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We prove that any two unitary operators with simple singular spectrum which contains the whole circle are unitarily equivalent up to a rank one operator.

Introduction.

This note is devoted to the spectral analysis of rank one perturbations of unitary and self-adjoint operators. We study the following question: given two cyclic (i.e., having simple spectrum) operators A and B, when is A equivalent to B up to a rank one perturbation? More precisely, when does there exist a unitary operator U such that rank $(UAU^* - B) = 1$? As usual, we are looking for an answer in terms of the spectra of A and B.

An analogous question for compact perturbations is answered by the Weyl-von Neumann Theorem [**K**]. It says that A is equivalent to $B + K$ for some compact K iff the essential spectra $\sigma_{\text{ess}}(A)$ and $\sigma_{\text{ess}}(B)$ coincide.

A necessary and sufficient condition for A and B to be equivalent up to a trace class operator, which was found by Carey and Pincus [**CP**], is more delicate and involves additional spectral invariants. In addition to the essential spectra, the isolated eigenvalues of A and B must now obey certain rules.

In this paper we make the last step down the ladder and study the case when A and B are equivalent up to a rank one perturbation. A general necessary and sufficient condition in these settings seems out of reach: It is impossible to formulate in any reasonable terms. However, it is still possible to achieve a good assessment of the situation by fully describing the most important particular case.

It is quite well understood how isolated eigenvalues of an operator behave under rank one perturbations. On the other hand, by the Weyl-von Neumann Theorem, if A is equivalent to B up to a rank one perturbation, then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$. In particular, rank one perturbations do not affect absolutely continuous spectrum. Hence, it seems reasonable to restrict our attention to the case when A and B have singular spectrum and

$$\sigma(A) = \sigma_{\text{ess}}(A) = \sigma(B) = \sigma_{\text{ess}}(B)$$

(where $\sigma(A)$ and $\sigma(B)$ denote the spectra of A and B respectively). Under this restriction we are able to give a complete answer to our question.

Denote $\mathbb{T} = \{|z| = 1\}$. We will say that operators U and V are completely non-equivalent if there are no non-trivial closed invariant subspaces H_1 and H_2 of U and V respectively such that the restriction of U on H_1 is unitarily equivalent to the restriction of V on H_2. Our main result is:

Theorem 1. *Let U and V be completely non-equivalent singular unitary cyclic operators such that $\sigma(U) = \sigma(V) = \mathbb{T}$. Then U and V are equivalent up to a rank one perturbation. I.e., there exist a unitary operator $\mathcal{U}$ and a rank one operator $\mathcal{V}$ such that*

$$\mathcal{U}U\mathcal{U}^* = V + \mathcal{V}.$$

We will prove this result in the next section. One can easily adapt the proof to replace unitary U and V with self-adjoint ones, and $\mathbb{T}$ with $\mathbb{R}$. An analogous result for the case of pure point operators follows from a function theoretic construction suggested by Aleksandrov [**A**] (see also [**P**]). However, the general case requires a different approach. Our main tool in the next section is the spectral shift function, which was originally introduced by Lifshits for finite rank perturbations and later studied by Krein in the trace class situation (see [**BY**] for the history of this notion and further references).

The following question naturally arises: Under which conditions on a closed set $K \subset \mathbb{T}$, can we replace $\mathbb{T}$ in the statement of Theorem 1 with K? Such necessary and sufficient conditions were formulated in [**P**] for the discrete self-adjoint case. They now can be proved to be valid for the general self-adjoint (unitary) case. We discuss this matter in the remark at the end of the next section.

Note, that if two unitary (self-adjoint) operators A and B satisfy

$$\sigma(A) = \sigma_{\text{ess}}(A) = \sigma(B) = \sigma_{\text{ess}}(B) = K$$

then, by the Weyl-von Neumann Theorem, they are equivalent up to a compact operator. Theorems 1 and 8 (below) refine this statement showing that, when K is sufficiently "good", the operators are much closer related.

To conclude the introduction, we would like to point out some function theoretical consequences of Theorem 1. Let $K\mu$ and $P\mu$ denote the Cauchy and Poisson integrals of the measure μ on $\mathbb{T}$ in the unit disk $\mathbb{D}$:

$$K\mu(z) = \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z} d\mu(\xi)$$

and

$$P\mu = \Re K\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi).$$

The results of [**P**] imply that Theorem 1 is equivalent to the following statement. For any two positive singular measures μ and ν on $\mathbb{T}$ such that $\mu \perp \nu$

and $\operatorname{supp}\mu = \operatorname{supp}\nu = \mathbb{T}$, there exist $f \in L^1(\mu), ||f|| = 1, f > 0$ μ-a.e. and $g \in L^1(\nu), ||g|| = 1, g > 0$ ν-a.e. such that

$$Kf\mu = \frac{1-\theta}{1+\theta} = (Kg\nu)^{-1}$$

for some inner function θ. From this statement one can obtain the following Corollary.

