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Some years ago, in D'Hoker and Phong (1989) studied the functional determinants of Laplacian on Mandelstam diagrams. They considered some renormalizations of the functional determinants of Laplacian on Mandelstam diagrams and explored their applications in String Theory. Recently, on quite a different subject, in Qing (1997) studied the renormalized energy for Ginzburg-Landau vortices on closed surfaces. In this paper we shall demonstrate how those two different renormalized functionals are related to each other.

1. Introduction.

Ginzburg and Landau in [**GL**] introduced the following variational problem in the study of super-conductivity

$$E_\epsilon[u] = \int \left\{ |\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2 \right\} dx$$

where u is a complex-valued function called a condensate wave function. In the seminal work [**BBH**] Bethuel, Brezis and Helein studied the asymptotic behavior of minimizers of E_ϵ and proved that the limit configuration of vortices of minimizers of E_ϵ minimizes the renormalized energy function, which was derived through renormalization of E_ϵ. After their works, many mathematicians have been attracted to the study of Ginzburg-Landau problems. In [**BBH**] it assumes that the domain is bounded and star-shaped. Later Struwe in [**Str1**], [**Str2**] extended to any smooth bounded domain. Soon after, first in Bethuel and Riviere [**BR**] studied the problem with magnetic fields on a bounded domain in R^2. Then in Qing [**Q**] studied the problem on a closed surface. From geometric point of view it is simply the variational problem of Yang-Mills-Higgs functional with $U(1)$ gauge group. The functional is defined for a unitary connection A on a hermitian line bundle L over a closed surface M with metric g and a smooth section s of L as

$$E_\epsilon[A, s] = \int_M \left\{ |d_A s|^2 + \frac{1}{4\epsilon^2}(1-|s|^2)^2 + |F_A|^2 \right\} dx, \tag{1.1}$$

where F_A is the curvature 2-form of A and $d_A s$ is the covariant derivative of s. It was proved that:

Theorem A (Qing). *For any sequence of minimizers* $(A_{\epsilon_k}, s_{\epsilon_k})$ *for* E_{ϵ_k} *as* $\epsilon_k \to 0$, *there exists a subsequence* $(A_{\epsilon_j}, s_{\epsilon_j})$ *whose "vortices" (not precisely defined)*

$$p_i^{\epsilon_j} \to p_i^* \quad \text{for } i = 1, 2, \ldots, d$$

and $\{p_i^*\}_{i=1}^d$ *minimizes the renormalized energy*

$$D[\{p_i\}] = 2\pi \sum_{i \neq k}^{d} G(p_k, p_i) + 2\pi \sum_{k=1}^{d} G(p_k, p_k) + 2\pi \sum_{k=1}^{d} R(p_k) \tag{1.2}$$

where $G(\cdot, p)$ *is the Green function with its pole at* p, $G(p,p)$ *is the regular part of Green function (please see* [**Q**]*) and* R *is the solution of*

$$\Delta R + R = \sum_{k=1}^{d} G(\cdot, p_k) - \frac{2\pi d}{\text{vol}(M)} \text{ on } M. \tag{1.3}$$

An important and interesting question is how the renormalized energy depends on the Riemannian metrics g. It turned out that the renormalized energy $D[\{p_i\}]$ can be better understood if we use Arakelov Green functions.

Mandelstam diagrams are surfaces whose curvature are all concentrated at isolated points. On regular compact surface Polyakov-Ray-Singer [**Po**], [**RS**], [**OPS**], [**Ch**] formula for log determinant of Laplacian is

$$\log \frac{\det \Delta_1}{\det \Delta_0} = -\frac{1}{12\pi} \int \{|\nabla u|^2 + 2K_0 u\} dv_0$$

where $g_1 = e^{2u} g_0$ and K is the Gaussian curvature of g_0. In D'Hoker and Phong [**DP**] studied the log determinants of Laplacian on Mandelstam diagrams and adopted a renormalization to define an unambiguous notion of log determinant for Mandelstam diagrams. In particular, their renormalized log determinant still enjoys the additive law as expected.

In this note, we first review some facts about the Arakelov Green functions. Then we will carry out the computation of $D[\{p_i\}]$ via the Arakelov Green functions and show the following main result:

Theorem B. *Suppose that* (M, g) *is a closed surface with genus larger than one. Let* g_0 *be the metric of constant curvature and volume one, and* $g = e^{2\phi} g_0$. *And suppose that* L *is a Hermitian line bundle over* M *with degree* $d = -\chi(M)$. *Then*

$$D[\{p_k\}] = -12\pi \log \frac{\det\Delta_{g_M}}{\det\Delta_{g_0}} + 4\pi^2 \frac{\chi^2(M)}{\text{vol}(M)} + \frac{2\pi\chi(M)}{\text{vol}(M)} \int_M H_{g_0} dv + \frac{2\pi\chi(M)}{\text{vol}(M)} \int_M \phi dv + 2\pi \sum_k \phi(p_k) + 2\pi \sum_k \mu(p_k).$$

where g_M is the so-called Mandelstam metric whose curvature are all concentrated at $\{p_1, p_2, \ldots, p_d\}$, H_{g_0} solves (2.3) and μ solves (3.3) in this note.

