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# SUBMODULES OF THE HARDY SPACE OVER POLYNOMIAL ALGEBRAS

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**The classical Hardy space $H^2$ has a natural structure of a module over the algebra of polynomials $\mathbb{C}[z]$. In this setting the theorem of Beurling describes all closed $\mathbb{C}[z]$-submodules of $H^2$. In this paper we prove a Beurling-type theorem for $H^2$ as a module over a finitely generated polynomial algebra.**

## 1. Introduction.

The celebrated theorem of Beurling [3] states that any closed $z$-invariant subspace of the Hardy space $H^2$ in the unit disk is of the form $M = g \cdot H^2$ where $g$ is a classical inner function. The result of Apostol, Bercovici, Foias, and Pearcy [2] emphasized the importance of a Beurling type theorem for the Bergman space $A^2$ in the unit disk, and such a theorem was proved in 1996 by by Aleman, Richter, and Sundberg [1]. Their approach uses the concept of wandering property introduced by Halmos in 1961 [7]. More precisely the result in [1] states that if $M$ is a closed $z$-invariant subspace of $A^2$, then the set $M \ominus zM$ generates $M$ in the sense that $M$ is the minimal closed $z$-invariant subspace of $A^2$ which contains $M \ominus zM$. This gives rise to the following interpretation of a Beurling type theorem.

Let $\mathbb{C}[z]$ stand, as usual, for the ring of polynomials. Then both the Hardy and Bergman spaces have a natural $\mathbb{C}[z]$-module structure. Any $z$-invariant subspace corresponds to a $\mathbb{C}[z]$-submodule in this setting. Thus a Beurling type theorem describes a constructive way of obtaining a generating set of closed submodules. In the Hardy setting for a closed $\mathbb{C}[z]$-submodule $M = g \cdot H^2$, $M \ominus zM$ has dimension 1 and is spanned by $g$.

This leads to the following general question. Let $A$ be a subalgebra of $H^\infty$; then both Hardy and Bergman spaces can be considered as modules over $A$. Then given a closed $A$-submodule $M$ of the Hardy (Bergman) space, how can one describe a canonical procedure of finding a set of generators of $M$? In particular, is every closed $A$-submodule finitely generated, and if $A_0 = \{f \in A : f(0) = 0\}$, must $M \ominus A_0 M$ generate $M$ as an $A$-submodule? We single out zero to follow the classical route for a canonical construction, but we could replace it with any point $w$ in the unit disk, and all the results of this paper would remain valid.

Beurling's theorem for the Hardy space and Aleman, Richter, and Sundberg's theorem for the Bergman setting give an affirmative answer to the last question when $A = \mathbb{C}[z]$. If the algebra $A$ is singly generated by a classical inner function $g$, $A = \mathbb{C}[g]$, the result also holds in $H^2$. This follows from the Wold Decomposition theorem (see [**9**] for details).

In general, we say that an algebra $A$ has the *wandering property* if for any closed $A$-submodule $M$, the set $M \ominus A_0M$ is a generating set of $M$. It was proved in [**4**], [**8**] that if a singly generated subalgebra of $H^\infty$, $A = \mathbb{C}[g]$, where $g \in H^\infty$, has the wandering property in $H^2$ or $A^2$, then $g$ is a composition of a bounded univalent function and a classical inner function.

In this paper we deal with subalgebras $A$ generated by more than one element, focusing on polynomial subalgebras, $A = \mathbb{C}[p_1, \dots, p_d]$ where the generators $p_1, \dots, p_d$ are themselves polynomials. We consider the case when polynomials $p_1, \dots, p_d$ satisfy the following two conditions:

$$\begin{aligned} &1)\ \text{greatest common divisor } (\deg\ p_1, \dots, \deg\ p_d) = 1 \\ &2)\ |p_1'(z)| + \cdots + |p_d'(z)| > 0,\ z \in \mathbb{C}. \end{aligned} \tag{1}$$

Our first result states that any closed submodule of $H^2$ over such an algebra is finitely generated.

**Theorem 1.** *Let $A = \mathbb{C}[p_1, \dots, p_d]$ be a finitely generated polynomial subalgebra of $H^\infty$ whose polynomial generators $p_1, \dots, p_d$ satisfy* (1). *Then every closed $A$-submodule of $H^2$ is finitely generated.*

At the same time we show that such algebras almost never have the wandering property in the Hardy space. In other words, there is a closed submodule $M$ of $H^2$ such that the set $M \ominus A_0M$ does not generate $M$. Nevertheless, it turns out that for polynomial algebras satisfying the conditions (1) only a finite number of elements need to be added to $M \ominus A_0M$ in order to obtain a generating set. In fact, the proof of Proposition 3 below describes a canonical procedure for finding these additional generators. The maximum possible number of additional generators is called the deficiency of the algebra and denoted by $\mathcal{D}(A)$ (the maximum is taken over all closed $A$-submodules of $H^2$). We express the deficiency in terms of geometric characteristics of part of an algebraic curve in $\mathbb{C}^d$ associated with the algebra $A$. Let $A = \mathbb{C}[p_1, \dots, p_d]$ and $p_1, \dots, p_d$ satisfy (1). Consider the map:

$$\begin{gathered} \mathcal{P} : \Delta \to \mathbb{C}^d \\ \mathcal{P}(z) = (p_1(z), \dots, p_d(z)). \end{gathered} \tag{2}$$

Write $\mathcal{P}(\Delta) = \Gamma$. The condition 2) of (1) implies that $\Gamma$ has no other singularities but self-intersections. We will show in Section 2 that (1) guarantees that the number of self-intersections is finite. Let $\rho$ equal the number of self-intersections of $\Gamma$. The curve $\Gamma$ is reducible at every point of self-intersection. Some of the components could be tangent (of course, the order of tangency is finite). We refer to this tangency as self-tangency of $\Gamma$. Let $m$ be equal to the highest order of self-tangency of $\Gamma$. Since we could have at most finite number of self-tangencies, $m$ is finite. Finally, let $\beta$ be the first Betti number of $\Gamma$ (recall that the first Betti number is the rank of the first homology group with coefficients in $\mathbb{Z}$). The following result gives an upper bound for the deficiency:

**Theorem 2.** *Let $A$ be a subalgebra of $\mathbb{C}[z]$ generated by polynomials $p_1$, … ,$p_d$ which satisfy (1). Let $\beta$, $\rho$, and $m$ be as described above then,*

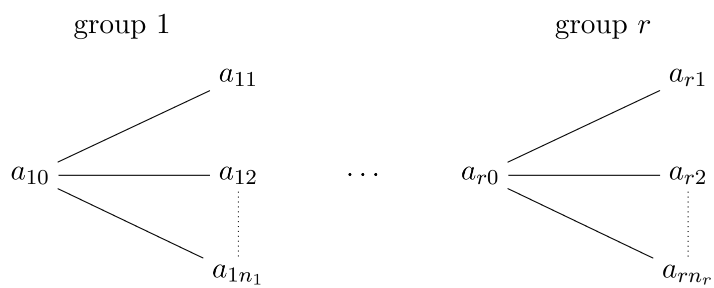
$$\mathcal{D}(A) \;\le\; 2(m+1)\beta(\beta+\rho).$$

In fact, we prove a little more general result, but for the sake of presentation clarity we do not mention it here in full generality.

In the case of some special algebras called level set algebras, the upper bound given by Theorem 2 could be sharpened. For such algebras we give a lower bound for the deficiency. It is possible to prove (though we do not do it here) that for level set algebras the lower bound given by Theorem 3 below is also the (sharpened) upper bound and, therefore, the deficiency is equal to this bound.

