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For a compact Riemann surface X of genus $g > 1$, $\mathrm{Hom}(\pi_1(X), \mathrm{PU}(p,q))/\mathrm{PU}(p,q)$ is the moduli space of flat $\mathrm{PU}(p,q)$-connections on X. There are two integer invariants, d_P, d_Q, associated with each $\sigma \in \mathrm{Hom}(\pi_1(X), \mathrm{PU}(p,q))/\mathrm{PU}(p,q)$. These invariants are related to the Toledo invariant τ by $\tau = 2\frac{qd_P - pd_Q}{p+q}$. This paper shows, via the theory of Higgs bundles, that if $q = 1$, then $-2(g-1) \le \tau \le 2(g-1)$. Moreover, $\mathrm{Hom}(\pi_1(X), \mathrm{PU}(2,1))/\mathrm{PU}(2,1)$ has one connected component corresponding to each $\tau \in \frac{2}{3}\mathbb{Z}$ with $-2(g-1) \le \tau \le 2(g-1)$. Therefore the total number of connected components is $6(g-1)+1$.

1. Introduction.

Let X be a smooth projective curve over $\mathbb{C}$ with genus $g > 1$. The deformation space

$$\mathbb{C}\mathcal{N}_B = \mathrm{Hom}^+(\pi_1(X), \mathrm{PGL}(n,\mathbb{C}))/\,\mathrm{PGL}(n,\mathbb{C})$$

is the space of equivalence classes of semi-simple $\mathrm{PGL}(n,\mathbb{C})$-representations of the fundamental group $\pi_1(X)$. This is the $\mathrm{PGL}(n,\mathbb{C})$-Betti moduli space on X [**22**, **23**, **24**]. A theorem of Corlette, Donaldson, Hitchin and Simpson relates $\mathbb{C}\mathcal{N}_B$ to two other moduli spaces, $\mathbb{C}\mathcal{N}_{DR}$ and $\mathbb{C}\mathcal{N}_{\mathrm{Dol}}$—the $\mathrm{PGL}(n,\mathbb{C})$-de Rham and the $\mathrm{PGL}(n,\mathbb{C})$-Dolbeault moduli spaces, respectively [**3**, **5**, **11**, **21**]. The Dolbeault moduli space consists of holomorphic objects (Higgs bundles) over X; therefore, the classical results of analytic and algebraic geometry can be applied to the study of the Dolbeault moduli space.

Since $\mathrm{PU}(p,q) \subset \mathrm{PGL}(n,\mathbb{C})$, $\mathbb{C}\mathcal{N}_B$ contains the space

$$\mathcal{N}_B = \mathrm{Hom}^+(\pi_1(X), \mathrm{PU}(p,q))/\,\mathrm{PU}(p,q).$$

The space $\mathcal{N}_B$ will be referred to as the $\mathrm{PU}(p,q)$-Betti moduli space which similarly corresponds to some subspaces $\mathcal{N}_{DR}$ and $\mathcal{N}_{\mathrm{Dol}}$ of $\mathbb{C}\mathcal{N}_{DR}$ and $\mathbb{C}\mathcal{N}_{\mathrm{Dol}}$, respectively. We shall refer to $\mathcal{N}_{DR}$ and $\mathcal{N}_{\mathrm{Dol}}$ as the $\mathrm{PU}(p,q)$-de Rham and the $\mathrm{PU}(p,q)$-Dolbeault moduli spaces.

The Betti moduli spaces are of great interest in the field of geometric topology and uniformization. In the case of $p = q = 1$, Goldman analyzed

$\mathcal{N}_B$ and determined the number of its connected components to be $4g-3$ [6]. Hitchin subsequently considered $\mathcal{N}_{\mathrm{Dol}}$ in the case of $p=q=1$ and determined its topology [11].

In this paper, we analyze $\mathcal{N}_{\mathrm{Dol}}$ for the case of $p=2, q=1$ and determine its number of connected components. In addition, we produce a new algebraic proof, via the Higgs-bundle theory, of a theorem by Toledo on the bounds of the Toledo invariant [26, 27].