Corollary 2. *For any two singular measures μ and ν on $\mathbb{T}$ such that $\mu \perp \nu$ and $\operatorname{supp}\mu = \operatorname{supp}\nu = \mathbb{T}$, there exists an inner function θ such that $\theta(\xi) = 1$ μ-a.e. and $\theta(\xi) = -1$ ν-a.e. Moreover, for some positive functions $f \in L^1(\mu)$ and $g \in L^1(\nu)$*

$$\{\theta = 1\} = \{d(f\mu)/dm = \infty\}$$

and

$$\{\theta = -1\} = \{d(g\nu)/dm = \infty\},$$

where m is the normalized Lebesgue measure on $\mathbb{T}$.

With some additional effort one can deduce from the last statement the following:

Corollary 3. *For any singular measure μ on $\mathbb{T}$ and any $\phi \in L^\infty(\mu)$ there exists $\psi \in H^\infty$ such that $\psi(\xi) = \phi(\xi)$ for μ-a.e. ξ and $||\psi||_{H^\infty} = ||\phi||_{L^\infty(\mu)}$.*

The Krein-Lifshits spectral shift for a unitary pair.

We first define the Krein-Lifshits spectral shift for the rank one perturbation problem of unitary operators. This definition is very similar to the one given in [**S**] or [**P**] for the self-adjoint case.

Let U_1 be a unitary cyclic operator, v, $||v|| = 1$ its cyclic vector. Then we can consider the family of unitary rank one perturbations of U_1:

$$U_\alpha = U_1 + (\alpha - 1)(\cdot, U_1^{-1}v)v \qquad \alpha \in \mathbb{T}. \tag{1}$$

Denote by μ_α the spectral measure of v for U_α

$$\mu_\alpha(B) = (v, E_B(U_\alpha)v)$$

for any Borel $B \in \mathbb{T}$, where $E_B(U_\alpha)$ is the spectral projection of U_α. Note, that since $||v|| = 1$, all μ_α's are probability measures.

Since

$$K\mu_\alpha = ((U_\alpha + z)(U_\alpha - z)^{-1}v, v)$$

for $z \in \mathbb{D}$, after simple computations one can obtain

$$K\mu_\alpha = \frac{(\alpha - 1) + (\alpha + 1)K\mu_1}{(\alpha + 1) + (\alpha - 1)K\mu_1} \tag{2}$$

(e.g., [**Ar**]).

If $\alpha \in \mathbb{T}$ then $(1+\alpha)/(1-\alpha)$ is imaginary, i.e., is equal to ic for some real c. Define the spectral shift function $u \in L^\infty(\mathbb{T})$ for the perturbation problem $(U_1 \mapsto U_\alpha)$ as

$$u = \pi/2 + \arg(K\mu_1 - ic) \tag{3}$$

where *arg* stands for the principal branch of the argument taking values in $[0; 2\pi)$. All analytic functions in this paper are naturally defined a.e. on $\mathbb{T}$ by their non-tangential boundary values.

Next, we discuss some elementary properties of the spectral shift. Since such properties are well known (see for instance [**MP**], [**S**] or [**P**]), we do not supply all the proofs here.

The first important property (which follows from (2)) is that (3) can be extended to

$$u = \frac{\pi}{2} + \arg(K\mu_1 - ic) = \frac{\pi}{2} - \arg(K\mu_\alpha + ic). \tag{4}$$

Formula (4) shows connections between the spectral shift u and the spectral measures μ_1 and μ_α. Since in this paper we will mostly operate with spectral measures, we will often call u satisfying (4) the spectral shift of the pair of measures $(\mu_1; \mu_\alpha)$ and u satisfying (3) (for some real c) a spectral shift of μ_1.

For a pair of probability measures μ_1 and μ_α there exists at most one real c and function u satisfying (4). If a pair of measures $(\mu_1; \mu_\alpha)$ possesses a spectral shift, i.e., there exist u (and c) to satisfy (4), then there exist unitary operators U_1 and U_α satisfying (1) such, that μ_1 and μ_α are their spectral measures.

Since μ_1 is a singular positive measure (U_1 is a singular operator), the function u takes only two values – 0 and π – on $\mathbb{T}$. For each non-constant non-negative $L^\infty(\mathbb{T})$-function $u, ||u||_\infty \leq \pi$ there exist unique $c \in \mathbb{R}$ and a pair of probability measures $(\nu; \gamma)$ for which u is the spectral shift. If u takes only values 0 and π, then ν and γ are singular.

Suppose we have a sequence of functions $u_k \geq 0, ||u_k||_\infty \leq \pi$ which converge in measure to the function u_0 as $k \to \infty$. For each u_k, $k \geq 0$ there exist pairs of measures $(\mu_k; \nu_k)$ and constants c_k satisfying (4). Then $\mu_k \to \mu_0, \nu_k \to \nu_0$ in $*$-weak topology and $c_k \to c_0$ as $k \to \infty$.

We will say that two measures μ and ν are equivalent ($\mu \sim \nu$) if there exists a positive μ-a.e. function $f \in L^1(\mu)$ such that $\nu = f\mu$. We will denote by $\mu|_I$ the restriction of μ on the set I.

