Pacific Journal of Mathematics

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Volume 194 No. 1 May 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 194, No. 1, 2000

GODBILLON–VEY CLASSES OF SYMPLECTIC FOLIATIONS

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Dedicated to the memory of Professor Shukichi Tanno

Each transversally oriented foliation has the Godbillon–Vey characteristic class, and regular Poisson structures define symplectic foliations. In this note, we shall give a new interpretation and the explicit formula for a representative of the Godbillon–Vey characteristic classes of symplectic foliations in the context of Poisson geometry.

1. Introduction and statement of result.

It is known that a Poisson structure with constant rank on a manifold, called a regular Poisson structure, defines the symplectic foliation on the manifold. For each transversely oriented foliation, we have the famous Godbillon-Vey characteristic class. When the symplectic foliations of regular Poisson structures are transversely oriented, they have the Godbillon-Vey characteristic classes. In this note we shall give a formula defining their Godbillon-Vey classes in terms of Poisson structure. The main result is:

Theorem. *Let (M^n, π) be an oriented manifold with a constant rank $2m$ Poisson structure π. Take a volume form vol_M of M. By the map ϕ, we mean the isomorphism induced by vol_M between multi-vector fields and forms. Then we claim that:*

(1) *The $q(= n - 2m)$-form $\alpha = \phi(\pi^m)$ gives a transverse orientation of the symplectic foliation of π.*
(2) *Take a q-vector field $\bigstar(\pi^m)$ on M with the property $\pi^m \wedge \bigstar(\pi^m) = \phi^{-1}(1)$. Then the 1-form $\beta = \phi([-\bigstar(\pi^m), \pi^m]_S)$ gives a representative of the Godbillon-Vey class by $\beta \wedge (d\beta)^q$ and*

$$\phi^{-1}(\beta \wedge (d\beta)^q) = (d\phi([-\bigstar(\pi^m), \pi^m]_S))^q \lrcorner [-\bigstar(\pi^m), \pi^m]_S,$$

where $[\cdot, \cdot]_S$ is the Schouten bracket on M and $\lrcorner$ is the inner derivation.

Acknowledgments. The author is grateful to Prof. T. Mizutani, Prof. H. Sato, and Prof. A. Weinstein for their helpful suggestions and discussions. The author also expresses his sincere thanks to Prof. C.-M. Marle and Prof. P. Libermann at Universite Pierre et Marie Curie (Univ. Paris VI) and Prof. M. Cahen and Prof. S. Gutt at the Universite Libre de Bruxelles for giving him chances to visit them and for discussions at their seminars. The author's special thanks go to the referee for his/her several kind and precise comments.

2. Generalized divergence of multi-vector fields.

Let vol_M be a volume form of an orientable manifold M. Then we get the $C^\infty(M)$-linear isomorphism $\phi : \bigwedge^u(\mathrm{T}M) \longrightarrow \bigwedge^{n-u}(\mathrm{T}^*M)$ by

$$\phi(U) = \iota_U \mathrm{vol}_M = U \lrcorner\, \mathrm{vol}_M, \text{ i.e., } \langle \phi(U), V\rangle := \langle \mathrm{vol}_M, U \wedge V\rangle$$

for all $V \in \bigwedge^{n-u}(\mathrm{T}M)$, where $\langle\ ,\ \rangle$ is the natural pairing. Define $\mathrm{base}_M \in \bigwedge^n(\mathrm{T}M)$ by $\phi^{-1}(1)$.

We may consider similarly the interior multiplication with respect to forms, and we have

$$\phi^{-1}(\alpha) = (-1)^{a(n+1)} \alpha \lrcorner\, \mathrm{base}_M$$

where a is the degree of α. We recall the following elementary formula.

$$\phi^{-1}\left(\phi(T) \wedge \phi(U)\right) = (-1)^{(n+t)(u+1)} \phi(T) \lrcorner\, U = (-1)^{(n+1)(n+u)} \phi(U) \lrcorner\, T$$

holds for each $T \in \bigwedge^t(\mathrm{T}M)$ and $U \in \bigwedge^u(\mathrm{T}M)$.

Let us consider the map $\psi := \phi^{-1} \circ d \circ \phi$. The next lemma says ψ can be expressed by the Schouten bracket and ψ-image of lower degree multi-vector fields.