An element $\sigma \in \mathrm{Hom}^+(\pi_1(X), \mathrm{PU}(p,q))$ defines a flat principal $\mathrm{PU}(p,q)$-bundle P over X. Such a flat bundle may be lifted to a principal $\mathrm{U}(p,q)$-bundle $\hat{P}$ with a Yang-Mills connection D [2, 3, 5, 11, 21]. Let E be the rank-$(p+q)$ vector bundle associated with $(\hat{P}, D)$. The second cohomology $\mathrm{H}^2(X,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, so one may identify the Chern class $c_1(E) \in \mathrm{H}^2(X,\mathbb{Z})$ with an integer, the degree of E. Suppose we impose the additional condition

$$0 \leq \deg(E) < n.$$

Then the above construction gives rise to a unique obstruction class $o_2(E) \in \mathrm{H}^2(X, \pi_1(\mathrm{U}(p,q)))$ [25]. The obstruction class is invariant under the conjugation action of $\mathrm{PU}(p,q)$; therefore, one obtains the obstruction map:

$$o_2 : \mathrm{Hom}^+(\pi_1(X), \mathrm{PU}(p,q))/\,\mathrm{PU}(p,q) \longrightarrow \mathrm{H}^2(X, \pi_1(\mathrm{U}(p,q))) \cong \mathbb{Z}\times\mathbb{Z}.$$

The maximum compact subgroup of $\mathrm{U}(p,q)$ is $\mathrm{U}(p)\times\mathrm{U}(q)$. Hence topologically E is a direct sum $E_P \oplus E_Q$ with

$$\deg(E) = \deg(E_P) + \deg(E_Q).$$

The obstruction class $o_2(E)$ is then $(\deg(E_P), \deg(E_Q)) \in \mathbb{Z}\times\mathbb{Z}$. Associated with σ is the Toledo invariant τ which relates to $d_P = \deg(E_P)$ and $d_Q = \deg(E_Q)$ by the formula [7, 26, 27]

$$\tau = 2\frac{\deg(E_P \otimes E_Q^*)}{p+q} = 2\frac{q d_P - p d_Q}{p+q}.$$

This explains why the Toledo invariant of a $\mathrm{PU}(2,1)$ representation cannot be an odd integer [7]. The main result presented here is the following:

Theorem 1.1. $\mathrm{Hom}^+(\pi_1(X), \mathrm{PU}(2,1))/\,\mathrm{PU}(2,1)$ *has one connected component for each* $\tau \in \frac{2}{3}\mathbb{Z}$ *with* $-2(g-1) \leq \tau \leq 2(g-1)$. *Therefore the total number of connected components is* $6(g-1)+1$.

We shall also provide a new proof en route to the following theorem:

Theorem 1.2 (Toledo). *Suppose* $\sigma \in \mathrm{Hom}^+(\pi_1(X), \mathrm{PU}(p,1))$ *and* τ *is the Toledo class of* σ. *Then*

$$-2(g-1) \leq \tau \leq 2(g-1).$$

Moreover $\tau = \pm 2(g-1)$ *implies* σ *is reducible.*

These results are related to the results of Domic and Toledo [4, 26, 27] and, as being pointed out to the author recently, are also related to the work of Gothen [8] which computed the Poincaré polynomials for the components of $\mathrm{Hom}(\pi_1(X), \mathrm{PSL}(3,\mathbb{C}))/\,\mathrm{PSL}(3,\mathbb{C})$, where $\deg(E)$ is coprime to 3.

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2. Backgrounds and Preliminaries.

In this section, we briefly outline the constructions of the Betti, de Rham and Dolbeault moduli spaces. For details, see [2, 3, 5, 11, 12, 18, 21, 22, 23, 24].

2.1. The Betti Moduli Space. The fundamental group $\pi_1(X)$ is generated by $S = \{A_i, B_i\}_{i=1}^{g}$, subject to the relation

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = e.$$

Denote by I and [I] the identities of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{PGL}(n,\mathbb{C})$, respectively. Define

$$R : \mathrm{PGL}(n,\mathbb{C})^{2g} \longrightarrow \mathrm{PGL}(n,\mathbb{C})$$
$$\mathcal{R} : \mathrm{GL}(n,\mathbb{C})^{2g} \longrightarrow \mathrm{GL}(n,\mathbb{C})$$

to be the commutator maps:

$$(X_1, Y_1, \ldots, X_g, Y_g) \xrightarrow{R,\mathcal{R}} \prod_{i=1}^{g} X_i Y_i X_i^{-1} Y_i^{-1}.$$

