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For $p \in (0,1)$, let Q_p be the subspace consisting of Möbius bounded functions in the Dirichlet-type space. Based on the study of the multipliers in Q_p, we establish the corona theorem for Q_p.

Introduction.

Let $\triangle$ and $\partial\triangle$ be the unit disk and circle in the finite complex plane, respectively. Also let dm and $d\theta$ be the Lebesgue measures on $\triangle$ and $\partial\triangle$, separately. Denote by $g(z,w) = \log|(1-\overline{w}z)/(w-z)|$ the Green function of $\triangle$. Also denote by A or H^∞ the set of analytic or bounded analytic functions on $\triangle$. For $p \in (-1,\infty)$ suppose that D_p is the Dirichlet-type space of functions $f \in A$ with

$$\|f\|_{D_p} = |f(0)| + \left[\iint_\triangle |f'(z)|^2(1-|z|)^p dm(z)\right]^{\frac{1}{2}} < \infty.$$

We are interested in the space of Möbius bounded functions in D_p. For our purpose, we write Q_p as this space. Indeed, Q_p is a Banach space of functions $f \in D_p$ with the norm

$$\|f\|_{Q_p} = |f(0)| + \sup_{w\in\triangle} \|f\circ\phi_w - f(w)\|_{D_p} < \infty,$$

where $\phi_w(z) = (w-z)/(1-\overline{w}z)$. From [**AuLaXiZh**], [**AuStXi**] and [**AuXiZh**] it follows that Q_p is equal to the space of functions $f \in D_p$ satisfying

$$\sup_{w\in\triangle} \iint_\triangle |f'(z)|^2[g(z,w)]^p dm(z) < \infty.$$

To better understand Q_p, we would like to remind the reader of some known facts.

a) $p \in (1,\infty)$. D_p is the Bergman space B^2_{p-2} with weight $(1-|z|)^{p-2}$ [**Steg2**]. Q_p is the Bloch space $\mathcal{B}$. Here $\mathcal{B}$ is actually composed by functions $f \in A$ obeying

$$\sup_{z\in\triangle}(1-|z|)|f'(z)| < \infty.$$

See [**AuLa**] for $p > 1$ and in particular [**Xi**] for $p = 2$.

b) $p = 1$. D_p is the classical Hardy space H^2 and

$$Q_p = BMOA = BMO(\partial\triangle) \cap H^2.$$

$BMO(\partial\triangle)$ is the usual space of functions $f \in L^2_{\text{loc}}(\partial\triangle)$ with bounded mean oscillation on $\partial\triangle$, namely,

$$\sup_{I \subset \partial\triangle} |I|^{-1} \int_I |f(e^{i\theta}) - f_I|^2 d\theta < \infty,$$

where the supremum is taken over all subarcs $I \subset \partial\triangle$ of the arc length $|I|$ and

$$f_I = |I|^{-1} \int_I f(e^{i\theta}) d\theta.$$

See either [**Ba**] or [**Ga**, Chapter VI].

c) $p \in (0, 1)$. D_p is the fractional Dirichlet space and $Q_p = Q_p(\partial\triangle) \cap H^2$. Hereafter, $Q_p(\partial\triangle)$ is the class of functions $f \in L^2_{\text{loc}}(\partial\triangle)$ with

$$\|f\|_{Q_p(\partial\triangle)} = \left| \int_{\partial\triangle} f(e^{i\theta}) d\theta \right| + \sup_{I \subset \partial\triangle} \left[|I|^{-p} \int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} d\theta d\varphi \right]^{\frac{1}{2}} < \infty,$$

where the supremum, as in b), ranges over all subarcs $I \subset \partial\triangle$ of the arc length $|I|$. See [**EsXi**]. It is not hard to see that $Q_p(\partial\triangle)$ strictly increases (in p) to $BMO(\partial\triangle)$, and so does Q_p at most up to $BMOA$ [**AuXiZh**].

d) $p = 0$. D_p becomes the classical Dirichlet space $\mathcal{D}$. In this case, Q_p is defined as $\mathcal{D}$.

e) $p \in (-1, 0)$. D_p is the fractional Dirichlet space (with negative exponent), certainly a subspace of $\mathcal{D} \cap H^\infty$, but Q_p contains complex constants only [**NiXi**].

From the above examples it is seen that the space Q_p, $0 < p < 1$ is of independent interest. In this paper, we concentrate on the corona problem for Q_p, $0 < p < 1$. Generally speaking, as to Q_p, $0 \le p < \infty$, the corona problem can be formulated as following:

The Q_p, $p \in [0, \infty)$ Corona Problem. *Given functions $f_1, f_2, \dots, f_n \in Q_p$. What are necessary and sufficient conditions on these functions such that for any function $g \in Q_p$ there exist functions $g_1, g_2, \dots, g_n \in Q_p$ to ensure the identity $f_1g_1 + f_2g_2 + \cdots + f_ng_n = g$?*

To the best of our knowledge, this problem has been answered respectively by J.M. Ortega and J. Fàbrega [**OrFa1**] in the case $p > 1$, V.A. Tolokonnikov [**To**] in the case $p = 1$ (for another proof, see [**OrFa2**]) and A. Nicolau [**Ni**] in the case $p = 0$. So, a natural question comes up: How is the case $0 < p < 1$? Here we say that this problem has also a complete answer in the last case. More precisely, we have:

The Q_p, $p\in(0,1)$ Corona Theorem. *Let $p\in(0,1)$ and let $f_1, f_2, \dots, f_n \in Q_p$. Then the corona problem is solvable in Q_p if and only if $f_1, f_2, \dots, f_n$ are multipliers of Q_p and satisfy the condition* $\inf_{z\in\triangle}\sum_{k=1}^{n}|f_k(z)|>0$.

