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An incompressible bounded surface F on the boundary of a compact, connected, orientable 3-manifold M is arc-extendible if there is a properly embedded arc γ on $\partial M - \text{Int}F$ such that $F \cup N(\gamma)$ is incompressible, where $N(\gamma)$ is a regular neighborhood of γ in ∂M . Suppose for simplicity that M is irreducible and F has no disk components. If M is a product $F \times I$, or if $\partial M - F$ is a set of annuli, then clearly F is not arc-extendible. The main theorem of this paper shows that these are the only obstructions for F to be arc-extendible.

Suppose F is a compact incompressible surface on the boundary of a compact, connected, orientable, irreducible 3-manifold M . Let F' be a component of $\partial M - \text{Int}F$ with $\partial F' \neq \emptyset$. We say that F is *arc-extendible* (in F') if there is a properly embedded arc γ in F' such that $F \cup N(\gamma)$ is incompressible. In this case γ is called an *extension arc* of F . We study the problem of which incompressible surfaces on the boundary M are arc-extendible. The result will be used in [H], in which it is shown that a compact, orientable 3-manifold M contains arbitrarily many disjoint, non-parallel, non-boundary parallel, incompressible surfaces, if and only if M has at least one boundary component of genus greater than or equal to two.

Denote by I the unit interval $[0, 1]$. We say that M is a product $F \times I$ if there is a homeomorphism $\varphi : M \cong F \times I$ with $\varphi(F) = F \times 1$. Note that in this case $F' = \partial M - \text{Int}F$, and F is not arc-extendible. A surface F is *diskless* if it has no disk component. An incompressible surface with a disk component is always arc-extendible, unless the disk lies on a sphere component of ∂M . Thus to avoid trivial cases, we will only consider arc-extension of diskless surfaces.

Theorem 1. *Let F be a diskless, compact, incompressible surface on the boundary of a compact, connected, orientable, irreducible 3-manifold M , and let F' be a non-annular component of $\partial M - \text{Int}F$ with $\partial F' \neq \emptyset$. Then either F is arc-extendible in F' , or M is a product $F \times I$.*

The proof of the theorem involves some deep results about incompressible surfaces related to Dehn surgery and 2-handle additions. It breaks down into three cases. The case that F' is a thrice punctured sphere is treated

in Theorem 4, which shows that if the surface obtained by gluing F and F' along one of the boundary curves of F' is compressible for all the three boundary curves of F' , then M must be a product. The second case is that F' is parallel into F (see below for definition). A similar result as above holds in this case. Theorem 9 shows that in the remaining case there is an arc γ intersecting some circle C in F' at one point, so that all but at most three Dehn twists of γ along C are extension arcs of F . Moreover, in this case the extension arc γ of F can be chosen to have endpoints on any prescribed components of $\partial F'$. See Theorem 10 below.

Note that the connectedness and irreducibility of M are irrelevant to the compressibility of surfaces on ∂M . However, this does make the conclusion of the theorem simpler. If we drop these assumptions from the theorem, the conclusion should be changed to “Either F is arc-extendible in F' , or the summand M' of the component of M containing F' is a product $F' \times I$, with $\partial M' - \text{Int}F'$ a component of F' .”

Given a simple closed curve α on a surface S on the boundary of M , we use $M[\alpha]$ to denote the manifold obtained by adding a 2-handle to M along the curve α . More explicitly, $M[\alpha]$ is the union of M and a $D^2 \times I$, with the annulus $(\partial D^2) \times I$ glued to a regular neighborhood $N(\alpha)$ of α on ∂M . Use $S[\alpha]$ to denote the surface on $\partial M[\alpha]$ corresponding to S , i.e., $S[\alpha] = (S - N(\alpha)) \cup (D^2 \times \partial I)$. The following two lemmas are very useful in dealing with incompressible surfaces. Various versions of Lemma 2 have been proved by Przytycki [Pr], Johansson [Jo], Jaco [Ja], and Scharlemann [Sch]. The lemma as stated is due to Casson and Gordon [CG].

Lemma 2 (The Handle Addition Lemma [CG]). *Let α be a simple closed curve on a surface S on the boundary of an orientable irreducible 3-manifold M , such that S is compressible and $S - \alpha$ is incompressible. Then $S[\alpha]$ is incompressible in $M[\alpha]$, and $M[\alpha]$ is irreducible.*

Lemma 3 (The Generalized Handle Addition Lemma). *Let S be a surface on the boundary of an orientable 3-manifold M , let γ be a 1-manifold on S , and let α be a circle on S disjoint from γ . Suppose $S - \gamma$ is compressible and $S - (\gamma \cup \alpha)$ is incompressible. If D is a compressing disk of $S[\alpha]$ in $M[\alpha]$, then there is a compressing disk D' of $S - \alpha$ in M such that $\partial D' \cap \gamma \subset \partial D \cap \gamma$.*

Proof. This is essentially [Wu2, Theorem 1]. The theorem there stated that $\partial D' \cap \gamma$ has no more points than $\partial D \cap \gamma$, but the proof there gives the stronger conclusion that $\partial D' \cap \gamma \subset \partial D \cap \gamma$. □