If $E \subset \mathbb{R}$ we will write that

$$p.\ v. \int_E d\mu(x) < \infty \quad (> -\infty)$$

if

$$\liminf_{\epsilon\to 0} \int\limits_{E\setminus(-\epsilon;\epsilon)} d\mu(x) < \infty \quad (\limsup > -\infty).$$

Let u_1 and u_2 be the phase shifts of the pairs of measures (μ_1, ν_1) and (μ_2, ν_2) respectively. Then, as follows from the definition,

$$\exp\left(iK\left(u_1 - \frac{\pi}{2}\right) + d_1\right) = K\mu_1 - ic_1 = (K\nu_1 + ic_1)^{-1} \tag{5}$$

for some real c_1 and d_1 and

$$\exp\left(iK\left(u_2 - \frac{\pi}{2}\right) + d_2\right) = K\mu_2 - ic_2 = (K\nu_2 + ic_2)^{-1} \tag{6}$$

for some real c_2 and d_2.

If I is an open subset of $\mathbb{T}$ and $u_1 = u_2$ on I then, $\mu_1|_I \sim \mu_2|_I$ and $\nu_1|_I \sim \nu_2|_I$. More generally we have the following lemma.

Lemma 4 ([**P**]). *Let spectral shift functions u_1, u_2 satisfy* (5) *and* (6) *for some singular measures $\mu_{1,2}, \nu_{1,2}$ and real constants $c_{1,2}, d_{1,2}$. Let $K \subset \mathbb{T}$ be a measurable set. Put $c = \exp(d_1 - d_2)$. Then*

(i)
$$p.\ v. \int\limits_{[-\pi;\pi]} \left(u_1\left(e^{i(t+x)}\right) - u_2\left(e^{i(t+x)}\right)\right)\frac{dt}{t} < \infty$$

for μ_1-a.e. point $e^{ix} \in K$ iff the restriction of μ_1^s on K is absolutely continuous with respect to μ_2;

$$p.\ v. \int\limits_{[-\pi;\pi]} \left(u_1\left(e^{i(t+x)}\right) - u_2\left(e^{i(t+x)}\right)\right)\frac{dt}{t} > -\infty$$

for ν_1-a.e. point $e^{ix} \in K$ iff the restriction of ν_1^s on K is absolutely continuous with respect to ν_2;

(ii) *if μ_2^s-a.e. $x \in K$ is a Lebesgue point of $u_1 - u_2$ and*

$$p.\ v. \int\limits_{[-\pi;\pi]} \left(u_1\left(e^{i(t+x)}\right) - u_2\left(e^{i(t+x)}\right)\right)\frac{dt}{t} = f(x) < \infty$$

for μ_2^s-a.e. $e^{ix} \in K$ then the restriction of μ_1^s on K is equal to the restriction of $ce^f\mu_2^s$ on K; if ν_2^s-a.e. $x \in K$ is a Lebesgue point of $u_1 - u_2$ and

$$p.\ v. \int\limits_{[-\pi;\pi]} \left(u_1\left(e^{i(t+x)}\right) - u_2\left(e^{i(t+x)}\right)\right)\frac{dt}{t} = f(x) > -\infty$$

for ν_2^s-a.e. $e^{ix} \in K$ then the restriction of ν_1^s on K is equal to the restriction of $ce^{-f}\nu_2^s$ on K.

When one of the measures $\mu_{1,2}$ is a point mass, Lemma 4 gives (see [**P**]):

Lemma 5 ([**MP**])**.** *Let u be the spectral shift of a pair of measures $(\mu;\nu)$. The measure μ has a point mass at e^{ix} iff*

$$\int_{x-1}^{x+1} \left(\pi\chi_{\{e^{it}:t\in(x;x+1)\}}(e^{iy}) - u(e^{iy})\right)\frac{dy}{y-x} < \infty. \tag{7}$$

The measure ν has a point mass at x iff

$$\int_{x-1}^{x+1} \left(\pi\chi_{\{e^{it}:t\in(x-1;x)\}}(e^{iy}) - u(y)\right)\frac{dy}{x-y} < \infty. \tag{8}$$

We will also need the following important example.

Example 6. Let $E\subset\mathbb{T}$ be a closed set, $|E|=0$. Denote $E=\mathbb{T}\backslash\cup I_n$ where I_n are disjoint open arcs. Suppose $\mu>0$ be a measure such that *supp* $\mu=E$ and

$$\frac{d\mu}{dm}=\infty$$

at every point of E. Let u be a spectral shift of μ. Then there exist a measure ν and real constants c and d such that:

$$\exp\left(iK\left(u-\frac{\pi}{2}\right)+d\right) = K\mu - ic = (K\nu+ic)^{-1}. \tag{9}$$

We claim that ν is a pure point measure with at most one point mass at each I_n.

Indeed, since the derivative of μ with respect to the Lebesgue measure is infinite on E, $\Re K\mu$ tends to ∞ non-tangentially at every point of E. This together with (9) implies that $K\nu$ tends to $-ic$ non-tangentially at every point of E. Since ν is a singular measure, $|K\nu|$ tends to ∞ ν-a.e. Therefore $\nu(E)=0$. Since, by (9), ν is concentrated at those points where $K\mu-ic$ tends to 0 and $K\mu$ is analytic on $\mathbb{T}$ outside of E, ν is pure point. Since μ is positive, $\Im K\mu$ is monotonic on every I_n, and therefore ν has at most one point mass at each of them.