The major tool used in arguing this theorem is a modified (p-) Carleson measure used in the study of the Q_p spaces combined with T. Wolff's proof (cf. [**Ga**, pp. 325-327]) of L. Carleson's corona theorem [**Ca**]. In Sections 1 and 2, we develop a full discussion on the multipliers of Q_p which is necessary for the corona problem. In Section 3, based on our results of the previous two sections, we reformulate and show the Q_p corona theorem by considering the surjective property of multiplication operator acting on n-copies of Q_p. There $\overline{\partial}$-estimates will be adapted sufficiently.

Throughout this paper, the letters $C, C_p, C_1, C_2, \dots$ and so on, stand for positive different constants which are independent of all points $z, w, \zeta \in \triangle$ and all subarcs $I \subset \partial\triangle$ without a particular remark.

1. Multipliers of Q_p, $p \in (0,\infty)$.

This section focuses our attention on determining the multipliers of Q_p, $p \in (0,\infty)$.

In what is going on, for a given Banach space X, denote by $\mathcal{M}(X)$ the set of multipliers of X, i.e.,

$$\mathcal{M}(X)=\{f\in X : M_f g = fg \in X \ \text{ whenever } \ g\in X\}.$$

For D_p, $p\in(0,1)$, D. Stegenga used the strong capacity inequality due to V.G. Mazya and D.R. Adams to generalize the classical Carleson measure from H^2 to D_p and hence to characterize $\mathcal{M}(D_p)$ [**Steg2**]. Unfortunately, we are at least now unable to give a similar description of the Carleson-type measure on Q_p (cf. [**ArFiPe**]) and so unable to directly follow Stegenga's method in order to describe $\mathcal{M}(Q_p)$. But, observing that Q_p behaves like $BMOA$ and $\mathcal{B}$, so we may borrow some ideas from [**Steg1**], [**OrFa1**] and [**BrSh**] to reveal what $\mathcal{M}(Q_p)$ is for each $p\in(0,1)$ and even for each $p\in[1,\infty)$. To this end, we here introduce a modified Carleson measure in terms of the geometric concept.

For $p\in(0,\infty)$ we say that a complex Borel measure μ given on $\triangle$ is a p-Carleson measure provided

$$\|\mu\|_p=\sup_{I\subset\partial\triangle}\frac{|\mu|(S(I))}{|I|^p}<\infty, \tag{1.1}$$

where the supremum is taken over all subarcs $I\subset\partial\triangle$. From now on, suppose that $|I|$ stands for the normalized arc length of I, i.e., $|I|\le 1$, and that $S(I)$ means the Carleson square based on I. When $p=1$, we get the standard definition of the original Carleson measure. As in [**Ga**, p. 239], any positive p-Carleson measure has an integral representation.

Lemma 1.1 (ASX)**.** *Let $p \in (0, \infty)$ and let μ be a positive Borel measure on $\triangle$. Then μ is a p-Carleson measure if and only if*

$$\sup_{w \in \triangle} \iint_{\triangle} \left(\frac{1 - |w|}{|1 - \overline{w} z|^2} \right)^p d\mu(z) < \infty.$$

Proof. See Lemma 2.1 in [**AuStXi**]. □

This lemma supplies a geometric way to characterize Q_p-functions.

Theorem 1.2 (ASX)**.** *Let $p \in (0, \infty)$ and $f \in A$ with*

$$d\mu_{f,p}(z) = |f'(z)|^2 (1 - |z|)^p dm(z).$$

Then $f \in Q_p$ if and only if $\mu_{f,p}$ is a p-Carleson measure.

Proof. See Theorem 1.1 in [**AuStXi**]. □

Regarding the multipliers of Q_p we have:

Theorem 1.3. *Let $p \in (0, \infty)$. Then:*

(i) *$f \in \mathcal{M}(Q_p)$ implies that $f \in H^\infty$ and*

$$\iint_{S(I)} |f'(z)|^2 (1 - |z|)^p dm(z) \leq \frac{C|I|^p}{\log^2 \frac{2}{|I|}} \tag{1.2}$$

for all Carleson squares $S(I)$.

(ii) *$f \in \mathcal{M}(Q_p)$ if $f \in H^\infty$ and $|f'(z)|^2 (1 - |z|)^p \log^2(1 - |z|) dm(z)$ is a p-Carleson measure.*

Proof. Step 1: (i). Assume that $f \in \mathcal{M}(Q_p)$ holds. Since Q_p is a subspace of $\mathcal{B}$, any function $g \in Q_p$ has the following growth

$$|g(z)| \leq C \log \frac{2}{1 - |z|}, \ z \in \triangle. \tag{1.3}$$

Observe that for a fixed $w \in \triangle$, the function $g_w(z) = \log \frac{2}{1 - \bar{w} z}$ belongs to Q_p with $\sup_{w \in \triangle} \|g_w\|_{Q_p} \leq C_p$ (owing to both Theorem 5.4 in [**EsXi**] and Corollary 2.2 in [**AuStXi**]). By (1.3) we have

$$|f(z) g_w(z)| \leq C \log \frac{2}{1 - |z|}, \ z \in \triangle,$$

which shows $f \in H^\infty$.

Concerning (1.2), we argue as follows. Because $f \in \mathcal{M}(Q_p)$, it follows from Theorem 1.2 (ASX) that for any Carleson square $S(I)$,

$$\iint_{S(I)} |(f g_w)'(z)|^2 (1 - |z|)^p dm(z) \leq C|I|^p,$$

and so that by $f \in H^\infty$,

$$\iint_{S(I)} |f'(z)|^2 |g_w(z)|^2 (1 - |z|)^p dm(z) \leq C|I|^p.$$

Note that if $w=(1-|I|)e^{i\theta}$ and $e^{i\theta}$ is taken as the center of I then for all $z\in S(I)$,

$$\frac{1}{C}\log\frac{2}{|I|}\le |g_w(z)|\le C\log\frac{2}{|I|}.$$

Whence (1.2) is forced to come out.