The group

$$\{\zeta \mathrm{I} : \zeta \in \mathbb{C}, \zeta^n = 1\}$$

is isomorphic to $\mathbb{Z}_n$. The space $\mathcal{R}^{-1}(\mathbb{Z}_n)$ is identified with the representation space $\mathrm{Hom}(\Gamma, \mathrm{GL}(n,\mathbb{C}))$, where Γ is the central extension [2, 11]:

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow 0.$$

Each element $\rho \in \mathcal{R}^{-1}(\mathbb{Z}_n)$ acts on $\mathbb{C}^n$ via the standard representation of $\mathrm{GL}(n,\mathbb{C})$. The representation ρ is called reducible (irreducible) if its action on $\mathbb{C}^n$ is reducible (irreducible). A representation ρ is called semi-simple if it is a direct sum of irreducible representations. Let $\zeta_1 = e^{2\pi \mathrm{i}/n}$ and define

$$\mathbb{C}\mathcal{M}_B(c) = \{\sigma \in \mathcal{R}^{-1}(\zeta_1^c \mathrm{I}) : \sigma \text{ is semi-simple}\}/\,\mathrm{GL}(n,\mathbb{C}),$$

$$\mathbb{C}\mathcal{M}_B = \bigcup_{c=0}^{n-1} \mathbb{C}\mathcal{M}_B(c),$$

$$\begin{aligned}\mathbb{C}\mathcal{N}_B(c) &= \mathbb{C}\mathcal{M}_B(c)/\operatorname{Hom}(\pi_1(X), \mathbb{C}^*)\\ &= \operatorname{Hom}^+(\pi_1(X), \mathrm{PGL}(n,\mathbb{C}))/\mathrm{PGL}(n,\mathbb{C}).\end{aligned}$$

Fix p, q such that $p+q=n$. Denote by $\mathcal{R}_U$ the restriction of $\mathcal{R}$ to the subgroup $\mathrm{U}(p,q)^{2g}$. Define

$$\mathcal{M}_B(c) = \{\sigma \in \mathcal{R}_U^{-1}(\zeta_1^c \mathrm{I}) : \sigma \text{ is semi-simple}\}/\mathrm{U}(p,q),$$

$$\mathcal{M}_B = \bigcup_{c=0}^{n-1} \mathcal{M}_B(c).$$

Note the center of $\mathrm{U}(p,q)$ is $\mathrm{U}(1)$ and is contained in the center of $\mathrm{GL}(n,\mathbb{C})$. It follows that $\mathcal{M}_B(c) \subset \mathbb{C}\mathcal{M}_B(c)$. Define

$$\mathcal{N}_B(c) = \mathcal{M}_B(c)/\operatorname{Hom}(\pi_1(X), \mathrm{U}(1))$$

$$\mathcal{N}_B = \mathcal{M}_B/\operatorname{Hom}(\pi_1(X), \mathrm{U}(1)) = \operatorname{Hom}^+(\pi_1(X), \mathrm{U}(p,q))/\mathrm{U}(p,q).$$

All the spaces constructed here that contain the symbols $\mathcal{M}_B$ or $\mathcal{N}_B$ will be loosely referred to as Betti moduli spaces. The subspace of irreducible elements of a Betti moduli space will be denoted by an s superscript. For example, $\mathbb{C}\mathcal{M}_B^s$ denotes the subspace of irreducible elements of $\mathbb{C}\mathcal{M}_B$.

2.2. The de Rham Moduli Space. Suppose P is a principal $\mathrm{GL}(n,\mathbb{C})$-bundle on X, E its associated vector bundle of rank n and $\mathcal{G}_{\mathbb{C}}(E)$ the group of $\mathrm{GL}(n,\mathbb{C})$-gauge transformations on E. A connection is called Yang-Mills (or central) if its curvature is central [2]. The gauge group $\mathcal{G}_{\mathbb{C}}(E)$ acts on the space of $\mathrm{GL}(n,\mathbb{C})$-connections on E and preserves the subspace of Yang-Mills connections. Fix E with $\deg(E) = c$. The de Rham moduli space $\mathbb{C}\mathcal{M}_{DR}(c)$ on E is defined to be the $\mathcal{G}_{\mathbb{C}}(E)$-equivalence classes of Yang-Mills connections.