Lemma 2.1. *For all multi-vector fields T and U, we have*

$$\psi(T \wedge U) = (-1)^u [T, U]_S + (-1)^u \psi(T) \wedge U + T \wedge \psi(U)$$

where u is the degree of U, i.e., $U \in \bigwedge^u(\mathrm{T}M)$.

Proof. From the definition of ϕ, $\langle \phi(T), U\rangle = \langle \mathrm{vol}_M, T \wedge U\rangle = \langle T \lrcorner\, \mathrm{vol}_M, U\rangle$. If U is a 1-vector field, then we have

$$\begin{aligned}
\psi(T \wedge U) &= \phi^{-1} \circ d \circ \phi(T \wedge U) = \phi^{-1} \circ d(T \wedge U \lrcorner\, \mathrm{vol}_M) \\
&= \phi^{-1} \circ d(U \lrcorner\, T \lrcorner\, \mathrm{vol}_M) = \phi^{-1} \circ (\mathcal{L}_U - \iota_U \circ d)(T \lrcorner\, \mathrm{vol}_M) \\
&= \phi^{-1} \circ \mathcal{L}_U (T \lrcorner\, \mathrm{vol}_M) - \phi^{-1} \circ \iota_U \circ d(T \lrcorner\, \mathrm{vol}_M) \\
&= \phi^{-1}([U, T]_{\mathrm{S}} \lrcorner\, \mathrm{vol}_M + \mathrm{div}(U) T \lrcorner\, \mathrm{vol}_M) \\
&\quad - \phi^{-1} \circ \iota_U \circ \phi \circ \phi^{-1} \circ d \circ \phi(T) \\
&= [U, T]_{\mathrm{S}} + \mathrm{div}(U) T - \psi(T) \wedge U
\end{aligned}$$

where $\mathcal{L}_U$ is the Lie differentiation with respect to U. We can complete the proof by the induction on degree of U. □

Remark 2.1. For each vector field X, $\psi(X)$ is equal to the divergence with respect to vol_M. For this reason, we may call the map ψ the generalized divergence. If we change our view point on the equation in Lemma 2.1, the Schouten bracket is characterized by some volume form and the mapping ψ. This idea is shown in [**2**]. The derivation rule of the Schouten bracket we use in this note is the bracket operation from left (or right) is a left (or right) derivation respectively, i.e., $[T, U\wedge W]_{\mathrm{S}} = [T,U]_{\mathrm{S}}\wedge W + (-1)^{(t-1)u} U \wedge [T,W]_{\mathrm{S}}$, where $T\in \bigwedge^t(\mathrm{T}M)$ and $U\in\bigwedge^u(\mathrm{T}M)$.

Remark 2.2. Let π be a 2-vector field. A. Weinstein ([**4**]) defines the modular vector field of π with respect to a density. It is the same as $\psi(\pi)$ if we take our vol_M as density.

3. Godbillon-Vey forms of symplectic foliations.

Transversely oriented foliations have secondary characteristic classes which are called the Godbillon-Vey class. We recall the definition of Godbillon-Vey class in accordance with [**1**] and [**3**]. It is well known that a codimension q foliation of M^n corresponds with an involutive rank $(n-q)$ distribution $\mathfrak{D}$. If the foliation is transversely oriented, then we have a q-form α which is of course locally decomposable, i.e., $\alpha = \omega_1\wedge\cdots\omega_q$ for some local 1-forms $\omega_1,\ldots,\omega_q$ and satisfies $d\alpha = \beta\wedge\alpha$ for some 1-form β. Let $\mathcal{I} := \left\{\tau\in\bigwedge^1(\mathrm{T}^*M) \mid \tau\wedge\alpha = 0\right\}$. The distribution $\mathfrak{D}$ of $\mathfrak{F}$ is characterized by

$$\mathfrak{D} = \left\{X\in\wedge^1(\mathrm{T}M) \;\mid\; \langle X,\tau\rangle = 0 \text{ for } \tau\in\mathcal{I}\right\} .$$

Conversely, if there is a q-form α which is locally decomposable, we have the distribution $\mathfrak{D}$ defined by α as

$$\mathfrak{D} := \left\{X\in\wedge^1(\mathrm{T}M) \;\mid\; \langle X,\tau\rangle = 0 \text{ for } \tau\in\mathcal{I}\right\}$$

where $\mathcal{I} := \left\{\tau\in\bigwedge^1(\mathrm{T}^*M) \;\mid\; \tau\wedge\alpha = 0\right\}$. If α satisfies $d\alpha = \beta\wedge\alpha$ for some 1-form β, then the distribution $\mathfrak{D}$ is involutive and the foliation of $\mathfrak{D}$ is transversely orientable.