We first study the case that the surface F' in Theorem 1 is a thrice punctured sphere. Let $\alpha_1, \alpha_2, \alpha_3$ be the boundary curves of F' . Since F' is a component of $\partial M - \text{Int}F$, we have $\alpha_i \subset \partial F$ for $i = 1, 2, 3$. Note that

if $\text{Int}F \cup \text{Int}F' \cup \alpha_i$ is incompressible for some i , then for any essential arc γ on F' with $\partial\gamma \subset \alpha_i$, the surface $F \cup N(\gamma)$ is incompressible. Hence the following theorem proves Theorem 1 in the case that F' is a thrice punctured sphere. However, it should be noticed that a similar statement is false if we drop the assumption that F' is a sphere with three holes.

Theorem 4. *Let F be a diskless compact incompressible surface on the boundary of a compact, connected, orientable, irreducible 3-manifold M , and let F' be a component of $\partial M - \text{Int}F$ which is a punctured sphere with $\partial F' = \alpha_1 \cup \alpha_2 \cup \alpha_3$. If $\text{Int}F \cup \text{Int}F' \cup \alpha_i$ is compressible for $i = 1, 2, 3$, then M is a product $F \times I$.*

Proof. We fix some notation. Write $\widehat{F} = F \cup F'$. Denote by \widehat{F}_i the surface obtained by gluing $\text{Int}F$ and $\text{Int}F'$ along α_i , i.e., $\widehat{F}_i = \text{Int}F \cup \text{Int}F' \cup \alpha_i$. Similarly, write $\widehat{F}_{ij} = \text{Int}F \cup \text{Int}F' \cup \alpha_i \cup \alpha_j$.

First notice that F' is incompressible. This is because each simple closed curve on F' is isotopic to one of the $\alpha_i \subset F$, and because F is incompressible and diskless. Since $\text{Int}F \cap \text{Int}F' = \emptyset$, the surface $\text{Int}F \cup \text{Int}F'$ is incompressible.

Let M' be the component of a maximal compression body of ∂M in M which contains \widehat{F} . Then a surface on the boundary of M' is compressible in M' if and only if it is compressible in M . Notice that if $M \neq M'$, then M' is never a product $F \times I$, so if the theorem is true for M' , it is true for M . Hence after replacing M by M' if necessary, we may assume without loss of generality that M is a compression body.

We claim that the curves $\alpha_1, \alpha_2, \alpha_3$ are mutually nonparallel on \widehat{F} , that is, no component of F is an annulus with both boundary components on F' . If two curves α_1, α_2 , say, are parallel on \widehat{F} , then the surface $\text{Int}F \cup \text{Int}F' = \widehat{F} - \alpha_1 \cup \alpha_2 \cup \alpha_3$ is incompressible if and only if $\widehat{F}_1 = \widehat{F} - \alpha_2 \cup \alpha_3$ is incompressible. However, by assumption \widehat{F}_1 is compressible, and we have shown that $\text{Int}F \cup \text{Int}F'$ is incompressible. Hence the claim follows.

Since \widehat{F}_i is compressible, and $\widehat{F}_i - \alpha_i = \text{Int}F \cup \text{Int}F'$ is incompressible, we can apply the Handle Addition Lemma (Lemma 2) to \widehat{F}_i and α_i to conclude that after adding a 2-handle along α_i , the surface $\widehat{F}_i[\alpha_i]$ is incompressible in $M[\alpha_i]$, and $M[\alpha_i]$ is irreducible.

Consider the surface $\widehat{F}[\alpha_1]$. Notice that after adding the 2-handle, the surface F' becomes an annulus on $\widehat{F}[\alpha_1]$ with boundary $\alpha_2 \cup \alpha_3$, so the two curves α_2, α_3 are parallel on $\widehat{F}[\alpha_1]$. Thus, $\widehat{F}_1[\alpha_1] = \widehat{F}[\alpha_1] - \alpha_2 \cup \alpha_3$ being incompressible in $M[\alpha_1]$ implies that $\widehat{F}[\alpha_1] - \alpha_2$ is incompressible in $M[\alpha_1]$. With the above notation, this says that $\widehat{F}_{13}[\alpha_1]$ is incompressible in $M[\alpha_1]$.