Let $\mathcal{M}_{DR}(c)$ denote the space of $\mathrm{U}(p,q)$-gauge equivalence classes of $\mathrm{U}(p,q)$-central connections on E. In other words, $\mathcal{M}_{DR}(c)$ is constructed as $\mathbb{C}\mathcal{M}_{DR}(c)$, but with $\mathrm{U}(p,q)$ replacing $\mathrm{GL}(n,\mathbb{C})$. Since the center of $\mathrm{U}(p,q)$ is contained in the center of $\mathrm{GL}(n,\mathbb{C})$, $\mathcal{M}_{DR}(c) \subset \mathbb{C}\mathcal{M}_{DR}(c)$.

The space of $\mathbb{C}^*$-gauge equivalence classes of $\mathbb{C}^*$-connections on X is $\mathrm{H}^1(X, \mathbb{C}^*)$ which acts on $\mathbb{C}\mathcal{M}_{DR}(c)$ [2]. Denote the quotient $\mathbb{C}\mathcal{N}_{DR}(c)$. This action corresponds to the action of $\operatorname{Hom}(\pi_1(X), \mathbb{C}^*)$ on $\mathbb{C}\mathcal{M}_B(c)$ and the quotient $\mathbb{C}\mathcal{N}_{DR}(c)$ corresponds to $\mathbb{C}\mathcal{N}_B(c)$. Similarly, the space of $\mathrm{U}(1)$-gauge equivalence classes of $\mathrm{U}(1)$-connections on X is $\mathrm{H}^1(X, \mathrm{U}(1))$ which acts on $\mathcal{M}_{DR}(c)$ and the quotient is denoted by $\mathcal{N}_{DR}(c)$. Define

$$\mathbb{C}\mathcal{M}_{DR} = \bigcup_{c=-\infty}^{\infty} \mathbb{C}\mathcal{M}_{DR}(c), \quad \mathbb{C}\mathcal{N}_{DR} = \bigcup_{c=-\infty}^{\infty} \mathbb{C}\mathcal{N}_{DR}(c)$$

$$\mathcal{M}_{DR} = \bigcup_{c=-\infty}^{\infty} \mathcal{M}_{DR}(c), \quad \mathcal{N}_{DR} = \bigcup_{c=-\infty}^{\infty} \mathcal{N}_{DR}(c).$$

All the spaces constructed here that contain the symbols $\mathcal{M}_{DR}$ or $\mathcal{N}_{DR}$ will be loosely referred to as de Rham moduli spaces. A central connection is irreducible if $(E, D) = (E_1 \oplus E_2, D_1 \oplus D_2)$ implies $\operatorname{rank}(E_1) = 0$ or $\operatorname{rank}(E_2) = 0$. The subspace of irreducible elements of a de Rham moduli space will be denoted by an s superscript.

Theorem 2.1. *The moduli space $\mathbb{C}\mathcal{M}_B(c)$ is homeomorphic to $\mathbb{C}\mathcal{M}_{DR}(c)$.*

Proof. See [**3**, **5**, **11**]. □

Consider all the objects we have defined so far with subscripts B or DR. With Theorem 2.1, one can verify the following: Suppose two objects have subscripts B or DR. Then the two objects are homeomorphic if they only differ in subscripts. For example, $\mathcal{N}_B(c)$ is homeomorphic to $\mathcal{N}_{DR}(c)$.

Since the maximum compact subgroup of $\mathrm{U}(p,q)$ is $\mathrm{U}(p) \times \mathrm{U}(q)$, $(E, D) \in \mathcal{M}_{DR}$ implies E is a direct sum of a $\mathrm{U}(p)$ and a $\mathrm{U}(q)$-bundle:

$$E = E_p \oplus E_q,$$

where the ranks of E_p and E_q are p and q, respectively. Therefore, associated to each (E, D) are the invariants

$$d_P = \deg(E_P) \text{ and } d_Q = \deg(E_Q),$$

with

$$d_P + d_Q = \deg(E) = c.$$

The Toledo invariant τ is [**7**, **26**, **27**]

