Pacific Journal of Mathematics

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Volume 194 No. 2 June 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 194, No. 2, 2000

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In this paper we will show that every semialgebraic semicone of codimension at least one is the tangent semicone to an algebraic variety.

Introduction.

The definition of the tangent cone $C(V, p)$ to an algebraic variety V at a point p was given by Whitney more than 40 years ago as one of the tools to get information about the geometric shape of a variety near a singular point. While the complex case has been widely and successfully studied, including from a computational point of view, only recently have some first attempts been made to elucidate the situation in the real case. For example in [**O-W-3**] (see also [**O-W-1**] and [**O-W-2**]), theorems are proven relating the tangent cone of a surface in $\mathbf{R}^3$ to its Nash fiber (the set of limits of tangent spaces at smooth points), and many examples are presented showing how the real case differs from the complex case.

Since the tangent cone to a real algebraic variety is a semialgebraic set, a question which, in our opinion, is very natural is that of investigating which semialgebraic cones of $\mathbf{R}^n$ can be realized as tangent cones to real algebraic subsets of $\mathbf{R}^n$. Partial results in this direction were proven in [**F-F**]; there it is shown that any closed semialgebraic cone of codimension at least one in $\mathbf{R}^n$ admitting a presentation with only "few" polynomial inequalities is the tangent cone to some real algebraic variety in $\mathbf{R}^n$. In particular, this holds for every semialgebraic cone of codimension at least one in $\mathbf{R}^3$.

In this paper we show that the same result is true in general for all closed semialgebraic cones of codimension at least one, without any restrictive hypothesis on the number of inequalities. Actually this is obtained as a corollary of a more general result stating that any closed semialgebraic semicone (i.e., a union of rays) of codimension at least one in $\mathbf{R}^n$ is the tangent semicone (i.e., a union of limits of secant rays) to a suitable real algebraic variety in $\mathbf{R}^n$.

1. Preliminaries.

Let V denote a real algebraic subset of $\mathbf{R}^n$ and let p be a point of V. The tangent cone $C(V,p)$ at p to V is the set of points $u \in \mathbf{R}^n$ such that there exist a sequence $x_m \in V$ converging to p and a sequence of real numbers t_m such that $\lim_{m\to\infty} t_m(x_m - p) = u$. For notational simplicity, we will always take $p = 0$ and denote $C(V,0)$ by $C(V)$.

By definition $C(V)$ is the union of lines which are limits of secant lines $\overline{0x_m}$ when $x_m \in V$ tends to 0, so it is a cone with vertex at 0, i.e., a set such that for every $u \in C(V)$, $u \neq 0$, the whole line through u and 0 is contained in $C(V)$. It is known (see e.g., [**O-W-2**]) that $C(V)$ is a closed semialgebraic subset of $\mathbf{R}^n$ with $\dim C(V) \leq \dim_0 V$; and, in general, it is not algebraic.

The tangent cone to a variety V gives information about the behavior of "tangent directions" to V at a singular point. If, for instance, $V = \{(x,y,z) \in \mathbf{R}^3 \mid z^3 = x^2 + y^2\}$, then $C(V)$ is the z-axis, though points on the negative z-semiaxis look scarcely related to the geometric shape of V. In this sense, a notion that seems to give better geometric information about V is that of the tangent semicone, i.e., the union of limits of secant rays starting from 0. Precisely, the tangent semicone $C^+(V)$ at 0 to V is the set of points $u \in \mathbf{R}^n$ such that there exist a sequence $x_m \in V$ converging to p and a sequence of real positive numbers t_m such that $\lim_{m\to\infty} t_m x_m = u$. For example the tangent semicone to $V = \{(x,y,z) \in \mathbf{R}^3 \mid z^3 = x^2 + y^2\}$ is the positive z-semiaxis, which shows that $C^+(V)$ can be properly contained in $C(V)$.

If $B \subseteq \mathbf{R}^n$ and $0 \in B$, we will say that B is a semicone (with vertex at the origin) if for every $y \in B, y \neq 0$, the whole ray starting from 0 and passing through y is contained in B. So the tangent semicone $C^+(V)$, which is semialgebraic, is a semicone.