By assumption \widehat{F}_3 is compressible in M . Let D be a compressing disk of \widehat{F}_3 in M . Then ∂D is disjoint from $\alpha_1 \cup \alpha_2$, because $\partial D \subset \widehat{F}_3$. Also, ∂D is not isotopic to α_1 in \widehat{F}_{13} , otherwise α_1 would bound a disk in M , contradicting the assumption that F is diskless and incompressible. We have shown that $\widehat{F}_{13}[\alpha_1]$ is incompressible in $M[\alpha_1]$, so D is not a compressing disk of $\widehat{F}_{13}[\alpha_1]$ in $M[\alpha_1]$, and hence ∂D must bound a disk in $\widehat{F}_{13}[\alpha_1]$. This is true if and only if ∂D is coplanar to α_1 on \widehat{F}_{13} , that is, either ∂D is parallel to α_1 , or it bounds a once punctured torus T in \widehat{F}_{13} which contains α_1 as a nonseparating curve. The first possibility has been ruled out, so the second must be true. Let \widehat{T} be the torus $T \cup D$. Since we have assumed above that M is a compression body, either (i) \widehat{T} is parallel to a boundary component of M , or (ii) \widehat{T} bounds a solid torus, or (iii) \widehat{T} lies in a 3-ball and bounds a cube with a knotted hole. But since T lies on ∂M , a pair of generators of $H_1(T)$ cannot both be null homologous in M , hence (iii) cannot happen.

If \widehat{T} is parallel to a boundary component T_0 of M , then after adding the 2-handle, the surface $\widehat{T}[\alpha_1]$ becomes a sphere which separates T_0 from $\widehat{F}[\alpha_1]$, hence is a reducing sphere of $M[\alpha_1]$, which contradicts the irreducibility of $M[\alpha_1]$. Similarly, if \widehat{T} bounds a solid torus V but α_1 is not a longitude of V , then after adding the 2-handle the manifold would have a lens space or $S^2 \times S^1$ summand, which again contradicts the irreducibility of $M[\alpha_1]$. (Note that $M[\alpha_1]$ cannot be a lens space because it has some boundary components.)

We have now shown that there is a compressing disk D of \widehat{F}_3 in M which cuts the manifold into two pieces, one of which is a solid torus V which contains α_1 as a longitude, but is disjoint from α_2 . Let D_1 be a meridian disk of V . Then $\partial D_1 \cap \alpha_1$ is a single point, and ∂D_1 is disjoint from α_2 because ∂V is disjoint from α_2 . Notice that ∂D_1 is not coplanar to α_2 , for if ∂D_1 were parallel to α_2 then α_2 would also intersect α_1 , and if ∂D_1 were to bound a once punctured torus containing α_2 then ∂D_1 would be a separating curve on ∂M , so it would intersect α_1 in an even number of points, either case leading to a contradiction. Thus, after adding a 2-handle to M along α_2 , the disk D_1 remains a compressing disk of $\widehat{F}[\alpha_2]$. Since the two curves α_1 and α_3 are parallel in $\widehat{F}[\alpha_2]$, and since D_1 intersects α_1 in a single point, we can isotope D_1 to another disk D_2 in $M[\alpha_2]$ so that it intersects each of α_1 and α_3 in a single point. We are looking for such a disk in M ; however D_2 is not necessarily the one because it may intersect the attached 2-handle.

Recall that the surface \widehat{F}_2 is compressible, but the surface $\widehat{F}_2 - \alpha_2 = \text{Int}F \cup \text{Int}F'$ is incompressible. Hence we can apply the Generalized Handle Addition Lemma (Lemma 3, with $S = \widehat{F}$, $\gamma = \alpha_1 \cup \alpha_3$, and $\alpha = \alpha_2$) to conclude that there is also a compressing disk D_3 of \widehat{F} in M , such that ∂D_3 is disjoint from α_2 , and $\partial D_3 \cap (\alpha_1 \cup \alpha_3)$ is a subset of $\partial D_2 \cap (\alpha_1 \cup \alpha_3)$.

The set $\partial D_3 \cap (\alpha_1 \cup \alpha_3)$ is nonempty, otherwise, since ∂D_3 is also disjoint from α_2 , D_3 would be a compressing disk of $\text{Int}F \cup \text{Int}F'$, contradicting the incompressibility of $\text{Int}F \cup \text{Int}F'$. Since $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is separating on \widehat{F} , the curve ∂D_3 can not intersect $\alpha_1 \cup \alpha_2 \cup \alpha_3$ at a single point. It follows that $\partial D_3 \cap (\alpha_1 \cup \alpha_3) = \partial D_2 \cap (\alpha_1 \cup \alpha_3)$, that is, ∂D_3 intersects each of α_1, α_3 in a single point. Such a disk is called a *bigon*.

Denote by D_{13} the bigon D_3 above. Interchanging the roles of α_1 and α_2 in the above argument, we get another compressing disk D_{23} of \widehat{F} in M , which is disjoint from α_1 , and intersects each of α_2, α_3 in a single point. Using the fact that no compressing disk of \widehat{F} would intersect $\alpha_1 \cup \alpha_2 \cup \alpha_3$ at a single point, we can modify D_{13} and D_{23} by a simple innermost circle outermost arc disk swapping argument, so that D_{13} and D_{23} are disjoint, and still have the same number of intersection points with each α_i . Cutting M along $D_{13} \cup D_{23}$, we get a submanifold M' of M , in which the surface F' becomes a disk $\widetilde{F}' \subset F'$, and the surface F becomes a surface $\widetilde{F} \subset F$. It is clear that one boundary component C of \widetilde{F} bounds a disk on $\partial M'$, namely the union of \widetilde{F}' and the two copies of $D_{13} \cup D_{23}$. Since F is incompressible, this curve C bounds a disk in F , so \widetilde{F} must be a disk. These disks together form a sphere boundary component of M' . Since M is connected and irreducible, M' must be a 3-ball, so it is a product $\widetilde{F} \times I$. Gluing back along D_{13} and D_{23} , we see that M is a product $F \times I$. This completes the proof of Theorem 4. \square