$$\tau = 2\frac{\deg(E_P \otimes E_Q^*)}{n} = 2\frac{q d_P - p d_Q}{n}.$$

The subspace of $\mathcal{M}_{DR}(c)$ with a fixed Toledo invariant τ is denoted by $\mathcal{M}_{DR}^{\tau}$. By the equivalence of Betti and de Rham moduli spaces, one may define the Toledo invariant on $\mathcal{M}_B(c)$. Denote by $\mathcal{M}_B^{\tau}$ the subspace of $\mathcal{M}_B(c)$ with a fixed Toledo invariant τ. The $\mathrm{H}^1(X, \mathrm{U}(1))$ action on $\mathcal{M}_{DR}(c)$ preserves $\mathcal{M}_{DR}^{\tau}$ and the quotient is denoted by $\mathcal{N}_{DR}^{\tau}$. In the Betti moduli space, the $\mathrm{Hom}(\pi_1(X), \mathrm{U}(1))$ action on $\mathcal{M}_B$ preserves $\mathcal{M}_B^{\tau}$, and the quotient is denoted by $\mathcal{N}_B^{\tau}$.

2.3. The Dolbeault Moduli Space. Let E be a rank n complex vector bundle over X with $\deg(E) = c$. Denote by Ω the canonical bundle on X. A holomorphic structure $\overline{\partial}$ on E induces holomorphic structures on the bundles $\mathrm{End}(E)$ and $\mathrm{End}(E) \otimes \Omega$. A Higgs bundle is a pair $(E_{\overline{\partial}}, \Phi)$, where $\overline{\partial}$ is a holomorphic structure on E and $\Phi \in \mathrm{H}^0(X, \mathrm{End}(E_{\overline{\partial}}) \otimes \Omega)$. Such a Φ is called a Higgs field. We denote the holomorphic bundle $E_{\overline{\partial}}$ by V.

Define the slope of a Higgs bundle (V, Φ) to be

$$s(V) = \deg(V)/\operatorname{rank}(V).$$

For a fixed Φ, a holomorphic subbundle $W \subset V$ is said to be Φ-invariant if $\Phi(W) \subset W \otimes \Omega$. A pair (V, Φ) is stable (semi-stable) if $W \subset V$ is Φ-invariant implies

$$s(W) < (\leq) s(V).$$

A Higgs bundle is called poly-stable if it is a direct sum of stable Higgs bundles of the same slope [**11**, **22**].

The gauge group $\mathcal{G}_{\mathbb{C}}(E)$ acts on holomorphic structures by pull-back and on Higgs fields by conjugation. Moreover the $\mathcal{G}_{\mathbb{C}}(E)$ action preserves stability, poly-stability and semi-stability. The Dolbeault moduli space $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$ on E (with $\deg(E) = c$), is the $\mathcal{G}_{\mathbb{C}}(E)$-equivalence classes of poly-stable (or S-equivalence classes of semi-stable [**18**]) Higgs bundles (V, Φ) on X [**11**, **12**, **18**, **22**]. A Higgs bundle is called reducible if it is poly-stable but not stable. Let

$$\mathbb{C}\mathcal{M}_{\mathrm{Dol}} = \bigcup_{c=-\infty}^{\infty} \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c).$$

If $D \in \mathbb{C}\mathcal{M}_{DR}(c)$, then for any Hermitian metric h on E, there is a decomposition,

$$D = D_A + \Psi,$$

where D_A is compatible with h and Ψ is a 1-form with coefficients in $\mathfrak{p}$. The $(0, 1)$ part of D_A determines a holomorphic structure $\overline{\partial}_A$ on E while the $(1, 0)$ part of Ψ is a section of the bundle $End(E) \otimes \Omega$. There exists a metric h such that the pair

$$(V, \Phi) = (E_{\overline{\partial}_A}, \Psi^{1,0})$$

so constructed is a poly-stable Higgs bundle [**11**, **21**, **22**]. Therefore this construction gives a map

$$f : \mathbb{C}\mathcal{M}_{DR}(c) \longrightarrow \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c).$$

Theorem 2.2 (Corlette, Donaldson, Hitchin, Simpson). *The map f is a homeomorphism.*

Proof. See [**3**, **5**, **11**, **21**]. □

3. The $\mathrm{U}(p, q)$-Yang-Mills Connections.

Assume $p \geq q$ and $p + q = n$. From the previous section, we know that $\mathcal{M}_{DR} \subset \mathbb{C}\mathcal{M}_{DR}$. Let $D \in \mathbb{C}\mathcal{M}_{DR}(c)$ be a $\mathrm{GL}(n, \mathbb{C})$-Yang-Mills connection on a rank n vector bundle

$$E \longrightarrow X.$$

Proposition 3.1. *D is a $\mathrm{U}(p,q)$-Yang-Mills connection if and only if its corresponding Higgs bundle $(V,\Phi) \in \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$ satisfies the following two conditions:*

1) *V is decomposable into a direct sum:*

$$V = V_P \oplus V_Q,$$

where V_P, V_Q are of rank p, q, respectively.