Step 2: (ii). Assume that f meets the hypotheses of (ii). Since all $g\in Q_p$ always obey (1.3), with the help of Theorem 1.2 (ASX) we read that for any subarc $I\subset\partial\triangle$,

$$\begin{aligned}&\iint_{S(I)}|(fg)'(z)|^2(1-|z|)^p dm(z)\\ &\le C\iint_{S(I)}|f'(z)|^2(1-|z|)^p\log^2(1-|z|)dm(z)\\ &+C\iint_{S(I)}|g'(z)|^2(1-|z|)^p dm(z)\\ &\le C|I|^p.\end{aligned}$$

In other words, $fg\in Q_p$, i.e., $f\in\mathcal{M}(Q_p)$. The proof is complete. $\square$

Denote by $\mathcal{M}_b(Q_p)$ the set of functions $f\in H^\infty$ satisfying (1.2). Then $\mathcal{M}(Q_p)\subset\mathcal{M}_b(Q_p)$. Moreover, we have:

Corollary 1.4. *Let $p\in[1,\infty)$. Then $\mathcal{M}(Q_p)=\mathcal{M}_b(Q_p)$.*

Proof. This corollary says that $f\in\mathcal{M}(Q_p)$ if and only if $f\in H^\infty$ and (1.2) holds for any Carleson square $S(I)$. Since the case $p=1$ is essentially known [**OrFa1**], we only need to check the case $p>1$. In fact, from Theorem 1.3 (ii) it yields that we suffice to show a proposition: If $f\in\mathcal{M}(Q_p)$ then $|f'(z)|^2(1-|z|)^p\log^2(1-|z|)dm(z)$ is a p-Carleson measure. To the end, for a given subarc I of $\partial\triangle$ let $\mathcal{D}_n(I)$ represent the set of 2^n subarcs of length $2^{-n}|I|$ obtained by n successive bipartition of I. For each $J\in\mathcal{D}_n(I)$ write $T(J)$ for the top half Carleson box of $S(J)$, i.e.,

$$T(J)=\left\{z\in S(J):\frac{z}{|z|}\in J,\ 1-|J|<|z|<1-\frac{|J|}{2}\right\}.$$

Then

$$S(I)=\bigcup_{n=0}^{\infty}\bigcup_{J\in\mathcal{D}_n(I)}T(J).$$

Thus, by Theorem 1.3 (i),

$$\begin{aligned}
&\iint_{S(I)} |f'(z)|^2(1-|z|)^p \log^2(1-|z|)dm(z) \\
&= \sum_{n=0}^{\infty} \sum_{J\in\mathcal{D}_n(I)} \iint_{T(J)} |f'(z)|^2(1-|z|)^p \log^2(1-|z|)dm(z) \\
&\le \sum_{n=0}^{\infty} \sum_{J\in\mathcal{D}_n(I)} \log^2\frac{2}{|J|} \iint_{T(J)} |f'(z)|^2(1-|z|)^p dm(z) \\
&\le C\sum_{n=0}^{\infty} \sum_{J\in\mathcal{D}_n(I)} \left(\log^2\frac{2}{|J|}\right)\left(\frac{|J|^p}{\log^2\frac{2}{|J|}}\right) \\
&\le C\sum_{n=0}^{\infty} |I|^p 2^{(1-p)n} \\
&\le C|I|^p.
\end{aligned}$$

This just reaches our aim. □

This corollary actually provides a new characterization for the multiplier space of the Bloch space (cf. [**BrSh**, **Zhu1**]). Its proof is partially inspired by [**Ja**]. Observe that the above demonstration does not work for the case $p \in (0,1)$. However, we hope the similar result is true. Therefore we pose:

Conjecture 1.5. *Let $p \in (0,1)$. Then $\mathcal{M}(Q_p) = \mathcal{M}_b(Q_p)$.*

Remark 1.6. In the same method as arguing Lemma 1.1 (ASX) (cf. Lemma 2.1 in [**AuStXi**]), we can prove that for a positive Borel measure μ on $\triangle$ and $p \in (0,\infty)$, $\mu(S(I)) = O(|I|^p \log^{-2}\frac{2}{|I|})$ if and only if

$$\sup_{w\in\triangle} \log^2(1-|w|) \iint_{\triangle} \left(\frac{1-|w|}{|1-\overline{w}z|^2}\right)^p d\mu(z) < \infty.$$

Consequently, (1.2) is equivalent to

$$\sup_{w\in\triangle} \log^2(1-|w|) \iint_{\triangle} |f'(z)|^2[1-|\phi_w(z)|]^p dm(z) < \infty.$$

2. Boundary behaviour of $\mathcal{M}(Q_p), p \in (0,1)$.

From a viewpoint of the boundary behaviour, this section continues the discussion about the space $\mathcal{M}(Q_p), p \in (0,1)$.

D. Stegenga's characterization on $\mathcal{M}(BMO(\partial\triangle))$ was first made by a logarithmic BMO-function on $\partial\triangle$. Later his result was extended to $\mathcal{M}(BMOA)$. In contrast with $\mathcal{M}(BMOA)$, some basic properties of $\mathcal{M}(Q_p)$, $p \in (0,1)$ have been worked out in advance. So it is hoped that

Stegenga's description on $\mathcal{M}(BMO(\partial\triangle))$ can be carried to $\mathcal{M}(Q_p(\partial\triangle))$ and even to $\mathcal{M}(Q_p)$, $p \in (0,1)$ because Q_p possesses its own boundary behaviour, but also because the proof of the Q_p corona theorem wants the boundary behaviour. To this end, we quote a theorem from [**NiXi**].

Theorem 2.1 (NX)**.** *Let $p \in (0,1)$ and $f \in L^2_{\mathrm{loc}}(\partial\triangle)$ with*

$$d\mu_{\hat{f},p}(z) = |\nabla \hat{f}(z)|^2 (1-|z|)^p dm(z).$$

Then $f \in Q_p(\partial\triangle)$ if and only if $\mu_{\hat{f},p}$ is a p-Carleson measure. Hereafter ∇ means the gradient operator and $\hat{f}$ Poisson's extension of f.

