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# ON A SHARP MOSER–AUBIN–ONOFRI INEQUALITY FOR FUNCTIONS ON $S^2$ WITH SYMMETRY

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**We show that for $\alpha \geq \frac{1}{2}$, the following inequality holds:**

$$\frac{\alpha}{2}\int_{-1}^{1}(1-x^2)|g'(x)|^2dx+\int_{-1}^{1}g(x)dx-\log\frac{1}{2}\int_{-1}^{1}e^{2g(x)}dx \geq 0,$$

**for every function $g$ on $(-1,1)$ satisfying $\|g\|^2 = \int_{-1}^{1}(1-x^2)|g'(x)|^2dx < \infty$ and $\int_{-1}^{1}e^{2g(x)}xdx = 0$. This improves a result of Feldman et al., 1998, and answers a question of Chang and Yang in the axially symmetric case.**

## 1. Introduction.

On $S^2$ let $J_\alpha$ denote the functional on the Sobolev space $H^{1,2}(S^2)$ defined by

$$J_\alpha(g) = \alpha\int_{S^2}|\nabla g|^2dw + 2\int_{S^2}gdw - \log\int_{S^2}e^{2g}dw.$$

Here $dw$ denotes the Lebesgue measure on the unit sphere, normalized to make $\int_{S^2}dw = 1$. The famous Moser-Trudinger inequality says that $J_1$ is bounded below by a non-positive constant $C_1$. Later Onofri [6] showed that $C_1$ can be taken to be 0. (Another proof was also given by Osgood-Phillips-Sarnack [7].) On the other hand, if we restrict $J_\alpha$ to the class of $\mathcal{G}$ of functions $g$ for which $e^{2g}$ has centre of mass equal to 0, that is $\int_{S^2}e^{2g}\vec{x}dw = 0$, then Aubin in [2] showed that for $\alpha \geq \frac{1}{2}$, the functional $J_\alpha$ is again bounded below by a non-positive constant $C_\alpha$. In [3] and [4] A. Chang and P. Yang showed that $C_\alpha = 0$ for $\alpha$ close enough to 1. This led them to the following

**Conjecture.** Let $\mathcal{G}$ denote the functions in $H^{1,2}(S^2)$ for which $\int_{S^2}e^{2g}\vec{x}dw = 0$. If $\alpha \geq \frac{1}{2}$, then $\inf_{g\in\mathcal{G}}J_\alpha(g) = 0$.

In this note, we prove this conjecture in the axially symmetric case. We note that Feldman, Froese, Ghoussoub and Gui [5] proved that the above conjecture holds for the axially symmetric case when $\alpha > \frac{16}{25} - \epsilon$ for some small $\epsilon$. They also gave an example which says the inequality is not true if $\alpha < \frac{1}{2}$. It is also known that $J_\alpha(g) \geq 0$ if $g$ is an even function, i.e., $g(\vec{x}) = g(-\vec{x})$ on $S^2$. (See [7].)

Let $\theta$ and $\varphi$ denote the usual angular coordinates on the sphere, and define $x = cos(\theta)$. Axially symmetric functions depend on $x$ only. For such functions, it is well-known (see [5]) that the functional $J_\alpha$ can be written as

$$I_\alpha(g) := \frac{\alpha}{2}\int_{-1}^{1}(1-x^2)|g'(x)|^2dx + \int_{-1}^{1} g(x)dx - \log\frac{1}{2}\int_{-1}^{1} e^{2g(x)}dx.$$

The set $\mathcal{G}$ is then replaced by

$$\mathcal{G}_r := \left\{ g | \int_{-1}^{1}(1-x^2)|g'(x)|^2 < \infty, \int_{-1}^{1} e^{2g(x)}xdx = 0 \right\}.$$

It is proved in [5, Proposition 3.1] that any critical point $g$ of $I_\alpha$ restricted to $\mathcal{G}_r$ satisfies the following differential equation

$$\alpha((1-x^2)g')' - 1 + \frac{2}{\lambda}e^{2g} = 0, \quad \lambda = \int_{-1}^{1} e^{2g}dx. \tag{1.1}$$

The main result of this note is the following:

**Theorem 1.1.** *If $\alpha \geq \frac{1}{2}$, then the only critical points of the functional $I_\alpha$ restricted to $\mathcal{G}_r$ are constant functions.*

As a consequence, the above theorem implies that the Conjecture of Chang and Yang is true in the axially symmetric case.

