$(cz + d)^{-s} G(Vz, s; \chi; r, h)$$
$$= \chi(c)\overline{\chi}(b)\big\{G(z, s; \overline{\chi}; R, H) + (e(-s) - 1)g(z, s; \chi; R, H)\big\}$$

$$
\begin{aligned}
&= \chi(c)\overline{\chi}(b)\Big\{G(z,s;\overline{\chi};R,H) \\
&\quad + (e(-s)-1)e(s/2) \\
&\quad\quad \cdot \big(\overline{\chi}(R_1)e(-R_1H_1/k)\psi(s;\chi,-R_2,-H_2) + h(z,s;\chi;R,H)\big)\Big\}.
\end{aligned}
$$

And from (3.5) we conclude that

$$
\begin{aligned}
&\Gamma(s)(cz+d)^{-s}G(Vz,s;\chi;r,h) \\
&= \chi(c)\overline{\chi}(b)\Big\{\Gamma(s)G(z,s;\overline{\chi};R,H) \\
&\quad - 2i\Gamma(s)\sin(\pi s)\overline{\chi}(R_1)e(-R_1H_1/k)\psi(s;\chi,-R_2,-H_2) \\
&\quad + e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big(-H_1(\mu c - kc + j + [R_1])/k \\
&\quad - H_2(k - kd + d\mu + [R_2] + [(dj+\rho)/c] - \nu)/k\big) \\
&\quad\quad \cdot \chi([R_2] + [(dj+\rho)/c] - \nu)\overline{\chi}(c\mu + j + [R_1])f(z,s;R,H)\Big\}
\end{aligned}
$$

as desired. □

4. Generalizations of Bernoulli Functions.

To introduce the analogues of Dedekind sums, we need some facts about Bernoulli functions and one of their character generalizations. The Bernoulli polynomials $B_n(x)$ are generated by ([1], p. 264)

$$\frac{ue^{xu}}{e^u - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{u^n}{n!} \qquad (|u| < 2\pi). \tag{4.1}$$

The Bernoulli functions $\overline{B}_n(x)$ are defined by

$$\overline{B}_n(x+m) = B_n(x), \tag{4.2}$$

where $0 \le x < 1$ and m is an arbitrary integer, except in the case $n = 1$ and $x = 0$, where we define $\overline{B}_1(0+m) = 0$. In [2], Berndt defines the generalized Bernoulli functions $\overline{B}_n(x,\chi)$. For our purposes, we are most interested in the following property that can serve as a definition.

$$\overline{B}_n(x,\chi) = k^{n-1}\sum_{h=1}^{k-1}\overline{\chi}(h)\overline{B}_n\left(\frac{x+h}{k}\right). \tag{4.3}$$

The alternating generalized Bernoulli functions $\overline{B}^*_n(x,\chi)$, $n \geq 1$, $-\infty < x < \infty$, are defined by

$$\overline{B}^*_n(x,\chi) = k^{n-1}\sum_{h=1}^{k-1}(-1)^h\overline{\chi}(h)\overline{B}_n\left(\frac{x+h}{k}\right). \tag{4.4}$$

It is natural to define $\overline{B}^*_0(x,\chi) = 1$. By (4.2), the alternating generalized Bernoulli functions are also periodic with period k. We also define the alternating generalized Bernoulli numbers $\overline{B}^*_n(\chi) = \overline{B}^*_n(0,\chi)$.

We use the following properties of finite character sums in the sequel.

Proposition 1. (i) *If χ is a nonprincipal character of modulus k, then*

$$\sum_{j=0}^{k-1}\chi(j) = 0.$$

(ii) *If k is odd and χ is an even character, or if k is even, then*

$$\sum_{j=0}^{k-1}(-1)^j\chi(j) = 0.$$

Proof. The first property is well-known (see for example [**1**], p. 136).

For (ii), we let

$$\begin{aligned} \mathrm{S} = \sum_{j=0}^{k-1}(-1)^j\chi(j) &= \sum_{j=1}^{k-1}(-1)^j\chi(j) \\ &= \sum_{j=0}^{k-1}(-1)^{k-j}\chi(k-j) \\ &= \chi(-1)\sum_{j=0}^{k-1}(-1)^{k-j}\chi(j). \end{aligned} \tag{4.5}$$

If k is odd and χ is even, we deduce from (4.5) that

$$\mathrm{S} = -\sum_{j=1}^{k-1}(-1)^j\chi(j) = -\mathrm{S},$$

and thus $\mathrm{S} = 0$.

If k is even, then $\chi(j) = 0$ for even j and we have that

$$\sum_{j=0}^{k-1}(-1)^j\chi(j) = -\sum_{j=0}^{k-1}\chi(j) = 0$$

from part (i). □

We use Theorem 2 to give transformation formulas for specific functions that result from specification of the parameters (r_1, r_2) and (h_1, h_2). In our applications we put $r_1 = r_2 = 0$. From (2.7) we conclude that

$$\begin{aligned}(4.6)\qquad &\lim_{s\to 0} \Gamma(s)G(z,s;\chi;(0,0),h)\\ &= G(\overline{\chi})\big(A(z,0;\chi;(0,0),h) + A(z,0;\chi;(0,0),-h)\big).\end{aligned}$$

We put $A(z;\chi;h) = A(z,0;\chi;(0,0),h)$. Now suppose that $\{2h_1/k\} = 0$ and $k \mid 2h_2$. Then

$$\begin{aligned}(4.7)\qquad &A(z;\chi;-h)\\ &= \sum_{m=1}^{\infty} e(-mh_1/k)\chi(m) \sum_{n>-h_2} \chi(n)e(mz(n+h_2)/k)(n+h_2)^{-1}\\ &= \sum_{m=1}^{\infty} e(mh_1/k)\chi(m) \sum_{n>-h_2} \chi(n)e(mz(n+h_2)/k)(n+h_2)^{-1}.\end{aligned}$$

We replace n by $n - 2h_2$ in (4.7) and deduce that

$$\begin{aligned}(4.8)\qquad &A(z;\chi;-h)\\ &= \sum_{m=1}^{\infty} e(mh_1/k)\chi(m) \sum_{n>h_2} \chi(n)e(mz(n-h_2)/k)(n-h_2)^{-1}\\ &= A(z;\chi;h).\end{aligned}$$

Under the newly assumed conditions, we conclude from (4.6) and (4.8) that

$$(4.9)\qquad \lim_{s\to 0} \Gamma(s)G(z,s;\chi;(0,0),h) = 2G(\overline{\chi})A(z;\chi;h).$$

Consider the function $A(z;\chi;(lk/2, mk/2))$ for integers l and m. Let $l' \equiv l \pmod 2$ and $m' \equiv m \pmod 2$. From a calculation using the definition of $G(z,s;\chi;(0,0),h)$, we have that

$$A(z;\chi;(lk/2, mk/2)) = A(z;\chi;(l'k/2, m'k/2)).$$

We make the following definition.