As mentioned in the introduction, we are interested in studying which semialgebraic subsets of $\mathbf{R}^n$ are tangent cones or semicones to an algebraic subvariety of $\mathbf{R}^n$; we recall that all the cones and semicones will be assumed to have vertex at the origin.

For semialgebraic subsets A of dimension 1 the answer is easy. If A is a cone, then it is a finite union of lines, so it is algebraic, and it is both the tangent cone and the tangent semicone to itself (which is true for any algebraic cone). If A is a semicone, then it is a finite union of rays and every ray l can be realized as a tangent semicone: By a suitable change of coordinates, we can assume $l = \{x \in \mathbf{R}^n \mid x_1 = \cdots = x_{n-1} = 0, x_n \geq 0\}$, so l is the tangent semicone to $V = \{x \in \mathbf{R}^n \mid x_n^3 = \sum_{i=1}^{n-1} x_i^2\}$. It is then enough to make use of the following:

Property of the union: If $A \subset \mathbf{R}^n$ is a finite union of closed semialgebraic semicones A_i such that, for each i, there exists an algebraic subset $V_i \subset \mathbf{R}^n$ such that $A_i = C^+(V_i)$, then $C^+(\bigcup_i V_i) = \bigcup_i C^+(V_i) = A$.

Before dealing with the question for any A, let us end the section by recalling some further properties of cones and semicones in $\mathbf{R}^n$.

First of all every cone is a semicone, and every semicone B uniquely determines a sort of "associated cone", that is $B \cup (-B)$, where $-B = \{-x \mid x \in B\}$.

If $A \subset \mathbf{R}^n$ is a cone, it is a straightforward consequence of the definition of a cone that the ideal $\mathcal{I}(A) \subseteq \mathbf{R}[x_1, \ldots, x_n]$ consisting of all the polynomial functions vanishing on A is homogeneous (recall that $\mathcal{I}$ is said to be homogeneous if for any $f \in \mathcal{I}$, all the homogeneous parts of f belong to $\mathcal{I}$). As is well known ([**C-L-O**]), the algebraic subvariety of $\mathbf{R}^n$ defined by a homogeneous ideal I is a cone, since I admits a finite number of homogeneous generators.

Incidentally, we recall that every real algebraic variety can be defined by a single equation, since $V(f_1, \ldots, f_r) = V(\sum_{i=1}^r f_i^2)$.

The next lemma contains some facts that will be useful later on; we will denote by $\overline{A}^Z$ the Zariski closure of a set A.

Lemma 1.1.

(1) *The Zariski closure of a semicone is a cone.*
(2) *The irreducible components of an algebraic cone are cones.*

Proof. (1) Let A be a semicone. Since $\overline{A}^Z = \overline{A \cup (-A)}^Z$, we can assume A is a cone. Therefore $\mathcal{I}(\overline{A}^Z) = \mathcal{I}(A)$ is homogeneous, hence the variety it defines, that is $\overline{A}^Z$, is a cone.

(2) Let Y be an irreducible component of an algebraic cone X. In order to prove that $\mathcal{I}(Y)$ is homogeneous, we show that for every $f \in \mathcal{I}(Y)$ each homogeneous component f_i of f belongs to $\mathcal{I}(Y)$.

Assume by contradiction that $E = \{i \mid f_i \notin \mathcal{I}(Y)\} \neq \emptyset$; let us denote by i_0 the minimum of E.

If $W = \overline{X \setminus Y}^Z$, then $X = Y \cup W$ and there exists $g \in \mathcal{I}(W)$ such that $g \notin \mathcal{I}(Y)$. The function fg belongs to the ideal $\mathcal{I}(X)$, which is homogeneous because X is a cone; so each homogeneous component $(fg)_k \in \mathcal{I}(X)$.