There is some ambiguity in choosing α and β, but it is known that the closed $(2q+1)$-form $\beta\wedge(d\beta)^q$ is unique up to an exact $(2q+1)$-form, and the cohomology class $[\beta\wedge(d\beta)^q]$ is uniquely determined. The cohomology class $[\beta\wedge(d\beta)^q]\in\mathrm{H}^{2q+1}_{\mathrm{DR}}(M,\mathbf{R})$ is called the Godbillon-Vey class of the given foliation, and $\beta\wedge(d\beta)^q$ is often called the Godbillon-Vey form in this note.

Let us take a Poisson tensor field π on an n-dimensional manifold M with constant rank $2m$ so that $[\pi,\pi]_{\mathrm{S}} = 0$, $\pi^m\neq 0$, and $\pi^{m+1} = 0$. Then we have a q-form $\alpha = \phi(\pi^m/m!)$, where ϕ is the isomorphism introduced in Section

2 and $q = n - 2m$. Locally, we can find a frame field $\{X_1, X_2, \dots, X_n\}$ such that

$$\pi = X_1 \wedge X_2 + X_3 \wedge X_4 + \cdots + X_{2m-1} \wedge X_{2m} \quad \text{and} \quad \langle \mathrm{vol}_M, X_1 \wedge \cdots \wedge X_n \rangle = 1$$

under the assumption that the codimension $q \geq 1$. Let $\{\theta^1, \theta^2, \dots, \theta^n\}$ be the corresponding dual frame field. Then we have $\pi^m = m! X_1 \wedge \cdots X_{2m}$ and $\alpha = \phi(\pi^m/m!) = \theta^{2m+1} \wedge \cdots \wedge \theta^n$.

Let $[X_a, X_b] = \sum_{c=1}^{n} \lambda_{ab}^c X_c$ $(a, b = 1, \dots, n)$. Then $d\theta^c = -\frac{1}{2} \sum_{a,b=1}^{n} \lambda_{ab}^c \theta^a \wedge \theta^b$ for $c = 1, \dots, n$. But, the Poisson condition $[\pi, \pi]_{\mathrm{S}} = 0$ is equivalent to $\lambda_{ij}^K = 0$ for each $i, j = 1, 2, \dots, 2m$ and $K = 2m+1, \dots, n$, and yields the following Proposition.

Proposition 3.1. *The distribution* $\mathfrak{D} = \{H_f \mid f \in C^\infty(M)\}$ *consisting of the Hamiltonian vector fields of the Poisson tensor* π*, which defines the symplectic foliation* $\mathfrak{F}$*, is characterized by* α *in the sense of Section 2 as* $\mathfrak{D} = \{X \in \bigwedge^1(\mathrm{T}M) \mid \langle X, \tau \rangle = 0 \ \text{for} \ \tau \in \mathcal{I}\}$*, where* $\mathcal{I} := \big\{\tau \in \bigwedge^1(\mathrm{T}^*M) \mid \tau \wedge \alpha = 0\big\} =$ *linear span of* $\theta^{2m+1}, \dots, \theta^n$*.*

Let us denote the q-form $\phi(\pi^m/m!)$ by α. Then α gives a transversal orientation of the symplectic foliation π from Proposition 3.1. Let $\bigstar(\pi^m)$ be a q-vector field on M with the property $\langle \mathrm{vol}_M, \pi^m \wedge \bigstar(\pi^m) \rangle = 1$. We have the main theorem below.