Definition 1. For z in the upper half-plane and χ a primitive character of modulus k, let

$$\begin{aligned}A_1(z;\chi) &= A(z;\chi;(k/2,k/2)),\\ A_2(z;\chi) &= A(z;\chi;(k/2,0))\end{aligned}$$

and

$$A_3(z;\chi) = A(z;\chi;(0,k/2)).$$

5. Transformations of $A_1(z;\chi)$.

We set $h=(k/2,k/2)$ and $H=(k(d-b)/2,k(a-c)/2)$ for the rest of this section. For the following argument, we assume $a\equiv d\equiv 0 \pmod{k}$. Now from Theorem 2 and (4.9), we arrive at the following formula.

$$
\begin{aligned}
&G(\overline{\chi})A_1(Vz;\chi) \\
&= \overline{\chi}(b)\chi(c)\Big\{G(\chi)A(z;\overline{\chi};H) \\
&\quad + \frac{1}{2}\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1} e\big((b-d)(c\mu-ck+j)/2 \\
&\quad + (c-a)(k+d\mu-dk+[dj/c]-\nu)/2\big) \\
&\qquad \cdot \overline{\chi}(c\mu+j)\chi([dj/c]-\nu) \\
&\qquad \cdot \int_C u^{-1}\frac{e^{-u(c\mu+j)(cz+d)/c}}{e^{-ku(cz+d)}-e(k/2)}\frac{e^{u(\nu+\{dj/c\})}}{e^{ku}-e((c-a)k/2)}du\Big\}.
\end{aligned}
\tag{5.1}
$$

With some simplification, (5.1) becomes

$$
\begin{aligned}
&G(\overline{\chi})A_1(Vz;\chi) \\
&= \overline{\chi}(b)\chi(c)\Big\{G(\chi)A(z;\overline{\chi};H) \\
&\quad + \frac{1}{2}\sum_{j=1}^{c}(-1)^{(b-d)j+(c-a)[dj/c]}\sum_{\mu=0}^{k-1}(-1)^{(k-\mu)}\overline{\chi}(c\mu+j) \\
&\qquad \cdot \sum_{\nu=0}^{k-1}(-1)^{(c-a)(k-\nu)}\chi([dj/c]-\nu) \\
&\qquad \cdot \int_C u^{-1}\frac{e^{-u(c\mu+j)(cz+d)/c}}{e^{-ku(cz+d)}-(-1)^k}\frac{e^{u(\nu+\{dj/c\})}}{e^{ku}-(-1)^{(c-a)k}}du\Big\}.
\end{aligned}
\tag{5.2}
$$

Assume that k is even and thus a and d are even, b and c are odd. We use the residue theorem and (4.1) to evaluate the integral in (5.2). Thus

$$
\begin{aligned}
&\int_C u^{-1}\frac{e^{-ku(c\mu+j)(cz+d)/(ck)}}{e^{-ku(cz+d)}-1}\frac{e^{ku(\nu+\{dj/c\})/k}}{e^{ku}-1}du \\
&= \pi i\Big\{-\frac{1}{cz+d}B_2\left(\frac{\nu+\{dj/c\}}{k}\right)-(cz+d)B_2\left(\frac{c\mu+j}{ck}\right) \\
&\quad + 2B_1\left(\frac{c\mu+j}{ck}\right)B_1\left(\frac{\nu+\{dj/c\}}{k}\right)\Big\}.
\end{aligned}
\tag{5.3}
$$

And so we conclude from (5.2) and (5.3), noting the parities of a, b, c and d, that

$$
\begin{aligned}
(5.4)\qquad & G(\overline{\chi})A_1(Vz;\chi)\\
&= \overline{\chi}(b)\chi(c)\Bigg\{G(\chi)A_1(z;\overline{\chi})\\
&\quad + \frac{\pi i}{2}\sum_{j=1}^{c}(-1)^{j+[dj/c]}\sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{\chi}(c\mu+j)\\
&\quad\cdot\sum_{\nu=0}^{k-1}(-1)^{\nu}\chi([dj/c]-\nu)\\
&\quad\cdot\Bigg\{-\frac{1}{cz+d}B_2\left(\frac{\nu+\{dj/c\}}{k}\right)-(cz+d)B_2\left(\frac{c\mu+j}{ck}\right)\\
&\quad + 2B_1\left(\frac{c\mu+j}{ck}\right)B_1\left(\frac{\nu+\{dj/c\}}{k}\right)\Bigg\}\Bigg\}.
\end{aligned}
$$

We must evaluate the triple sums in (5.4). Let us first investigate the sum on μ and the contribution of $B_2\left((\nu+\{dj/c\})/k\right)/(cz+d)$. In the first line of the next calculation we make crucial use of the fact that k is even. Since $ad-bc=1$, we have $(b,k)=1$ and thus

$$
\begin{aligned}
\sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{\chi}(c\mu+j) &= \sum_{\mu \bmod k}(-1)^{\mu}\overline{\chi}(c\mu+j)\\
&= \sum_{\mu \bmod k}(-1)^{-b\mu-bj}\overline{\chi}(c(-b\mu-bj)+j)\\
&= (-1)^{j}\sum_{\mu \bmod k}(-1)^{\mu}\overline{\chi}(\mu).
\end{aligned}
$$