**Definition 1.** Let $\{a_k\}_{k=1}^N$ be a collection of $N$ distinct points in $\Delta$. Combine the points into $r$ groups, each group containing $n_j+1$ points, $j = 1,\dots,r$.

(3)




For each group we pick a base point $a_{j0}$. If zero is among $\{a_k\}_{k=1}^n$, then for convenience we choose zero to be in the first group and designate 0 to be the base point, $a_{10}$. Let $A$ be the collection of all polynomials which satisfy the following conditions:

$$P(a_{k0}) \;=\; P(a_{ki}),\ k=1,\dots,r,\ i=1,\dots,n_k.$$

Then $A$ is a subalgebra of $\mathbb{C}[z]$. We call a subalgebra of this form a *level set algebra*.

**Theorem 3.** *Let $A$ be a level set algebra. If $a_{10}=0$, then $\mathcal{D}(A)\ge N-1$. If $a_{10}\ne 0$, then $\mathcal{D}(A)\ge N$.*

The structure of this paper is as follows. First, we prove that the $H^2$-closure of a polynomial subalgebra which satisfies the conditions of (1) has a special form, called *point evaluation* type. This is the main result of Section 2. Further, we show that every closed $A$-submodule of $H^2$ is of the form $gM$ where $g$ is a classical inner fuction and $M$ is a point evaluation type subspace of $H^2$ of finite codimension which is also a finitely generated $A$-submodule. Theorem 1 immediately follows from this result. This is done in Section 3. Finally, Section 4 is devoted to proving upper and lower estimates for the deficiency given by Theorems 2 and 3.

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## 2. Subalgebras of Point Evaluation type.

In this section we prove that $H^2$-closures of finitely generated polynomial subalgebras of $H^\infty$ have a special form.

**Definition 2.** Let $a_1, \dots, a_n$ be a finite collection of points inside the open unit disk in the complex plane and $M$ be a closed subspace of $H^2$ consisting of functions $f$ which satisfy the following conditions:

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j} c_{jk}^{i} f^{(k)}(a_j) = 0 \quad \text{where} \quad c_{jk}^{i} \in \mathbb{C}, \quad i = 1, \dots, t.$$

We call such a subspace $M$ a subspace of Point Evaluation type, or *P.E.* subspace. If $A$ is a subalgebra of $H^\infty$ such that the $H^2$-closure of $A$ is a P.E. subspace, we call $A$ a subalgebra of point evaluation type, or P.E. algebra.

Let $M$ be a P.E. subspace generated by a single condition:

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j} c_{jk} f^{(k)}(a_j) = 0.$$

Write $g(z)$ as:

$$g(z) = \sum_{j=1}^{n}\sum_{k=0}^{r_j} \frac{\overline{c_{jk}} k! \ z^k}{(1 - \overline{a_j} z)^{k+1}}. \tag{4}$$

It readily follows that the $\mathrm{codim}_{H^2} M = 1$ and $M^\perp$ is spanned by $g(z)$. Now let $M$ be a P.E. subspace generated by $t$ independent conditions at points $a_1, \dots, a_n \in \Delta$, and let $M_i$ be the P.E. subspace generated by the $i^{th}$ condition, $i = 1, \dots, t$. For each i there is a function $g_i$, given by (4), such that $M_i^\perp$ is spanned by $g_i$. It is readily apparent that $M^\perp$ is spanned by $\{g_1(z), \dots, g_t(z)\}$. It also follows that a closed subspace $M$ of $H^2$ is a P.E. subspace if $M^\perp$ is spanned by a finite set of functions each in the form of (4). This immediately leads to the fact that the finite intersection of P.E. subspaces is again a P.E. subspace and to the following results.

**Lemma 1.** *If $M$ is any closed subspace of $H^2$ containing a P.E. subspace, $M_{pe}$, then $M$ is also a P.E. subspace.*

*Proof.* Since $M^\perp \subseteqq M_{pe}^\perp$, $M^\perp$ is spanned by functions which are linear combinations of elements of $M_{pe}^\perp$, and hence linear combination of function of the form (4). Thus $M$ is a P.E. subspace. □

**Lemma 2.** *A closed subspace $M$ is a P.E. subspace if and only if it contains a $z$-invariant subspace of $H^2$ of finite codimension.*

*Proof.* Let $M$ be a P.E. subspace generated by the following conditions at points $a_1, \dots, a_n$:

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j} c_{jk}^i f^{(k)}(a_j) = 0; \quad i = 1, \dots, \eta. \tag{5}$$

Let $B(z)$ be the Blaschke product:

$$B(z) \;=\; \prod_{j=1}^{n}\left(\frac{a_j - z}{1 - \overline{a_j}\cdot z}\right)^{r_j+1}. \tag{6}$$

Any $f \in B \cdot H^2$ trivially satisfies the conditions (5). Hence, $B \cdot H^2 \subset M$ and, thus, $M$ contains a $z$-invariant subspace of finite codimension.

Conversely, let $M$ be a closed subspace containing a $z$-invariant subspace of finite codimension. By Beurling's theorem any $z$-invariant subspace of finite codimension is in the form $g \cdot H^2$ where $g$ is a finite Blaschke product. Let $b_1, \dots, b_n$ be the zero set of $g$ with multiplicity $r_1, \dots, r_n$ respectively. It is clear that $g \cdot H^2$ is a P.E. subspace which is determined by the conditions

$$f^{(i)}(b_j) = 0,\ i = 1, \dots, r_j,\ j = 1, \dots, n.$$

Thus, by Lemma 1, $M$ is a P.E. subspace. □

**Lemma 3.** *Let $a_1, \dots, a_n \in \mathbb{C}$ be distinct points and $|a_1| \le \cdots \le |a_n|$, and $M \subseteq \mathbb{C}[z]$ be the collection of all of polynomials satisfying the single condition:*

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j} c_{jk} P^{(k)}(a_j) = 0, \quad \textit{where } \ c_{jk} \in \mathbb{C} \textit{ and } c_{jr_j} \neq 0.$$

*If $|a_n| \ge 1$, then $M$ is dense in $H^2$. If $|a_n| < 1$, then*

$$\overline{M} = \left\{ f \in H^2 : f \ \textit{ satisfies } \ \sum_{j=1}^{n}\sum_{k=0}^{r_j} c_{jk} f^{(k)}(a_j) = 0 \right\}.$$

*Proof.* The case $|a_n| < 1$ is clear since $H^2$ convergence implies uniform convergence on compacta of all derivatives, so let $|a_n| \ge 1$.