Proof. See Theorem 2.1 in [**NiXi**]. □

In addition, we cite a lemma of D. Stegenga which played an important role in capturing a boundary behaviour of multipliers from the Hardy space to the Bergman space [**Steg2**].

Lemma 2.2 (S)**.** *Let I and J be arcs on $\partial\triangle$ centered at e^{is_0} with $|J| \geq 3|I|$ and let $f \in L^1_{\mathrm{loc}}(\partial\triangle)$. For $p \in (-1,1)$, there is a constant C_p independent of f, I and J such that*

$$\begin{aligned} &\iint_{S(I)} |\nabla \hat{f}|^2 (1-|z|^2)^p dm(z) \\ &\leq C_p \left\{ \int_J \int_J \frac{|f(e^{is}) - f(e^{it})|^2}{|e^{is} - e^{it}|^{2-p}} ds dt \right. \\ &\qquad \left. + |I|^{2+p} \left(\int_{|t| \geq |J|/3} |f(e^{i(t+s_0)}) - f_J| dt/t^2 \right)^2 \right\}. \end{aligned} \tag{2.1}$$

Proof. It follows from taking $\alpha = \frac{1-p}{2}$ in Lemma 3.2 of [**Steg2**]. □

After holding the previous propositions, we can state the main result of this section.

Theorem 2.3. *Let $p \in (0,1)$. Then:*

(i) *$f \in \mathcal{M}(Q_p(\partial\triangle))$ implies that $f \in L^\infty(\partial\triangle)$ and*

$$\iint_{S(I)} |\nabla \hat{f}(z)|^2 (1-|z|)^p dm(z) \leq \frac{C|I|^p}{\log^2 \frac{2}{|I|}} \tag{2.2}$$

for all Carleson squares $S(I)$, equivalently,

$$\int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} d\theta d\varphi \leq \frac{C|I|^p}{\log^2 \frac{2}{|I|}} \tag{2.3}$$

for all subarcs $I \subset \partial\triangle$.

(ii) *$f \in \mathcal{M}(Q_p(\partial\triangle))$ if $f \in L^\infty(\partial\triangle)$ and $|\nabla \hat{f}(z)|^2 (1-|z|)^p \log^2(1-|z|) dm(z)$ is a p-Carleson measure.*

Proof. A careful reading at the argument of Theorem 1.3 indicates that it is enough to verify the equivalence between (2.2) and (2.3). Note that the details of proving $f \in L^\infty(\partial\triangle)$ in (i) can be easily given by means of [**Zhu1**]. First, let (2.2) be true. In this case, we will handle (2.3) using A. Nicolau's approach [**Ni**].

From now on, for $m = 1, 2, \dots$, denote by mI the subarc of $\partial\triangle$ with the same center as I and with the length $m|I|$. Clearly, there is a constant C_p depending on p only such that

$$\int_I \int_I \frac{|f(e^{is}) - f(e^{it})|^2}{|e^{is} - e^{it}|^{2-p}} ds dt \le C_p \int_0^{|I|} \frac{1}{t^{2-p}} \left[\int_{2I} |f(e^{i(\theta+t)}) - f(e^{i\theta})|^2 d\theta\right] dt$$

for any subarc I with $|I| < \frac{1}{2}$. So, without loss of generality, assume that $I = [0, |I|]$ with $|I| \le \frac{1}{4}$. Then by the Minkowski inequality for integrals,

$$\begin{aligned}\left[\int_{2I} |f(e^{i(\theta+t)}) - f(e^{i\theta})|^2 d\theta\right]^{\frac{1}{2}} &\le 2\int_{1-t}^{1} \left[\int_{3I} \left|\frac{\partial \hat{f}}{\partial n}(ue^{is})\right|^2 ds\right]^{\frac{1}{2}} du \\ &\quad + 2\int_0^t \left[\int_{3I} \left|\frac{\partial \hat{f}}{\partial \theta}((1-t)e^{is})\right|^2 ds\right]^{\frac{1}{2}} du \\ &= \mathrm{Int}_1 + \mathrm{Int}_2,\end{aligned}$$

where $\frac{\partial f}{\partial n}$ resp. $\frac{\partial f}{\partial \theta}$ is the directional derivative of f relative to the radius resp. the argument.

Making use of Hardy's inequality [**Stei**, p. 272], we obtain

$$\begin{aligned}\int_0^{|I|} \frac{(\mathrm{Int}_1)^2}{t^{2-p}} dt &\le C_p \int_0^{|I|} t^p \left[\int_{3I} \left|\frac{\partial \hat{f}}{\partial n}((1-t)e^{is})\right|^2 ds\right] dt \\ &\le C_p \iint_{S(3I)} |\nabla \hat{f}(z)|^2 (1-|z|)^p dm(z).\end{aligned}$$

Meanwhile, Int_2 obeys

$$\begin{aligned}\int_0^{|I|} \frac{(\mathrm{Int}_2)^2}{t^{2-p}} dt &\le C_p \int_0^{|I|} t^p \left[\int_{3I} \left|\frac{\partial \hat{f}}{\partial \theta}(re^{is})\right|^2 ds\right] dt \\ &\le C_p \iint_{S(3I)} |\nabla \hat{f}(z)|^2 (1-|z|)^p dm(z).\end{aligned}$$

Putting these inequalities in order, we see that (2.3) is valid.