Below, F, F' and M will be as in Theorem 1. Using Theorem 4 we may assume that F' is not a thrice punctured sphere. A curve C' on F' is *∂ -nonseparating* if (i) C' is not parallel to a boundary curve on F' , and (ii) there is a proper arc γ in F' intersecting C' in a single point. A sub-surface G' of F' is *parallel into F* if there is a product $G' \times I \subset M$ such that $G' = G' \times 0$, and $G' \times 1 \subset F$. Similarly, a curve C' on F' is *parallel into F* if there is an embedded annulus $A \subset M$ with $\partial A = C' \cup C$, where $C \subset F$.

Lemma 5. *If F' is compressible, then there is a ∂ -nonseparating curve C' on F' which is not parallel into F .*

Proof. Let D be a compressing disk of F' . If ∂D is nonseparating on F' , let C' be a curve in F' that intersects ∂D in one point. Then C' is nonseparating, hence ∂ -nonseparating on F' . We want to show that C' is not parallel into F . Otherwise, let A be an annulus with $\partial A = C' \cup C$, where $C \subset F$. Then $A \cap D$ is a proper 1-manifold on D . But $\partial(A \cap D) = (\partial A) \cap \partial D$ is a single point, which is absurd. Hence C' is the curve required.

Now assume that ∂D is separating on F' , cutting F' into F'_1 and F'_2 . Choose a simple loop C_i on F'_i as follows. If F'_i is nonplanar, then it contains a pair of nonseparating curves intersecting each other in one point, at least one of which is not null-homologous in M . Choose this one as C_i . If F'_i is planar, choose C_i to be isotopic to a boundary curve of F' . Note that since F is incompressible and diskless, C_i is not null-homotopic in M . Also notice that in all cases there is a properly embedded arc γ on one of the F'_i which intersects $C_1 \cup C_2$ in one point.

Now choose a band $B = I \times I$ on F' such that $B \cap \partial D = I \times \frac{1}{2}$, $B \cap C_1 = I \times 0$, $B \cap C_2 = I \times 1$, and B is disjoint from the arc γ above. Such a band exists because γ is a nonseparating arc on F'_i . Let C' be the band sum of C_1 and C_2 , that is, $C' = (C_1 \cup C_2 - I \times \{0, 1\}) \cup (\{0, 1\} \times I)$. Then C' intersects γ in one point. Since C' intersects ∂D essentially in two points, it is not parallel to any boundary component on F' . Therefore C' is ∂ -nonseparating.

We want to show that C' is not parallel into F . Using the property that C_i are not null-homotopic in M , one can show by an innermost circle argument that C' is not null-homotopic in M . Now suppose that there is an annulus A in M with $\partial A = C' \cup C$, where $C \subset F$. Since C' is not null-homotopic in M , A is incompressible in M . By surgery along an innermost circle of $D \cap A$ one can eliminate all circle intersections of $A \cap D$. Since $\partial(A \cap D)$ consists of two points, $A \cap D$ is a single arc, which has endpoints on the same component of ∂A , hence it cuts off a disk D' from A . Assume without loss of generality that $D' \cap F'$ is on F'_1 . Let D'' be the disk on D bounded by $(A \cap D) \cup (B \cap D)$, and let $B_1 = B \cap F'_1$. Then $D' \cup D'' \cup B_1$ is a disk with boundary C_1 , which contradicts the fact that C_1 is not null-homotopic in M . Therefore, C' is not parallel into F . □

Lemma 6. *Suppose F' is incompressible, and is not a thrice punctured sphere. Then either (i) there is a ∂ -nonseparating curve C' on F' which is not parallel into F , or (ii) F' is parallel into F .*

Proof. Since F' is not a thrice punctured sphere, one can easily find a ∂ -nonseparating curve α_0 on F' . Assume that (i) is not true, so all ∂ -nonseparating curves are parallel into F . We want to show that F' is parallel into F .

Since α_0 is parallel into F , the annulus $N(\alpha_0)$ is also parallel into F . It is an incompressible annulus because α_0 is essential on F' and F' is incompressible. Among all connected incompressible surfaces in $\text{Int}F'$ which contain α_0 and are parallel into F , choose G' such that the complexity $(\chi(G'), |\partial G'|)$ is minimal in the lexicographic order, where $\chi(G')$ is the Euler characteristic of G' , and $|\partial G'|$ is the number of boundary components of G' . All incompressible sub-surfaces of F' have Euler characteristics bounded below by $\chi(F')$, hence such G' does exist.

