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Peter Symonds

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When a finite group G acts faithfully on a graded integral domain S which is an algebra over a field k, such as a polynomial ring, we consider S as a kG-module. We show that S is asymptotically mostly projective in each degree, and also that it is in fact mostly free in an appropriate sense. Similar results also hold for filtered algebras, such as power series rings.

1. Introduction.

Let $S = \bigoplus_{n=0}^{\infty} S_n$ be a graded algebra over a field k. We suppose that S is finitely generated over k as a k-algebra and that the homogeneous components S_n are finite dimensional vector spaces over k. Let G be a finite group of grading preserving automorphisms of S (so G acts faithfully). We are concerned with the structure of S as a kG-module.

The classical theory of Hilbert and Serre asserts that for large n, $\dim_k S_n$ is given by a function

$$\phi_S(n) = c_{d-1}(n)n^{d-1} + c_{d-2}(n)n^{d-2} + \cdots + c_0(n),$$

where the $c_i(n)$ are rational valued functions periodic in n, i.e., $\phi_i(n + p) = \phi_i(n)$ for some integer p (see Section 2). If c_{d-1} is assumed not to be identically zero then d is equal to the dimension of the ring in various senses. If S is a polynomial ring then d is equal to the number of variables.

From now on, we assume that S is an integral domain. Let P_n denote the maximal projective summand of S_n (defined up to isomorphism).

Theorem 1.1. $\dim_k(S_n/P_n)$ *is bounded by a polynomial in* n *of degree* $d-2$.

Thus S_n is mostly projective, and if S is a polynomial ring then the non-projective part grows like a polynomial ring in one fewer variables.

In fact S is mostly free, although the individual S_n do not have to contain a free module at all; the different projectives can occur in different degrees. To explain this let $R = S^G$, the ring of invariants.

Theorem 1.2. *S contains a free kG-submodule F of rank 1, a sum of homogeneous pieces, such that the product map $R \otimes_k F \to S$ is injective. Denote its image by $RF = \bigoplus_n (RF)_n$. Then RF is a free summand of S and $\dim_k(S_n/(RF)_n)$ is bounded by a polynomial of degree $d-2$.*

Of course, the first theorem is a corollary to the second. Versions of these theorems were proven by Howe [**4**] in characteristic 0 and by Bryant [**2**, **3**] for polynomial rings.

Section 2 contains the main proof, except for some technical details which appear in Section 3. Section 4 proves similar results for filtered algebras.

2. Main Proof.

Proof. We can assume that k is a splitting field for G, since if a kG-module contains a free or projective summand after extension of scalars then it did so before. Let Q_S (resp. Q_R) denote the fields of fractions of S (resp. R), so $Q_S \cong Q_R \otimes_R S$. By the Normal Basis Theorem, Q_S is a free module of rank 1 over $Q_R G$; let e be a generator. Then, over kG, e generates a free submodule E of rank 1 such that $Q_S \cong Q_R \otimes_k E$. Now there is an $r \in R$ such that $re \in S$. Let F be the kG-module generated by re, so $F \subset S$ and $F = rE \cong kG$. Also the product map $R \otimes_k F \to RF \subset S$ is injective.

We claim that F can be assumed to be a sum of homogeneous pieces, $F = \bigoplus_i F_{n_i}$. The proof of this plausible statement is somewhat delicate, and we postpone it to the next section.

Let $x_1, \ldots, x_s$ be homogeneous generators for S as a k-algebra. Then $x_i = \sum_j \frac{a_{ij}}{b_{ij}} e_j$, where $a_{i,j}, b_{i,j} \in R$ and the e_j form a homogeneous k-basis for F. By writing $b_{i,j} x_i = \sum_j a_{i,j} e_j$ and taking the homogeneous component of this equation in some degree where $b_{i,j} x_i$ is non-zero, we see that we may assume that the $b_{i,j}$ are homogeneous. Let $\alpha \in R_a$ be the product of all the $b_{i,j}$. Then each $x_i \in \alpha^{-1} RF$, so $S \subset \alpha^{-1} RF$.