Second, (2.3) $\Rightarrow$ (2.2). Since $|e^{i\theta} - e^{i\varphi}| \le |I|$ for $e^{i\theta}, e^{i\varphi} \in I$, (2.3) induces

$$\int_I \int_I |f(e^{i\theta}) - f(e^{i\varphi})|^2 d\theta d\varphi \le \frac{C|I|^2}{\log^2 \frac{2}{|I|}},$$

consequently,

$$\frac{1}{|I|}\int_I |f(e^{i\theta}) - f_I| d\theta \le \frac{C}{\log\frac{2}{|I|}}. \tag{2.4}$$

For the sake of simplicity, let $J = 3I$ and $s_0 = 0$ in Lemma 2.2 (S). Then we have

$$\begin{aligned}
&\int_{|t|\ge|J|/3} |f(t) - f_J| dt/t^2\\
&\le \sum_{k=0}^{\infty}\int_{3^{k-1}|J|\le|t|\le 3^k|J|} |f(t) - f_J| dt/t^2\\
&\le C\sum_{k=0}^{\infty}(3^k|J|)^{-2}\int_{|t|\le 3^k|J|} |f(t) - f_{3^{k+1}I}| dt\\
&+ C\sum_{k=0}^{\infty}(3^k|J|)^{-1}|f_{3^{k+1}I} - f_J|\\
&= \mathrm{Int}_1 + \mathrm{Int}_2.
\end{aligned}$$

To control Int_1 and Int_2 we may suppose that $3^{-(N+1)} \le |I| < 3^{-N}$ for some positive integer N. At the moment, $|I|\log\frac{3}{|I|} \approx N3^{-N}$. Hereafter $a \approx b$ means that there are two absolute constants C_1 and C_2 to insure $C_1 a \le b \le C_2 a$. As a consequence of (2.4), we get

$$\begin{aligned}
\mathrm{Int}_1 &\le C\sum_{k=0}^{\infty}\frac{1}{(3^k|I|)\log\frac{3}{3^{(k+1)}|I|}}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{(3^{k-N})(N+1-k)}\\
&\le \frac{C3^N}{N}\\
&\le \frac{C}{|I|\log\frac{3}{|I|}},
\end{aligned}$$

which results from

$$\frac{N+1}{N+1-k} \le 3^{1+\frac{3k}{4}}, k = 0, 1, 2, \ldots N.$$

As to Int_2, we use the preceding idea of bounding Int_1 as well as an elementary estimate involved in Lemma 1.1 of [**Ga**, Chapter VI] to obtain

$$\begin{aligned}\mathrm{Int}_2 &\le C\sum_{k=0}^{\infty}\frac{k+1}{(3^k|I|)\log\frac{3}{3^{(k+1)}|I|}}\\ &\le \frac{C}{|I|\log\frac{3}{|I|}}.\end{aligned}$$

Combining those inequalities above, we immediately see

$$\int_{|t|\ge|J|/3}|f(t)-f_J|dt/t^2 \le \frac{C}{|I|\log\frac{3}{|I|}}.$$

Finally, (2.1) and (2.3) imply (2.2). Now, the proof is complete. □

As a matter of fact, a boundary behaviour of $\mathcal{M}(Q_p)$, $p\in(0,1)$ is built up.

Corollary 2.4. *Let $p\in(0,1)$ Then $f\in\mathcal{M}(Q_p)$ only if $f\in H^\infty$ and (2.3) holds for any subarc $I\subset\partial\triangle$.*

Proof. In this case, $Q_p=Q_p(\partial\triangle)\cap H^2$, so this corollary yields from Theorem 2.3 (i). □

Upon defining $\mathcal{M}_b(Q_p(\partial\triangle))$ as the set of functions $f\in L^\infty(\partial\triangle)$ satisfying (2.3), we have $\mathcal{M}(Q_p(\partial\triangle))\subset\mathcal{M}_b(Q_p(\partial\triangle))$, and hence we raise:

Conjecture 2.5. *Let $p\in(0,1)$. Then $\mathcal{M}(Q_p(\partial\triangle))=\mathcal{M}_b(Q_p(\partial\triangle))$.*

Remark 2.6. Modifying the proof of Theorem 2.1 in [**NiXi**] will show that under the restriction $p\in(0,1)$, (2.3) is equivalent to

$$\sup_{w\in\triangle}\log^2(1-|w|)\int_{\partial\triangle}\int_{\partial\triangle}\frac{|f(e^{i\theta})-f(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^{2-p}}\left(\frac{1-|w|}{|e^{i\theta}-w||e^{i\varphi}-w|}\right)^p d\theta d\varphi$$
$$<\infty.$$

It is worth comparing this result with Theorem 2.9 in [**OrFa2**].

3. Proof of the Q_p, $p\in(0,1)$ corona theorem.

In this section, we present a proof of the Q_p, $p\in(0,1)$ corona theorem. Moreover, we assert that Carleson's H^∞ corona theorem can be extended to $\mathcal{M}_b(Q_p)$, $p\in(0,1)$.

We start with introducing some auxiliary notation. For a natural number n let $(f_1,f_2,\dots,f_n)\in A\times A\times\cdots\times A$. Define a linear operator on $A\times A\times\cdots\times A$ via

$$\mathcal{M}_{(f_1,f_2,\dots,f_n)}(g_1,g_2,\dots,g_n)=\sum_{k=1}^{n} f_k g_k, \quad (g_1,g_2,\dots,g_n)\in A\times A\times\cdots\times A.$$

Also, for $z=x+iy$ define

$$\bar\partial=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right); \quad \partial=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right).$$

Let g be $\mathcal{C}^1$ and bounded on $\triangle$. Then the inhomogeneous Cauchy-Riemann equation, i.e., $\bar\partial$-equation

$$\bar\partial f=g \tag{3.1}$$

has a standard solution

$$f(z)=\frac{1}{\pi}\iint_{\triangle}\frac{g(w)}{z-w}dm(w) \tag{3.2}$$

on $\triangle$. It is easy to verify that the convolution f defined by (3.2) is continuous on the finite complex plane and that f is $\mathcal{C}^2$ on $\triangle$. Furthermore, this solution is employed to verify the Q_p, $p\in(0,1)$ corona theorem. In order to be self-complete, we reformulate the theorem in terms of operator theory.