Note that $F = \{j \mid g_j \notin \mathcal{I}(Y)\} \neq \emptyset$, because $g \notin \mathcal{I}(Y)$; so let $j_0 = \min\ F$ and $k_0 = i_0 + j_0$. Consider the homogeneous component

$$(fg)_{k_0} = f_{i_0} g_{j_0} + \sum_{i+j=k_0, i<i_0} f_i g_j + \sum_{i+j=k_0, j<j_0} f_i g_j.$$

We easily get that $f_{i_0} g_{j_0} \in \mathcal{I}(Y)$, which is absurd since neither f_{i_0} nor g_{j_0} belong to $\mathcal{I}(Y)$ while $\mathcal{I}(Y)$ is prime. □

Finally note that if A is a closed semialgebraic cone in $\mathbf{R}^n$, then the image $\pi(A)$ of $A\setminus\{0\}$ in $\mathbf{R}P^{n-1}$ by the canonical projection $\pi:\mathbf{R}^n\setminus\{0\}\to\mathbf{R}P^{n-1}$ is a semialgebraic subset of $\mathbf{R}P^{n-1}$; so we have that

$$\pi(A)=\bigcup_{i=1}^{s}\{X\in\mathbf{R}P^{n-1}\mid \phi_i(X)=0,\psi_{ij}(X)\geq 0\quad j=1,\dots,m_i\}$$

where ϕ_i and ψ_{ij} are regular functions on $\mathbf{R}P^{n-1}$. Recall that, given a regular function η on $\mathbf{R}P^{n-1}$, then $\eta=\frac{P}{Q}$ with P,Q homogeneous polynomials of the same degree. So $V(\eta)=V(PQ)$ and $\{\eta\geq 0\}=\{PQ\geq 0\}$. Hence A can be presented as

$$A=\bigcup_{i=1}^{s}\{x\in\mathbf{R}^n\mid f_i(x)=0,h_{i1}(x)\geq 0,\dots,h_{im_i}(x)\geq 0\}$$

where f_i,h_{ij} are homogeneous polynomials of even degree.

2. Basic tools.

One of the results proved in [**F-F**] is that, if A is a closed semialgebraic cone of codimension at least one in $\mathbf{R}^n$, with vertex at the origin, and admitting a presentation with only one inequality, say $A=\{x\in\mathbf{R}^n\mid f(x)=0,h(x)\geq 0\}$ with f,h homogeneous polynomials, then A is the tangent cone to the variety $V(g)$, where $g(x)=f(x)^2-h(x)^r$ for a suitable odd integer r.

In order to prove the same result when the presentation of A contains more inequalities, the first natural idea would be that of iterating the previous procedure; but the iteration fails since $g=f^2-h$ may no longer be homogeneous. It is therefore necessary to study more closely the relation between $C(V(g))$ and $C(V(f))$ when f is not homogeneous; this is precisely what we do in this paragraph.

For simplicity, for any function $h:\mathbf{R}^n\to\mathbf{R}$, we will use the following notations:

$$V^{\geq}(h)=\{x\in\mathbf{R}^n\mid h(x)\geq 0\}\quad\text{and}\quad V^{>}(h)=\{x\in\mathbf{R}^n\mid h(x)>0\}.$$

Moreover, for any $f\in\mathbf{R}[x_1,\dots,x_n]$ we will denote by $\operatorname{ord}(f)$ the order of f, i.e., the degree of the lowest degree homogeneous part of f.

We recall that a function $f:\mathbf{R}^n\to\mathbf{R}$ is said to be semialgebraic if its graph is a semialgebraic subset of $\mathbf{R}^{n+1}$.

Proposition 2.1. *Let $f:\mathbf{R}^n\to\mathbf{R}$ be a continuous semialgebraic function, $f(0)=0$. Then there exists a constant $l(f)>0$ such that, for any homogeneous polynomial $h\in\mathbf{R}[x_1,\dots,x_n]$ of degree d with $d>2l(f)$, we have*

$$C^+(V(g))\subseteq C^+(V(f))\cap V^{\geq}(h),$$

where $g=f^2-h$.

Proof. Fix a rational number $s > 1$. For any $x \in \mathbf{R}^n$, let $B(x, r)$ denote the open ball of radius r centered at x. Let

$$U = \bigcup_{x \in V(f) \cap B(0, \frac{1}{2})} B(x, \|x\|^s).$$

Then U is an open neighborhood of $V(f) \cap B(0, \frac{1}{2}) \setminus \{0\}$ in $\mathbf{R}^n$. Furthermore, U is semialgebraic: it is the projection to $\mathbf{R}^n$ of the semialgebraic set

$$\left\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \mid \|x - y\| < \|x\|^s, x \in V(f) \cap B\left(0, \frac{1}{2}\right)\right\}.$$

Since $V(f) \cap (B(0, \frac{1}{2}) \setminus U) = \{0\}$, by a Łojasiewicz inequality there exist positive real constants $l = l(f)$ and C such that $|f(x)| \geq C\|x\|^l$ for all $x \in (\mathbf{R}^n - U) \cap B(0, \frac{1}{4})$.