But this is zero by Proposition 1 and hence, so is the contribution of $B_2\left((\nu+\{dj/c\})/k\right)/(cz+d)$. A similar argument shows that, by summing on ν, the contribution of $(cz+d)B_2\left((c\mu+j)/(ck)\right)$ is zero. Next observe that what remains in the triple sum is unchanged if $B_1\left((\nu+\{dj/c\})/k\right)$ is replaced by $\overline{B}_1\left((\nu+\{dj/c\})/k\right)$, since $d\equiv 0 \pmod{k}$. Then by the definition in (4.4),

$$
\begin{aligned}
(5.5)\qquad & \sum_{\nu=0}^{k-1}(-1)^{\nu}\chi([dj/c]-\nu)\overline{B}_1\left(\frac{\nu+\{dj/c\}}{k}\right)\\
&= \sum_{\nu=0}^{k-1}(-1)^{\nu}\chi([dj/c]-\nu)\overline{B}_1\left(\frac{\nu+\{dj/c\}}{k}\right)
\end{aligned}
$$

$$= (-1)^{[dj/c]} \sum_{\nu=0}^{k-1} (-1)^{\nu} \chi(-\nu) \overline{B}_1 \left(\frac{\nu + dj/c}{k} \right)$$
$$= \chi(-1)(-1)^{[dj/c]} \overline{B}_1^* (dj/c, \overline{\chi}) .$$

Observe that the sum in (5.4) is unchanged if $B_1 ((c\mu + j)/(ck))$ is replaced by $\overline{B}_1 ((c\mu + j)/(ck))$. Then we conclude from this fact and from (5.5) that the triple sum in (5.4) can be written as

$$\chi(-1)\pi i \sum_{j=1}^{c} (-1)^j \overline{B}_1^* (dj/c, \overline{\chi}) \sum_{\mu=0}^{c-1} (-1)^{\mu} \overline{\chi}(c\mu + j) \overline{B}_1 \left(\frac{c\mu + j}{ck} \right) . \tag{5.6}$$

Now we put $c\mu + j = n$, where $1 \le n \le ck$. Since $d \equiv 0 \pmod{k}$, and $\overline{B}_1^*(x, \chi)$ has period k, (5.6) becomes

$$\chi(-1)\pi i \sum_{n=1}^{ck} (-1)^n \overline{\chi}(n) \overline{B}_1^* \left(\frac{dn}{c}, \overline{\chi} \right) \overline{B}_1 \left(\frac{n}{ck} \right) . \tag{5.7}$$

Definition 2. Let $(c, d) = 1$ with $c > 0$. The alternating Dedekind character sum $s^*(d, c; \chi)$ is defined by

$$s^*(d, c; \chi) = \sum_{n \mod ck} (-1)^n \chi(n) \overline{B}_1^* \left(\frac{dn}{c}, \chi \right) \overline{B}_1 \left(\frac{n}{ck} \right) .$$

With Definition 2 and the fact that (5.7) is the value of the sum in (5.4), we have proved the following for $a \equiv d \equiv 0 \pmod{k}$. The proof for $b \equiv c \equiv 0 \pmod{k}$ is analogous.

Theorem 3. *Let k be even and $\operatorname{Im}(z) > 0$. Let $Vz = (az + b)/(cz + d)$, where a, b, c and d are integers with $c > 0$ and $ad - bc = 1$. If $a \equiv d \equiv 0 \pmod{k}$, then*

$$G(\overline{\chi}) A_1(Vz; \chi) = \overline{\chi}(b)\chi(c) \big\{ G(\chi) A_1(z; \overline{\chi}) + \chi(-1)\pi i s^*(d, c; \overline{\chi}) \big\}; \tag{5.8}$$

if $b \equiv c \equiv 0 \pmod{k}$, then

$$G(\overline{\chi}) A_1(Vz; \chi) = \overline{\chi}(a)\chi(d) \big\{ G(\overline{\chi}) A_1(z; \chi) + \chi(-1)\pi i s^*(d, c; \chi) \big\}. \tag{5.9}$$

The regular Dedekind sums possess a reciprocity law. We next prove a reciprocity law for $s^*(d, c; \chi)$ analogous to that of the regular Dedekind sums.

Theorem 4. *Let k be even and let $c, d > 0$, $(c, d) = 1$ with either c or $d \equiv 0 \pmod{k}$. Then*

$$s^*(c, d; \chi) + s^*(d, c; \overline{\chi}) = -\chi(-1) \overline{B}_1^*(\chi) \overline{B}_1^*(\overline{\chi}).$$

Proof. By symmetry, we may assume without loss of generality that $d \equiv 0 \pmod{k}$. Given c and d as above, we can find a and b so that $a \equiv 0 \pmod{k}$ and $ad - bc = 1$. Let $Vz = (az+b)/(cz+d)$ and $Tz = -1/z$. Then set $Wz = VTz = (bz-a)/(dz-c)$. If we replace z by Tz in (5.8), we see that

$$G(\overline{\chi})A_1(Wz;\chi) = \overline{\chi}(b)\chi(c)\big\{G(\chi)A_1(Tz;\overline{\chi}) + \chi(-1)\pi i s^*(d,c;\overline{\chi})\big\}. \tag{5.10}$$

We apply (5.9) with V replaced by W to find that

$$\begin{aligned} &G(\overline{\chi})A_1(Wz;\chi) \\ &= \overline{\chi}(b)\chi(-c)\big\{G(\overline{\chi})A(z;\chi;H') + \chi(-1)\pi i s^*(-c,d;\chi)\big\}, \end{aligned}$$

where $H' = ((a-c)k/2, (b-d)k/2)$. But since $a-c$ and $b-d$ are both odd we write this as

$$\begin{aligned} &G(\overline{\chi})A_1(Wz;\chi) \\ &= \overline{\chi}(b)\chi(-c)\big\{G(\overline{\chi})A_1(z;\chi) + \chi(-1)\pi i s^*(-c,d;\chi)\big\}. \end{aligned} \tag{5.11}$$

Finally, we apply (5.8) with V replaced by T and χ replaced by $\overline{\chi}$ to see that

$$G(\chi)A_1(Tz;\overline{\chi}) = \chi(-1)\big\{G(\overline{\chi})A_1(z;\chi) + \chi(-1)\pi i s^*(0,1;\chi)\big\}. \tag{5.12}$$

We replace $G(\chi)A_1(Tz;\overline{\chi})$ in (5.10) with (5.12) and combine the result with (5.11) to conclude that

$$\begin{aligned} &\overline{\chi}(b)\chi(c)\pi i s^*(0,1;\chi) + \chi(-1)\overline{\chi}(b)\chi(c)\pi i s^*(d,c;\overline{\chi}) \\ &= \overline{\chi}(b)\chi(c)\pi i s^*(-c,d;\chi). \end{aligned} \tag{5.13}$$

We can divide by $\pi i\overline{\chi}(b)\chi(c)$ since $(b,k) = (c,k) = 1$. Thus (5.13) becomes

$$s^*(0,1;\chi) + \chi(-1)s^*(d,c;\overline{\chi}) = s^*(-c,d;\chi). \tag{5.14}$$