**Theorem 1.2.** *If $\alpha \geq \frac{1}{2}$, then $I_\alpha(g) \geq 0$ for $g \in \mathcal{G}_r$.*

The rest of the paper is devoted to the study of (1.1). To this end, we need some notations and some basic facts.

Let $g$ be a solution of (1.1). Following [5], we set

$$G = (1-x^2)g'.$$

Then $G$ satisfies (see [5])

$$\alpha G' - 1 + \frac{2}{\lambda}e^{2g} = 0, \tag{1.2}$$

and

$$\begin{cases} (1-x^2)G'' + \frac{2}{\alpha}G - 2GG' = 0 \\ G(-1) = G(1) = 0. \end{cases} \tag{1.3}$$

We also need some facts about the Legendre's polynomials.

Let $P_n(x)$ be the $n-$th Legendre polynomial, i.e., $P_n$ satisfies

$$((1-x^2)P_n')' + \lambda_n P_n = 0, \lambda_n = n(n+1), n = 0, 1, \dots .$$

Note that

$$P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2-1), \dots$$

Moreover (see [1])

$$|P_n'(x)| \le \frac{1}{2}\lambda_n, \int_{-1}^{1} P_n^2 = \frac{2}{2n+1}. \tag{1.4}$$

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## 2. Proof of Theorem 1.1.

In this section, we shall prove Theorem 1.1.

Let

$$G(x) = \beta x + a_2 \frac{1}{2}(3x^2 - 1) + \sum_{k=3}^{\infty} a_k P_k(x),$$

$$G_2 = \sum_{k=3}^{\infty} a_k P_k(x)$$

and

$$b_k^2 = a_k^2 \int_{-1}^{1} P_k^2, k \ge 2.$$

We first derive some equalities:

$$\int_{-1}^{1} (1-x^2)(G')^2 = \left(\frac{2}{\alpha} - 1\right) \int_{-1}^{1} G^2, \tag{2.1}$$

$$\int_{-1}^{1} P_1 G = \frac{2}{3}\beta, \tag{2.2}$$

$$\int_{-1}^{1} (1-x^2)\frac{e^{2g}}{\lambda} = \frac{2}{3}(1-\alpha\beta), \tag{2.3}$$

$$\int_{-1}^{1} P_k G = -\frac{2}{\alpha\lambda_k} \int_{-1}^{1} (1-x^2) P_k' \frac{e^{2g}}{\lambda}, k \ge 2, \tag{2.4}$$

$$\int_{-1}^{1} G^2 = \left(6 - \frac{2}{\alpha}\right) \frac{2}{3}\beta, \tag{2.5}$$

$$\frac{2}{3}\beta\left(4\beta + \left(7 - \frac{2}{\alpha}\right)\left(\frac{2}{\alpha} - 6\right)\right) = \int_{-1}^{1} (1-x^2)(G_2')^2 - 6\int_{-1}^{1} G_2^2, \tag{2.6}$$

$$\int_{-1}^{1}(1-x^2)(G_2')^2 - 6\int_{-1}^{1} G_2^2 = \sum_{k=3}^{\infty}(\lambda_k - 6)b_k^2. \tag{2.7}$$

*Proofs of* (2.1)-(2.7). Multiplying (1.3) by $G$ and integrating over $[-1,1]$, we obtain (2.1). The relation (2.2) follows by definition. Multiplying (1.2) by $\int_{-1}^{x} P_k(s)ds, k \geq 1$ and integrating over $[-1,1]$ we obtain (2.3) and (2.4). Multiplying (1.3) by $x$ and integrating from $-1$ to $1$ we obtain (2.5). To show (2.6), we just need to use (2.1), (2.5) and the definition of $G_2$. The equality (2.7) follows from definition. □

We will show $\beta = 0$, which implies $G = 0$ by (2.5). Our basic strategy is to show that if $\beta \neq 0$, then

$$\beta = \frac{1}{\alpha},$$

which will lead to a contradiction.