Let $h \in \mathbf{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d. Using Euler's formula and induction on the degree, we get that there exists a constant M such that $|h(x)| \leq M\|x\|^d$ for every x.

If $d > 2l$, let $q = d - 2l$. Then $g \geq (C^2 - M\|x\|^q)\|x\|^{2l}$ on $(\mathbf{R}^n - U) \cap B(0, \frac{1}{4})$. So, for r sufficiently small, g is positive on $(\mathbf{R}^n - U) \cap B(0, r)$ except at 0 and consequently

$$(V(g) \setminus \{0\}) \cap B(0, r) \subseteq U \cap B(0, r). \tag{1}$$

From the last inclusion we get that $C^+(V(g)) \subseteq C^+(V(f))$. In fact, let $u \in C^+(V(g))$; since $C^+(V(g))$ is a semicone, we can suppose that $\|u\| = 1$. Let $\{x_i\} \in V(g) \setminus \{0\}$ be a sequence converging to 0 such that $\lim_{i \to \infty} \frac{x_i}{\|x_i\|} = u$. By (1) and for any i big enough, there exists $y_i \in V(f) \setminus \{0\}$ such that $\|x_i - y_i\| < \|y_i\|^s$ with $\|y_i\| < r < 1$. Any limit point y_0 of the bounded sequence $\{y_i\}$ satisfies $\|y_0\| \leq \|y_0\|^s$ and $\|y_0\| < 1$, hence $\lim_{i \to \infty} y_i = 0$. Moreover we have

$$\begin{aligned} \left\| \frac{x_i}{\|x_i\|} - \frac{y_i}{\|y_i\|} \right\| &= \frac{\|(\|y_i\| - \|x_i\|)x_i + \|x_i\|(x_i - y_i)\|}{\|x_i\|\|y_i\|} \\ &\leq \frac{2\|x_i - y_i\|}{\|y_i\|} \leq 2\|y_i\|^{s-1}, \end{aligned}$$

hence $\lim_{i \to \infty} \frac{y_i}{\|y_i\|} = u$. This proves that $u \in C^+(V(f))$.

The inclusion $C^+(V(g)) \subseteq V^{\geq}(h)$ follows easily from $V(g) \subseteq V^{\geq}(h)$. □

Lemma 2.2. *Let $f \in \mathbf{R}[x_1, \dots, x_n]$, with* $\operatorname{ord}(f) = d$. *Assume $h \in \mathbf{R}[x_1, \dots, x_n]$ is a homogeneous polynomial with* $\deg h > 2d$ *and define $g = f^2 - h$. Let x be a point in $\mathbf{R}^n$ such that $h(x) > 0$ and suppose that b is a real positive number such that $f(bx) = 0$. Then there exist a point y arbitrarily near x and a real number t_0 with $0 < t_0 < b$ such that $t_0 y \in V(g)$.*

Proof. Denote by f_0 the lowest degree homogeneous part of f. Since $g(bx) < 0$, we can find y arbitrarily near x with $f_0(y) \neq 0$ and $g(by) < 0$. Let $\lambda(t) = g(ty)$. The lowest degree term of $\lambda(t)$ is $t^{2d} f_0(y)^2$, therefore $\lambda(t) > 0$ for $t > 0$ sufficiently small. Since $\lambda(b) < 0$, by the Intermediate Value Theorem $\lambda(t_0) = 0$ for some t_0 with $0 < t_0 < b$. □

Proposition 2.3. *Suppose that $f \in \mathbf{R}[x_1, \dots, x_n]$ is a polynomial of order d and $h \in \mathbf{R}[x_1, \dots, x_n]$ is a homogeneous polynomial with $\deg h > 2d$. Let $g = f^2 - h$. Then $C^+(V(g)) \supseteq C^+(V(f)) \cap V^>(h)$.*