Our main tool in the proof of Theorem 1 is the following lemma.

Lemma 7. *Let μ,ν be singular measures on $\mathbb{T}$, $I\subset\mathbb{T}$ be an open arc, $E\subset I$ be a closed set, $|E|=0$. Suppose that $I\subset \operatorname{supp}\,\mu$ and $I\subset\operatorname{supp}\,\nu$.*

Then for any $\epsilon>0$ there exist closed subsets F and G of I, and measures μ' and ν' satisfying the following conditions:

(1) *$|F|=|G|=0$, $\mu(G)=\nu(F)=0$ and $E\subset F$,*
(2) *$\mu'\sim\mu|_F$ and $\nu'\sim\nu|_G$,*
(3) *the pair (μ',ν') possesses a phase shift u,*

(4) *$u = \pi$ on $\mathbb{T} \setminus I$ and*

$$\int_I (\pi - u(\xi))dm(\xi) < \epsilon.$$

Remark. In the statement of Lemma 7, (4) can be replaced with:
$(4')$ $u = 0$ on $\mathbb{T} \setminus I$ and

$$\int_I u(\xi)dm(\xi) < \epsilon.$$

If $x, y \in \mathbb{T}, x \neq y$, we will denote by $(x; y)$ the open arc going from x to y counterclockwise.

Proof. Denote $F_0 = E$. WLOG

$$\frac{d(\mu|_{F_0})}{dm} = \infty \tag{10}$$

everywhere on $F_0 = E$ (if this is not true, we can always choose an equivalent measure with this property). Put $\mu_0 = \mu|_{F_0}$.

Step 1.

Chose $c \in \mathbb{R}$ so that

$$\{\Im K\mu_0 > c\} \subset I \tag{11}$$

and

$$|\{\Im K\mu_0 > c\}| < \epsilon/2. \tag{12}$$

There exist a spectral shift function u_0 and a measure ν_0 satisfying

$$\exp\left(iK\left(u_0 - \frac{\pi}{2}\right) + d_0\right) = K\mu_0 - ic_0 = (K\nu_0 + ic_0)^{-1} \tag{13}$$

for some real c_0 and d_0. Suppose $F_0 = I \setminus \cup I_n^0$ where $I_n^0 = (x_n^0; y_n^0)$ are disjoint open arcs. Condition (10), in the same way as in Example 6, implies that $\nu_0 = \sum \alpha_k^0 \delta_{z_k^0}$ where $z_k^0 \in I_{n_k}$ for some sequence n_k.

Now we will replace point masses z_k^0 with "pieces" of the measure ν.

For each k let $V_k^0 \subset I_{n_k}^0$ be a neighborhood of z_k^0 such that

$$|\cup V_k^0| < \epsilon/4 \tag{14}$$

and

$$\int_{\{x : e^{ix} \in \cup V_k^0\}} \frac{1}{|x - y|} dx < 1/2 \tag{15}$$

for any $y \in F_0$.

Choose a closed set $H_1, |H_1| = 0$ so that $\nu(H_1 \cap V_k^0) > 0$ for any k. Consider the measure $\gamma_1 \sim \nu|_{H_1}$ chosen so that $d\gamma_1/dm = \infty$ at any point of H_1.

Let v_1 be a phase shift function of the measure γ_1. In each neighborhood V_k^0 choose points a_k^0 and b_k^0 so that $z_k^0 \in (a_k^0; b_k^0)$ and

$$\int_{\{\arg a_k^0 < x < \arg a_k^0 + 1\}} \frac{v_1(e^{ix})dx}{x - \arg a_k^0} = \infty, \tag{16}$$

$$\int_{\{\arg b_k^0 - 1 < x < \arg b_k^0\}} \frac{(\pi - v_1(e^{ix}))dx}{\arg b_k^0 - x} = \infty \tag{17}$$

and ν has no point masses at a_k^0, b_k^0. Note that it is always possible to choose a_k^0 and b_k^0 satisfying (16) and (17) because γ_1 has non-zero mass on V_k^0.

Put $G_1 = H_1 \cap \cup(a_k^0; b_k^0)$. Define the function u_1 to be equal to $-v_1$ on each interval $(a_k^0; b_k^0)$ and to u_0 outside of $\cup(a_k^0; b_k^0)$. Then u_1 is the phase shift of a pair of measures $(\mu_1; \nu_1)$ such that $\mu_1 = f_1\mu|_{F_0} + \sum \alpha_k^1 \delta - z_k^1$ and $\nu_1 = g_1\nu|_{G_1}$ for some positive functions $f_1 \in L^1(\mu|_{F_0})$, $g_1 \in L^1(\nu|_{G_1})$ and for some sequence of points $\{z_k^1\} \subset \cup V_k^0 \setminus G_1$.