If all boundary components of G' are parallel to some boundary components on F' , then either G' is contained in a collar of $\partial F'$, or $F' - \text{Int}G' = \partial F' \times I$. The first case does not happen because G' contains the ∂ -nonseparating curve α_0 , which by definition is not parallel to any boundary curve on F' . In the second case F' is isotopic to G' , so it is parallel into F , and we are done. Hence we may assume that some boundary curve β of G' is not parallel to any boundary curve on F' .

We want to find a ∂ -nonseparating curve α' which intersects β essentially in one or two points. If β is nonseparating on F' , choose α' to be any curve on F' that intersects β in a single point. Then α' is nonseparating, hence ∂ -nonseparating on F' . If β separates F' into F'_1 and F'_2 , choose an essential arc α'_i on F'_i with $\partial\alpha'_1 = \partial\alpha'_2 \subset \beta$. Moreover, if F'_i is nonplanar, choose α'_i to be nonseparating on F'_i . Then $\alpha' = \alpha'_1 \cup \alpha'_2$ is ∂ -nonseparating, and intersects β essentially in two points, as required.

Isotope α' so that it intersects $\partial G'$ minimally. The geometric intersection number between α' and β is 1 or 2, so $\alpha' \cap \partial G' \neq \emptyset$. Since α' is ∂ -nonseparating, by our assumption above it is parallel into F , so there is an annulus A with $\partial A = \alpha' \cup \alpha$, where $\alpha \subset F$. Isotope A rel α' so that it intersects $(\partial G') \times I$ minimally. Since G' is incompressible, $(\partial G') \times I$ consists of incompressible annuli in M , hence $A \cap ((\partial G') \times I)$ has no trivial circles. Since F and F' are also incompressible, one can show that $A \cap ((\partial G') \times I)$ has no trivial arcs on A either. Therefore $A \cap ((\partial G') \times I)$ consists of vertical arcs $(\alpha' \cap \partial G') \times I$. These arcs cut A into several squares $\alpha'_i \times I$, where each α'_i is the closure of a component of $\alpha' - \partial G'$. Choose i so that α'_i lies outside of G' . Let H be the component of $F' - \text{Int}G'$ that contains α'_i . Then $G'' = G' \cup N(\alpha'_i)$ is a surface parallel into F , and $\chi(G'') = \chi(G') - 1$. The arc α'_i is essential on H , so the only case that some boundary component γ of G'' bounds a disk on F' is when H is an annulus, and γ is the boundary of the disk obtained by cutting H along α'_i . Since F and F' are incompressible and M is irreducible, both ends of the annulus $\gamma \times I \subset G'' \times I \subset M$ bound disks on $F \cup F'$, which together with $\gamma \times I$ bounds a 3-ball in M . It follows that $G' \cup H$ is parallel into F . Since $G' \cup H$ has the same Euler characteristic as G' but fewer boundary components, this contradicts the choice of G' . Therefore $\partial G''$ consists of essential curves on F' . Since F' is incompressible, G'' is also incompressible. Since $\chi(G'') < \chi(G')$, this again contradicts the choice of G' . \square

Given a simple closed curve α and a proper arc γ on F' , denote by $\tau_\alpha^n \gamma$ the curve obtained from γ by Dehn twisting n times along α , and by $N(\tau_\alpha^n \gamma)$ a regular neighborhood of $\tau_\alpha^n \gamma$ on ∂M . Suppose T is a fixed torus boundary component of a 3-manifold M . Denote by $M(r)$ the manifold obtained by Dehn filling on T along a slope r on T , that is $M(r)$ is obtained by gluing a solid torus V to M along T so that the curve r on T bounds a meridian

disk in V . Denote by $\Delta(r_1, r_2)$ the minimal geometric intersection number between two slopes r_1, r_2 . The following two theorems will be used in the proof of Theorem 9, which proves Theorem 1 in the case that F' contains a ∂ -nonseparating curve which is not parallel into F .

Lemma 7 ([Wu2, Theorem 1]). *Let T be a torus component on the boundary of a 3-manifold M , and let S be an incompressible surface on $\partial M - T$. Suppose there is no incompressible annulus in M with one boundary component on each of S and T . If S is compressible in $M(r_1)$ and $M(r_2)$, then $\Delta(r_1, r_2) \leq 1$. In particular, S is incompressible in all but at most three $M(r)$.*

The following lemma can be found in [M], [Sh] or [CGLS, Theorem 2.4.3].