Theorem 3.1. *Let $p\in(0,1)$ and $(f_1,f_2,\dots,f_n)\in A\times A\times\cdots\times A$. Then the following are equivalent:*

(i) $\mathcal{M}_{(f_1,f_2,\dots,f_n)}$ *maps* $Q_p\times Q_p\times\cdots\times Q_p$ *onto* Q_p.
(ii) $(f_1,f_2,\dots,f_n)\in\mathcal{M}(Q_p)\times\mathcal{M}(Q_p)\times\cdots\times\mathcal{M}(Q_p)$ *with*

$$\sigma=\inf_{z\in\triangle}\sum_{k=1}^{n}|f_k(z)|>0. \tag{3.3}$$

Proof. Step 1. (i) $\Rightarrow$ (ii). Suppose (i) is true. Evidently, it is enough to check (3.3). For this, we use the open map theorem to get a uniform constant C_0 such that to $g\in Q_p$ there correspond $g_1,g_2,\dots,g_n\in Q_p$ with $\|g_k\|_{Q_p}\le C_0\|g\|_{Q_p}$ and

$$g=\sum_{k=1}^{n} f_k g_k.$$

Further, by (1.3) and $g(z)=\log\frac{1-ze^{-i\theta}}{2}$ we obtain another uniform constant C_1 to ensure

$$\left|\log\frac{1-ze^{-i\theta}}{2}\right|\le C_1\log\frac{2}{1-|z|}\sum_{k=1}^{n}|f_k(z)|,$$

which implies (3.3).

Step 2. (ii) $\Rightarrow$ (i). This direction is more difficult. Let now (ii) hold. We need only to show that if $g\in Q_p$ then $g\in\mathcal{M}_{(f_1,f_2,\dots,f_n)}(Q_p\times Q_p\times\cdots\times Q_p)$

whenever $(f_1, f_2, \dots, f_n) \in \mathcal{M}(Q_p) \times \mathcal{M}(Q_p) \times \cdots \times \mathcal{M}(Q_p)$ satisfies (3.3). It is well known that

$$h_k = \frac{\bar{f}_k}{\sum_{k=1}^n |f_k|^2}$$

is a group of nonanalytic solutions of the equation: $\sum_{k=1}^n f_k h_k = 1$. However, if we can find functions $b_{j,k}$ $(j, k = 1, 2, \dots, n)$ defined on $\triangle \cup \partial\triangle$ to guarantee $b_{j,k} \in Q_p(\partial\triangle)$ and

$$\bar{\partial} b_{j,k} = g h_j \bar{\partial} h_k \tag{3.4}$$

on $\triangle$, then

$$g_j = g h_j + \sum_{k=1}^n (b_{j,k} - b_{k,j}) f_k$$

just meet the requirements: $\sum_{k=1}^n f_k g_k = g$ and $g_j \in Q_p$. Note that $g h_j \in Q_p(\partial\triangle)$ can be figured out from the following argument (cf. (3.6)). Obviously, we require only to prove that $\bar{\partial} b = gh$ (where $b = b_{j,k}$ and $h = h_j \bar{\partial} h_k$) admits $Q_p(\partial\triangle)$-solution. After making an elementary calculation related to (3.3) (cf. [**Ga**, p. 326]), we reach

$$|h(z)|^2 \le C_\sigma \sum_{k=1}^n |f_k'(z)|^2, \ z \in \triangle, \tag{3.5}$$

where C_σ is a constant depending only on σ. Notice that the constants appeared in this section may rely on n, but not the functions involved in the argument. It is normal to take (3.2) (in which b and gh substitute for f and g, respectively) as the desired solution. Certainly, we cannot help checking whether or not such a function defined by (3.2) belongs to $Q_p(\partial\triangle)$.

From (3.4), (3.5), $g \in Q_p$, $f_k \in \mathcal{M}(Q_p)$, Theorem 1.2 (ASX) and Theorem 1.3 (i) it yields readily that for any Carleson square $S(I)$,

$$\begin{aligned} &\iint_{S(I)} |\bar{\partial} b(z)|^2 (1-|z|)^p \, dm(z) \\ &= \iint_{S(I)} |g(z)h(z)|^2 (1-|z|)^p \, dm(z) \\ &\le C_\sigma \sum_{k=1}^n \iint_{S(I)} |g'(z) f_k(z)|^2 (1-|z|)^p \, dm(z) \\ &\quad + C_\sigma \sum_{k=1}^n \iint_{S(I)} |(g f_k)'(z)|^2 (1-|z|)^p \, dm(z) \\ &\le C|I|^p. \end{aligned} \tag{3.6}$$

For convenience, write $B(f)$ for the Beurling transform of a function f. So $\partial b = B(gh)$ and

$$
\begin{aligned}
&\iint_{S(I)} |\partial b(z)|^2 (1-|z|)^p dm(z) \\
&\le 2 \iint_{S(I)} |B(gh\chi_{S(2I)})(z)|^2 (1-|z|)^p dm(z) \\
&\quad + 2 \iint_{S(I)} |B((1-\chi_{S(2I)})gh)(z)|^2 (1-|z|)^p dm(z) \\
&\le 4 \iint_{\triangle} |B(gh\chi_{S(2I)})(z)|^2 (1-|z|)^p dm(z) \\
&\quad + 4 \iint_{S(I)} \left[\iint_{\triangle\setminus S(2I)} \frac{|g(w)h(w)|}{|w-z|^2} dm(w) \right]^2 (1-|z|)^p dm(z) \\
&= \mathrm{Int}_1 + \mathrm{Int}_2,
\end{aligned}
$$

where $\chi_{S(2I)}$ is the characteristic function of $S(2I)$.