From the definition of $\overline{B}_1^*(x,\chi)$, and the fact that k is even, we see that

$$\begin{aligned} \overline{B}_1^*(-x,\chi) &= \sum_{h=1}^{k-1}(-1)^h\overline{\chi}(h)\overline{B}_1\left(\frac{h-x}{k}\right) \\ &= -\sum_{h=1}^{k-1}(-1)^h\overline{\chi}(h)\overline{B}_1\left(\frac{x-h}{k}\right) \\ &= -\sum_{h=1}^{k-1}(-1)^{k-h}\overline{\chi}(k-h)\overline{B}_1\left(\frac{x-k+h}{k}\right) \\ &= -\overline{\chi}(-1)\sum_{h=1}^{k-1}(-1)^h\overline{\chi}(h)\overline{B}_1\left(\frac{x+h}{k}\right) \\ &= -\chi(-1)\overline{B}_1^*(x,\chi). \end{aligned}$$

It follows that

$$s^*(-c,d;\chi) = -\chi(-1)s^*(c,d;\chi). \tag{5.15}$$

Lastly, we calculate $s^*(0,1;\chi)$ from Definition 2 and (4.4). We have

$$\begin{aligned} s^*(0,1;\chi) &= \sum_{n \pmod k} (-1)^n \chi(n)\overline{B}_1^*(0,\chi)\overline{B}_1\left(\frac{n}{k}\right) \\ &= \overline{B}_1^*(\chi) \sum_{n \pmod k} (-1)^n \chi(n)\overline{B}_1\left(\frac{n}{k}\right) \\ &= \overline{B}_1^*(\chi)\overline{B}_1^*(\overline{\chi}). \end{aligned} \tag{5.16}$$

The theorem follows upon the application of (5.15) and (5.16) in the equation (5.14). □

Now assume that k is odd. To evaluate the integral in (5.2), we need to consider separately two cases depending on the parity of $c-a$. If $c-a$ is odd, the integral in (5.2) has the value $\pi i/2$ by the residue theorem. Thus, in this case, (5.2) becomes

$$\begin{aligned} &G(\overline{\chi})A_1(Vz;\chi) \\ &\quad = \overline{\chi}(b)\chi(c)\Bigg\{G(\chi)A(z;\overline{\chi};H) \\ &\qquad + \frac{\pi i}{4}\sum_{j=1}^{c}(-1)^{(b-d)j+(c-a)[dj/c]}\sum_{\mu=0}^{k-1}(-1)^{(k-\mu)}\overline{\chi}(c\mu+j) \\ &\qquad \cdot \sum_{\nu=0}^{k-1}(-1)^{(c-a)(k-\nu)}\chi([dj/c]-\nu)\Bigg\}. \end{aligned} \tag{5.17}$$

We need to separate two subcases depending on whether $b-d$ is even or odd. In the case where $c-a$ is odd and $b-d$ is even, (5.17) becomes

$$\begin{aligned} &G(\overline{\chi})A_1(Vz;\chi) \\ &= \overline{\chi}(b)\chi(c)\Bigg\{G(\chi)A_3(z;\overline{\chi}) \\ &\quad + \frac{\pi i}{4}\sum_{j=1}^{c}(-1)^{[dj/c]}\sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{\chi}(c\mu+j)\sum_{\nu=0}^{k-1}(-1)^{\nu}\chi([dj/c]-\nu)\Bigg\}. \end{aligned}$$

Since k is odd, we no longer have that

$$\sum_{\mu=0}^{k-1}(-1)^{\mu}\chi(\mu) = \sum_{\mu \bmod k}(-1)^{\mu}\chi(\mu).$$

Thus the more elegant simplifications of the previous case are not generally possible. So we define the more complicated sum

$$g(d,c;\chi)=\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{[dj/c]+\mu+\nu}\overline{\chi}(c\mu+j)\chi([dj/c]-\nu).$$

Remark 1. The notation is devised so that sums named with a g are those that contain a sum that is close to the Gauss sum $G(k/2,\chi)=\sum_{j=0}^{k-1}(-1)^j\chi(j)$, for k odd.

We can now give the following result.

Theorem 5. *Let k be odd and* $\mathrm{Im}\,(z)>0$*. Let $Vz=(az+b)/(cz+d)$, where a, b, c and d are integers with $c>0$, $c-a$ odd, $b-d$ even and $ad-bc=1$. Let $a\equiv d\equiv 0 \pmod k$. Then*

$$G(\overline{\chi})A_1(Vz;\chi)=\overline{\chi}(b)\chi(c)\left\{G(\chi)A_3(z;\overline{\chi})+\frac{\pi i}{4}g(d,c;\chi)\right\}.$$

We consider the case with $b-d$ odd. Then (5.17) becomes

$$\begin{aligned}G(\overline{\chi})A(Vz;\chi;h)=\overline{\chi}(b)\chi(c)\Bigg\{&G(\chi)A(z;\overline{\chi};H)\\&+\frac{\pi i}{4}\sum_{j=1}^{c}(-1)^{j+[dj/c]}\sum_{\mu=0}^{k-1}(-1)^{\mu}\overline{\chi}(c\mu+j)\\&\cdot\sum_{\nu=0}^{k-1}(-1)^{\nu}\chi([dj/c]-\nu)\Bigg\}.\end{aligned}\tag{5.18}$$

We define

$$g_1(d,c;\chi)=\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{j+[dj/c]+\mu+\nu}\overline{\chi}(c\mu+j)\chi([dj/c]-\nu).$$

And we have proved the following theorem.