Theorem 3.2. *We can define the global* 1*-form* $\beta := \phi\left([-\bigstar(\pi^m), \pi^m]_S\right)$*.* β *satisfies* $d\alpha = \beta \wedge \alpha$ *and hence the Godbillon-Vey characteristic class of the symplectic foliation of* π *is given by the cohomology class of* $\beta \wedge (d\beta)^q$ *and the tangential pullback* $\phi^{-1}\left(\beta \wedge (d\beta)^q\right)$ *of our Godbillon-Vey form is expressed as*

$$\underbrace{\gamma \lrcorner \gamma \lrcorner \cdots \gamma \lrcorner}_{q\text{-times}} ([-\bigstar(\pi^m), \pi^m]_S)$$

where γ *is the* 2*-form given by* $\phi \circ \psi[-\bigstar(\pi^m), \pi^m]_S$*.*

Proof. Set $\mu = \mathrm{vol}_M$ and $U = \bigstar(\pi^m)$. Then

$$\beta \wedge \alpha = (-1)^{n-1} \alpha \wedge ([U, \pi^m]_{\mathrm{S}} \lrcorner \mu) = -(\alpha \lrcorner [U, \pi^m]_{\mathrm{S}}) \lrcorner \mu.$$

Here the last step uses only linear algebra. On the other hand, $d\alpha = \psi(V) \lrcorner \mu$, where $V = \pi^m/m!$, it remains to show that $-\alpha \lrcorner [U, \pi^m]_{\mathrm{S}} = \psi(V)$. By Lemma 2.1, we have

$$[U, V]_{\mathrm{S}} = -\psi(U) \wedge V - U \wedge \psi(V).$$

Thus,

$$-\alpha \lrcorner [U, V]_{\mathrm{S}} = \alpha \lrcorner (\psi(U) \wedge V) + \alpha \lrcorner (U \wedge \psi(V)).$$

We now claim that

$$\alpha \lrcorner (\psi(U) \wedge V) = 0 \tag{1}$$

and

$$\alpha \lrcorner (U \wedge \psi(V)) = \langle \alpha, U\rangle \psi(V) = \psi(V)/m! \tag{2}$$

To see these, we write $\pi = X_1 \wedge X_2 + \cdots + X_{2m+1} \wedge X_{2m}$ locally, where $X_1, X_2, \ldots, X_n$ is a local frame for the tangent bundle of M, so that $V = X_1 \wedge X_2 \wedge \cdots \wedge X_{2m}$. Then for any $j = 1, \ldots, 2m$ and any local vector field W of degree $q-1$,

$$\langle X_j \lrcorner \alpha,\, W\rangle \;=\; \langle \alpha,\, X_j \wedge W\rangle \;=\; \langle \mu,\, V \wedge X_j \wedge W\rangle \;=\; 0.$$

It follows from this and the definition of the operator $Y \mapsto \alpha \lrcorner Y$ that $\alpha \lrcorner (\psi(U) \wedge V) = 0$. Let $\theta = \psi(\pi)$ be the modular vector field. Then it follows from Lemma 2.1 and $\pi^{m+1} = 0$ that $\theta \wedge V = 0$ which implies that $\theta \lrcorner \alpha = 0$. Consequently we get (2) and thus also the fact that $d\alpha = \beta \wedge \alpha$. We got finally a 1-form β given by

$$\beta = \phi([-\bigstar(\pi^m), \pi^m]_{\mathrm{S}}).$$

From the definition, this β defines the Godbillon-Vey class by $[\beta \wedge (d\beta)^{n-2m}]$. Since our hope is to see the Godbillon-Vey class in the context of Poisson geometry, we would like to pull back the Godbillon-Vey form above by ϕ to obtain a Tangential Godbillon-Vey multi-vector field, TGV, hereafter.

$$\begin{aligned} TGV &= \phi^{-1}\left(\beta \wedge (d\beta)^q\right) = (-1)^{(n+1)(1+2q)} \beta \wedge (d\beta)^q \lrcorner \, \mathrm{base}_M \\ &= (-1)^{n+1} \underbrace{d\beta \lrcorner \cdots d\beta \lrcorner}_{q\text{-times}} (\beta \lrcorner \, \mathrm{base}_M) = \underbrace{d\beta \lrcorner \cdots d\beta \lrcorner}_{q\text{-times}} (\phi^{-1}\beta) \end{aligned}$$

and we have

$$d\beta = \phi \circ \phi^{-1} \circ d \circ \phi\left([-\bigstar(\pi^m), \pi^m]_{\mathrm{S}}\right) = \phi\left(\psi([-\bigstar(\pi^m), \pi^m]_{\mathrm{S}})\right).$$

□

Remark 3.1. There is some ambiguity in choosing $\bigstar(\pi^m)$ with the property $\pi^m \wedge \bigstar(\pi^m) = \mathrm{base}_M$. If we take an arbitrary riemannian metric $\langle \cdot, \cdot \rangle$ on M, then we may choose such a multi-vector $\bigstar(\pi^m)$ by $*(\pi^m)$, where $*$ is the Hodge-star operator with respect to the metric.