Thus

$$(RF)_n \subset S_n \subset \alpha^{-1}(RF)_{n+a},$$

and so, identifying RF with $R \otimes_k F$, we have

$$\bigoplus_i R_{n-n_i} \otimes F_{n_i} \subset S_n \subset \alpha^{-1} \bigoplus_i R_{n+a-n_i} \otimes F_{n_i}.$$

In particular, the dimension of $S_n/(RF)_n$ is bounded by the difference in the dimensions of the two sides, i.e., by

$$\sum_i (\phi_R(n+a-n_i) - \phi_R(n-n_i)) \dim_k F_{n_i}.$$

But

$$\phi_R(n+a-n_i) - \phi_R(n-n_i) = c_{d-1}(n+a-n_i)(n+a-n_i)^{d-1} - c_{d-1}(n-n_i)(n-n_i)^{d-1} + \text{lower degree terms},$$

and c_{d-1} is periodic, with period dividing a (see 3.1), so the n^{d-1} term cancels and we are done. □

3. Technical Details.

The form of $\phi_S(n)$ given above is not quite the standard one, although it is quoted in [**4**]. The usual references deal with a module over a polynomial ring which has all the variables in degree 1, and then all the coefficients of ϕ are constants. To deduce the version given in the introduction, note that if S is generated by $x_1, \dots, x_s$ then it is a finitely generated module over $k[x_1, \dots, x_s]$. By taking suitable powers y_i of the x_i we can get all the y_i in the same degree b, and S will still be finitely generated over $k[y_1, \dots, y_s]$. For $0 \le j \le b-1$, set $T_j = \bigoplus_{l=0}^{\infty} S_{j+lb}$. Then $R \cong \bigoplus_j T_j$ as a $k[y_1, \dots, y_s]$-module, and after regrading each T_j so that each y_i can have degree 1, we can apply the usual theory ([**1**] 11.2, [**5**] VII Theorem 41) to each T_j and sum the results. It is the summation that leads to the periodic coefficients.

Lemma 3.1 ([**4**]). *If R is an integral domain* (*as it always is for us*), $c = \gcd\{r \in \mathbb{Z} | R_r \neq 0\}$ *and* $\phi_R(n) = c_{d-1}(n)n^{d-1} + \cdots + c_0(n)$, *then there is a constant b such that*

$$c_{d-1}(n) = \begin{cases} b, & \text{if } n|c, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $0 \neq \alpha \in R_a$, then multiplication by α embeds R_n in R_{n+a}, so for large n, $\phi_R(n) \le \phi_R(n+a)$. Now consider the limit of $\phi_R(n)/n^{d-1}$ as $n \to \infty$ through elements of the same residue class modulo the period of c_{d-1} to see that $c_{d-1}(n) \le c_{d-1}(n+a)$. This, together with the periodicity, implies the result. □

Now we prove the claim made in Section 2.

Proposition 3.2. *The free module $F \subset S$ can be assumed to be a sum of homogeneous pieces in such a way that the product map $Q_R \otimes_k F \to Q_S$ is still an isomorphism.*

Proof. For each simple kG-module V, let $T_V = \operatorname{Hom}_{kG}(V, S)$, a graded R-module. Now $\operatorname{soc} F$ is a direct sum of simples. Let $\operatorname{soc}_V(F)$ denote the sum of those isomorphic to V, so $\operatorname{soc}_V(F) = V^1 \oplus \cdots \oplus V^s$, where $V^i \cong V$, and let P_{V^i} be a projective summand of F with $\operatorname{soc}(P_{V^i}) = V^i$. The inclusions of the V^i in S give us s homomorphisms $f^i \in T_V$, which are linearly independent over R.

Lemma 3.3. *Let $f^1, \dots, f^s$ be elements of a graded R-module T which are linearly independent over R. Write each f^j as a sum of its homogeneous components; $f^j = \sum_k f^j_k$, $f^j_k \in T_k$. Then for each j there is an integer k_j such that $f^1_{k_1}, \dots, f^s_{k_s}$ are linearly independent over R.*

Proof. For each $0 \leq t \leq s$, let P_t be the claim that there exist integers $k_1, \dots, k_t$ such that $f^1_{k_1}, \dots, f^t_{k_t}, f^{t+1}, \dots, f^s$ are linearly independent over R. P_0 is true by hypothesis and we want P_s. We give a proof by induction on t, so assume P_t.

If P_{t+1} is false, then for each $k \in \mathbb{Z}$ we can find $u_k, r^i_k \in R$, $u_k \neq 0$, such that

$$u_k f^{t+1}_k = r^1_k f^1_{k_1} + \cdots + r^t_k f^t_{k_t} + r^{t+2}_k f^{t+2} + \cdots + r^s_k f^s.$$

Let u be the product of the u_k for which $f^{t+1}_k \neq 0$. Then $u f^{t+1} = \sum_k (\frac{u}{u_k}) u_k f^{t+1}_k$, contradicting P_t. □

Applying this to the $\{f^i\} \subset T_V$ we obtain homogeneous $\{\bar{f}^i\} \subset T_V$, $\bar{f}^i \in T_{a_i}$, say, linearly independent over R.