Theorem 6. *Let k be odd and* $\mathrm{Im}(z)>0$*. Let $Vz=(az+b)/(cz+d)$, where a, b, c and d are integers with $c>0$, $c-a$ odd, $b-d$ odd and $ad-bc=1$. Let $a\equiv d\equiv 0 \pmod k$. Then*

$$G(\overline{\chi})A_1(Vz;\chi)=\overline{\chi}(b)\chi(c)\left\{G(\chi)A_1(z;\overline{\chi})+\frac{\pi i}{4}g_1(d,c;\chi)\right\}.$$

We consider the case where k, c and a are each odd. From the residue theorem and (4.1) we deduce that (5.2) becomes

$$G(\overline{\chi})A_1(Vz;\chi)=\overline{\chi}(b)\chi(c)\Big\{G(\chi)A_2(z;\overline{\chi})\tag{5.19}$$

$$- \frac{\pi i}{2} \sum_{j=1}^{c} (-1)^j \sum_{\mu=0}^{k-1} (-1)^\mu \overline{\chi}(c\mu + j)$$
$$\cdot \sum_{\nu=0}^{k-1} \chi([dj/c] - \nu) B_1 \left(\frac{\nu + \{dj/c\}}{k} \right) \Bigg\}.$$

Recall that, as in a prior calculation, the triple sum is unchanged if $B_1\left((\nu + \{dj/c\})/k\right)$ is replaced by $\overline{B}_1\left((\nu + \{dj/c\})/k\right)$, since $d \equiv 0 \pmod{k}$. By (4.3),

$$\sum_{\nu=0}^{k-1} \chi([dj/c] - \nu) \overline{B}_1 \left(\frac{\nu + \{dj/c\}}{k} \right) = \sum_{\nu=0}^{k-1} \chi(-\nu) \overline{B}_1 \left(\frac{\nu + dj/c}{k} \right) = \chi(-1) \overline{B}_1 (dj/c, \overline{\chi}). \tag{5.20}$$

From (5.20), we rewrite (5.19) as

$$G(\overline{\chi}) A_1(Vz; \chi) = \overline{\chi}(b)\chi(c) \Bigg\{ G(\chi) A_2(z; \overline{\chi}) + \frac{\pi i}{2} \sum_{j=1}^{c} (-1)^j \overline{B}_1 (dj/c, \overline{\chi}) \sum_{\mu=0}^{k-1} (-1)^\mu \overline{\chi}(c\mu + j) \Bigg\}. \tag{5.21}$$

In the spirit of the paper, we make the following definition. Note that, as before, since k is odd, further simplification is not generally possible.

Definition 3. For $c > 0$, define

$$s_1^*(d, c; \chi) = \sum_{j=1}^{c} (-1)^j \overline{B}_1 (dj/c, \overline{\chi}) \sum_{\mu=0}^{k-1} (-1)^\mu \overline{\chi}(c\mu + j).$$

We have now proved the following result.

Theorem 7. *Let k be odd and $\mathrm{Im}(z) > 0$. Let $Vz = (az + b)/(cz + d)$, where a, b, c and d are integers with $c > 0$, c odd, a odd and $ad - bc = 1$. Let $a \equiv d \equiv 0 \pmod{k}$. Then*

$$G(\overline{\chi}) A_1(Vz; \chi) = \overline{\chi}(b)\chi(c) \left\{ G(\chi) A_2(z; \overline{\chi}) + \frac{\pi i}{2} s_1^*(d, c; \chi) \right\}.$$

Let us make the following definitions.

Definition 4. For $c > 0$, define

$$s_2^*(d, c; \chi) = \sum_{n \pmod{ck}} \chi(n) \overline{B}_1^* \left(\frac{dn}{c}, \chi \right).$$

Definition 5. For $c > 0$, let

$$s_3^*(d,c;\chi) = \sum_{j=1}^{c}(-1)^j\chi(j)\sum_{\mu=0}^{k-1}\overline{B}_1^*\left(\frac{d(c\mu+j)}{c},\chi\right).$$

The next results are applications of the same methods to the second part of Theorem 2. The proof of Theorem 8 is similar to those already presented and will not be given here.

Theorem 8. *Let k be odd and $\mathrm{Im}(z) > 0$. Let $Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$, $c - a$ odd and $ad - bc = 1$. Let $b \equiv c \equiv 0 \pmod{k}$. If b and d are odd, then*

$$G(\overline{\chi})A_1(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_3(z;\chi) + \frac{\pi i}{2}\chi(-1)s_2^*(d,c;\chi)\right\};$$

if $b - d$ is odd, then

$$G(\overline{\chi})A_1(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_1(z;\chi) + \frac{\pi i}{2}\chi(-1)s_3^*(d,c;\chi)\right\}.$$

Theorem 9. *Let k be odd and $\mathrm{Im}(z) > 0$. Let $Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$, c and a both odd and $ad - bc = 1$. Let $b \equiv c \equiv 0 \pmod{k}$. Then*

$$G(\overline{\chi})A_1(Vz;\chi) = \overline{\chi}(a)\chi(d)G(\overline{\chi})A_2(z;\chi).$$

Proof. If we consider (5.2) and the residue theorem under these conditions we find that

$$G(\overline{\chi})A_1(Vz;\chi) = \overline{\chi}(a)\chi(d)\Bigg\{G(\overline{\chi})A_2(z;\chi) + \frac{\pi i}{4}\sum_{j=1}^{c}(-1)^j\chi(j)\sum_{\mu=0}^{k-1}(-1)^\mu\sum_{\nu=0}^{k-1}\overline{\chi}([dj/c]+d\mu-\nu)\Bigg\}.$$

But the sum on ν is zero by Proposition 1. This then completes the proof. □

6. Transformations of $A_2(z;\chi)$.

In this section we continue the development of the new alternating character analogues of Dedekind sums. Many of the calculations are identical to those in the previous section and will therefore not be included. Throughout this section $h = (k/2, 0)$, and thus $H = (dk/2, -ck/2)$.

We begin with the relevant definitions of the sums that arise in the formulas.

Definition 6. For $c > 0$, define the following sums.

$$s_4^*(d,c;\chi) = \sum_{n \pmod{ck}} \chi(n)\overline{B}_1^*\left(\frac{dn}{c},\chi\right)\overline{B}_1\left(\frac{n}{ck}\right),$$

$$s_5^*(d,c;\chi) = \sum_{n \pmod{ck}} (-1)^n\chi(n)\overline{B}_1\left(\frac{dn}{c},\chi\right)\overline{B}_1\left(\frac{n}{ck}\right),$$

$$g_2(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{[dj/c]+\nu}\overline{\chi}(\mu)\chi([dj/c]-\nu)\overline{B}_1\left(\frac{\mu}{ck}\right),$$

$$g_3(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{j+[dj/c]+\nu}\overline{\chi}(\mu)\chi([dj/c]-\nu)\overline{B}_1\left(\frac{\mu}{ck}\right),$$

$$g_4(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{[dj/c]+\nu}\chi(j)\overline{\chi}([dj/c]+d\mu-\nu)\overline{B}_1\left(\frac{\mu}{ck}\right)$$

and

$$g_5(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{j+[dj/c]+\nu}\chi(j)\overline{\chi}([dj/c]+d\mu-\nu)\overline{B}_1\left(\frac{\mu}{ck}\right).$$