Remark 3.2. The final statement of our Theorem 3.2 makes sense if the rank of our Poisson structure is equal to the dimension of the manifold, in other words, if the Poisson structure comes from a symplectic structure, namely, if $\pi = -\omega^{-1}$. We may consider an arbitrary volume form on M.

Then it is of the form $\mathrm{vol}_M = f\dfrac{\omega^m}{m!}$ with $f \neq 0$ and $2m = \dim M$. Since

$$\begin{aligned}\beta &= -\phi([1/(m!f), \pi^m]_{\mathrm{S}}) = -1/(m-1)!\phi([1/f, \pi]_{\mathrm{S}} \wedge \pi^{m-1}) \\ &= -f/(m-1)![1/f, \pi]_{\mathrm{S}} \rfloor \underbrace{\pi \rfloor \cdots \rfloor \pi \rfloor}_{(m-1)\text{-times}} \frac{\omega^m}{m!} \\ &= -f[1/f, \pi]_{\mathrm{S}} \rfloor \omega = -fH_{1/f} \rfloor \omega = d\log|f|\end{aligned}$$

we see $[\beta] = 0 \in \mathrm{H}^1_{\mathrm{DR}}(M, \mathbf{R})$.

3.1. Codimension 1 symplectic foliations.

Let us consider the codimension 1 symplectic foliations, namely the full rank Poisson structures of manifolds of odd dimension $n = 2m+1$ as the restricted cases. We already know that the tangential object corresponding to a Godbillon-Vey form is given as

$$\begin{aligned}TGV &= (-1)^{n+1}\beta \rfloor d\beta \rfloor \mathrm{base}_M = \beta \rfloor \left(\phi^{-1} \circ d \circ \phi([-\bigstar(\pi^m), \pi^m]_{\mathrm{S}})\right) \\ &= \phi[\xi, \pi^m]_{\mathrm{S}} \rfloor \psi[\xi, \pi^m]_{\mathrm{S}}\end{aligned}$$

where $\xi := -\bigstar(\pi^m)$. Thus, we have the following result.

Theorem 3.3. *Let π be a full rank Poisson structure on a manifold M^{2m+1}. Take a vector field ξ satisfying $\pi^m \wedge \xi = -\mathrm{base}_M$. Then the Godbillon-Vey class corresponds to*

$$TGV = \phi[\xi, \pi^m]_S \rfloor \psi[\xi, \pi^m]_S.$$

From Lemma 2.1, we have

$$\begin{aligned}[\xi, \pi^m]_{\mathrm{S}} &= \psi(\pi^m) \wedge \xi - \psi(\xi)\pi^m \quad \text{using } \psi(\pi^m \wedge \xi) = -\psi(\mathrm{base}_M) = 0 \\ \psi[\xi, \pi^m]_{\mathrm{S}} &= [\xi, \psi(\pi^m)]_{\mathrm{S}} - [\psi(\xi), \pi^m]_{\mathrm{S}} \\ &\qquad\qquad\qquad \text{using the equation above and } \ \psi \circ \psi = 0 \\ \psi(\pi^k) &= k\pi^{k-1} \wedge \psi(\pi) \qquad (k = 2, 3, \dots)\end{aligned}$$

and the generalized divergence of each 2-vector field is obtained by

$$\psi(X \wedge Y) = -\psi(X)Y + \psi(Y)X - [X, Y].$$

Remark 3.3. Using the relations above, we can split TGV into parts of π and ξ. Thus, we would say we can understand the Godbillon-Vey form or Godbillon-Vey class in the context of Poisson geometry.