Below we assume that $\beta \neq 0$.

Next we obtain some inequalities.

From (2.3) we have

$$\frac{1}{\alpha} - \beta > 0. \tag{2.8}$$

By definition we have

$$\begin{aligned} b_k^2 = a_k^2 \int_{-1}^{1} P_k^2 &= \frac{(\int_{-1}^{1} GP_k)^2}{\int_{-1}^{1} P_k^2} \\ &\leq \frac{2k+1}{2}\left(\frac{2}{\alpha\lambda_k}\int_{-1}^{1}(1-x^2)|P_k'|\frac{e^{2g}}{\lambda}\right)^2 \\ &\leq \frac{2k+1}{2}\left(\frac{2}{\alpha\lambda_k}\frac{\lambda_k}{2}\frac{2}{3}(1-\alpha\beta)\right)^2. \end{aligned}$$

Hence we obtain

$$b_k^2 \leq \frac{2(2k+1)}{9}\left(\frac{1}{\alpha} - \beta\right)^2, k \geq 2. \tag{2.9}$$

Similarly we obtain

$$\frac{3}{5}|a_2| \leq \frac{1}{\alpha} - \beta. \tag{2.10}$$

From (2.6) (since $\beta > 0$),

$$4\beta + \left(7 - \frac{2}{\alpha}\right)\left(\frac{2}{\alpha} - 6\right) \geq 0.$$

Since $\alpha \geq 0.5$, we have

$$\beta \geq \frac{1}{4}\left(7-\frac{2}{\alpha}\right)\left(6-\frac{2}{\alpha}\right) \geq 1.5. \tag{2.11}$$

From (2.6) and (2.8), we have

$$\frac{4}{\alpha}+\left(7-\frac{2}{\alpha}\right)\left(\frac{2}{\alpha}-6\right) \geq 0$$

which implies that

$$\alpha \leq 0.537.$$

From (2.6) we have

$$\begin{aligned}
&\frac{2}{3}\beta\left(4\beta+\left(7-\frac{2}{\alpha}\right)\left(\frac{2}{\alpha}-6\right)\right)\\
&=\int_{-1}^{1}(1-x^2)(G_2')^2-6\int_{-1}^{1}G_2^2\\
&\geq \frac{1}{2}\int_{-1}^{1}(1-x^2)(G_2')^2\\
&\geq \frac{1}{2}\left[\int_{-1}^{1}(1-x^2)(G')^2-\frac{4}{3}\beta^2-\frac{12}{5}a_2^2\right]\\
&\geq \frac{1}{2}\left[\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right)\frac{2}{3}\beta-\frac{4}{3}\beta^2-\frac{12}{5}a_2^2\right].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{2}{3}\beta\left[\frac{5}{\alpha}+\left(7-\frac{2}{\alpha}\right)\left(\frac{2}{\alpha}-6\right)-\frac{1}{2}\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right)\right]\\
&\geq \frac{10}{3}\beta\left(\frac{1}{\alpha}-\beta\right)-\frac{6}{5}a_2^2\\
&\geq \frac{10}{3}\beta\left(\frac{1}{\alpha}-\beta\right)-\frac{6}{5}\times\frac{25}{9}\left(\frac{1}{\alpha}-\beta\right)^2\\
&\geq \frac{10}{3}\left(2\beta-\frac{1}{\alpha}\right)\left(\frac{1}{\alpha}-\beta\right).
\end{aligned} \tag{2.12}$$