Theorem 10. *Let χ be a primitive character with modulus k and let $\mathrm{Im}(z) > 0$. Let $Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$ and $ad - bc = 1$. Let $a \equiv d \equiv 0 \pmod{k}$. If k is even, then*

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(b)\chi(c)\big\{G(\chi)A_3(z;\overline{\chi}) + \chi(-1)\pi i s_4^*(d,c;\chi)\big\}.$$

If k is odd and c is even, then

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(b)\chi(c)\left\{G(\chi)A_2(z;\overline{\chi}) + \chi(-1)\pi i s_5^*(d,c;\chi)\right\}.$$

If k is odd, c is odd and d is even, then

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(b)\chi(c)\left\{G(\chi)A_3(z;\overline{\chi}) + \frac{\pi i}{2}g_2(d,c;\chi)\right\}.$$

If k, c and d are odd, then

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(b)\chi(c)\left\{G(\chi)A_1(z;\overline{\chi}) - \frac{\pi i}{2}g_3(d,c;\chi)\right\}.$$

Theorem 11. *Let χ be a primitive character with modulus k and let $\mathrm{Im}(z) > 0$. Let $Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$ and $ad - bc = 1$. Let $b \equiv c \equiv 0 \pmod{k}$. If k is even, then*

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(a)\chi(d)\big\{G(\overline{\chi})A_2(z;\chi) + \chi(-1)\pi i s_5^*(d,c;\chi)\big\}.$$

If k is odd and c is even, then

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(a)\chi(d)\big\{G(\overline{\chi})A_2(z;\chi) + \chi(-1)\pi i s_5^*(d,c;\chi)\big\}.$$

If k is odd, c is odd and d is even, then

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_3(z;\chi) - \frac{\pi i}{2} g_4(d,c;\chi)\right\}.$$

If k, c and d are odd, then

$$G(\overline{\chi})A_2(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_1(z;\chi) - \frac{\pi i}{2} g_5(d,c;\chi)\right\}.$$

7. Transformations of $A_3(z;\chi)$.

We now turn to the application of Theorem 2 where $h = (0, k/2)$ and thus $H = (-bk/2, ak/2)$. As in the previous section, the calculations are identical with those in Section 5.

We define the following analogues of Dedekind sums.

Definition 7. For $c > 0$, let the following definitions be given.

$$g_6(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{j+\mu+\nu}\overline{\chi}(c\mu+j)\chi([dj/c]-\nu)\overline{B}_1\left(\frac{\nu+\{dj/c\}}{k}\right),$$

$$g_7(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{j+\mu+\nu}\chi(j)\overline{\chi}([dj/c]+d\mu-\nu)\overline{B}_1\left(\frac{\nu+\{dj/c\}}{k}\right),$$

$$g_8(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{[dj/c]+\mu+\nu}\chi(j)\overline{\chi}([dj/c]+d\mu-\nu)$$

and

$$g_9(d,c;\chi) = \sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}(-1)^{j+[dj/c]+\mu+\nu}\chi(j)\overline{\chi}([dj/c]+d\mu-\nu).$$

Theorem 12. *Let χ be a primitive character with modulus k and let $\mathrm{Im}(z) > 0$. Let $Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$ and $ad - bc = 1$. Let $a \equiv d \equiv 0 \pmod{k}$. If k is even, then*

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(b)\chi(c)\big\{G(\chi)A_3(z;\overline{\chi}) + \chi(-1)\pi i s_5^*(d,c;\chi)\big\}.$$

If k is odd and a is even, then

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(b)\chi(c)\left\{G(\chi)A_1(z;\overline{\chi}) - \frac{\pi i}{2}g_6(d,c;\chi)\right\}.$$

If k is odd, a is odd and b is even, then

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(b)\chi(c)\left\{G(\chi)A_2(z;\overline{\chi}) + \frac{\pi i}{4}g(d,c;\chi)\right\}.$$

If k, a and b are odd, then

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(b)\chi(c)\left\{G(\chi)A_1(z;\overline{\chi}) - \frac{\pi i}{2}g_1(d,c;\chi)\right\}.$$

Theorem 13. *Let χ be a primitive character with modulus k and let $\mathrm{Im}(z) > 0$. Let $Vz = (az+b)/(cz+d)$, where a, b, c and d are integers with $c > 0$ and $ad - bc = 1$. Let $b \equiv c \equiv 0 \pmod{k}$. If k is even, then*

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(a)\chi(d)\big\{G(\overline{\chi})A_2(z;\chi) + \chi(-1)\pi i s_5^*(d,c;\chi)\big\}.$$

If k is odd and a is even, then

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_2(z;\chi) - \frac{\pi i}{2}g_7(d,c;\chi)\right\}.$$

If k is odd, a is odd and b is even, then

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_3(z;\chi) + \frac{\pi i}{4}g_8(d,c;\chi)\right\}.$$

If k, a and b are odd, then

$$G(\overline{\chi})A_3(Vz;\chi) = \overline{\chi}(a)\chi(d)\left\{G(\overline{\chi})A_1(z;\chi) + \frac{\pi i}{4}g_9(d,c;\chi)\right\}.$$

8. The functions $A_1(z;\chi)$, $A_2(z;\chi)$ and $A_3(z;\chi)$.

We investigate the functions $A_i(z;\chi)$, $i = 1,2,3$, further by direct calculations of the series involved. The proofs of the theorems about $A_1(z;\chi)$ are given; the proofs of the others are similar and are therefore omitted.

From the definition

$$\begin{aligned}(8.1)\quad & G(\overline{\chi})A_1(z;\chi) \\ &= G(\overline{\chi})\sum_{m=1}^{\infty}(-1)^m\chi(m)\sum_{n>k/2}\chi(n)e\left(\frac{mz(n-k/2)}{k}\right)(n-k/2)^{-1} \\ &= \sum_{m=1}^{\infty}\sum_{n>k/2}(-1)^m G(\overline{\chi})\chi(m)\chi(n)e\left(\frac{mz(n-k/2)}{k}\right)(n-k/2)^{-1}.\end{aligned}$$

Now, if k is odd, then (8.1) can be rewritten as

$$\begin{aligned} G(\overline{\chi})A_1(z;\chi) \\ = 2\sum_{m,n=1}^{\infty}(-1)^m\chi(m)\chi\left(n+\frac{k+1}{2}\right)G(\overline{\chi}) \\ \cdot e\left(\frac{mz(2n+1)}{2k}\right)(2n+1)^{-1}. \end{aligned} \tag{8.2}$$

And if k is even, then (8.1) becomes

$$\begin{aligned} G(\overline{\chi})A_1(z;\chi) \\ = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^m\chi(m)\chi\left(n+\frac{k}{2}\right)G(\overline{\chi})e\left(\frac{mzn}{k}\right)n^{-1}. \end{aligned} \tag{8.3}$$

With the application of (2.3), we can absorb one of the characters into the Gauss sum. Thus we are able to sum one of the series in a conventional manner.