Corollary 3.1. *Assume that the situation is the same as that of Theorem 3.3.*

(1) *If π is a unimodular, namely, if $\psi(\pi) = 0$, then $TGV = 0$ and the Godbillon-Vey class is zero.*
(2) *If we can find ξ to be a Poisson vector field, namely, $[\xi, \pi]_S = 0$, then $TGV = 0$ and the Godbillon-Vey class is zero.*

(3) *If we can find a divergence-free* ξ*, namely,* $\psi(\xi) = 0$*, then*

$$TGV = \phi(\psi(\pi^m) \wedge \xi) \rfloor [\xi, \psi(\pi^m)]_S.$$

Proof. (1) We use the same notation in this section. Then $\phi(\psi(\xi)\pi^m) = \psi(\xi)\theta^{2m+1}$ and

$$[\psi(\xi), \pi^m]_{\mathrm{S}} = \sum_{j=1}^{2m} [\psi(\xi), X_j]_{\mathrm{S}} X_1 \wedge \cdots \widehat{X_j} \cdots \wedge X_{2m},$$

thus $TGV = \phi(\psi(\xi)\pi^m) \rfloor [\psi(\xi), \pi^m]_{\mathrm{S}} = 0$. (This is already stated in [4].)

(2) and (3) are the direct corollary of Theorem 3.3. □

As an analogy of exact symplectic structures, there is a notion of exact Poisson structures. We recall the definition.

Definition 3.1. A Poisson structure π is called exact if there is a vector field Y satisfying $[Y, \pi]_{\mathrm{S}} = \pi$.

Remark 3.4. 1) In Definition 3.1, there is no restriction on the rank of π.

2) If π is exact in the sense of Definition 3.1, then π is exact in the sense of Poisson cohomology theory.

3) There is some ambiguity choosing Y in Definition 3.1 up to Poisson vector fields.

Corollary 3.2. *Let* (M^{2m+1}, π) *be a full-rank regular Poisson manifold. We assume* π *is exact by a vector field* Y*. We assume furthermore* $\pi^m \wedge Y \neq 0$ *everywhere. Then it turns out that the Godbillon-Vey class is zero, in fact* $TGV = 0$.

Proof. We can find a volume form vol_M so that $\langle \mathrm{vol}_M, \pi^m \wedge Y \rangle = 1$ and thereby we may take $-Y$ as ξ in Theorem 3.3. Then, using the same notation in the proof of Theorem 3.3, we have

$$\begin{aligned} \beta &= \phi([-\xi, \pi^m]_{\mathrm{S}}) = \phi([Y, \pi^m]_{\mathrm{S}}) \\ &= \phi(m\pi^{m-1} \wedge [Y, \pi]_{\mathrm{S}}) = \phi(m\pi^m) = m\alpha, \end{aligned}$$

and we see

$$\beta \wedge d\beta = m\alpha \wedge m\, d\alpha = m^2 \alpha \wedge \beta \wedge \alpha = 0.$$

□

3.2. 3-dimensional case.

If $\dim M = 3$, then the situation is very simple. From Theorem 3.3, we see that a Godbillon-Vey form corresponds to

$$TGV = \phi^{-1}(\beta \wedge d\beta) = \phi[\xi, \pi]_{\mathrm{S}} \rfloor \psi[\xi, \pi]_{\mathrm{S}}.$$

Since $\phi[\xi,\pi]_{\mathrm{S}}$ is a 1-form and $\psi[\xi,\pi]_{\mathrm{S}}$ is a 1-vector field because of $\dim M = 3$,

$$\begin{aligned}\phi^{-1}(\beta\wedge d\beta) &= \psi[\xi,\pi]_{\mathrm{S}} \lrcorner\, \phi[\xi,\pi]_{\mathrm{S}} = \psi[\xi,\pi]_{\mathrm{S}} \lrcorner\, \phi[\xi,\pi]_{\mathrm{S}} \lrcorner\, \mathrm{vol}_M \\ &= \frac{[\xi,\pi]_{\mathrm{S}}\wedge\psi([\xi,\pi]_{\mathrm{S}})}{\mathrm{base}_M}.\end{aligned}$$

Since $[\xi,\pi]_{\mathrm{S}}$ is 2-vector field and $\dim M = 3$, we see that $[\xi,\pi]_{\mathrm{S}}\wedge[\xi,\pi]_{\mathrm{S}} = 0$, and applying the map ψ to this equation, we get

$$0 = \psi([\xi,\pi]_{\mathrm{S}}\wedge[\xi,\pi]_{\mathrm{S}}) = [[\xi,\pi]_{\mathrm{S}},[\xi,\pi]_{\mathrm{S}}]_{\mathrm{S}} + 2\psi([\xi,\pi]_{\mathrm{S}})\wedge[\xi,\pi]_{\mathrm{S}}.$$

We therefore have the special result for 3-dimensional manifolds.