Let P(z)$\in \mathbb{C}$[z]. Consider the following sequence of polynomials:

$$R_N(z)=\left(\frac{K}{Q_N(a_n)}\cdot Q_N(z)\right)\left((z-a_n)^{r_n}\cdot\prod_{j=1}^{n-1}(z-a_j)^{r_j+1}\right), \tag{7}$$

where K is a constant to be determined shortly and

$$Q_N(z)=\begin{cases}(\lambda z)^N, & \text{if } |a_n|>1\\ 1-(1-\frac{z}{a_n})(1+\cdots+(\frac{z}{a_n})^N+\frac{N-1}{N}(\frac{z}{a_n})^{N+1}+\cdots & \\ \qquad\cdots+\frac{1}{N}(\frac{z}{a_n})^{2N-1}), & \text{if } |a_n|=1,\end{cases}$$

where $\lambda$ is a complex number which satisfies $|\lambda a_n|>1,\ \ |\lambda|<1$. It is easily seen that $|Q_N(a_n)|\geq 1$. Now, it follows from (7) that

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}R_N^{(k)}(a_j)=c_{nr_n}\left(K\cdot r_n!\prod_{j=1}^{n-1}(a_n-a_j)^{r_j+1}\right). \tag{8}$$

Consider the following sequence of polynomials:

$$P_N(z)=P(z)+R_N(z). \tag{9}$$

Note that (9) implies

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}P_N^{(k)}(a_j)=\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}P^{(k)}(a_j)+\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}R_N^{(k)}(a_j),$$

and, therefore, we have by (8)

$$\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}P_N^{(k)}(a_j)=\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}P^{(k)}(a_j)+c_{nr_n}\left(K\cdot r_n!\prod_{j=1}^{n-1}(a_n-a_j)^{r_j+1}\right). \tag{10}$$

Set

$$K=\frac{-\left(\sum_{j=1}^{n}\sum_{k=0}^{r_j}c_{jk}P^{(k)}(a_j)\right)}{c_{nr_n}\cdot r_n!\prod_{j=1}^{n-1}(a_n-a_j)^{r_j+1}}.$$

It is easily seen that $K$ is independent of $N$, and that $P_N\in M$.

It follows from the definition of $Q_N(z)$ that $\|Q_N(z)\|_{H^2}\to 0$ as $N\to\infty$. Hence,

$$\|P_N(z)-P(z)\|_{H^2}=\|R_N(z)\|_{H^2}.$$

It also follows from (10) that for all $N$

$$\|P_N(z)-P(z)\|_{H^2}\leq\left\|\frac{K}{Q_N(a_n)}(z-a_n)^{r_n}\prod_{j=1}^{n-1}(z-a_j)^{r_j+1}\right\|_{H^\infty}\cdot\|Q_N(z)\|_{H^2}.$$

Since $\|Q_N(z)\|_{H^2} \to 0$ as $N \to \infty$ and $|Q_N(a_n)| \geq 1$ for all $N$, it follows immediately that

$$\|P_N(z) - P(z)\|_{H^2} \to 0, \ N \to \infty.$$

Hence, we have for any polynomial $P(z)$, $P(z) \in \overline{M}$ and, therefore, $\overline{M} = H^2$. □

**Definition 3.** Let $p_1, \dots, p_d$ be a collection of polynomials in $\mathbb{C}[z]$. Let $\mathcal{P} : \mathbb{C} \to \mathbb{C}^d$ be the mapping (2). If there is a finite collection of points $a_1, \dots, a_n \in \mathbb{C}$ such that $\mathcal{P}$ is injective on $\mathbb{C}\backslash\{a_1, \dots, a_n\}$ we say that the polynomials $p_1, \dots, p_d$ almost separate points.

The following Lemma is probably not new. However, the authors could not find a reference in the literature, and so included a detailed proof below.

**Lemma 4.** *Let $p_1, \dots, p_d \in \mathbb{C}[z]$. If the greatest common divisor* $(\deg p_1, \dots, \deg p_d) = 1$, *then the polynomials $p_1, \dots, p_d$ almost separate points.*

*Proof.* We need to show that the number of points in $\mathbb{C}^d$ with more than one preimage is finite. Let S=$\{w \in \mathbb{C}^d : \text{card}\,\{\mathcal{P}^{-1}(w)\} > 1\}$. Suppose that card $\{S\}$ is not finite, then there exists a sequence of distinct points $\{w_r\}_{r=1}^{\infty} \in \mathbb{C}^d$ and a sequence of points $\{(z_{1r}, z_{2r})\}_{r=1}^{\infty} \in \mathbb{C}^2$ such that $z_{1r} \neq z_{2r}$ and $\mathcal{P}(z_{1r}) = \mathcal{P}(z_{2r}) = w_r$.

Write

$$R_i(z_1, z_2) = \frac{p_i(z_1) - p_i(z_2)}{z_1 - z_2}, \ i = 1, \dots, d. \tag{11}$$

Each $R_i(z_1, z_2)$ is a polynomial in $z_1$ and $z_2$. Since the collection of polynomials $R_i$ share an infinite number of zeros, Bezout's theorem [5, p. 178] implies that there is an irreducible algebraic manifold of degree 1 belonging to the algebraic variety generated by $R_1, \dots, R_n$. Let $q(z_1, z_2)$ be a non-constant polynomial generating the corresponding ideal, $\mathcal{I}$. Since each $R_i(z_1, z_2)$ is in $\mathcal{I}$, we have

$$R_i(z_1, z_2) = q(z_1, z_2) M_i(z_1, z_2), \quad i = 1, \dots, d. \tag{12}$$

Write the homogeneous decomposition of the polynomials $R_i(z_1, z_2)$, $M_i(z_1, z_2)$, and $q(z_1, z_2)$:

$$R_i(z_1, z_2) = \sum_{j=0}^{\deg\ R_i} r_{i,j}(z_1, z_2),$$

$$M_i(z_1, z_2) = \sum_{j=0}^{\deg\ M_i} m_{i,j}(z_1, z_2), \tag{13}$$

$$q(z_1,z_2)=\sum_{j=0}^{\deg\ q} q_j(z_1,z_2),$$

where $r_{i,j}, m_{i,j}$, and $q_j$ are all homogeneous polynomials in $z_1, z_2$ of degree $j$. Now for each $i$ the relations (12) and (13) yield

$$R_i(z_1,z_2)=\sum_{j=0}^{\deg\ R_i} r_{i,j}(z_1,z_2)=\sum_{j=0}^{\deg\ q} q_j(z_1,z_2)\cdot\sum_{j=0}^{\deg\ M_i} m_{i,j}(z_1,z_2). \tag{14}$$

Let $k_0=\deg\ q(z_1,z_2)$ and $k_i=\deg\ M_i(z_1,z_2)$. In particular, (11) and (14) imply

$$r_{i,\deg\ R_i}=q_{k_0}(z_1,z_2)\cdot m_{i,k_i}(z_1,z_2)=C\prod_{j=1}^{\deg\ P_i-1}(z_1-\zeta_i^j z_2)$$

where $C\in\mathbb{C}$ is some constant and $\zeta_i$ is the (deg $P_i$)-root of unity. Since each $r_{i,\deg\ R_i}$ is divisible by $q_{k_0}$, the polynomial $q_{k_0}(z_1,z_2)$ divides all the products

$$\prod_{j=1}^{\deg\ P_i-1}(z_1-\zeta_i^j z_2),\ i=1,\dots,d.$$

This is not possible since $(\deg\ p_1,\dots,\deg\ p_d)=1$. Hence, card $\{S\}$ must be finite. □

**Corollary.** *If polynomials $p_1,\dots,p_d$ satisfy the relations* (1), *then they almost separate points.*

We are now ready to prove the main result of this section.

**Proposition 1.** *Let $p_1,\dots,p_d\in\mathbb{C}[z]$ such that*

(1) *the polynomials $p_1,\dots,p_d$ almost separate points*
(2) $|p_1'(z)|+\cdots+|p_d'(z)|>0,\ z\in\mathbb{C},$

*then $\mathbb{C}[p_1,\dots,p_d]$ is a P.E. algebra.*

*Proof.* Let $\mathcal{P}:\mathbb{C}\to\mathbb{C}^d$ be the mapping (2), and $\hat{\Gamma}=\mathcal{P}(\mathbb{C})$. Let $\{a_1,\dots,a_n\}$ be the finite number of points where the map $\mathcal{P}$ fails to be injective. Since $\mathcal{P}$ is injective except at a finite number of points, then $\hat{\Gamma}$ has at most a finite number of self-intersections. Let $\{\zeta_1,\dots,\zeta_t\}$ be those points. Given a polynomial $q(z)\in\mathbb{C}[z]$ we define $Q:\hat{\Gamma}\to\mathbb{C}$ by $Q(\omega)=q\circ\mathcal{P}^{-1}(\omega)$. The question is: When is $Q$ a well-defined analytic map?