Proof. Pick $x \in C^+(V(f)) \cap V^>(h)$. Then there exist a sequence of points x_i converging to x and a sequence of positive numbers a_i converging to 0 such that $a_i x_i \in V(f) \cap V^>(h)$. By Lemma 2.2, there exist points y_i converging to x and positive numbers t_i converging to 0 such that $t_i y_i \in V(g)$. Therefore $x \in C^+(V(g))$. □

3. Main theorem.

We are now ready to prove the main result of this paper:

Theorem 3.1. *Let A be a closed semialgebraic semicone in $\mathbf{R}^n$, $\dim A < n$. Then there exists a polynomial function $F \in \mathbf{R}[x_1, \dots, x_n]$ such that A is the tangent semicone to the algebraic variety $V(F) = \{x \in \mathbf{R}^n \mid F(x) = 0\}$.*

Proof. The theorem will be proved by induction on the dimension of A. If $\dim A = 1$, A is a finite union of rays, hence $V(F)$ can be found as seen in Section 1.

Assume now $\dim A > 1$ and that the theorem holds for any closed semialgebraic semicone of dimension $< \dim A$.

Claim. Without any loss of generality, we may assume that

$$A = \{x \in \mathbf{R}^n \mid f(x) = 0, h_1(x) \geq 0, \dots, h_m(x) \geq 0\}$$

with

(i) $f, h_1, \dots, h_m$ homogeneous polynomials of positive degree,
(ii) $\{x \in \mathbf{R}^n \mid f(x) = 0, h_1(x) > 0, \dots, h_m(x) > 0\} \neq \emptyset$,
(iii) $\overline{A}^Z$ irreducible.

Proof of the Claim. By the property of the union recalled in Section 1, it is enough to prove that A is a finite union of subsets with the properties stated in the claim.

Observe that A can be seen as $A = \bigcup_{i=1}^{p} A_i$, where each A_i is a closed semialgebraic semicone such that $A_i \setminus \{0\} \subseteq \{l_i > 0\}$ for a suitable linear function l_i. Set $C_i = A_i \cup (-A_i)$. So $A = \bigcup_{i=1}^{p}(C_i \cap \{l_i \geq 0\})$ is a finite union of subsets of the form $C \cap \{l \geq 0\}$, where C is a closed semialgebraic cone and l is a linear function. By Section 1, C is a finite union of subsets presented as

$$B = \{x \in \mathbf{R}^n \mid f(x) = 0, h_1(x) \geq 0, \dots, h_m(x) \geq 0\},$$

where $f, h_1, \dots, h_m$ are homogeneous polynomials of even degree.

Let $X = \overline{B}^Z$, $X = X_1 \cup \ldots \cup X_p$ the decomposition of X into irreducible components and $B_i = B \cap X_i$. Evidently $B = \bigcup_{i=1}^{p} B_i$ and X_i is the Zariski closure of B_i. By Lemma 1.1, X and its irreducible components X_i are cones, so for each i there exists a homogeneous polynomial f_i of even degree such that $X_i = V(f_i)$. So

$$B_i = \{x \in \mathbf{R}^n \mid f_i(x) = 0, h_1(x) \geq 0, \dots, h_m(x) \geq 0\}.$$

Since we can assume that h_j does not identically vanish on B_i for any j and since $\overline{B_i}^Z$ is irreducible, then $\dim(B_i \cap V(h_j)) < \dim B_i$ for every j. This implies that $\{x \in \mathbf{R}^n \mid f_i(x) = 0, h_1(x) > 0, \dots, h_m(x) > 0\} \neq \emptyset$, because otherwise $B_i = \bigcup_{j=1}^{m}(B_i \cap V(h_j))$, which is absurd.