Corollary 3.3. *Consider a nowhere vanishing Poisson structure π on 3-dimensional manifold M. Choose a vector field ξ satisfying $\pi\wedge\xi = -\mathrm{base}_M$. Then a Godbillon-Vey form is given by*

$$-\frac{1}{2}\frac{[[\xi,\pi]_S,[\xi,\pi]_S]_S}{\mathrm{base}_M}\ \mathrm{vol}_M.$$

4. Examples.

We show some concrete examples of regular Poisson structures on 3-dimensional manifolds and their tangential Godbillon-Vey fields.

4.1. 3-dimensional tori.

We shall consider Poisson structures on the 3-dimensional torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$. Poisson structures and related objects on $\mathbf{T}^3$ may be lifted on the universal covering space $\mathbf{R}^3$, where we may consider $\mathbf{Z}^3$-invariant Poisson structures and objects. Let $x = (x^1,x^2,x^3)$ be the canonical coordinates of $\mathbf{R}^3$ and $D_j = \dfrac{\partial}{\partial x^j}$ for $j = 1,2,3$. Each 2-vector field on $\mathbf{T}^3$ is of form

$$\pi = a_1(x)D_2\wedge D_3 + a_2(x)D_3\wedge D_1 + a_3(x)D_1\wedge D_2$$

where $a_j(x)$ is $\mathbf{Z}^3$-invariant for $j = 1,2,3$. The Poisson condition for π is

$$\begin{vmatrix} a_1 & a_2 \\ [D_3,a_1]_{\mathrm{S}} & [D_3,a_2]_{\mathrm{S}}\end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ [D_1,a_2]_{\mathrm{S}} & [D_1,a_3]_{\mathrm{S}}\end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ [D_2,a_3]_{\mathrm{S}} & [D_2,a_1]_{\mathrm{S}}\end{vmatrix} = 0.$$

Rank 2 is equivalent to ${a_1}^2 + {a_2}^2 + {a_3}^2 \neq 0$.

We take the foliations generated by $\pi = (D_1 + b_2D_2)\wedge(D_1 + b_3D_3)$ as concrete examples. Then the Poisson condition is

$${b_3}^2[D_3,b_2]_{\mathrm{S}} + b_3[D_1,b_2]_{\mathrm{S}} - {b_2}^2[D_2,b_3]_{\mathrm{S}} - b_2[D_1,b_3]_{\mathrm{S}} = 0.$$

(This is just the integrability condition for the distribution.) If b_3 is a periodic function of x_3, then the Poisson condition is reduced to $b_3[D_3,b_2]_{\mathrm{S}}+$

$[D_1, b_2]_S = 0$. Take $dx^1 \wedge dx^2 \wedge dx^3$ as a volume form of $\mathbf{T}^3$. If we assume that b_3 is nonzero, then we may take $\xi = \frac{1}{b_3} D_2$. We have

$$[\xi, \pi]_S = \frac{[D_2, b_2]_S}{b_3} D_2 \wedge (D_1 + b_3 D_3)$$

and so

$$[[\xi, \pi]_S, [[\xi, \pi]_S]_S = 0.$$

Thus, the Godbillon-Vey classes of those Poisson structures are all zero.

4.2. Left invariant symplectic foliations of $PSL(2, \mathbf{R})/\Gamma$.

Let X_1, X_2, X_3 be the left-invariant vector fields of $SL(2, \mathbf{R})$ corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. Then we have the relation

$$[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = X_1.$$

Take a 2-vector field $\pi = p_1 X_2 \wedge X_3 + p_2 X_3 \wedge X_1 + p_3 X_1 \wedge X_2$. The Poisson condition for π is given by

$$p_1(-[X_2, p_3]_S + [X_3, p_2]_S) + p_2(-[X_3, p_1]_S - [X_1, p_3]_S) \\ + p_3(-[X_1, p_2]_S + [X_2, p_1]_S) + \left({p_1}^2 + 4p_2 p_3\right) = 0.$$