Indeed, on $F_0 = E$ we have $\mu_0 \sim \mu_1$ by (15) and Lemma 4. On $\mathbb{R} \setminus (F_0 \cup \bigcup[a_k^0; b_k^0])$ the measures μ_1 and ν_1 are not supported because u is locally constant there. Also, μ_1 and ν_1 do not have point masses at points a_k^0, b_k^0 because by (16) and (17) conditions (7) and (8) are not satisfied there. Since $u_1 = -v_1$ on each $(a_k^0; b_k^0)$, $\nu_1 \sim \nu_0$ on $\cup(a_k^0; b_k^0)$. Since $d\nu_0/dm = \infty$ on G_1, we have that $\mu|_{\cup(a_k^0; b_k^0)}$ is a discrete measure with point masses in $\cup(a_k^0; b_k^0) \setminus G_1$.

Step n, n is even.

After step $n-1$ we obtained a shift function u_{n-1} of a pair of measures (μ_{n-1}, ν_{n-1}) such that $\mu_{n-1} = f_{n-1}\mu|_{F_{n-2}} + \sum \alpha_k^{n-1}\delta_{z_k^{n-1}}$ and $\nu_{n-1} = g_{n-1}\nu|_{G_{n-1}}$, where $|F_{n-2}| = 0$, $F_{n-2} \supset E$, $|G_{n-1}| = 0$ and $\{z_k^{n-1}\} \subset \cup V_k^{n-1} \setminus G_{n-1}$. Note that the pair (μ_{n-1}, ν_{n-1}) "almost" satisfies the conditions of the lemma, except for the discrete part $\sum \alpha_k^{n-1}\delta_{z_k^{n-1}}$ of μ_{n-1}. We must, therefore, replace those point masses with "pieces" of μ.

We will do it in the same way as in Step 1. First we choose neighborhoods V_k^n of points z_k^n such that

$$\cup V_k^n \subset \cup V_k^{n-1}, \tag{18}$$

$$|\cup V_k^n| < \epsilon/2^{n+1} \tag{19}$$

and

$$\int_{\{x: e^{ix} \in \cup V_k^n\}} \frac{dx}{|x-y|} < 1/2^n \tag{20}$$

at each point $y \in F_{n-2} \cup G_{n-1}$. After that inside $\cup V_k^n$ we choose a closed set H_n such that $\mu(V_k^n \cap H_n) > 0$ for any k. Also let H_n satisfy

$$\mu(\cup_k V_k^n \setminus H_n) < 1/n. \tag{21}$$

We choose a measure $\gamma_n \sim \mu|_{H_n}$ so that $d\gamma_n/dm = \infty$ at each point of H_n. Denote by v_n a shift function of γ_n. Inside each neighborhood V_k^n choose points a_k^n and b_k^n so that $z_k^0 \in (a_k^0; b_k^0)$ and

$$\int_{\{\arg a_k^n < x < arg a_k^n + 1\}} \frac{v_n dx}{x - \arg a_k^n} = \infty, \tag{22}$$

$$\int_{\{\arg b_k^n - 1 < x < arg b_k^n\}} \frac{(\pi - v_n) dx}{\arg b_k^n - x} = \infty \tag{23}$$

and μ has no point masses at a_k^n, b_k^n.

Put $F_n = F_{n-2} \cup [\cup(a_k^n; b_k^n) \cap H_n]$. Define the spectral shift function u_n to be equal to v_n on $\cup(a_k^n; b_k^n)$ and to u_{n-1} elsewhere. Then in the same way as in Step 1 we can show that u_n is the shift function of the pair (μ_n, ν_n), where $\mu_n = f_n \mu|_{F_n}$ for some positive function $f_n \in L^1(\mu|_{F_n})$, and $\nu_n = g_n \nu|_{G_{n-1}} + \sum \alpha_k^n \delta_{z_k^n}$ for some positive function $g_n \in L^1(\nu|_{G_{n-1}})$, positive constants α_k^n and points z_k^n from $\cup V_k^n \setminus F_n$. The closed sets F_n and G_n satisfy $|G_n| = |F_n| = 0, F_n \supset E$.

Step n, n is odd.

Our construction is similar to the one we used for the even n. Essentially, we only have to replace μ with ν and ν with μ.

I.e., after step $n-1$ we obtained a shift function u_{n-1} of a pair of measures (μ_{n-1}, ν_{n-1}) such that $\mu_{n-1} = f_{n-1}\mu|_{F_{n-1}}$ and $\nu_{n-1} = g_{n-1}\nu|_{G_{n-2}} + \sum \alpha_k^{n-1} \delta_{z_k^{n-1}}$, where $|F_{n-1}| = 0$, $F_{n-1} \supset E$, $|G_{n-2}| = 0$, and $\{z_k^{n-1}\} \subset \cup V_k^{n-1} \setminus F_{n-1}$. First we choose neighborhoods V_k^n of points z_k^{n-1} satisfying (17), (18) and (19) for each $y \in F_{n-1} \cup G_{n-2}$. After that inside $\cup V_k^n$ we choose a closed set H_n satisfying (21) and such that $\nu(V_k^n \cap H) > 0$ for any k. We choose a measure $\gamma_n \sim \nu|_{H_n}$ so that $d\gamma_n/dm = \infty$ at each point of H_n. Denote by v_n a shift function of γ_n. Inside each neighborhood V_k^n choose points $a_k^n < z_k^n$ and $b_k^n > z_k^n$ so that

$$\int_{\{\arg a_k^n < x < \arg a_k^n + 1\}} \frac{\pi - v_n dx}{x - \arg a_k^n} = \infty, \tag{24}$$