Since $(1-|z|)^p$ is an A_2-weight for $p \in (0,1)$ (see [**CuFr**, p. 411]), it follows from (3.5) and (3.6) that

$$
\begin{aligned}
\mathrm{Int}_1 &\le C \iint_{\triangle} |B(gh\chi_{S(2I)})(z)|^2 (1-|z|)^p dm(z) \\
&\le C \iint_{\triangle} |(gh\chi_{S(2I)})(z)|^2 (1-|z|)^p dm(z) \\
&\le C \iint_{S(2I)} |g(z)h(z)|^2 (1-|z|)^p dm(z) \\
&\le C|I|^p.
\end{aligned}
$$

Due to $f_k \in \mathcal{M}(Q_p)$ once again, Theorem 1.3 (i) implies

$$
\iint_{S(I)} |f_k'(z)|^2 (1-|z|)^p dm(z) \le \frac{C|I|^p}{\log^2 \frac{2}{|I|}}.
$$

Accordingly, by (3.5), (3.6) and Hölder's inequality,

$$
\begin{aligned}
&\iint_{S(I)} |g(z)h(z)| dm(z) \\
&\le C \sum_{k=1}^{n} \iint_{S(I)} |g'(z) f_k(z)| dm(z) \\
&\quad + C \sum_{k=1}^{n} \iint_{S(I)} |(g f_k)'(z)| dm(z) \\
&\le C|I|,
\end{aligned}
$$

that is to say, $|g(z)h(z)|dm(z)$ is a 1-Carleson measure. This fact is applied to deduce

$$\begin{aligned}
&\mathrm{Int}_2\\
&\leq C\iint_{S(I)}\left[\sum_{k=1}^{\infty}\iint_{S(2^{k+1}I)\setminus S(2^kI)}\frac{|g(w)h(w)|}{|w-z|^2}dm(w)\right]^2(1-|z|)^p dm(z)\\
&\leq C\iint_{S(I)}\left[\sum_{k=1}^{\infty}\frac{1}{2^{2k}|I|^2}\iint_{S(2^{k+1}I)}|g(w)h(w)|dm(w)\right]^2(1-|z|)^p dm(z)\\
&\leq C|I|^p.
\end{aligned}$$

The above estimates on Int_k, $k=1,2$ tell us that

$$\iint_{S(I)}|\partial b(z)|^2(1-|z|)^p dm(z)\leq C|I|^p,$$

and so that

$$\iint_{S(I)}|\nabla b(z)|^2(1-|z|)^p dm(z)\leq C|I|^p.$$

The last inequality gives easily that b lies in $Q_p(\partial\triangle)$ (refer to the implication (2.2) $\Rightarrow$ (2.3)). This completes the proof. $\square$

We close this section with a version of the corona theorem on $\mathcal{M}_b(Q_p)$. To do so, we ought to explore $\mathcal{M}_b(Q_p(\partial\triangle))$-estimates for $\bar{\partial}$-equation. P. Jones' solution [**Jo**] of the $\bar{\partial}$-equation is suitable for our purpose.

Lemma 3.2 (J). *Let $d\mu(z)=g(z)dm(z)$ be a 1-Carleson measure on $\triangle$ with $\|\mu\|_1=1$. If for $z\in\triangle\cup\partial\triangle$ and $\zeta\in\triangle$,*

$$\begin{aligned}
&K(\mu,z,\zeta)\\
&\quad=\frac{2i}{\pi}\cdot\frac{1-|\zeta|^2}{(1-\bar{\zeta}z)(z-\zeta)}\exp\left[\iint_{|w|\geq|\zeta|}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}-\frac{1+\overline{w}z}{1-\overline{w}z}\right)d|\mu|(w)\right]
\end{aligned}$$

then

$$S(\mu)(z)=\iint_{\triangle}K(\mu,z,\zeta)d\mu(\zeta) \tag{3.7}$$

satisfies $S(\mu)\in L^1_{\mathrm{loc}}(\triangle)$ and $\bar{\partial}S(\mu)=g$. Moreover, if $z\in\partial\triangle$ then the above integral converges absolutely and obeys

$$\iint_{\triangle}|K(\mu,z,\zeta)|d|\mu|(\zeta)\leq C,$$

and hence $S(\mu)\in L^\infty(\partial\triangle)$, where C is a universal constant.

Proof. This lemma follows immediately from Theorem 1 in [**Jo**] and Caylay's transformation of $\triangle$ onto the half plane. $\square$

It is known that Carleson's corona theorem is available for $\mathcal{M}_b(Q_1)$ (cf. [**To**]). Next assertion illustrates that for $p \in (0,1)$, $\mathcal{M}_b(Q_p)$ has such a theorem, too.

Theorem 3.3. *Let $p \in (0,1)$ and $(f_1, f_2, \dots, f_n) \in A \times A \times \cdots \times A$. Then the following are equivalent:*

(i) $\mathcal{M}_{(f_1,f_2,\dots,f_n)}$ *maps* $\mathcal{M}_b(Q_p) \times \mathcal{M}_b(Q_p) \times \cdots \times \mathcal{M}_b(Q_p)$ *onto* $\mathcal{M}_b(Q_p)$.
(ii) $(f_1, f_2, \dots, f_n) \in \mathcal{M}_b(Q_p) \times \mathcal{M}_b(Q_p) \times \cdots \times \mathcal{M}_b(Q_p)$ *with* (3.3).

Proof. The implication (i) $\Rightarrow$ (ii) is evident. In fact, replacing Q_p by $\mathcal{M}_b(Q_p)$ and $\log(1 - e^{-i\theta}z)$ by 1 in the argument for (i) $\Rightarrow$ (ii) in Theorem 3.1 just arrives at our key point. As to the opposite implication (ii) $\Rightarrow$ (i), we may repeat almost whole proof of (ii) $\Rightarrow$ (i) in Theorem 3.1. The unique difference between two situations is the constructive solution of (3.4). Unlike there, our present solution should belong to $\mathcal{M}_b(Q_p(\partial\triangle))$. To this end, we prepare to take advantage of Lemma 3.2 (J). At the moment, it suffices to show a fact: If b is given by (3.7) where $d\mu = hdm(z) = h_j\bar{\partial}h_k dm(z)$ and $\|\mu\|_1 = 1$, then $b \in \mathcal{M}_b(Q_p(\partial\triangle))$. It will be done if we can prove that

$$\iint_{S(I)} |\nabla\tilde{b}(z)|^2(1-|z|)^p dm(z) \le \frac{C|I|^p}{\log^2 \frac{2}{|I|}} \tag{3.8}$$

for all Carleson squares $S(I)$, where

$$\tilde{b}(z) = \frac{2i}{\pi}\iint_{\triangle} \frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2} \exp\left[\iint_{|w|\ge|\zeta|}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta} - \frac{1+\overline{w}z}{1-\overline{w}z}\right)d|\mu|(w)\right]d|\mu|(\zeta).$$