The function $G(z,s;\chi;r,h)$ was defined to introduce a character analogue of the theta-functions since Berndt [**5**] showed that the generalized Eisenstein series defined without the characters gives the logarithms of the theta-functions. The theta-functions can be written as infinite products; it is therefore reasonable to ask if the functions $A_i(z;\chi)$ can be written as the logarithms of an infinite product.

Theorem 14. *Let* $\operatorname{Im}(z)>0$. *Then, if k is odd, we have*

$$G(\overline{\chi})A_1(z;\chi) = \log\prod_{h=1}^{k-1}\prod_{m=1}^{\infty}\left(\frac{1-e\left(\frac{h+mz}{k}\right)}{\left(1-e\left(\frac{h+mz}{2k}\right)\right)^2}\right)^{(-1)^{h+m}\overline{\chi}(h)\chi(m)};$$

and if k is even, then

$$\begin{aligned} G(\overline{\chi})A_1(z;\chi) \\ = -\log\prod_{h=1}^{k/2}\prod_{m=1}^{\infty}\left(1-e\left(\frac{(2m-1)z+2h-1}{k}\right)\right)^{\overline{\chi}(2h-1)\chi(2m-1)}. \end{aligned}$$

Proof. Let k be odd. From (8.2) and (2.3), we have

$$\begin{aligned} G(\overline{\chi})A_1(z;\chi) \\ = 2\sum_{h=1}^{k-1}(-1)^h\overline{\chi}(h)\sum_{m=1}^{\infty}(-1)^m\chi(m) \\ \cdot\sum_{n=1}^{\infty}e\left(\frac{(2n-1)(mz+h)}{2k}\right)(2n-1)^{-1} \end{aligned} \tag{8.4}$$

$$= 2\sum_{h=1}^{k-1}(-1)^h\overline{\chi}(h)\sum_{m=1}^{\infty}(-1)^m\chi(m)$$
$$\cdot\left\{\sum_{n=1}^{\infty} e\left(\frac{n(mz+h)}{2k}\right)\Big/ n - \frac{1}{2}\sum_{n=1}^{\infty} e\left(\frac{n(mz+h)}{k}\right)\Big/ n\right\}$$
$$= -2\sum_{h=1}^{k-1}\sum_{m=1}^{\infty}\log\left(1-e\left(\frac{mz+h}{2k}\right)\right)^{(-1)^{h+m}\overline{\chi}(h)\chi(m)}$$
$$+\sum_{h=1}^{k-1}\sum_{m=1}^{\infty}\log\left(1-e\left(\frac{mz+h}{k}\right)\right)^{(-1)^{h+m}\overline{\chi}(h)\chi(m)}$$
$$= \log\prod_{h=1}^{k-1}\prod_{m=1}^{\infty}\left(\frac{1-e\left(\frac{h+mz}{k}\right)}{\left(1-e\left(\frac{h+mz}{2k}\right)\right)^2}\right)^{(-1)^{h+m}\overline{\chi}(h)\chi(m)} + 2\pi i r(z),$$

where $r(z)$ is an integer. But $r(z)$ is also analytic on the upper half-plane since $G(\overline{\chi})A_1(z;\chi)$ and the logarithm above are. Thus $r(z)$ is constant on the upper half-plane.

Now we establish the value of r. Consider $z = iy$, $y > 0$. As y tends to infinity, $|A_1(iy,\chi)|$ becomes very small and the factors on the right side of (8.4) become very close to 1. Thus we conclude that for y large enough, $r = 0$. Since r is a constant, $r \equiv 0$.

The proof for k even is similar. □

Infinite product results exist for $A_2(z;\chi)$ and $A_3(z;\chi)$.

Theorem 15. *Let* $\mathrm{Im}(z) > 0$. *Then*

$$G(\overline{\chi})A_2(z;\chi) = \log\prod_{h=1}^{k-1}\prod_{m=1}^{\infty}\left(1-e\left(\frac{mz+h}{k}\right)\right)^{(-1)^{m+1}\overline{\chi}(h)\chi(m)}.$$

Theorem 16. *Let* $\mathrm{Im}(z) > 0$. *Then, if k is odd, we have*

$$G(\overline{\chi})A_3(z;\chi) = \log\prod_{h=1}^{k-1}\prod_{m=1}^{\infty}\left(\frac{1-e\left(\frac{h+mz}{k}\right)}{\left(1-e\left(\frac{h+mz}{2k}\right)\right)^2}\right)^{(-1)^{h}\overline{\chi}(h)\chi(m)};$$

and if k is even, then

$$G(\overline{\chi})A_1(z;\chi) = -\log\prod_{h=1}^{k-1}\prod_{m=1}^{\infty}\left(1-e\left(\frac{mz+h}{k}\right)\right)^{(-1)^{h}\overline{\chi}(h)\chi(m)}.$$

If we sum first on m instead, we obtain the following series representations for $G(\overline{\chi})A_i(z;\chi)$.

Theorem 17. *Let* $\operatorname{Im}(z) > 0$. *If* k *is odd, then*

$$G(\overline{\chi})A_1(z;\chi) = -2\sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=0}^{\infty}\frac{\chi(n+\frac{k+1}{2})}{(2n+1)\left(e\left(-\frac{z(2n+1)+2h}{2k}\right)+1\right)}.$$

If k *is even, then*

$$G(\overline{\chi})A_1(z;\chi) = -\sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=1}^{\infty}\frac{\chi(n+k/2)}{n\left(e\left(-\frac{zn+h}{k}\right)+1\right)}.$$

For all k,

$$G(\overline{\chi})A_2(z;\chi) = -\sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=1}^{\infty}\frac{\chi(n)}{n\left(e\left(-\frac{zn+h}{k}\right)+1\right)}.$$

If k *is odd, then*

$$G(\overline{\chi})A_3(z;\chi) = 2\sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=0}^{\infty}\frac{\chi(n+\frac{k+1}{2})}{(2n+1)\left(e\left(-\frac{z(2n+1)+2h}{2k}\right)-1\right)},$$

and if k *is even, then*

$$G(\overline{\chi})A_1(z;\chi) = -\sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=1}^{\infty}\frac{\chi(n+k/2)}{n\left(e\left(-\frac{zn+h}{k}\right)+1\right)}.$$

9. Some series relations.

We conclude this paper with applications of the transformation formulas to some series relations.