Since $(\frac{1}{\alpha}-\beta)\geq 0, \alpha \geq 0.5$ and $2\beta-\frac{1}{\alpha}\geq 0$, we conclude that (since $\beta>0$)

$$0 \leq \frac{5}{\alpha}+\left(7-\frac{2}{\alpha}\right)\left(\frac{2}{\alpha}-6\right)-\frac{1}{2}\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right) \leq 1 \tag{2.13}$$

which implies, by a simple computation, that

$$\alpha \leq 0.52. \tag{2.14}$$

Moreover since $\alpha \geq 0.5$ and $\beta \geq 1.5$, we obtain from (2.12) and (2.13) that

$$\frac{1}{\alpha} - \beta \leq \frac{\beta}{5(2\beta - \frac{1}{\alpha})} \leq \frac{\beta}{5}. \tag{2.15}$$

To obtain better estimates, we fix an integer $n \geq 3$. We have by (2.6) and (2.7)

$$\begin{aligned}
&\frac{2}{3}\beta\left(4\beta + \left(7 - \frac{2}{\alpha}\right)\left(\frac{2}{\alpha} - 6\right)\right) \\
&= \sum_{k=3}^{\infty}(\lambda_k - 6)b_k^2 \\
&= \sum_{k=3}^{n}(\lambda_k - 6)b_k^2 + \sum_{k=n+1}^{\infty}(\lambda_k - 6)b_k^2 \\
&\geq \sum_{k=3}^{n}(\lambda_k - 6)b_k^2 + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}\sum_{k=n+1}^{\infty}\lambda_k b_k^2 \\
&= \sum_{k=3}^{n}(\lambda_k - 6)b_k^2 \\
&\quad + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}\left(\frac{2}{3}\beta\left(\frac{2}{\alpha} - 1\right)\left(6 - \frac{2}{\alpha}\right) - \frac{4}{3}\beta^2 - \frac{12}{5}a_2^2 - \sum_{k=3}^{n}\lambda_k b_k^2\right) \\
&= \sum_{k=3}^{n}\left(\lambda_k - 6 - \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}\lambda_k\right)b_k^2 \\
&\quad + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}\left(\frac{2}{3}\beta\left(\frac{2}{\alpha} - 1\right)\left(6 - \frac{2}{\alpha}\right) - \frac{4}{3}\beta^2 - \frac{12}{5}a_2^2\right) \\
&= \sum_{k=3}^{n} 6\frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}}b_k^2 - \frac{12}{5}a_2^2\frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \\
&\quad + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}\left(\frac{2}{3}\beta\left(\frac{2}{\alpha} - 1\right)\left(6 - \frac{2}{\alpha}\right) - \frac{4}{3}\beta^2\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{2}{3}\beta\left(4\beta + \left(7 - \frac{2}{\alpha}\right)\left(\frac{2}{\alpha} - 6\right)\right) \\
&\qquad - \frac{\lambda_{n+1} - 6}{\lambda_{n+1}}\left(\frac{2}{3}\beta\left(\frac{2}{\alpha} - 1\right)\left(6 - \frac{2}{\alpha}\right) - \frac{4}{3}\beta^2\right) \\
&\geq \sum_{k=3}^{n} 6\frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}}b_k^2 - \frac{12}{5}a_2^2\frac{\lambda_{n+1} - 6}{\lambda_{n+1}}.
\end{aligned} \tag{2.16}$$

After some simple computations, the left hand of (2.16) equals to

$$12\beta\left(\frac{1}{\alpha}-2\right)+\frac{4\beta}{\lambda_{n+1}}\left[\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right)-\frac{2}{\alpha}\right] - 4\beta\left(1-\frac{2}{\lambda_{n+1}}\right)\left(\frac{1}{\alpha}-\beta\right).$$