Lemma 8. *Let T, S, M be as in Lemma 7, and assume further that M is irreducible. Suppose that there is an incompressible annulus A in M with one boundary component on S and the other a curve r_0 on T . Then either S is a torus and $M = S \times I$, or S remains incompressible in all $M(r)$ with $\Delta(r, r_0) > 1$.*

Theorem 9. *Let α be a ∂ -nonseparating curve on F' which is not parallel into F , and let γ be a proper arc on F' intersecting α in one point. Then $F_n = F \cup N(\tau_\alpha^n \gamma)$ is incompressible for all but at most three consecutive n 's.*

Proof. Let K be the knot obtained by pushing α slightly into M . There is an embedded annulus A_0 in M with $\partial A_0 = \alpha \cup K$. Consider the manifold $M_K = M - \text{Int}N(K)$, where $N(K)$ is a regular neighborhood of K in M . Let T be the torus $\partial N(K)$, and let (m, l) be the meridian-longitude pair on T such that $l = A_0 \cap T$. Denote by $M_K(p/q)$ the manifold obtained by Dehn filling on T along the slope $pm + ql$. The Dehn twist τ_α^{-n} on F' extends to a Dehn twist of M_K along the annulus $A = A_0 \cap M_K$, which sends the meridian slope m of T to the slope $m - nl$, so it extends to a homeomorphism $\varphi_n : M = M_K(1/0) \cong M_K(-1/n)$, which maps the curve $\tau_\alpha^n \gamma$ to the curve γ , and hence the surface F_n to the surface $F_0 = F \cup N(\gamma)$. It follows that φ_n is a homeomorphism of pairs

$$\varphi_n : (M, F_n) \rightarrow (M_K(-1/n), F_0).$$

Therefore to prove the theorem we need only show that for all but at most three consecutive integers n , the surface F_0 is incompressible in $M_K(-1/n)$.

Claim 1. *$T = \partial N(K)$ is incompressible in M_K , and M_K is irreducible.*

If D is a compressing disk of T in M_K , then ∂D must intersect the meridian m of K in one point, because otherwise after the trivial Dehn filling, $M = M_K(1/0)$ would contain a lens space or $S^2 \times S^1$ summand, contradicting the irreducibility of M . It follows that K , and hence α , bounds a disk

in M . In this case α is parallel to a trivial curve on F , which contradicts the assumption that α is not parallel into F . Similarly, if M_K is reducible, then since M is irreducible, K is contained in a ball in M , so α would be null-homotopic. Using Dehn's Lemma, we see that α bounds a disk in M , hence is parallel to a trivial circle in F , contradicting the assumption that α is not parallel into F .

Claim 2. F_0 is incompressible in M_K .

Recall that A denotes the annulus $A_0 \cap M_K$. Since α intersects γ in a single point, $A \cap F_0$ is a single arc C on the boundary curve α of A . Let D be a compressing disk of F_0 , chosen so that $|D \cap A|$, the number of components in $D \cap A$, is minimal. After disk swapping along disks on A bounded by innermost circles, we can assume that no component of $D \cap A$ is a trivial circle on A . Since T is incompressible by Claim 1, the annulus A is also incompressible, so $D \cap A$ contains no essential circle component on A either. Hence $D \cap A$ consists of arcs only. If some arc e of $D \cap A$ is parallel in A to a sub-arc on $C = A \cap F_0$, then after boundary compressing D along a disk Δ cut off by an outermost such arc we will get two disks D_1, D_2 with boundary on F_0 , at least one of which has boundary an essential curve on F_0 , hence is a compressing disk of F_0 . Since $|D_i \cap A| < |D \cap A|$, this contradicts the minimality of $|D \cap A|$. Therefore, each arc of $D \cap A$ is essential relative to C , in the sense that it is not parallel in A to an arc on C . See Figure 1(a). Notice that $|D \cap A| \neq 0$, otherwise D would be a compressing disk of F , contradicting the incompressibility of F .

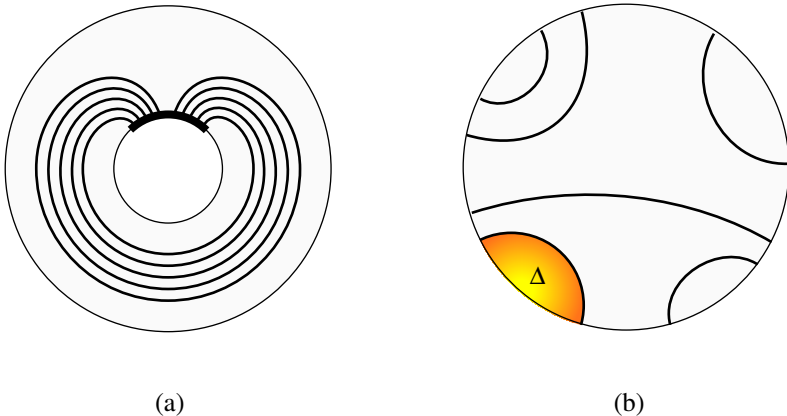


Figure 1.

Consider an outermost disk Δ on D , as shown in Figure 1(b). Then $\partial\Delta$ consists of two arcs e_1, e_2 , where e_1 is an arc on A which is essential relative

to C , and e_2 is an arc on F_0 with interior disjoint from C . Thus $e_2 \cap N(\gamma)$ consists of two arcs e'_2, e''_2 . Let t_1 be the subarc of C connecting the two ends of $e'_2 \cup e''_2$ on C , and let t_2 be the subarc on $\partial N(\gamma)$ connecting the other two ends of $e'_2 \cup e''_2$. Then $e'_2 \cup t_1 \cup e''_2 \cup t_2$ bounds a disk Δ' on $N(\gamma)$. Now $A' = \Delta \cup \Delta'$ is an annulus in M , with one boundary component $e_1 \cup t_1$ an essential circle on A , which is parallel to α , and the other component $\widehat{e}_2 \cup t_2$ a curve on F , where \widehat{e}_2 is the closure of $e_2 - (e'_2 \cup e''_2)$. This contradicts the assumption that α is not parallel into F .