$$\int_{\{\arg b_k^n - 1 < x < \arg b_k^n\}} \frac{v_n dx}{\arg b_k^n - x} = \infty \tag{25}$$

and ν has no point masses at a_k^n, b_k^n. Put $G_n = G_{n-2} \cup \cup(a_k^n; b_k^n)$. Define the spectral shift function u_n to be equal to v_n on $\cup(a_k^n; b_k^n)$ and to u_{n-1} elsewhere. Then in the same way as before we can show that u_n is the shift function of the pair (μ_n, ν_n), where $\nu_n = g_n \nu|_{G_n}$ for some closed set G_n, $|G_n| = 0$ and positive $g_n \in L^1(\nu|_{G_n})$, and $\mu_n = f_n \mu|_{F_{n-1}} + \sum \alpha_k^n \delta_{z_k^n}$ for some set F_{n-1}, $|F_{n-1}| = 0$, $, F_{n-1} \supset E$, positive function $f_n \in L^1(\mu|_{F_{n-1}})$, positive constants α_k^n and points z_k^n from $\cup V_k^n \setminus G_n$.

Conclusion of proof.

After step n of our construction we obtain the spectral shift function u_n of a pair of measures (μ_n, ν_n). Let u_n and $(\mu_n; \nu_n)$ satisfy

$$\exp\left(iK\left(u_n - \frac{\pi}{2}\right) + d_n\right) = K\mu_n - ic_n = (K\nu_n + ic_n)^{-1}$$

for some real c_n, d_n. Note that $\{u_n \neq u_{n-1}\} \subset \cup V_k^n$ and therefore by (19) the sequence u_n converges in measure to a function u. Let u be the spectral shift of a pair of measures (μ', ν'), i.e.,

$$\exp\left(iK\left(u - \frac{\pi}{2}\right) + d'\right) = K\mu' - ic' = (K\nu' + ic')^{-1}$$

for some real c', d'. We claim that (μ', ν') satisfies the conditions of the lemma with $F = \text{Clos} \cup F_{2n}$ and $G = \text{Clos} \cup G_{2n+1}$.

Indeed, $|F| = 0$ because

$$F = \text{Clos}\,[\cup F_{2n}] = \cup F_{2n} \cup F'$$

where

$$F' \subset [\cap_n \cup_k V_k^n] \setminus \cup_n H_n \subset \cap_n \cup_k V_k^n. \tag{26}$$

Since $| \cup_k V_k^n|$ tends to 0 as $n \to \infty$ by (19), $|F| = 0$.

Also, (26) and (21) imply that $\mu(F') = 0$. Therefore

$$\mu|_F = \mu|_{\cup F_{2n}} \tag{27}$$

and $\mu(G) = 0$. Hence we have to show that $\mu' \sim \mu|_{\cup F_{2n}}$.

Since $u_n \to u$ in measure, $\mu_n \to \mu'$ in $*$-weak topology, $c_n \to c'$ and $d_n \to d'$. Consider the sequence $\{\mu_{2n}\}$. Each measure μ_{2n} is equivalent to $\mu|_{F_{2n}}$ where $F_{2n} \subset F_{2n+2}$. Inequality (20) and Lemma 4 imply that

$$1 - \frac{1}{2^{n-1}} < \frac{d\mu_{2n+2}}{d\mu_{2n}}(y) < 1 + \frac{1}{2^{n-1}} \tag{28}$$

at any point y of F_{2n} for sufficiently big n (note that $\exp(d_{n-1} - d_n) \to 1$). Since

$$||\mu_{2n+2}|| = ||\mu_{2n}|| = ||\mu|_{F_{2n}}|| = 1,$$

(28) implies that for large n

$$||\mu_{2n+2} - \mu_{2n}|| < \frac{1}{2^{n-2}}.$$

Therefore $\mu_{2n} \to \mu'$ in norm. Since $\mu_{2n} \sim \mu|_{F_{2n}}$, $\mu' \sim \mu|_{\cup F_{2n}} = \mu|_F$.

Similarly, $|G| = 0$, $\nu(F) = 0$ and $\nu' \sim \nu|_G$. □

Proof of Theorem 1. Since U and V are completely non-equivalent, the spectral measures of U and V are mutually singular. Thus, for any pair of singular probability measures $(\mu; \nu)$ on $\mathbb{T}$ such that $\mu \perp \nu$ we have to show that there exists an equivalent pair $(\mu_0; \nu_0)$ possessing a phase shift u_0. We will do it by constructing u_0. The main part of our construction will consist of recursive applications of Lemma 7.

Let M and N be disjoint subsets of $\mathbb{T}$ such that $|M| = |N| = 0$, $\mu(M) = 1, \mu(\mathbb{T} \setminus M) = 0$ and $\nu(N) = 1, \nu(\mathbb{T} \setminus N) = 0$.

Step 1.

Let E_1 be a closed subset of M such that

$$\mu(M \setminus E) < 1. \tag{29}$$

By Lemma 7 there exist closed sets F_1 and G_1 such that $|F_1| = |G_1| = 0$, $\mu(G_1) = \nu(F_1) = 0$, $E_1 \subset F_1$ and $(f_1\mu|_{F_1}; g_1\nu|_{G_1})$ possesses a phase shift, for some positive summable f_1 and g_1.