Thus we have by (2.9), (2.10) and (2.16)

$$\begin{aligned}
&12\beta\left(\frac{1}{\alpha}-2\right)+\frac{4\beta}{\lambda_{n+1}}\left[\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right)-\frac{2}{\alpha}\right]\\
&\geq 4\beta\left(1-\frac{2}{\lambda_{n+1}}\right)\left(\frac{1}{\alpha}-\beta\right)-\frac{12}{5}a_2^2\frac{\lambda_{n+1}-6}{\lambda_{n+1}}\\
&\quad+6\sum_{k=3}^{n}\frac{\lambda_k-\lambda_{n+1}}{\lambda_{n+1}}\frac{2(2k+1)}{9}\left(\frac{1}{\alpha}-\beta\right)^2\\
&\geq 4\beta\left(1-\frac{2}{\lambda_{n+1}}\right)\left(\frac{1}{\alpha}-\beta\right)-\frac{20}{3}\frac{\lambda_{n+1}-6}{\lambda_{n+1}}\left(\frac{1}{\alpha}-\beta\right)^2\\
&\quad-\frac{4}{3}\sum_{k=3}^{n}\frac{\lambda_{n+1}-\lambda_k}{\lambda_{n+1}}(2k+1)\left(\frac{1}{\alpha}-\beta\right)^2\\
&\geq \left[4\beta\left(1-\frac{2}{\lambda_{n+1}}\right)\right.\\
&\qquad\left.-\frac{20}{3}\frac{\lambda_{n+1}-6}{\lambda_{n+1}}\left(\frac{1}{\alpha}-\beta\right)-\frac{4}{3}c_n\left(\frac{1}{\alpha}-\beta\right)\right]\left(\frac{1}{\alpha}-\beta\right)
\end{aligned} \tag{2.17}$$

where

$$c_n=\sum_{k=3}^{n}\frac{\lambda_{n+1}-\lambda_k}{\lambda_{n+1}}(2k+1).$$

Since $1/2<\alpha\leq 1$ and $\lambda_n>2$ for $n\geq 1$, we have

$$\begin{aligned}
&12\beta\left(\frac{1}{\alpha}-2\right)+\frac{4\beta}{\lambda_{n+1}}\left[\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right)-\frac{2}{\alpha}\right]-\frac{8\beta}{\lambda_{n+1}}\\
&=4\beta\left(\frac{1}{\alpha}-2\right)\left[3-\frac{4}{\lambda_{n+1}}\left(\frac{1}{\alpha}-1\right)\right]\\
&\leq 0.
\end{aligned}$$

Thus the left hand side of (2.17) satisfies

$$\text{LHS of (2.17)}\leq\frac{8\beta}{\lambda_{n+1}}. \tag{2.18}$$

We now claim

$$\frac{1}{\alpha}-\beta\leq\frac{4}{\lambda_n},\quad\forall n\geq 4. \tag{2.19}$$

By (2.18), we just need to show that the right hand side of (2.17) satisfies

$$\text{RHS of (2.17)} \geq 2\beta\left(\frac{1}{\alpha}-\beta\right). \tag{2.20}$$

We prove it by induction.

We first prove $n=4$. To this end, we iterate the inequality (2.17). Note that the right hand side of (2.17) with $n=3$ equals

$$\begin{aligned}
&\left[4\beta\left(1-\frac{2}{20}\right)-\frac{20}{3}\frac{20-6}{20}\left(\frac{1}{\alpha}-\beta\right)\right.\\
&\qquad\left.-\frac{4}{3}\frac{20-12}{20}\times 7\left(\frac{1}{\alpha}-\beta\right)\right]\left(\frac{1}{\alpha}-\beta\right)\\
&\geq\left[4\beta\frac{9}{10}-\frac{14}{3}\left(\frac{1}{\alpha}-\beta\right)-\frac{56}{15}\left(\frac{1}{\alpha}-\beta\right)\right]\left(\frac{1}{\alpha}-\beta\right)\\
&\geq\left[3.6\beta-\frac{126}{15}\left(\frac{1}{\alpha}-\beta\right)\right]\left(\frac{1}{\alpha}-\beta\right)\\
&\geq\left[3.6\beta-\frac{126}{15}\frac{\beta}{5}\right]\left(\frac{1}{\alpha}-\beta\right) \qquad \text{(by (2.15))}\\
&\geq 1.92\beta\left(\frac{1}{\alpha}-\beta\right).
\end{aligned} \tag{2.21}$$