Let $\zeta\in\hat{\Gamma}$ be any point such that the preimage of $\zeta$ under $\mathcal{P}$ contains a single point $z_0$. By assumption there is a polynomial $p_j$, $1\le j\le d$, so that $p_j'(z_0)\neq 0$. By the inverse mapping theorem [**6**, p. 17] there are open neighborhoods $U,\ V\subset\mathbb{C}$ of the points $z_0,\ p_j(z_0)=w_j$ respectively so that $p_j^{-1}:V\to U$ exists and is analytic. Let $D=\mathcal{P}(U)\cap\hat{\Gamma}$. It follows that $\mathcal{P}$ is injective on $U$. Define $\mathcal{P}_j^{-1} : D \to U$ by $\omega \to p_j^{-1}(\omega_j)$. This map is analytic since $p_j^{-1}(\omega_j)$ is analytic. Now $Q(\omega) = q \circ \mathcal{P}_j^{-1}(\omega) = q \circ p_j^{-1}(\omega_j) = Q(\omega_j)$ can be analytically extended in an open neighborhood $O_\zeta \subset \mathbb{C}^n$ of $\zeta$, by $\hat{Q} : O_\zeta \to \mathbb{C}$, $\hat{Q}(\omega) = Q(\omega_j)$. Thus, Q is analytic at any $\zeta$ such that the preimage of $\zeta$ under $\mathcal{P}$ is a single point.

Let $\zeta$ be one of the points $\{\zeta_1, \dots, \zeta_t\}$ so that $card\{\mathcal{P}^{-1}(\zeta)\} > 1$. Now we use a standard argument to find sufficient conditions for $q$ which guarantee that $Q$ is analytic at $\zeta$. Let $\mathcal{P}^{-1}(\zeta) = \{a_1, \dots, a_r\}$. For each $1 \le i \le r$ there is a polynomial $p_{j_i}$, $1 \le j_i \le d$, so that $p'_{j_i}(a_i) \ne 0$. Again by the inverse mapping theorem there are open neighborhoods $U_i, V_i \subset \mathbb{C}$ about the points $a_i,\ p_{j_i}(a_i) = w_{j_i}$ respectively which satisfy the following conditions:

(1) $p_{j_i}^{-1} : V_i \to U_i$ exists and is analytic,

(2) if $j_i = j_l$ then $V_i = V_l$,

and

(3) $\mathcal{P}$ is injective on $\cup_{i=1}^r (U_i \backslash a_i)$.

For each $i$ let $D_i = \hat{\Gamma} \cap \mathcal{P}(U_i)$ be the corresponding irreducible component of the curve $\Gamma$ in a neighborhood of $\zeta$. For each component $D_i$ of $\Gamma$ the function $f_i = q \circ \mathcal{P}^{-1}|_{D_i}$ is analytic on $D_i$ as shown above.

For $k = 1, \dots, d,\ i = 1, \dots, r$ write

$$\lambda_{ki} : O_\zeta \to \mathbb{C}$$
$$\lambda_{ki}(\omega) \to p_k(p_{j_i}^{-1}(\omega_{j_i})).$$

Locally $D_i$ is given by $\omega_k = \lambda_{ki}(\omega_{j_i}),\ k = 1, \dots, d$. For $s = 1, \dots, d,\ i = 1, \dots, r$ write

$$\phi_{s,i}(\omega) = \omega_s - \lambda_{si}(\omega_{j_i}).$$

Obviously $\phi_{s,i}$ are analytic in a neighborhood $O_{s,i}$ of $\zeta$ and $\phi_{s,i}$ vanishes on $D_i$. Since the intersection $\cap_i^r D_i$ consists of the single point $\zeta$, for each pair $D_k, D_i$ ,$k \ne i$, there is a $\phi_{s_{ki},k},\ 1 \le s_{ki} \le d$, such that $\phi_{s_{ki},k}$ is not identically zero on $D_i$. Let $\mu$ be the order of tangency between $D_i$ and $D_k$ at $\zeta$. We can choose $\phi_{s_{ki},k}$ so that the order of zero $\phi_{s_{ki},k}|_{D_i}$ has at $\zeta$ is maximal and, hence, is equal to $\mu + 1$. Thus, the point $\zeta$ is an isolated zero of $\phi_{s_{ki},k}|_{D_i}$ of order $\mu + 1$; passing if necessary to smaller neighborhoods we might assume that $\zeta$ is the only zero of $\phi_{s_{ki},k}|_{D_i}$.

Let $O_\zeta = \cap O_{s,i}$. For $1 \le t \le r$ write

$$\Upsilon_t(\omega) = \prod_{i=1 i \ne t}^{r} \phi_{s_{it},i}(\omega).$$

Then $\Upsilon_t$ is analytic in $O_\zeta$, vanishes on $D_i,\ i = 1, \dots, r,\ i \ne t$, and the restriction $\Upsilon_t|_{D_t}$ vanishes only at $\zeta$. Let $\nu_t$ be the order of zero $\Upsilon_t|_{D_t}$ has at $\zeta$. We obviously have

$$\nu_t \leq (r-1)(m+1), \tag{15}$$

where $m$ is the highest order of self tangency $\Gamma$ has at $\zeta$. Now write

$$F(\omega) = f_1(w_{j_1}) + \Upsilon_2(\omega)\cdot\Psi_2(\omega) + \cdots + \Upsilon_r(\omega)\cdot\Psi_r(\omega),$$

where the functions $\Psi_i(\omega),\ i=1,\dots,r$ are to be determined. Then we have

$$F|_{D_k}(\omega) = f_1|_{D_k} + \Upsilon_2\cdot\Psi_2|_{D_k} + \cdots + \Upsilon_r\cdot\Psi_r|_{D_k} = f_1|_{D_k} + \Upsilon_k\cdot\Psi_k|_{D_k}.$$

Now, $F|_{D_k} = f_k(\omega_{j_k})$ is equivalent to

$$f_k(\omega_{j_k}) = f_1(\lambda_{1_j,1}(\omega_{j_k})) + (\Upsilon_k|_{D_k})(\omega_{j_k})\Psi_k(\omega).$$

The last relation yields

$$\Psi_k(\omega) = \Psi_k(\omega_{j_k}) = \frac{f_k(\omega_{j_k}) - f_1(\lambda_{1_j,1}(\omega_{j_k}))}{\Upsilon_k|_{D_k}(\omega_{j_k})}. \tag{16}$$

The function $\Psi_k$ given by (16) is analytic at $\zeta$ if the order of zero the numerator has at $\zeta$ is greater than or equal to $\nu_k$, which by (15) does not exceed $(r-1)(m+1)$. Thus, if

$$\frac{D^\alpha}{D\omega_{j_k}^\alpha}[f_k(\omega_{j_k}) - f_1(p_{j_1}(p_{j_k}^{-1}(\omega_{j_k})))](\zeta) = 0,\ \alpha = 0,\dots,\nu_k - 1, \tag{17}$$

then $\Psi_k$ are analytic at $\zeta$.