Step n, n is even.

After step $n-1$ we obtained closed sets F_{n-1} and G_{n-1} of zero measure such that $\mu(G_{n-1}) = \nu(F_{n-1}) = 0$ and $(f_{n-1}\mu|_{F_{n-1}}; g_{n-1}\nu|_{G_{n-1}})$ possesses a phase shift u_{n-1}, for some positive summable functions f_n and g_n. Denote $\mathbb{T} \setminus (F_{n-1} \cup G_{n-1}) = \cup_k I_k^n$ where I_k^n are disjoint open arks.

Choose a closed subset E_n of N so that $E_n \subset \cup_k I_k^n$ and

$$\nu(N \setminus (G_{n-1} \cup E_n)) < 1/n. \tag{30}$$

Note that such a choice of E_n is possible because $\nu(F_{n-1}) = 0$. Since E_n is closed, there exist open arcs J_k^n such that $\operatorname{Clos} J_k^n \subset I_k^n$ $(k = 1, 2, \dots)$, $E_n \subset \cup J_k^n \subset \cup I_k^n$ and

$$\operatorname{dist}(\cup_k J_k^n, F_{n-1} \cup G_{n-1}) = \delta_n > 0. \tag{31}$$

By Lemma 7, for each k there exist closed sets $F_k^n, G_k^n \subset J_k^n$ such that $G_k^n \supset E_n \cap J_k^n$ and $(f_k^n\mu|_{F_k^n}; g_k^n\nu|_{G_k^n})$ possesses a spectral shift u_k^n. We choose u_k^n to satisfy condition 4) from the statement of Lemma 7 for $I = J_k^n$ and

$$\epsilon = \epsilon_k^n = \delta_n/2^{n+k} \tag{32}$$

if $u_{n-1} = \pi$ on I_k^n and choose u_k^n to satisfy condition 4′) if $u_{n-1} = 0$ on I_k^n (note that u_{n-1} is constant on each I_k^n).

Define u_n to be equal to u_k^n on each I_k^n. We claim that then u_n is a phase shift of a pair of measures $(f_n\mu|_{F_n}; g_n\nu|_{G_n})$ where f_n and g_n are positive summable functions, $F_n = F_{n-1} \cup \operatorname{Clos} \cup_k F_k^n$ and $G_n = G_{n-1} \cup \operatorname{Clos} \cup_k G_k^n$. Indeed, let u_n be the phase shift of a pair $(\mu_n; \nu_n)$. Recall that on each J_k^n, where u_{n-1} was equal to π, the functions u_k^n satisfy condition 4) from Lemma 7, and on those J_k^n, where u_{n-1} was equal to 0, the functions u_k^n satisfy condition 4′), with ϵ satisfying (32). Therefore

$$|\{u_n \neq u_{n-1}\}| < \delta_n/2^n < 1/2^{n-1}. \tag{33}$$

Also by (32) and (33)

$$\int\limits_{[-\pi;\pi]} |u_n(e^{i(t+x)}) - u_{n-1}(e^{i(t+x)})|\frac{dt}{|t|} < \sum_k \epsilon_n^k/\delta_n < 1/2^n \tag{34}$$

for any point $e^{ix} \in F_{n-1} \cup G_{n-1}$. Hence, by Lemma 4, $\mu_n|_{F_{n-1}} \sim \mu|_{F_{n-1}}$ and $\nu_n|_{F_{n-1}} \sim \nu|_{F_{n-1}}$. Also, on each J_k^n, since u_n is equal to u_k^n, we have that $\mu_n|_{I_k^n} \sim \mu|_{F_k^n}$ and $\nu_n|_{I_k^n} \sim \nu|_{G_k^n}$. Therefore $\mu_n = f_n\mu|_{F_n}$ and $\nu_n = g_n\nu|_{G_n}$ for some positive summable f_n and g_n. By construction $G_n \supset E_n$. Also, since $\mu(G_{n-1}) = \nu(F_{n-1}) = 0$, $\mu(G_k^n) = \nu(F_k^n) = 0$ (by condition (1) of Lemma 7), $\operatorname{Clos} \cup F_k^n \setminus \cup F_k^n \subset (F_{n-1} \cup G_{n-1})$ and $\operatorname{Clos} \cup G_k^n \setminus \cup G_k^n \subset (F_{n-1} \cup G_{n-1})$, we have $\mu(G_n) = \nu(F_n) = 0$.

Step n, n is odd.

The construction here is essentially the same as in the previous case.

After step $n-1$ we obtained closed sets F_{n-1} and G_{n-1} of zero measure such that $\mu(G_{n-1}) = \nu(F_{n-1}) = 0$ and $(f_{n-1}\mu|_{F_{n-1}}; g_{n-1}\nu|_{G_{n-1}})$ possesses a phase shift u_{n-1}, for some positive summable functions f_n and g_n. Let us again denote $\mathbb{T} \setminus (F_{n-1} \cup G_{n-1}) = \cup_k I_k^n$.