2. Arakelov Green Functions.

In this section we will introduce Mandelstam metrics, which are metrics whose curvature are all concentrated at isolated points, and Arakelov Green functions. We will also introduce the Liouville action which is the so-called log determinant of Laplacian on M due to Polyakov [**Po**], Ray and Singer [**RS**] (see also [**OPS**], [**Ch**]). It is interesting that the transform formula for Arakelov Green functions under conformal change of metrics naturally brings out the Liouville action.

Suppose M is a closed surface of genus larger than 1, and g_0 is the constant curvature metric with volume 1 on M. And suppose $\{p_1, p_2, \ldots, p_l\}$ be l distinct points on M, where $l = -\chi(M)$. The Green function with respect to g_0 is defined as

$$\begin{cases} \Delta G(\cdot, p) = 2\pi\delta(\cdot, p) - 2\pi \\ \displaystyle\int_M G(x,p)dv_0 = 0. \end{cases} \tag{2.1}$$

Then let

$$H_{g_0}(x) = \sum_{k=1}^{l} G(x, p_k). \tag{2.2}$$

Thus

$$\begin{cases} \Delta H_{g_0}(x) = 2\pi \displaystyle\sum_{k=1}^{l} \delta(\cdot, p) + 2\pi\chi(M) \\ \displaystyle\int_M H_{g_0}(x)dv_0 = 0. \end{cases} \tag{2.3}$$

A Mandelstam metric is

$$g_M = e^{-2H_{g_0}} g_0. \tag{2.4}$$

It is readily seen from (2.3) that the Mandelstam metric has its curvature all concentrated at $\{p_1, p_2, \ldots, p_l\}$.

Next, we introduce Arakelov Green functions for any given metric g. First, we recall the definition of the usual Green function with respect to g. It is defined as

$$\begin{cases} \Delta_g G(\cdot, p) = 2\pi\delta(\cdot, p) - \dfrac{2\pi}{\mathrm{vol}(M, g)} \\ \displaystyle\int_M G(x,p)dv = 0, \end{cases} \tag{2.5}$$

where dv is the volume element of g. The Arakelov Green function with respect to g is defined as

$$\begin{cases} \Delta_g G^A(\cdot, p) = 2\pi\delta(\cdot, p) - \dfrac{K}{\chi(M)} \\ \displaystyle\int_M G^A(x,p)K(x)dv = 0, \end{cases} \tag{2.6}$$

where K is the Gaussian curvature of the metric g. It is easily seen that the Arakelov Green function is the same as the usual Green function when the metric is of constant curvature. Solving for $G^A(x,p)$ in terms of the usual Green function gives

$$\begin{aligned} G^A(s,t) =& G(s,t) - \frac{1}{2\pi\chi(M)}\int_M G(s,x)K(x)dv \\ & - \frac{1}{2\pi\chi(M)}\int_M G(x,t)K(x)dv \\ & + \frac{1}{4\pi^2\chi^2(M)}\int_M\int_M G(x,y)K(x)K(y)dv_x dv_y. \end{aligned} \tag{2.7}$$

Before we give the transform formula for the Arakelov Green functions under conformal change of metrics, let us introduce the Liouville action

$$S(g,u) = \frac{1}{12\pi}\int_M \{|\nabla u|^2 dv + 2K_0 u\}dv, \tag{2.8}$$

which is related to the functional determinant by Polyakov-Ray-Singer formula [**Po**], [**RS**], [**OPS**], [**Ch**]. Because Gaussian curvature transforms as

$$\Delta\phi + K = K_\phi e^{2\phi}, \tag{2.9}$$

where K_ϕ is the Gaussian curvature of the metric $g = e^{2\phi}g_0$, we have the additive rule for Liouville action

$$S(g,u) = S(g, u-\phi) + S(g_0,\phi). \tag{2.10}$$

Then from (2.6), (2.7) and (2.9), one obtains

$$G^A_\phi(x,p) = G^A(x,p) - \frac{1}{\chi(M)}(\phi(x) + \phi(p)) + \frac{6}{\chi^2(M)}S(g_0,\phi). \tag{2.11}$$