The relations (17) are equivalent to certain linear relations between values of $q(z)$ and its derivatives of order not higher than $(m+1)(r-1)$ at points $a_1,\dots,a_r$ . Hence, if $q(z)$ satisfies a finite collection of P.E. conditions at the points $a_1,\dots,a_n$, then $Q(\omega) = q\circ\mathcal{P}^{-1}(\omega)$ is analytic on $\hat{\Gamma}$.

Our next step is to show that if $Q(\omega) = q\circ\mathcal{P}^{-1}(\omega)$ is analytic on $\hat{\Gamma}$, then $q(z)$ is in $\overline{\mathbb{C}[p_1,\dots,p_d]}$.

Choose $\rho > 0$ large enough so that the ball of radius $\rho$, $\Delta_\rho^d \subset \mathbb{C}^d$, contains $\overline{\mathcal{P}(\Delta)}$ as a compact subset. Since $\hat{\Gamma}\cap\Delta_\rho^d$ is a closed analytic set in $\Delta_\rho^d$, $Q$ is analytically extendable in $\Delta_\rho^d$ to $\hat{Q}(\omega) : \Delta_\rho^d \to \mathbb{C}$ [6, p. 212]. Write for $\hat{Q}$

$$\hat{Q}(\omega_1,\dots,\omega_d) = \sum_{(N_1,\dots,N_d)\in\mathbb{N}^d} C_N\cdot\omega_1^{N_1}\cdots\omega_d^{N_d},\ \omega\in\Delta_\rho^d.$$

This series converges uniformly on $\overline{\mathcal{P}(\Delta)}$, therefore,

$$q(z) = \sum_{(N_1,\dots,N_d)\in\mathbb{N}^d} C_N\cdot p_1^{N_1}(z)\cdots p_d^{N_d}(z),\ z\in\Delta$$

converges uniformly on $\overline{\Delta}$.

Hence, if $q(z)$ satisfies the mentioned above P.E. conditions at $a_1,\dots,a_n$ then $q(z)\in\overline{\mathbb{C}[p_1,\dots,p_d]}$. Thus by Lemmas 2 and 3, $\overline{\mathbb{C}[p_1,\dots,p_d]}$ contains a P.E. subspace and, hence, is a P.E. subspace. □

**Corollary 1.** *Let $A = \mathbb{C}[p_1, \dots , p_d]$ be a polynomial subalgebra satisfying the conditions* (1)*. Then $A$ is a P.E. subalgebra.*

*Proof.* The result follows from Proposition 1 and Lemma 4. □

**Corollary 2.** *Let $A = \mathbb{C}[p_1, \dots , p_d]$ be a polynomial subalgebra of $\mathbb{C}[z]$ satisfying the conditions* (1) *and let $\beta$ be the first Betti number of $\mathcal{P}(\Delta)$, where $\mathcal{P}$ is the mapping* (2)*. Then*

$$\operatorname{codim}_{H^2} A \leq (m+1)\beta(\beta+\rho),$$

*where $\rho$ is the number of self-intersections of $\mathcal{P}(\Delta)$ and $m$ is the highest order of self-tangency of $\mathcal{P}(\Delta)$.*

*Proof.* By Proposition 1 $\overline{A}$ is a P.E. subspace. Let $N$ be equal to the number of points defining the point evaluating conditions of $\overline{A}$. It follows directly from the proof of Proposition 1 that $N = \beta + \rho$. It also follows from this proof (relation (17)) that the highest order of the derivative involved in these point evaluation conditions does not exceed $\max_\zeta (m+1)(r_\zeta - 1)$ where the maximum is taken over all points $\zeta \in \mathcal{P}(\Delta)$ which are points of self-intersection and $r_\zeta = \operatorname{card}(\mathcal{P}^{-1}(\zeta))$. Since $r_\zeta - 1 \leq \beta$, we obtain that $\bar{A}$ contains the $z$-invariant subspace generated by the Blaschke product with zeros of order $(m+1)\beta$ at each of the $N = \beta + \rho$ evaluation points, and the result follows. □

## 3. Submodules over Point Evaluation Algebras, Proof of Theorem 1.

The main result of this section is that every closed submodule of $H^2$ over a P.E. algebra is finitely generated. We also show that such submodules are determined by an inner function and a P.E. subspace.

**Lemma 5.** *Let $A$ be a subspace of $H^\infty$ such that $\operatorname{closure}_{H^2} A$ is a P.E. subspace. If $\mathcal{F} \subset H^2$ be a subset such that the minimal $z$-invariant subspace containing $\mathcal{F}$ is $H^2$, then the closed subspace*

$$S = \operatorname{closure}\{fp : p \in A, f \in \mathcal{F}\}$$

*is a P.E. subspace.*

*Proof.* Let $\overline{A}$ be determined by the conditions:

$$\sum_{j=1}^{n} \sum_{k=0}^{r_j} c^i_{jk} f^{(k)}(a_j) = 0; \quad i = 1, \dots , \eta.$$

Write:

$$B(z) = \prod_{i=1}^{n} \left( \frac{a_i - z}{1 - \overline{a_i} \cdot z} \right)^{r_j+1}.$$

Then $B \cdot H^2 \subset \overline{A}$. Since the minimal $z$-invariant subspace containing $\mathcal{F}$ is $H^2$, there are polynomials $\{p_{i,m}\}$ and functions $\{f_{i,m}\} \subset \mathcal{F}$ such that

$$\sum_{i=1}^{m} f_{i,m} \cdot p_{i,m} \to 1 \tag{18}$$

in $H^2$ as $m \to \infty$. Let $q(z)$ be any polynomial in $B \cdot H^2$. We obviously have

$$f_{i,m} \cdot (q \cdot p_{i,m}) \in S, \ \forall i, m$$

since each polynomial $q \cdot p_{i,m} \in B \cdot H^2 \subset \overline{A}$ and S is closed. Further, (18) implies

$$\begin{aligned} &\left\| \sum_{i=1}^{m} f_i \cdot q \cdot p_{i,m} \ - \ q \right\|_{H^2} \\ &\qquad \leq \|q\|_{H^\infty} \cdot \left\| \sum_{i=1}^{m} f_i \cdot p_{i,m} - 1 \right\|_{H^2} \to 0, \quad \text{as} \quad m \to \infty. \end{aligned} \tag{19}$$

Now, (19) implies that $B \cdot H^2$ is contained in $S$. By Lemma 1, $S$ is a P.E. subspace. □

**Lemma 6.** *Let $A$ be a P.E. subspace in $H^2$ and $F$ be an outer function in $H^\infty$. Then*

$$\operatorname{codim}_{H^2} A = \operatorname{codim}_{H^2} F \cdot A. \tag{20}$$

*Proof.* Let $S = F \cdot A$ and $s_1, \dots, s_\eta$ be functions of the form (4) which span $A^\perp$ ($\operatorname{codim}_{H^2} A = \eta$). Since $F$ is cyclic in $H^2$, we have

$$H^2 = \overline{\text{span}\,\{s_1, \dots, s_\eta\} \oplus A} = \overline{\text{span}\,\{F \cdot s_1, \dots, F \cdot s_\eta\} + F \cdot A} = \overline{F \cdot H^2}. \tag{21}$$