Claim 3. *There is no incompressible annulus P in M_K with one boundary component C_1 on F_0 and the other component C_2 a curve on T which is disjoint from $l = A \cap T$.*

The proof is similar to that of Claim 2. Choose P so that $|P \cap A|$ is minimal. Using the fact that P is incompressible, one can show as above that $P \cap A$ has no trivial circle component. Note that since C_2 is disjoint from l , $P \cap A$ has no arc component with endpoints on $l = A \cap T$. If $P \cap A$ had some essential circle component, choose such a component t which is closest to l on A . By cutting and pasting along the annulus on A bounded by $t \cup l$, one would get another incompressible annulus P' which has fewer intersection components with A . As in the proof of Claim 2 one can eliminate all arc components of $P \cap A$ which on A are inessential relative to $C = A \cap F_0$. Hence $P \cap A$ consists of arcs with ends on C and are essential relative to C , as shown in Figure 1(a). Also, since P is disjoint from l , $P \cap A$ consists of inessential arcs on P . Now one can use a disk Δ cut off by an outermost arc on P , proceed as in the proof of Claim 2 to get an annulus with one boundary on α and the other on F , and get a contradiction. Finally, if $P \cap A = \emptyset$ then P extends to an annulus with one boundary on α and the other on F , contradicting the assumption that α is not parallel into F . This completes the proof of Claim 3.

We now continue with the proof of Theorem 9. We have shown that F_0 is incompressible in M_K . If there is no essential annulus in M_K with one boundary component on each of F_0 and T , then by Lemma 7 we know that F_0 is incompressible in $M_K(r)$ for all but at most three slopes r with mutual intersection number 1. In particular, it is incompressible in $M_K(-1/n)$ for all but at most two consecutive n 's, so the theorem follows. Now suppose there is an essential annulus P in M_K with one boundary component on F_0 and the other a curve r_0 on T . Since F_0 is not a closed surface, it is not a torus. Hence by Lemma 8, F_0 remains incompressible in $M_K(-1/n)$ unless $\Delta(-1/n, r_0) \leq 1$. By Claim 3, r_0 is not the longitude slope $0/1$, therefore, $\Delta(-1/n, r_0) \leq 1$ holds for at most three consecutive integers n . This completes the proof of Theorem 9. □

Proof of Theorem 1. By Theorem 4, Lemmas 5 and 6, and Theorem 9, we can now assume that F' is incompressible and is parallel into F . We want to show that either F is arc-extendible in F' , or M is a product $F \times I$. As in the proof of Theorem 4, we may assume without loss of generality that M is a compression body, so all closed incompressible surfaces of M are boundary parallel. Let $\alpha_1, \dots, \alpha_k$ be the boundary curves of F' . Let $F' \times I$ be a product in M such that $F' = F' \times 0$ and $F' \times 1 \subset F$. Write $\alpha_i^1 = \alpha_i \times 1$, which is a curve on F isotopic to α_i in M .

We have assumed above that F' is incompressible in M , so $\text{Int}F \cup \text{Int}F'$ is incompressible in M . Write $\widehat{F}_i = \text{Int}F \cup \text{Int}F' \cup \alpha_i$. If \widehat{F}_i is incompressible for some i , then $F \cup N(\gamma)$ is incompressible for any essential arc γ in F' with endpoints on α_i , and we are done. (Such an arc exists because F' is not an annulus or disk.) So assume that \widehat{F}_i is compressible for all i . By the Handle Addition Lemma (Lemma 2), after adding a 2-handle to M along α_i , the surface $\widehat{F}_i[\alpha_i]$ is incompressible, and $M[\alpha_i]$ is irreducible. Notice that since F' is incompressible, the curve $\alpha_i^1 = \alpha_i \times 1$ in F is essential on F . But after adding the 2-handle, α_i^1 bounds a disk in $M[\alpha_i]$, so it must also bound a disk on $\widehat{F}_i[\alpha_i]$ because $\widehat{F}_i[\alpha_i]$ is incompressible. By definition $\widehat{F}_i[\alpha_i]$ is obtained from $(\text{Int}F \cup \text{Int}F') - \text{Int}N(\alpha_i)$ by capping off the two copies of α_i with disks, hence $\alpha_i^1 \cup \alpha_i$ bounds an annulus A_i on \widehat{F}_i . Denote by A'_i the annulus $\alpha_i \times I \subset F' \times I \subset M$. Then $T_i = A_i \cup A'_i$ is a torus in M (T_i cannot be a Klein bottle since M is a compression body). Since we have assumed above that M is a compression body, either (i) T_i bounds a solid torus V_i , or (ii) it is parallel to some torus component of ∂M , or (iii) it lies in a ball and bounds a cube with knotted hole. Since F' is incompressible, the curve α_i is not null homotopic in M , hence (iii) does not happen. Since $M[\alpha_i]$ is irreducible, one can proceed as in the proof of Theorem 4 to show that T_i must bound a solid torus V_i with α_i a longitude. This is true for all i . It is now easy to see that M is a product $F \times I$. \square

The following theorem supplements Theorem 1. It says that in most cases there are extension arcs with endpoints on any prescribed boundary components of F' .