Lemma 3.4. *The evaluation map* $\mathrm{ev} : T_V \otimes_k V \to S$ *is injective.*

Proof. In fact $\mathrm{ev} : \mathrm{Hom}_{kG}(V, M) \otimes_k V \to M$ is injective for any kG-module M. This is because it factors through $\mathrm{soc}_V(M)$, which is a direct sum of V's, so we are reduced to proving the case $M = V$. But then ev is an isomorphism, since $\mathrm{Hom}_{kG}(V, V) \cong k$, by the assumption that k is a splitting field. □

Corollary. *The product map $R \otimes_k (\bigoplus_i \bar{f}^i(V)) \to S$ is injective.*

Now let $\bar{P}_{V^i}$ be the image of the projection of P_{V^i} to S_{a_i}. The projection map is injective on $\mathrm{soc}(P^i_V)$, by the construction of a_i, so $\bar{P}_{V^i} \cong P_{V^i}$ and $\mathrm{soc}(\bar{P}_{V^i}) = \bar{f}^i(V)$. Let $\bar{P}_V = \bigoplus_i \bar{P}_{V^i}$ and consider the product map $R \otimes_k \bar{P}_V \to S$. Since $\mathrm{soc}(R \otimes_k \bar{P}_V) = R \otimes \bigoplus_i \bar{f}^i(V)$, it is injective on the socle, so is injective.

Finally, we sum the $\bar{P}_V$ over the simples V to obtain $\bar{F}$, a free kG-module of rank 1, which is a sum of homogeneous pieces, as required. □

Remark. If G is a p-group, where p is the characteristic of k, then the proof is much simpler because $\mathrm{soc}(F) \cong k$. Under at least one of the projections of F onto its homogeneous components the image of $\mathrm{soc}(F)$ must be non-zero. Let $\bar{F}$ be the image of F under this projection. Then $\bar{F} \cong kG$ and $Q_R \otimes_k F \to Q_S$ is an isomorphism because it is injective on the socle, and both sides have the same dimension over Q_R.

This is enough to prove 1.1 for general G. For if $P = \mathrm{Syl}_p(G)$ then S is a direct summand of $\mathrm{Ind}^G_P \mathrm{Res}^G_P S$.

Remark. It is not hard to see that, given any degree m, the summands of $\bar{F}$ can be moved by multiplication by a scalar to lie in $T_{m+lc} = S_{m+lc} \oplus \cdots \oplus S_{m+(l+1)c-1}$, for some l. The argument of the proof of 1.2 now shows that the non-free part of T_n has dimension bounded by a polynomial of degree $d-2$ (cf. [**2**, **3**]).

4. Filtered Rings.

The case of filtered rings is slightly different. Consider the power series ring $k[[x]]$ in characteristic 2 and let the group of order 2 act by $x \mapsto x/(x+1) = x + x^2 + x^3 + \cdots$. The action on the associated graded ring is trivial, yet the action on $k[[x]]$ certainly contains free summands (the only alternative is trivial).

We consider finitely generated k-algebras S which are integral domains and have a filtration $S = I_0 \subset I_1 \subset I_2 \subset \cdots$. Each S/I_n is assumed to be finite dimensional over k, and $\cap I_n = \{0\}$. There is a finite group G of automorphisms of S, which preserves the filtration. The invariants are $R = S^G$ with the induced filtration $J_n = R \cap I_n$. Again there is a function

$$\chi_S(n) = c_d(n)n^d + c_{d-1}n^{d-1} + \cdots c_0(n),$$

where the c_i are periodic, such that $\dim_k(S/I_n) = \chi_S(n)$ for large n. If S is a power series ring, then d is equal to the number of variables.

As before there is a free kG-module of rank 1 in S, and the product map $R \otimes_k F \to S$ is injective. Since F is finite dimensional there is some integer f such that $F \cap I_f = 0$, so F injects into S/I_f.

For each n, let K_n be a vector space complement to J_n in R. Then the product map $K_n \otimes F \to S/I_{f+n}$ is injective, so its image, K_nF is a free summand of S/I_{f+n}.

Proceeding in the same way as before we can prove:

Theorem 4.1. $\dim_k((S/I_{f+n})/K_nF)$ *is bounded by a polynomial of degree* $d-1$.

So S is mostly free. Again, for a power series ring, the non-free part grows like a power series ring in one fewer variables.

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Department of Mathematics
U.M.I.S.T.
Manchester M60 1QD
England
E-mail address: psymonds@umist.ac.uk