So the B_i's fulfill conditions (i), (ii), (iii). Then our claim is proved by just remarking that $C \cap \{l \geq 0\}$ is a finite union of the semicones

$$B_i' = \{x \in \mathbf{R}^n \mid f_i(x) = 0, h_1(x) \geq 0, \dots, h_m(x) \geq 0, l \geq 0\},$$

which satisfy the requested properties (i), (ii), (iii); in particular $\overline{B_i'}^Z$ is irreducible, as $\overline{B_i'}^Z = \overline{B_i}^Z$. □

Assume that A is presented as in the Claim. If we denote $\deg f = d_0$, $\deg h_i = d_i$, let us now recursively define polynomials $g_0, \dots, g_m$ and odd positive integers $s_0, \dots, s_m$ as follows:

- $g_0 = f$ and $s_0 = 1$;
- if $i > 0$, let s_i be an odd positive integer such that

$$d_i s_i > 2 \max\{l(g_{i-1}), 2^{i-1} d_0\}$$

(where $l(g_{i-1})$ is the exponent in Proposition 2.1) and let $g_i = g_{i-1}^2 - h_i^{s_i}$.

We can apply Proposition 2.1 m times to conclude that $C^+(V(g_m)) \subseteq A$.

Let $A_0 = V(f)$ and, $\forall i = 1, \dots, m$, let $A_i = \{f = 0, h_1^{s_1} > 0, \dots, h_i^{s_i} > 0\}$; by (ii), $A_i \neq \emptyset$ for each i.

Let us prove by induction on i that $A_i \subseteq C^+(V(g_i))$. This is obvious if $i = 0$; if $i > 0$, we have by the inductive hypothesis that

$$A_i = A_{i-1} \cap \{h_i^{s_i} > 0\} \subseteq C^+(V(g_{i-1})) \cap \{h_i^{s_i} > 0\}.$$

Moreover one can easily check that $\operatorname{ord}(g_i) = 2^i d_0$ for any i, so that we can apply Proposition 2.3 to obtain that

$$C^+(V(g_{i-1})) \cap \{h_i^{s_i} > 0\} \subseteq C^+(V(g_i)).$$

Since $C^+(V(g_i))$ is closed, we get that $\overline{A_i} \subseteq C^+(V(g_i))$. In particular $A' = \overline{A_m} \subseteq C^+(V(g_m))$.

In general A' may be strictly contained in A, so we cannot conclude that A is contained in $C^+(V(g_m))$ and we need to consider the set $B = \overline{A \setminus A'}$. Since A' is a closed semialgebraic semicone, so is B.

We claim that $\dim B < \dim A$.

Assume by contradiction that $\dim B = \dim A$. Then $\overline{B}^Z \subseteq \overline{A}^Z$ and $\dim \overline{B}^Z = \dim \overline{A}^Z$, hence $\overline{B}^Z = \overline{A}^Z$ because $\overline{A}^Z$ is irreducible.

It is enough to prove that for each $x \in B$, at least one of $h_i(x) = 0$; in fact this implies that $h = h_1 \cdot h_2 \cdot \dots \cdot h_m$ vanishes on B, hence on $\overline{B}^Z = \overline{A}^Z$, which contradicts the fact that $A_m \neq \emptyset$.

Let us now show that for each $x \in B$, there exists an index i such that $h_i(x) = 0$. If not, $x \in A_m \cap B$. Therefore there exists an open neighborhood U of x in $\mathbf{R}^n$ on which $h_i > 0$ for each $i = 1, \dots, m$. Hence $U \cap V(f) \cap (A \setminus A') = \emptyset$. Since $A \subseteq V(f)$, we get that $U \cap (A \setminus A') = \emptyset$, which is absurd because $x \in B = \overline{A \setminus A'}$.

So, by induction, there exists a polynomial function G such that $C^+(V(G)) = B$. Then

$$A = A' \cup B \subseteq C^+(V(g_m)) \cup C^+(V(G)) \subseteq A.$$

Hence $A = C^+(V(F))$, where $F = g_m G$. □

Note that, if the tangent semicone to an algebraic variety V is a cone, then it coincides with the tangent cone to V. Therefore, as each cone is a semicone, from the previous Theorem we deduce the following:

Corollary 3.2. *Let A be a closed semialgebraic cone in $\mathbf{R}^n$,* $\dim A < n$. *Then there exists a polynomial function $F \in \mathbf{R}[x_1, \dots, x_n]$ such that A is the tangent cone to the algebraic variety $V(F) = \{x \in \mathbf{R}^n \mid F(x) = 0\}$.*

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Received August 7, 1998. This work was partially supported by GNSAGA (CNR), MURST and Eurocontract CHRX-CT94-0506.

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