This is because $zb(z)$ and $\tilde{b}(z)$ possess the same boundary values on $\partial\triangle$, but also (3.8) ensures (2.3) to be valid for $\tilde{b}(z)$ and so for $b(z)$. Note that the function

$$f_\zeta(w) = \frac{(1-|\zeta|^2)^{\frac{1}{2}}}{1-\bar{\zeta}w}$$

is in H^2 and its H^2-norm is independent of $\zeta \in \triangle$. Since μ is a classical Carleson measure,

$$Re\left(\iint_{|w|\ge|\zeta|}\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}d|\mu|(w)\right) \le 2\iint_{\triangle}\frac{1-|\zeta|^2}{|1-\overline{w}\zeta|^2}d|\mu|(w) \le C.$$

Further, the argument for Lemma 2.1 in [**Jo**] gives

$$\iint_{\triangle}\frac{1-|z\zeta|^2}{|1-\bar{\zeta}z|^2}\exp\left[-\iint_{|w|\ge|\zeta|}\frac{1-|z\overline{w}|^2}{|1-z\overline{w}|^2}d|\mu|(w)\right]d|\mu|(\zeta) \le 1.$$

Hence

$$\begin{aligned}
&\iint_{S(I)} |\nabla \tilde{b}(z)|^2 (1-|z|)^p dm(z)\\
&\le C \iint_{S(I)} \left[\iint_{\triangle} \frac{(1-|z|)^{\frac{p}{2}}}{|1-\overline{w}z|^2} d|\mu|(w) \right]^2 dm(z)\\
&\le C \iint_{S(I)} (1-|z|)^p \left[\iint_{S(2I)} \frac{d|\mu|(w)}{|1-\overline{w}z|^2} \right]^2 dm(z)\\
&+ C \iint_{S(I)} (1-|z|)^p \left[\iint_{\triangle\setminus S(2I)} \frac{d|\mu|(w)}{|1-\overline{w}z|^2} \right]^2 dm(z)\\
&= \mathrm{Int}_1 + \mathrm{Int}_2.
\end{aligned}$$

Define an integral operator by

$$(Tf)(z) = \iint_{\triangle} f(w) k(z,w) dm(z),$$

where

$$k(z,w) = \frac{(1-|z|)^{1-p/2}(1-|w|)^{-p/2}}{|1-\overline{w}z|^2}.$$

Using Shur's lemma (see also [**Zhu2**, p. 42]), we conclude that the operator T is bounded on $L^2(\triangle, dm)$. Accordingly, selecting $f(z) = (1-|z|)^{\frac{p}{2}}|h(z)|\chi_{S(2I)}(z)$ in Tf implies

$$\begin{aligned}
\mathrm{Int}_1 &\le C \iint_{\triangle} |Tf(z)|^2 dm(z)\\
&\le C \iint_{S(2I)} |h(z)|^2 (1-|z|)^p dm(z)\\
&\le \frac{C|I|^p}{\log^2 \frac{2}{|I|}}.
\end{aligned}$$

In the sequel, we will directly estimate Int_2 in a standard manner. Actually, one may assume that $|I| \le \frac{1}{3}$ and $z \in S(I)$. Hölder's inequality is employed to establish

$$\begin{aligned}
\iint_{S(I_{k+1})\setminus S(I_k)} \frac{d|\mu|(w)}{|1-\overline{w}z|^2} &\le \frac{1}{2^{2k}|I|^2} \iint_{S(2^{k+1}I)} |h(w)| dm(w)\\
&\le \frac{C}{2^k|I| \cdot \log \frac{2}{2^k|I|}}.
\end{aligned}$$

Also, suppose $2^{-(N+1)} \leq |I| < 2^{-N}$ for some positive integer N, so $|I| \log \frac{2}{|I|} \approx N 2^{-N}$ and

$$\begin{aligned}
\mathrm{Int}_2 &\leq C \iint_{S(I)} (1-|z|)^p \left(\sum_{k=1}^{N+1} \frac{1}{2^k |I| \cdot \log \frac{2}{2^k |I|}} \right)^2 dm(z) \\
&\leq C |I|^{2+p} \left(\sum_{k=1}^{N+1} \frac{1}{2^k |I| \cdot \log \frac{2}{2^k |I|}} \right)^2 \\
&\leq C |I|^{2+p} \left[\sum_{k=1}^{N+1} \frac{1}{2^{k-N}(2+N-k)} \right]^2 \\
&\leq C |I|^{2+p} \left(\frac{2^N}{N} \right)^2 \\
&\leq \frac{C|I|^p}{\log^2 \frac{2}{|I|}}.
\end{aligned}$$

Here we have used an elementary inequality below

$$\frac{N}{N-k} \leq 2^{2+\frac{k}{2}}, k = 0, 1, 2, \ldots, N-1.$$

Summing, we just arrive at (3.8). Therefore, the proof is complete. □

Remark 3.4. (i) When $p \in (0,1)$, the difference between Q_p-setting and $BMOA$-setting is obvious. Concerning the corona theorem, Q_p is more flexible than $BMOA$. However, as to the multipliers, $\mathcal{M}(Q_p)$ is harder and more complicated than $\mathcal{M}(BMOA)$ (cf. Sections 1 and 2). Very recently, our principal result has been generalized to the Q_p spaces on strictly pseudo-convex domains in $\mathbb{C}^n$ [**AnCa**].

(ii) It is still open whether the corona theorem is valid for $\mathcal{M}(\mathcal{B}) = \mathcal{M}(Q_p)$, $p \in (1, \infty)$.

(iii) It was pointed out in [**Ni**] and [**NiXi**] that the corona theorem remains true for the algebra $Q_p \cap H^\infty, p \in [0,1)$.

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