2) *The Higgs field decomposes into two maps:*

$$\Phi_1 : V_P \longrightarrow V_Q \otimes \Omega,$$

$$\Phi_2 : V_Q \longrightarrow V_P \otimes \Omega.$$

Proof. Suppose D is a $\mathrm{U}(p,q)$-Yang-Mills connection. Denote by h the Hermitian-Yang-Mills metric on (E,D). Then D decomposes as

$$D = D_A + \Psi,$$

where D_A is the part compatible with h. The Cartan decomposition ($\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$) for $\mathfrak{u}(p,q)$ is

$$\mathfrak{u}(p,q) = (\mathfrak{u}(p) \oplus \mathfrak{u}(q)) \oplus \mathfrak{p}.$$

If we take the standard representation of $\mathfrak{u}(p,q)$, then elements in $\mathfrak{k}$ are of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

where $a \in \mathfrak{u}(p), b \in \mathfrak{u}(q)$, respectively. The elements in $\mathfrak{p}$ are then of the form

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

where $b \in \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p), c \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$, respectively. Hence on local charts, D_A and Ψ have coefficients in $\mathfrak{k}$ and $\mathfrak{p}$, respectively. In particular, the connection D_A is reducible.

The Higgs bundle corresponding to D is $(E_{\overline{\partial}_A}, \Phi)$ where $\overline{\partial}_A$ is the $(0,1)$-part of D_A and Φ, the $(1,0)$-part of Ψ, is considered as a holomorphic bundle map:

$$\Phi : V \longrightarrow V \otimes \Omega.$$

Since D_A has coefficient in $\mathfrak{k}$, the holomorphic structure on V defined by $D_A^{0,1}$ is a direct sum:

$$V = V_P \oplus V_Q.$$

Since Ψ is block off-diagonal, Φ is also block off-diagonal implying Φ can be decomposed into two maps:

$$\Phi_1 : V_P \longrightarrow V_Q \otimes \Omega,$$

$$\Phi_2 : V_Q \longrightarrow V_P \otimes \Omega.$$

This proves the only if part of the proposition.

Suppose (V, Φ) is a Higgs bundle that satisfies the two conditions of Proposition 3.1. Let α be the constant gauge

$$\alpha = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix},$$

where $\mathrm{I}_p, \mathrm{I}_q$ are $p \times p, q \times q$ identity matrices, respectively. Then α acts on the space of holomorphic structures on E and fixes V. Moreover,

$$\alpha \Phi \alpha^{-1} = -\Phi$$

since Φ is of the form

$$\Phi = \begin{pmatrix} 0 & \Phi_1 \\ \Phi_2 & 0 \end{pmatrix}.$$

Hence by a theorem of Simpson, the corresponding Hermitian-Yang-Mills metric h is invariant under the action of α [**21**]. In other words, on local charts, h is a Hermitian matrix of the form

$$h = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

where a, d are Hermitian matrices of dimension $p \times p, q \times q$, respectively. Hence the corresponding Yang-Mills connection is

$$D = D_A + \Phi + \Phi^{\ddagger},$$

where $\Phi^{\ddagger}$ is the adjoint of Φ with respect to h. In local coordinates, D_A has coefficient of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

and $\Phi + \Phi^{\ddagger}$ is of the form

$$\begin{pmatrix} 0 & b \\ b^{\ddagger} & 0 \end{pmatrix}.$$

Hence D_A and $\Phi + \Phi^{\ddagger}$ have coefficients in $\mathfrak{u}(p) \oplus \mathfrak{u}(q)$ and $\mathfrak{p}$, respectively. This implies D is a $\mathrm{U}(p,q)$-Yang-Mills connection. □

Denote by $\mathcal{M}_{\mathrm{Dol}}(c)$ the subspace of $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$ satisfying the hypothesis of Proposition 3.1. Then $\mathcal{M}_{\mathrm{Dol}}(c)$ is homeomorphic to $\mathcal{M}_{DR}(c)$.