Theorem 18. *Let* $\alpha, \beta > 0$ *and* $\alpha\beta = \pi^2$. *If* k *is even, then*

$$\begin{aligned}&\sum_{h=1}^{k-1}\overline{\chi}(h)e(h/k)\sum_{n=1}^{\infty}\frac{\chi\left(n+\frac{k}{2}\right)}{n\left(e^{2\alpha n/k}+e(h/k)\right)}\\&=\overline{\chi}(-1)\sum_{h=1}^{k-1}\chi(h)e(h/k)\sum_{n=1}^{\infty}\frac{\overline{\chi}\left(n+\frac{k}{2}\right)}{n\left(e^{2\beta n/k}+e(h/k)\right)}\\&\quad+\pi i\overline{B}_1^*(\chi)\overline{B}_1^*(\overline{\chi}).\end{aligned}$$

Proof. We apply Theorem 3 with $V = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ to see that

$$G(\overline{\chi})A_1\left(-\tfrac{1}{z};\chi\right) = \overline{\chi}(-1)G(\chi)A_1(z;\overline{\chi}) + \pi i s^*(0,1;\overline{\chi}),$$

or, in series form:

$$-\sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=1}^{\infty}\frac{\chi\left(n+\frac{k}{2}\right)}{n\left(e\left(-\frac{n(-1/z)+h}{k}\right)+1\right)}$$
$$=-\overline{\chi}(-1)\sum_{h=1}^{k-1}\chi(h)\sum_{n=1}^{\infty}\frac{\overline{\chi}\left(n+\frac{k}{2}\right)}{n\left(e\left(-\frac{nz+h}{k}\right)+1\right)}+\pi i\overline{B}_1^*(\chi)\overline{B}_1^*(\overline{\chi}),$$

where we use the evaluation of $s^*(0,1;\overline{\chi})$ from the proof of the reciprocity theorem.

Next, set $z=\pi i/\alpha$ and multiply numerator and denominator by $e(h/k)$ to complete the proof. □

Corollary 1. *Let* $k=4$ *and* χ *be the primitive character of modulus* 4, *defined by*

$$\chi(n)=\begin{cases}0, & \textit{for } n \textit{ even },\\ 1, & \textit{for } n\equiv 1 \pmod 4,\\ -1, & \textit{for } n\equiv 3 \pmod 4.\end{cases}$$

Then

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n\left(e^{\pi n/2}+e^{-\pi n/2}\right)}=\frac{\pi}{16}.$$

Proof. In Theorem 18 set $\alpha=\beta=\pi$. Then we have

$$\sum_{h=1}^{3}\chi(h)i^h\sum_{n=1}^{\infty}\frac{\chi\left(n+2\right)}{n\left(e^{\pi n/2}+i^h\right)}$$
$$=-\sum_{h=1}^{3}\chi(h)i^h\sum_{n=1}^{\infty}\frac{\chi\left(n+2\right)}{n\left(e^{\pi n/2}+i^h\right)}-\frac{\pi i}{4}.$$

Or with some simplification

$$\sum_{n=1}^{\infty}\frac{\chi(n+2)}{n\left(e^{\pi n/2}+i\right)}+\sum_{n=1}^{\infty}\frac{\chi(n+2)}{n\left(e^{\pi n/2}-i\right)}=-\frac{\pi}{8}.$$

And finally

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n\left(e^{\pi n/2}+e^{-\pi n/2}\right)}=\frac{\pi}{16}.$$

□

Corollary 1 is also found in Ramanujan's Notebooks; see Entry 25(vii) on p. 295 of Berndt [6]. Preece [8] and Zucker [10] also give different proofs of this and related results.

Theorem 19. *Let $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$. If k is odd, then*

$$\sum_{h=1}^{k-1} \overline{\chi}(h) e(h/k) \sum_{j=1}^{\infty} \frac{\chi\left(j + \frac{k+1}{2}\right)}{(2j-1)(e^{\alpha(2j-1)/k} + e(h/k))}$$
$$= -\sum_{h=1}^{k-1} \chi(h) e(h/k) \sum_{j=1}^{\infty} \frac{\overline{\chi}\left(j + \frac{k+1}{2}\right)}{(2j-1)(e^{\beta(2j-1)/k} + e(h/k))}$$
$$- \frac{\pi i}{8} G\left(\frac{k}{2}, \chi\right) G\left(\frac{k}{2}, \overline{\chi}\right).$$

Proof. Here we apply Theorem 6 with $V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and use the evaluation

$$g_1(0, 1; \chi) = G\left(\frac{k}{2}, \chi\right) G\left(\frac{k}{2}, \overline{\chi}\right),$$

where $G(x, \chi)$ is the Gauss sum. The remainder of the calculations are similar to those in the proof of Theorem 18. □

The next corollary follows immediately from Theorem 19.

Corollary 2. *Let k be odd and χ be an odd, real-valued primitive character. Let $\alpha = \beta = \pi$. Then*

$$\sum_{h=1}^{k-1} \chi(h) e(h/k) \sum_{n=1}^{\infty} \frac{\chi\left(n + \frac{k+1}{2}\right)}{(2n-1)(e^{(2n-1)\pi/k} + e(h/k))} = -\frac{\pi i}{16} G\left(\frac{k}{2}, \chi\right)^2.$$

Corollary 3. *If $\left(\frac{j}{3}\right)$ represents the Legendre symbol, then*

$$\sum_{n=0}^{\infty} \frac{\left(\frac{n}{3}\right)}{(2n+1)\left(e^{(2n+1)\pi/3} + e^{-(2n+1)\pi/3} - 1\right)} = -\frac{\pi}{4\sqrt{3}}.$$

Proof. The result follows from Corollary 2 if we set $k = 3$ and $\chi(j)$ to be the Legendre symbol $\left(\frac{j}{3}\right)$ and then simplify the series involved. □

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Received August 27, 1998 and revised November 13, 1998.

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