Let us assume that p_j are constant a_j ($j = 1, 2, 3$), namely we assume that π is a left-invariant 2-vector field $\pi = a_1 X_2 \wedge X_3 + a_2 X_3 \wedge X_1 + a_3 X_1 \wedge X_2$. Since

$$[\pi, \pi]_S = 2({a_1}^2 + 4a_2 a_3) X_1 \wedge X_2 \wedge X_3$$

holds, π is a Poisson tensor if and only if ${a_1}^2 + 4a_2 a_3 = 0$. This condition determines 2-dimensional subalgebras of $\mathfrak{sl}(2, \mathbf{R})$. (If we consider the Lie group $SO(3)$ and the left-invariant vector fields $\{Y_1, Y_2, Y_3\}$ with Lie bracket relations of $[Y_1, Y_2] = 2Y_3$, $[Y_2, Y_3] = 2Y_1$ and $[Y_3, Y_1] = 2Y_2$, then the Poisson condition for a 2-vector field $\pi = p_1 Y_2 \wedge Y_3 + p_2 Y_3 \wedge Y_1 + p_3 Y_1 \wedge Y_2$ is given by

$$p_1([Y_2, p_3]_S - [Y_3, p_2]_S) + p_2([Y_3, p_1]_S - [Y_1, p_3]_S) \\ + p_3([Y_1, p_2]_S - [Y_2, p_1]_S) - 2\left({p_1}^2 + {p_2}^2 + {p_3}^2\right) = 0.$$

If p_j ($j = 1, 2, 3$) are constant, then π must be zero. This corresponds to the fact that $SO(3)$ has no 2-dimensional Lie subgroup.) Consider a left-invariant rank 2 Poisson tensor $\pi = a_1 X_2 \wedge X_3 + a_2 X_3 \wedge X_1 + a_3 X_1 \wedge X_2$ on $SL(2, \mathbf{R})$, namely we put the conditions ${a_1}^2 + {a_2}^2 + {a_3}^2 \neq 0$ and ${a_1}^2 + 4a_2 a_3 = 0$. Take the left-invariant volume form dual to $X_1 \wedge X_2 \wedge X_3$. Then we may take $-\xi = \bigstar\pi = b_1 X_1 + b_2 X_2 + b_3 X_3$ with the condition $b_3 = 1/a_3$ if $a_2 = 0$ or $b_2 = (1 - a_1 b_1 - a_3 b_3)/a_2$ if $a_2 \neq 0$.

We have

$$[-\xi, \pi]_{\mathrm{S}} = (-a_1b_2 + 2a_3b_1)X_1 \wedge X_2 + 2(a_2b_2 - a_3b_3)X_2 \wedge X_3 \\ + (a_1b_3 - 2a_2b_1)X_3 \wedge X_1$$

and

$$[[\xi, \pi]_{\mathrm{S}}, [\xi, \pi]_{\mathrm{S}}]_{\mathrm{S}} = 8\Big((2a_3b_1 - a_1b_2)(-2a_2b_1 + a_1b_3) \\ + (a_2b_2 - a_3b_3)^2\Big)X_1 \wedge X_2 \wedge X_3 \\ = 8X_1 \wedge X_2 \wedge X_3$$

under the Poisson condition ${a_1}^2 + 4a_2a_3 = 0$ and the condition about b_1, b_2 and b_3.

Since $SL(2, \mathbf{R})$ is homeomorphic to $S^1 \times \mathbf{R}^+ \times \mathbf{R}$, $\mathrm{H}^3(SL(2, \mathbf{R}), \mathbf{R}) = \{0\}$ and the Godbillon-Vey classes of codimension 1 foliations of $SL(2, \mathbf{R})$ are zero. However, our discussion makes sense if we restrict ourselves to $PSL(2, \mathbf{R})/\Gamma$, where Γ is some co-compact discrete subgroup of $PSL(2, \mathbf{R})$. Thus, we see that the Godbillon-Vey form of our Poisson structure is $-4 \times$ canonical volume form. Since the left-invariant volume form on $SL(2, \mathbf{R})$ is also right-invariant, we have $\psi(X_i) = 0$ for $i = 1, 2, 3$ and $\psi(\pi) = -a_1X_1 - 2a_3X_2 - 2a_2X_3$. We get the same result from direct computation of $\phi(\psi(\pi) \wedge \xi) \rfloor [\xi, \psi(\pi)]_{\mathrm{S}}$.

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Received September 9, 1998 and revised December 28, 1998. This research was partially supported by Grant-in-Aid for Scientific Research, The Priority Area of the infinite dimensional integrability systems (No. 08211207) and the Fundamental Research (C) (No. 08640084, 10640057), The ministry of Education, Science and Culture, Japan.

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