Theorem 10. *Let F, F', M be as in Theorem 1. Suppose M is not a product $F \times I$, and suppose F' is not parallel into F and is not a thrice punctured sphere. Then it contains an extension arc γ of F with endpoints on any prescribed components of $\partial F'$.*

Proof. We need to find a curve α on F' which is not parallel into F . If F' is nonplanar, then by the proof of Lemmas 5 and 6, we can find such an α (denoted by C' there) on F' which is actually nonseparating on F' . Hence given any boundary components ∂_1, ∂_2 of F' , (possibly $\partial_1 = \partial_2$), there is an arc γ with endpoints on ∂_1 and ∂_2 , intersecting α in one point. By Theorem

9, for all but at most three integers n , the arc $\gamma_n = \tau_\alpha^n \gamma$ is an extension arc of F .

Now suppose F' is planar with $|\partial F'| \geq 4$. First assume that ∂_1, ∂_2 are distinct boundary components of F' . By the proofs of Lemmas 5 and 6, there is a curve α which is a band sum of two boundary components of F' , such that α is not parallel into F . From the proofs one can see that we can always choose α to be the band sum of ∂_1 and ∂_3 , with $\partial_3 \neq \partial_1, \partial_2$. Hence there is an arc γ from ∂_1 to ∂_2 intersecting α in one point. We can then apply Theorem 9 to get an extension arc γ_n with one endpoint on each of ∂_1 and ∂_2 .

We now proceed to find an extension arc in F' with boundary on the same component ∂_1 of $\partial F'$. By the proof of Lemmas 5 and 6, we can choose the curve α above to be the band sum of ∂_2 and ∂_3 , with $\partial_1 \neq \partial_2, \partial_3$. Recall that α is not parallel into F . Choose an arc γ as follows. Let ∂'_2 be a curve on F' parallel to ∂_2 , let γ' be an arc connecting ∂'_2 to ∂_1 intersecting α in one point, and let Q be the sub-surface $N(\gamma' \cup \partial'_2)$ of F' . Then γ is the closure of the arc component of $\partial Q \cap \text{Int} F'$, that is, γ is the arc component of the frontier of Q in F' , see Figure 2 below. Consider the surface $F_0 = F \cup N(\gamma)$, and observe that F_0 is isotopic to the surface $F \cup Q$. After Dehn twisting along α , it is isotopic to the surface $F \cup N(\tau_\alpha^n \gamma)$; hence to show that all but at most three $\tau_\alpha^n \gamma$ are extension arcs of F in F' , we need only show that $F \cup Q$ is incompressible after all but at most three Dehn twists along α . Since $F \cup Q$ intersects α in a single arc, the argument in the proof of Theorem 9 is still valid, with the following easy modifications. We use the notations in that proof.

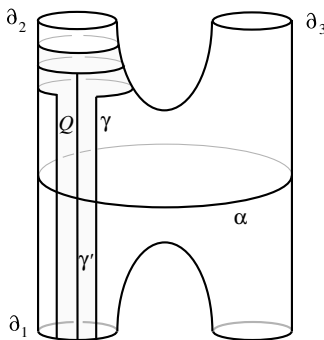


Figure 2.

The proof of Claim 2 needs the following modifications. (i) The arc e_2 on the boundary of the outermost disk Δ may be on Q . In this case, notice

that the other arc e_1 on $\partial\Delta$ is isotopic to an arc α_1 on α , and $e_2 \cup \alpha_1$ is isotopic in F' to the curve ∂_3 , so the fact that $e_1 \cup e_2$ bounds a disk Δ would imply that ∂_3 bounds a disk. Since ∂_3 is also on ∂F , this contradicts the fact that F is incompressible and diskless. (ii) The compressing disk D of $F \cup Q$ could be disjoint from the annulus A . But since F is incompressible, this would imply that ∂D lies on Q , hence is isotopic to ∂_2 , which would imply that ∂_2 bounds a disk, again contradicting the assumption that F is incompressible and diskless.

The proof of Claim 3 applies to show that the annulus P there can be modified to be disjoint from the annulus A . Then notice that the component of ∂P on $F \cup Q$ is either in F , or in Q and hence parallel to ∂_2 . Since $\partial_2 \subset F$, in either case P can be extended to an annulus with one boundary component on α and the other on F , which contradicts the assumption that α is not parallel into F .

The rest of the proof of Theorem 9 follows verbatim to show that $F \cup Q$ is incompressible after all but at most three Dehn twists along α . \square

Remark. Theorem 10 is not true if F' is a thrice punctured sphere.

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