Thus, if we denote $\sum_{k=1}^l G^A_\phi(x,p_k)$ by $H_g(x)$ for the metric g,

$$H_g(x) = H_{g_0}(x) + \phi(x) - \frac{1}{\chi(M)}\sum_{k=1}^l \phi(p_k) - \frac{6}{\chi(M)}S(g_0,\phi),$$

that is

$$H_{g_0}(x) + \phi(x) = H_g(x) + \frac{1}{\chi(M)}\sum_{k=1}^l \phi(p_k) + \frac{6}{\chi(M)}S(g_0,\phi). \tag{2.12}$$

Therefore, the Mandelstam metric can be rewritten in term of the metric g as follows:

$$g_M = e^{-2(H_g + \frac{1}{\chi(M)}\sum_{k=1}^{l}\phi(p_k) + \frac{6}{\chi(M)}S(g_0,\phi))}g. \tag{2.13}$$

3. Renormalizations.

Suppose that L is a Hermitian line bundle of degree l over the closed surface (M, g) still having $g = e^{2\phi}g_0$. To compute renormalized energy for the Ginzburg-Landau vortices is to compute the renormalized energy for canonical solutions of Yang-Mills with prescribed isolated singularity (please see [**Q**]). What one does is to renormalize the energy

$$Y(s, A) = Y(g, h) = \int_M |\nabla h|^2 dv + \int_M h^2 dv, \tag{3.1}$$

where g stands for the metric and h stands for the solution of

$$\begin{cases} \Delta_g h + h = 2\pi \displaystyle\sum_{k=1}^{l} \delta(\cdot, p_k) \\ \displaystyle\int_M h dv = -2\pi\chi(M). \end{cases} \tag{3.2}$$

In the following, we shall solve h in terms of Arakelov Green functions. First, let us denote the Gaussian curvature of g by K, and solve

$$\begin{cases} \Delta_g \mu + K = -h \\ \displaystyle\int_M \mu dv = 0. \end{cases} \tag{3.3}$$

Let $\omega = \phi + \mu$. By (3.3) we know that the Gaussian curvature of the metric $g_\omega = e^{2\omega}g_0$ is $-he^{-2\mu}$. Hence, its Arakelov Green function $G_\omega^A(x, p)$ satisfies

$$\begin{cases} \Delta_{g_\omega} G_\omega^A(x, p) = 2\pi\delta(x, p) + \dfrac{he^{-2\mu}}{\chi(M)} \\ \displaystyle\int_M G_\omega^A(x, p) he^{-2\mu} dv_{g_\omega} = 0, \end{cases}$$

which implies

$$\begin{cases} \Delta_{g_w} H_{g_\omega} = 2\pi \displaystyle\sum_{k=1}^{l} \delta(\cdot, p_k) - he^{-2\mu} \\ \displaystyle\int_M H_{g_\omega} h dv_g = 0. \end{cases} \tag{3.4}$$

Notice that $\Delta_{g_\omega} = e^{-2\mu}\Delta_g$ and $\delta_{g_\omega}(\cdot, p)e^{-2\mu} = \delta(\cdot, p)$. Thus, we arrive at

$$\Delta_g H_{g_\omega} + h = 2\pi \sum_{k=1}^{l} \delta(\cdot, p_k) \tag{3.5}$$

from (3.4). Comparing this to (3.2), we conclude that

$$h = H_{g_\omega} + C \tag{3.6}$$

for some constant C. Multiplying h to (3.6) and integrating both sides over M gives

$$\int_M h^2 dv = -2\pi\chi(M)C. \tag{3.7}$$

On the other hand, simply integrating both sides of (3.6) gives

$$\mathrm{vol}(M)C = -2\pi\chi(M) - \int_M H_{g_\omega} dv. \tag{3.8}$$

Now we are going to do the renormalization for both Yang-Mills action $Y(g, h)$ and Liouville action S. One of the key points is that the following metric-depending renormalization will produce an anomaly for both actions. In other words, the renormalizations performed on different metrics yield different normalized functionals. Therefore, one needs to keep track of the possible anomaly. Set $g_M = e^{2\sigma} g_w$ for simplicity, i.e.,

$$\sigma = -H_{g_\omega} - \frac{1}{\chi(M)} \sum_{k=1}^{l} \omega(p_k) - \frac{6}{\chi(M)} S(g_0, \omega) \tag{3.9}$$

in the light of (2.12). By (3.6), one has

$$\sigma = -h + C - \frac{1}{\chi(M)} \sum_{k=1}^{l} \omega(p_k) - \frac{6}{\chi(M)} S(g_0, \omega). \tag{3.10}$$