By using (2.18) and (2.17) again, we obtain

$$\frac{1}{\alpha}-\beta \leq \frac{8}{20}\frac{1}{1.92} < 0.25. \tag{2.22}$$

Similarly, by using (2.22), we have

$$\begin{aligned}
\text{RHS of (2.17)} &\geq \left[3.6\beta-\frac{126}{15}\times 0.25\right]\left(\frac{1}{\alpha}-\beta\right) \qquad \text{(by (2.22))}\\
&\geq 2\beta\left(\frac{1}{\alpha}-\beta\right) \qquad (\text{since } \beta\geq 1.5 \text{ by (2.11)}).
\end{aligned}$$

Thus (2.20) holds for $n=4$ and hence (2.19) holds for $n=4$.

Let us now assume that

$$\frac{1}{\alpha}-\beta \leq \frac{4}{\lambda_k}, \quad k=n\geq 4.$$

We observe that for $n\geq 4$

$$\begin{aligned}
c_n &= \sum_{k=3}^{n}(2k+1)-\frac{1}{\lambda_{n+1}}\sum_{k=3}^{n}\lambda_k(2k+1)\\
&= \sum_{k=3}^{n}(2k+1)-\frac{1}{\lambda_{n+1}}\sum_{k=3}^{n}k(k+1)(2k+1)
\end{aligned}$$

$$= \frac{1}{2}\lambda_{n+1} - 9 + \frac{36}{\lambda_{n+1}}.$$

Hence we have by (2.17)

(2.23)

$$\begin{aligned} &12\beta\left(\frac{1}{\alpha}-2\right)+\frac{4\beta}{\lambda_{n+1}}\left[\left(\frac{2}{\alpha}-1\right)\left(6-\frac{2}{\alpha}\right)-\frac{2}{\alpha}\right] \\ &\geq \left[4\beta\left(1-\frac{2}{\lambda_{n+1}}\right)\right. \\ &\quad \left.-\left(\frac{20}{3}\frac{\lambda_{n+1}-6}{\lambda_{n+1}}+\frac{4}{3}\left(\frac{1}{2}\lambda_{n+1}-9+\frac{36}{\lambda_{n+1}}\right)\right)\left(\frac{1}{\alpha}-\beta\right)\right]\left(\frac{1}{\alpha}-\beta\right). \end{aligned}$$

The right hand of (2.23) satisfies

$$\begin{aligned} &\text{RHS of (2.23)} \\ &\geq \left[4\beta\left(1-\frac{2}{\lambda_{n+1}}\right)+\frac{64}{3}\frac{1}{\lambda_n}-\frac{32}{\lambda_n\lambda_{n+1}}-\frac{8}{3}\frac{\lambda_{n+1}}{\lambda_n}\right]\left(\frac{1}{\alpha}-\beta\right). \end{aligned}$$

To show (2.20), we only need to show

$$\beta\left(1-\frac{4}{\lambda_{n+1}}\right) \geq -\frac{32}{3}\frac{1}{\lambda_n}+\frac{16}{\lambda_n\lambda_{n+1}}+\frac{4}{3}\frac{\lambda_{n+1}}{\lambda_n},$$

or

$$\beta \geq \frac{4}{3}\cdot\frac{\lambda_{n+1}^2-8\lambda_{n+1}+12}{\lambda_n(\lambda_{n+1}-4)}.$$

In view of the inductive assumption, it suffices to show

$$\frac{1}{\alpha} \geq \frac{4}{3}\cdot\frac{\lambda_{n+1}}{\lambda_n}\cdot\frac{\lambda_{n+1}-5}{\lambda_{n+1}-4}.$$

Because of (2.14), it is easy to verify that the above inequality holds for $n \geq 4$.

In conclusion, we have obtained (2.19).

Finally we can finish the proof of Theorem 1.1. In fact, if we let $n \to +\infty$ in (2.19), we obtain

$$\frac{1}{\alpha} - \beta = 0$$

which is a contradiction to (2.8).

This implies that $\beta = 0$ and therfore $G \equiv 0$. Hence $g' \equiv 0$, and $g \equiv$ Constant. □

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