Let $P_S$ be the orthogonal projection operator onto $S$. We now show that none of the functions

$$f_i \ = \ F \cdot s_i - P_S(F \cdot s_i), \quad i = 1, \dots, \eta$$

is zero, and that they are linearly independent. This is equivalent to the fact that no linear combination of the functions $F \cdot s_i$ lies within $\overline{F \cdot A}$. Suppose that there is a $Q(z) \in \overline{F \cdot A}$ and coefficients $b_i \in \mathbb{C}, \ i = 1, \dots, n$ such that

$$\sum_{i=1}^{\eta} b_i F \cdot s_i \ = \ Q. \tag{22}$$

Since $Q(z) \in \overline{F \cdot A}$, there is a sequence of polynomials $\{Q_j\}_{j=1}^{\infty}$ in $A$ such that

$$F \cdot Q_j \xrightarrow{H^2} Q \ \text{ as } \ j \to \infty. \tag{23}$$

Now (22) and (23) yield

$$\sum_{i=1}^{\eta} b_i F \cdot s_i = \lim_{j\to\infty} F \cdot Q_j.$$

Let $0 < r < 1$ be such that the points defining $A$ are contained in $\Delta_r$. Since $H^2$ convergence implies uniform convergence on compacta of all derivatives, the last relation implies

$$\left(\sum_{i=1}^{\eta} b_i \cdot s_i\right)\Big|_{\overline{\Delta}_r} = \lim_{j\to\infty} Q_j\Big|_{\overline{\Delta}_r}. \tag{24}$$

Let $r \to 1$ in (24). We obtain $\sum_{i=1}^{\eta} b_i s_i$ is in $S$, which implies $b_1 = \cdots = b_\eta = 0$. This and (21) imply (20). □

For a subset $\mathcal{F} \subset H^2$ we denote by $[\mathcal{F}]_A$ the smallest closed $A$-submodule of $H^2$ which contains $\mathcal{F}$.

**Lemma 7.** *Let $A$ be a P.E. subalgebra and $\mathcal{F} \subset H^2$ be such that the minimal $z$-invariant subspace of $H^2$ containing $\mathcal{F}$ is $H^2$. Let $M = [\mathcal{F}]_A$ . If $h(z) \in M$ is analytic in $\overline{\Delta}$ and $\mu$ is equal to the sum of the orders of zeros $h$ has in $\Delta$, then*

$$\operatorname{codim}_M\ h \cdot A = \operatorname{codim}_{H^2} A - \operatorname{codim}_{H^2} M + \mu.$$

*Proof.* Let $h = gF$ be the canonical factorization of $h$, where $g$ is inner and $F$ is outer. Since $h$ is analytic in $\overline{\Delta}$, $g$ is a finite Blaschke product and $F \in H^\infty$. By Lemmas 5 and 6, $\overline{F \cdot A} = S$ is a P.E. subspace and $\operatorname{codim}_{H^2} S = \operatorname{codim}_{H^2} A$.

Since $\operatorname{codim}_{H^2}\ g \cdot H^2 = \mu$ and multiplication by an inner function $g$ is an isometry on $H^2$, the following equality holds

$$H^2 = g \cdot H^2 \oplus (g \cdot H^2)^\perp = (g \cdot H^2)^\perp \oplus g \cdot S^\perp \oplus g \cdot S.$$

Since $\overline{h \cdot A} = g \cdot S$, we have

$$\operatorname{codim}_{H^2} h \cdot A \ = \ \operatorname{codim}_{H^2} S \ + \ \operatorname{codim}_{H^2}\ g \cdot H^2 \ = \operatorname{codim}_{H^2} A \ + \ \mu.$$

Now, the inclusion $h \cdot A \subset M$ implies

$$\operatorname{codim}_M h \cdot A = \operatorname{codim}_{H^2} A - \operatorname{codim}_{H^2} M + \mu.$$

□

**Proposition 2.** *Let $A$ be a P.E. subalgebra and $M$ be a closed $A$-submodule $H^2$. Then there are an inner function $g$ and a finitely generated closed $A$-submodule of $H^2$, $M'$, which is also a P.E. subspace such that*

$$M \ = \ g \cdot M' \quad \textit{and} \quad [M']_{\mathbb{C}[z]} = H^2.$$

*Proof.* Consider the $z$-invariant subspace $\mathcal{M}$ of $H^2$ generated by $M$ (that is $\mathcal{M}$ is the minimal closed $z$-invariant subspace of $H^2$ which contains $M$). There is an inner function $g$ such that $\mathcal{M} = gH^2$. Write

$$M' = \{h : gh \in M\}.$$

It easily follows that $M'$ is a closed $A$-submodule of $H^2$. Since the whole space $H^2$ is the minimal $z$-invariant subspace which contains $M'$, Lemma 5 and the obvious relation $M' = \text{closure}\{hp : p \in A, h \in M'\}$ imply that $M'$ is a P.E. subspace. By Lemma 1 $M'$ contains a $z$-nvariant subspace of finite codimension. Let $B$ be the finite Blaschke product which generates this $z$-invariant subspace. Then $B \in M'$ and by Lemma 7 the codimension $\text{codim}_{M'} BA$ is finite. This implies that $M'$ is finitely generated as an $A$-module. ☐

*Proof of Theorem 1.* The result follows directly from Proposition 2. ☐

## 4. Proofs of Theorems 2 and 3.

**Definition 4.** Let $A$ be a subalgebra of $H^\infty$. The number $\mathcal{D}(A)$ defined as

$$\mathcal{D}(A) = \sup_S \{\text{codim}_S (S \ominus A_0 S)A\},$$

where $A_0 = \{p \in A | p(0) = 0\}$ and $S$ runs over all closed $A$-submodules of $H^2$, is called the dediciency of $A$. If $\mathcal{D}(A) < \infty$, we say that $A$ is an algebra of finite deficiency (in $H^2$).

**Proposition 3.** *Let $A$ be a P.E. subalgebra of $H^\infty$ satisfying $\eta$ independent conditions,*

$$\sum_{j=1}^{n} \sum_{k=0}^{r_j} c^i_{jk} f^{(k)}(a_j) = 0 \quad \textit{where} \quad c^i_{jk} \in \mathbb{C}, \quad i = 1, \dots, \eta.$$

*Then $\mathcal{D}(A) \le \mu + \eta$, where $\mu = \sum_{j=1}^{n} r_j + 1$.*

*Proof.* Let $S$ be a closed $A$-submodule of $H^2$. By Proposition 2 we have $S = g \cdot M$, where $g$ is an inner function and $M$ is a finitely generated $A$ submodule which is also a P.E. subspace such that the minimal $z$-invariant subspace containing $M$ is $H^2$. Since multiplication by an inner function is an isometry, it follows that

$$\text{codim}_S (S \ominus A_0 S)A = \text{codim}_M (M \ominus A_0 M)A. \tag{25}$$

Let $\{h_1, \dots, h_t\}$ be a set of generators for $M$. Now, $\overline{A_0}$ is generated by the same conditions as $\overline{A}$ with one additional condition $f(0) = 0$. Therefore, Lemma 5 implies that

$$A_0 M = \text{closure}(\text{span}\{h_i \cdot p : p \in A_0, i = 1, \dots, t\})$$

is a P.E. subspace. This subspace contains the $z$-invariant subspace generated by the Blaschke product,

$$B(z) = z \cdot \prod_{j=1}^{n} \left( \frac{z - a_j}{1 - \overline{a_j} \cdot z} \right)^{r_j+1}.$$

Thus $A_0 M^{\perp}$ is contained in the span of the following functions

$$\left\{ 1, \frac{k! \cdot z^k}{(1 - \overline{a_j} z)^{k+1}} : \ j = 1, \dots, n \ k = 0, \dots, r_j \right\}. \tag{26}$$