Then, let M_ϵ be the surface M with the disks $\{d_{g_\omega}(x, p_k) < \epsilon\}$ removed, where d_{g_ω} is the distance function of metric g_ω. Therefore, the renormalized Liouville action is

$$\begin{aligned} S_r(g_\omega, \sigma) & \\ &= \lim_{\epsilon \to 0} \left\{ \frac{1}{12\pi} \left\{ \int_{M_\epsilon} |\nabla\sigma|^2 dv_w - 2\int_{M_\epsilon} he^{-2\mu}\sigma dv_{g_\omega} \right\} + \frac{\chi(M)}{6} \log\frac{1}{\epsilon} \right\} \end{aligned} \tag{3.11}$$

where

$$\begin{aligned}(3.12)\quad &\int_M he^{-2\mu}\sigma dv_{g_\omega}\\ &= \int_M h\left(-h + C - \frac{1}{\chi(M)}\sum_{k=1}^{l}\omega(p_k) - \frac{6}{\chi(M)}S(g_0,\omega)\right)dv\\ &= -\int_M h^2 dv - 2\pi\chi(M)\left(C - \frac{1}{\chi(M)}\sum_{k=1}^{l}\omega(p_k) - \frac{6}{\chi(M)}S(g_0,\omega)\right).\end{aligned}$$

Hence,

$$\begin{aligned}S_r(g_\omega,\sigma) &= \lim_{\epsilon\to 0}\left\{\frac{1}{12\pi}\int_{M_\epsilon}\{|\nabla h|^2 + h^2\}dv + \frac{\chi(M)}{6}\log\frac{1}{\epsilon}\right\}\\ &\quad + \frac{1}{6}\chi(M)C - \frac{1}{3}\sum_{k=1}^{l}\omega(p_k) - 2S(g_0,\omega).\end{aligned} \tag{3.13}$$

Let us denote this by

$$Y_r(g,h) = \lim_{\epsilon\to 0}\left\{\frac{1}{12\pi}\int_{M_\epsilon}\{|\nabla h|^2 dv + h^2\} + \frac{\chi(M)}{6}\log\frac{1}{\epsilon}\right\}. \tag{3.14}$$

Then, in the light of (3.1) and (3.7), we have

$$\begin{aligned}&S_r(g_\omega,\sigma)\\ &= \frac{1}{12\pi}Y_r(g,h) + \frac{\chi(M)}{6}C - \frac{1}{3}\sum_k\omega(p_k) - 2S(g_0,\omega)\\ &= \frac{1}{12\pi}Y_r(g,h) - \frac{\pi\chi^2(M)}{3\,\mathrm{vol}(M)} - \frac{\chi(M)}{6\,\mathrm{vol}(M)}\int_M H_{g_\omega}dv\\ &- \frac{1}{3}\sum_k\omega(p_k) - 2S(g_0,\omega),\\ &= \frac{1}{12\pi}Y_r(g,h) - \frac{\pi\chi^2(M)}{3\,\mathrm{vol}(M)} - \frac{1}{3}\sum_k\omega(p_k) - 2S(g_0,\omega)\\ &- \frac{\chi(M)}{6\,\mathrm{vol}(M)}\left(\int_M H_{g_0}dv + \int_M \phi dv - \frac{\mathrm{vol}(M)}{\chi(M)}\sum_k\omega(p_k) - \frac{6\,\mathrm{vol}(M)}{\chi(M)}S(g_0,\omega)\right)\\ &= \frac{1}{12\pi}Y_r(g,h) - \frac{\pi\chi^2(M)}{3\,\mathrm{vol}(M)} - \frac{\chi(M)}{6\,\mathrm{vol}(M)}\left(\int_M H_{g_0}dv + \int_M\phi dv\right)\\ &- \frac{1}{6}\sum_k\omega(p_k) - S(g_0,\omega).\end{aligned}$$

To have an unambiguous notion of determinants for a Mandelstam diagram, one needs to take off the conformal anomaly. In D'Hoker and Phong [**DP**] proved that

$$S(g_0, -H_{g_0}) = S(g_0, \omega) + S_r(g_\omega, \sigma) - \frac{1}{6}\sum_{k=1}^{l} \omega(p_k). \tag{3.15}$$

In the same spirit, one can carefully check the difference between $Y_r(g, h)$ and the renormalized energy $D[\{p_k\}]$ obtained in [**Q**], and conclude

$$D[\{p_k\}] = Y_r(g, h) - 2\pi \sum_{k=1}^{l} \mu(p_k). \tag{3.16}$$

Because the disks removed on M are measured in the metric g_ω not g and the scale at any point p differs by the conformal factor $e^{\mu(p)}$ for the two metrics. Thus

$$\begin{aligned} D[\{p_k\}] = {} & 12\pi S(g_0, -H_{g_0}) + 4\pi^2 \frac{\chi^2(M)}{\mathrm{vol}(M)} + \frac{2\pi\chi(M)}{\mathrm{vol}(M)} \int_M H_{g_0} dv \\ & + \frac{2\pi\chi(M)}{\mathrm{vol}(M)} \int_M \phi dv + 2\pi \sum_k \phi(p_k) + 2\pi \sum_k \mu(p_k). \end{aligned} \tag{3.17}$$

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