Choose a closed subset E_n of M so that $E_n \subset \cup_k I_k^n$ and

$$\mu(M \setminus (F_{n-1} \cup E_n)) < 1/n. \tag{35}$$

After that in the same way as above we construct the phase shift function u_n of a pair $(\mu_n; \nu_n) = (f_n\mu|_{F_n}; g_n\nu|_{G_n})$. The function u_n satisfies (33) and (34). The sets F_n and G_n are zero-measure closed sets such that $F_n \supset E_n$ and $\mu(G_n) = \nu(F_n) = 0$.

Conclusion of proof.

This part is similar to the corresponding part in the proof of Lemma 7.

By (33) u_n tend in measure to a function u_0. Let u_0 be the shift function of a pair $(\mu_0; \nu_0)$. Then μ_n and ν_n weakly converge to μ_0 and ν_0 respectively. Since $u_n \to u_0$, condition (34) and Lemma 4 imply that

$$1 - 1/2^{n+1} < c_n \frac{f_n}{f_{n+1}} < 1 + 1/2^{n+1} \tag{36}$$

μ_n-a.e. for some $c_n \to 1$. Since all our measures have norm 1, (36) implies that $\mu_n \to \mu_0$ in norm. Since $\mu_n \sim \mu|_{F_n}$, $\mu_0 \ll \mu$. Since (by (36)) $\mu(F_n) > 1 - 1/2^{n-1}$ and

$$\mu_n = \frac{f_n}{f_{n+1}} \mu_{n+1}|_{F_n}, \tag{37}$$

where functions f_n and f_{n+1} satisfy (36), $\mu \ll \mu_0$. Hence $\mu \sim \mu_0$. Similarly $\nu \sim \nu_0$. $\square$

Remark. A closed set K can replace $\mathbb{T}$ in the statement of Theorem 1 iff it is not very "porous". More precisely, the following generalization of Theorem 1 can be proved. We will state it in self-adjoint settings.

Definition. We will say that two disjoint sets of real numbers $\mathcal{A}$ and $\mathcal{B}$ are well-mixed if they satisfy the following conditions:

(1) For any two points $x, y \in \mathcal{A}$ each of the sets $(x; y)$ and $\mathbb{R}\backslash[x; y]$ contains at least one point from $\mathcal{B}$.
(2) For any two points $x, y \in \mathcal{B}$ each of the sets $(x; y)$ and $\mathbb{R}\backslash[x; y]$ contains at least one point from $\mathcal{A}$.

We will denote by $\sigma_{p.p.}(A)$ the set of eigenvalues of the operator A.

Theorem 8. *Let $K \subset \mathbb{R}$ be a closed set with no isolated points. Denote by $I_1, I_2, \dots$ the disjoint open intervals $I_k = (x_k; y_k)$ such that $K = \mathbb{R} \setminus \bigcup I_n$. Denote $\partial I = \{x_1, y_1, x_2, y_2, \dots\}$.*

Then the following two conditions are equivalent:

(i) *Any two completely non-equivalent cyclic self-adjoint operators A and B such that the sets*
$$(\sigma_{p.p.}(A) \setminus \sigma_{p.p.}(B)) \cap \partial I$$
and
$$(\sigma_{p.p.}(B) \setminus \sigma_{p.p.}(A)) \cap \partial I$$
are well-mixed and $\sigma(A) = \sigma(B) = K$ are unitarily equivalent up to a rank one perturbation.

(ii) *If $y \in K \setminus \{x_1, y_1, x_2, y_2, \dots\}$ then*
$$\int\limits_{(y-1;y+1)\backslash K} \frac{dx}{|y-x|} < \infty;$$
if $y = x_k$ or $y = y_k$ for some $k \in \mathbb{N}$ then
$$\int\limits_{(y-1;y+1)\backslash (K \cup I_k)} \frac{dx}{|y-x|} < \infty.$$

In particular if the spectrum of A and B is an interval $[a; b]$ such that
$$\{a, b\} \not\subset (\sigma_{p.p.}(A) \setminus \sigma_{p.p.}(B)) \text{ and } \{a, b\} \not\subset (\sigma_{p.p.}(B) \setminus \sigma_{p.p.}(A)),$$
then A is equivalent to B up to a rank one perturbation.

Theorem 8 was proved in [**P**] for pure point operators. One can easily combine the proof with the methods of the present paper to prove the general case. Here is another (simpler) way to generalize Theorem 1. We prohibit our operators to have eigenvalues at the endpoints of the complementary intervals.

Theorem 9. *Let $K \subset \mathbb{R}$ be a closed set. Denote by $I_1 = (x_1; y_1), I_2 = (x_2; y_2), \dots$ disjoint open intervals such that $K = \mathbb{R} \setminus \bigcup I_n$. Let A and B be two completely non-equivalent self-adjoint cyclic operators. Suppose that $\sigma(A) = \sigma(B) = K$ and $\sigma_{p.p.}(A) \cap \{x_1, y_1, x_2, y_2, \dots\} = \sigma_{p.p.}(B) \cap \{x_1, y_1, x_2, y_2, \dots\} = \emptyset$. Then A and B are equivalent up to a rank one perturbation.*

This theorem was also proved in [**P**] in the pure point case.

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