If $f \in M \ominus A_0 M$, then $f$ is a linear combination of functions (25), and, therefore, can have no more than $\mu = \sum_{j=1}^{n} (r_j + 1)$ zeros. Hence,

$$\operatorname{codim}_M (M \ominus A_0 M) A \ \le \ \operatorname{codim}_M f \cdot A. \tag{27}$$

Since $f$ is analytic in the closed unit disk, Lemma 7 yields

$$\operatorname{codim}_M f \cdot A \ \le \ \operatorname{codim}_{H^2} A \ - \ \operatorname{codim}_{H^2} M \ + \ \mu. \tag{28}$$

Now (27) and (28) imply

$$\operatorname{codim}_M (M \ominus A_0 M) A \ \le \ \operatorname{codim}_{H^2} A + \mu \le \mu + \eta.$$

□

*Proof of Theorem 2.* It follows from Corollary 1 to Proposition 1 that $\mathbb{C}[p_1, \dots, p_d]$ is a P.E. algebra, and from Corollary 2 - that both $\mu$ and $\eta$ do not exceed $(m+1)\beta(\beta + \rho)$. Thus, by Proposition 3, for any closed $A$-submodule $S$ of $H^2$ we have

$$\operatorname{codim}_S (S \ominus A_0 S) A \le 2(m+1)\beta(\beta + \rho).$$

□

**Remark.** The upper bound for the deficiency of a polynomial algebra given by Theorem 1 is not sharp. It is possible to prove that in the case of a level set algebra the deficiency does not exceed $N$ - the number of points defining the algebra. This estimate is sharp by Theorem 2.

Before proving Theorem 3 we mention that every level set algebra is a finitely generated polynomial algebra.

**Proposition 4.** *Any level set algebra $A$ can be written in the form $A = \mathbb{C}[p_1, \dots, p_t]$ for some set of polynomials $p_1, \dots, p_t$.*

*Proof.* Let $A$ be a level set algebra. Consider the following polynomials,

$$P_0(z) = \prod_{j=1}^{r}\prod_{i=0}^{n_j}(z - a_{ji}),$$

$$q_{ji}(z) = \frac{P_0(z)}{(z - a_{ji})},\ j = 1, \dots, r,\ i = 0, \dots, n_j,$$

$$P_j(z) = \sum_{i=0}^{n_j} \frac{q_{ji}(z)}{q_{ji}(a_{ji})},\ j = 1, \dots, r,$$

and

$$P_{r+k}(z) = z^k P_0(z),\ k = 1, \dots, N = \deg\ P_0.$$

Then,

(29) $$P_j(a_{ki}) = \begin{cases} 1, & \text{if } k = j,\ 1 \le j, k \le r \\ 0, & \text{if } k \ne j,\ 1 \le j, k \le r \\ 0, & \text{if } j \ge r+1,\ 1 \le k \le r \end{cases}$$

and, therefore $P_j \in A,\ j = 0, \dots, r+N$.

Let us show that

$$A \ = \ \mathbb{C}[P_0, \dots, P_{r+N}].$$

Obviously $\mathbb{C}[P_0, \dots, P_{r+N}] \subset A$. To prove the inverse inclusion note that

(30) $$z^n P_0 \in \mathbb{C}[P_0, \dots ., P_{r+N}] \ \text{ for any } \ n \ge 0.$$

Indeed, for $n \le N = \deg\ P_0$ it is true since $P_{r+n} = z^n P_0$. Further,

(31) $$z^k \cdot P_0^2 = z^{N+k} P_0(z) + T,$$

where $T \in$ span$\{P_0, zP_0, \dots, z^{N+k-1}P_0\}$. Now proceed by induction in $m$. Assume $z^k P_0(z) \in \mathbb{C}[P_0, \dots, P_{r+N}]$ for $k \le N + n - 1 = m$, then we have

$$z^{m-N} P_0^2 = (z^{m-N}P_0)(P_0) \in \mathbb{C}[P_0, \dots, P_{r+N}]$$

by the induction hypothesis, and, thus by (31), $z^m P_0 \in \mathbb{C}[P_O, \dots, P_{r+N}]$.

Let $Q \in A$, $Q(a_{10}) = \cdots = Q(a_{1n_1}) = c_1, \dots, Q(a_{r0}) = \cdots = Q(a_{rn_r}) = c_r$. Then by (29) and (30) we have

$$Q(z) - \sum_{j=1}^{r} c_j P_j(z) \ = \ s(z)P_0(z) \ \in \ \mathbb{C}[P_0, \dots, P_{r+N}].$$

□

**Remark.** Let $A$ be a level set algebra and $P_0, \dots, P_r$ be the polynomials from Proposition 4. It is easily seen that the first Betti number of the image of $\Delta$ under the corresponding map (2), $\mathcal{P}:\mathbb{C}\to\mathbb{C}^{r+N}$ given by $z\to(P_1(z),\dots,P_r(z),P_0(z),\dots,z^{N-1}P_0(z))$, is equal to

$$\beta=n_1+\cdots+n_r=N-r.$$

The number of points of self-intersections in this case is equal to r.

**Definition 5.** Let $A$ be a subalgebra of $H^\infty$and $M$ be a closed $A$-submodule of $H^2$. We say that $M$ has the $A$- codimension 1 property if

$$\dim(M\ominus A_0M)=1.$$

**Lemma 8.** *Let $A$ be a level set algebra determined by the points $a_1,\dots,a_N$ $=a_{10},\dots,a_{1n_1},a_{20},\dots,a_{2n_2},\dots,a_{rn_r}$ and $M$ be a closed $A$-submodule given by*

$$f(a_{k0})=\alpha_{ki}f(a_{ki}),\ i=1,\dots,n_k,\ k=1,\dots,r,$$

*where*

$$\alpha_{ki}\neq 0\ i=1,\dots,n_k,\ k=1,\dots,r.$$

*Then $M$ has the $A$-codimension 1 property.*

*Proof.* Let $M_0$ be an $A$-submodule determined by the following conditions

$$f(0)=0,\ f(a_{k0})=\alpha_{ki}f(a_{ki}),\ i=1,\dots,n_k,\ k=1,\dots,r.$$

We will show that $M_0\subset A_0M$. First, consider the case when $a_{10}=0$. Let $h\in M_0$. Denote by $m_0,\dots,m_{n_1}$ the orders of zeros which h has at $a_{10},\dots,a_{1n_1}$ respectively. Write

$$P_0(z)=z^{m_0}(z-a_{11})^{m_1}\cdots(z-a_{1n_1})^{m_{n_1}},$$

$$P_j(z)=\frac{P_0(z)}{(z-a_{1j})^{m_j}},\ j=0,1,\dots,n_1,$$

and define $w_{10},\dots,w_{rn_r}$ by

$$w_{ki}=\log\left(\frac{1}{P_0(a_{ki})}\right),\ k=2,\dots,r,\ i=0,\dots,n_k$$

$$w_{1j}=\log\left(\frac{h^{(m_j)}(a_{1j})}{m_j!P_j(a_{1j})}\right),\ j=0,\dots,n_1.$$

Finally, let $q(z)$ be a polynomial which interpolates $(w_{10},\dots,w_{rn_r})$ at $(a_{10},\dots,a_{rn_r})$. Then $\hat{h}=P_0e^{q(z)}$ is in the closure of $A_0$ in the disk-algebra metric. The function $\phi=h/\hat{h}$ is holomorphic in $\Delta$ and satisfies the conditions of $M$. Hence $h=\hat{h}\phi\in\overline{A_0M}$. Thus $M_0\subset\overline{A_0M}$. Since $\operatorname{codim}_{H^2}M_0=\operatorname{codim}_{H^2}M+1$, it follows that $\overline{A_0M}=M_0$ and that $M$ has codimension 1 property.

The case of $a_{10}\neq 0$ follows in a similar manner. ∎

*Proof of Theorem* 3. Let $A$ be any level set algebra defined by $r$ groups of total $N$ points.

*Case* 1: $a_{10} = 0$.

For the sake of notational simplicity we renumerate the points $\{a_k\}_{k=0}^N$ defining $A$ as $a_{10}, \dots, a_{1n_1}, a_{20}, \dots, a_{rn_r}$ and use either notation when convenient. Let

$$g(z) \;=\; 1 \;+ \sum_{k=1}^{N-1} \frac{c_k}{1-\overline{a_k}z}, \quad c_k \in \mathbb{C}. \tag{32}$$

We will choose $c_k$ in some way so that $g$ has exactly $N-1 = \beta+\rho-1$ zeros in $\Delta$ and then will show that there is an $A$-submodule $M$ such that $M \ominus A_0 M$ is generated by $g$. Let $b_l,\ l = 1, \dots, N-1$, be a fixed set of distinct points in $\Delta$. Consider the following system of equations linear in $c_i,\ i = 1, \dots, N-1$,

$$\begin{gathered} g(b_1) = 0 \\ \vdots \\ g(b_{N-1}) = 0. \end{gathered} \tag{33}$$

Write

$$D = \left[\frac{1}{1-\overline{a_k}b_l}\right]_{l,k=1}^{N-1},$$

and let $C = (c_1, \dots, c_{N-1})$ and $\mathbf{1} = (1, \dots, 1)$ be $(N-1)$-dimensional complex vectors. Then the system (33) can be written as

$$DC^t \;=\; -\mathbf{1}^t. \tag{34}$$

The Determinant $\mathrm{Det}(D)$ is an analytic function in $(b_1, \dots, b_{N-1})$ in the polydisc $\Delta^{N-1}$. When $a_l = b_l,\ l = 1, \dots, N-1$ it is well known that $D$ is invertible and, hence, $\mathrm{Det}(D) \neq 0$. (For the explicit expression cf [**10**].) Since $\mathrm{Det}(D)$ is analytic in the $b_l,\ l = 1, \dots, N-1$, there exists an open neighborhood, $O_a$, of the point $a = (a_1, \dots, a_{N-1}) \in \mathbb{C}^{N-1}$ such that $\mathrm{Det}(D)|_{O_a} \neq 0$. Pick $(b_1, \dots, b_{n_1}, a_{n_1+1}, \dots, a_{N-1}) \in O_a$ such that $b_j \neq a_k$ and $b_j \neq b_k$ for all $j, k \leq n_1$. Now (34) has a unique solution, $(\hat{c}_1, \dots, \hat{c}_{N-1})$, and the corresponding $\hat{g}$ vanishes at $b_1, \dots, b_{n_1}, a_{n_1+1}, \dots, a_{N-1}$. Notice that this implies that the none of the constants $\hat{c}_i = 0$, since a function in the form (32) has $N-1$ zeros only when all $c_1, \dots, c_{N-1}$ do not vanish.

We now show that it is possible to choose constants $\{\alpha_{kj}, \lambda_{kj}\},\ k = 1, \dots, r,\ j = 1, \dots, n_k$, such that $\hat{g}$ is represented as

$$\hat{g}(z) = 1 + \sum_{i=1}^{n_1} \frac{\lambda_{1i}\alpha_{1i}}{1-\overline{a_{1i}}z} + \sum_{k=2,j}^{r,n_k} \lambda_{kj}\left(\frac{1}{1-\overline{a_{k0}}z} - \frac{\alpha_{kj}}{1-\overline{a_{kj}}}\right). \tag{35}$$

Since $\hat{g}(a_{1i}) \neq 0$, write,

$$\alpha_{1i} = \hat{g}(a_{10})/\hat{g}(a_{1i}),\ i = 1, \dots, n_1, \qquad (36)$$

$$\lambda_{1i} = \hat{c}_{1i}/\alpha_{1i},\ i = 1, \dots, n_1.$$

For $k = 2, \dots, r$, $j = 1, \dots, n_k$, write:

$$\lambda_{kj} = \hat{c}_{k0}/n_k,$$

$$\alpha_{ji} = -n_k \hat{c}_{kj}/\hat{c}_{k0}.$$

Now (35) is easy to verify.

Let $M$ be an $A$-submodule defined by the following conditions

$$P(a_{k0}) = \alpha_{ki} P(a_{ki}),\ k = 1, \dots, r,\ i = 1, \dots, n_k. \qquad (37)$$

Note that

$$(A_0 M)^{\perp} = \operatorname{span}\left\{1, \frac{1}{1 - \overline{a_{k0}}z} - \frac{\alpha_{ki}}{1 - \overline{a_{ki}}z} \,\middle|\, k = 1, \dots, r,\ i = 1, \dots, n_k\right\}, \qquad (38)$$

and that $\hat{g}$ is a linear combination of function (38).

If $k \geq 2$, $\hat{g}(a_{ki}) = 0$, i=1, ... ,$n_k$. Thus $\hat{g}$ satisfies (37) for these $k$ trivially. When $k = 1$ it follows from (36) that $\hat{g}$ satisfies (37) and, therefore, it is in $M$. Now, we have $\hat{g} \in M \ominus A_0 M$. By Lemma 8 $M$ has the $A$-codimension 1 property and, therefore, it follows that $\hat{g}$ generates $M \ominus A_0 M$. By Lemma 7

$$\operatorname{codim}_M\ (M \ominus A_0 M)A = \operatorname{codim}_M\ \hat{g}A = N - 1,$$

since the $\operatorname{codim}_{H^2} A = \operatorname{codim}_{H^2} M$. Thus, $\mathcal{D}(A) \geq N - 1$.

*Case* 2: $a_{10} \neq 0$.

This case is almost identical to the first so we only sketch the details. We now choose $\hat{g}$ along with non-zero constants $(\hat{c}_1, \dots, \hat{c}_N)$ as above with the only difference that $b_j = a_j,\ j = 1, \dots, n_1$. Hence $\hat{g}$ vanishes at all $N = \beta + \rho$ points defining $A$. Write

$$\lambda_{ki} = \hat{c}_{k0}/n_k$$

$$\alpha_{ki} = -n_k \hat{c}_{ki}/\hat{c}_{k0},$$

for $k = 1, \dots, r$. It follows that

$$\hat{g}(z) = 1 + \sum_{k=1,i=0}^{r,n_k} \frac{\hat{c}_{ki}}{1 - \overline{a_{ki}}z} = 1 + \sum_{k=1,i=1}^{r,n_k} \lambda_{ki} \left( \frac{1}{1 - \overline{a_{k0}}z} - \frac{\alpha_{ki}}{1 - \overline{a_{ki}}} \right).$$

Let $M$ be an $A$-submodule $M$ determined by

$$P(a_{k0}) = \alpha_{ki} P(a_{ki})\ k = 1, \dots, r\ i = 1, \dots, n_k.$$

It follows at once that $\hat{g} \in M \ominus A_0 M^{\perp}$ and, since $M$ has the $A$-codimension 1 property we are done, $\mathcal{D}(A) \geq N$. □

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