The invariants d_P, d_Q and τ on (E, D) translate to invariants on the corresponding $\mathrm{U}(p,q)$-Higgs bundles $(V_P \oplus V_Q, \Phi)$:

$$d_P = \deg(V_P), \quad d_Q = \deg(V_Q), \quad \tau = 2\frac{q d_P - p d_Q}{n}.$$

The subspace of $\mathcal{M}_{\mathrm{Dol}}(c)$ with a fixed Toledo invariant τ is denoted by $\mathcal{M}^{\tau}_{\mathrm{Dol}}$.

4. Group Actions and Kähler Structures on $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$.

4.1. The Action of line bundles. The space of holomorphic line bundles, $\mathrm{H}^1(X, \mathcal{O}^*)$, acts freely on $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ as follows:

$$\mathrm{H}^1(X, \mathcal{O}^*) \times \mathbb{C}\mathcal{M}_{\mathrm{Dol}} \longmapsto \mathbb{C}\mathcal{M}_{\mathrm{Dol}},$$
$$(L, (V, \Phi)) \longmapsto (V \otimes L, \Phi \otimes 1),$$

where 1 is the identity map on L. An immediate consequence is:

Proposition 4.1. *If $c_1 \equiv c_2 \bmod n$, then $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c_1)$ is homeomorphic to $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c_2)$.*

4.2. The Action of $\mathrm{H}^0(X, \Omega)$. The vector space $\mathrm{H}^0(X, \Omega)$ acts freely on $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ as follows:

$$\mathrm{H}^0(X, \Omega) \times \mathbb{C}\mathcal{M}_{\mathrm{Dol}} \longmapsto \mathbb{C}\mathcal{M}_{\mathrm{Dol}},$$
$$(\phi, (V, \Phi)) \longmapsto (V, \Phi + \phi \mathrm{I}).$$

The actions of $\mathrm{H}^1(X, \mathcal{O}^*)$ and $\mathrm{H}^0(X, \Omega)$ commute and the quotient is defined to be

$$\mathbb{C}\mathcal{N}_{\mathrm{Dol}} = \mathbb{C}\mathcal{M}_{\mathrm{Dol}}/(\mathrm{H}^1(X, \mathcal{O}^*) \times \mathrm{H}^0(X, \Omega)).$$

The $\mathrm{H}^1(X, \mathcal{O}^*)$ action preserves the subspaces $\mathcal{M}_{\mathrm{Dol}}(c)$ and $\mathcal{M}^{\tau}_{\mathrm{Dol}}$. The quotients are defined to be

$$\mathcal{N}_{\mathrm{Dol}}(c) = \mathcal{M}_{\mathrm{Dol}}(c)/\mathrm{H}^1(X, \mathcal{O}^*),$$
$$\mathcal{N}^{\tau}_{\mathrm{Dol}} = \mathcal{M}^{\tau}_{\mathrm{Dol}}/\mathrm{H}^1(X, \mathcal{O}^*).$$

All the spaces constructed so far that contain the symbols $\mathcal{M}_{\mathrm{Dol}}$ or $\mathcal{N}_{\mathrm{Dol}}$ will be loosely referred to as the Dolbeault moduli spaces. The subspace of stable Higgs bundles of a Dolbeault moduli space will be denoted by an s superscript. For example, $\mathbb{C}\mathcal{M}^{s}_{\mathrm{Dol}}$ will denote the subspace of irreducible elements of $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$.

Remark 1. The Betti, de Rham and Dolbeault moduli spaces $\mathbb{C}\mathcal{M}_B$, $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ and $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ constructed here are variations of those of Simpson's [**22**, **23**, **24**].

With Theorems 2.1 and 2.2, one can obtain the following equivalence relations between the various Betti, de Rham and Dolbeault moduli spaces.

Corollary 4.2. *Suppose $\mathcal{M}^{\tau}_{DR} \subset \mathcal{M}_{DR}(c)$. Then one obtains the following commutative diagram:*

$$\begin{array}{ccccc}
\mathcal{M}^{\tau}_{B} & \longrightarrow & \mathcal{M}_B(c) & \longrightarrow & \mathbb{C}\mathcal{M}_B(c) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}^{\tau}_{DR} & \longrightarrow & \mathcal{M}_{DR}(c) & \longrightarrow & \mathbb{C}\mathcal{M}_{DR}(c) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}^{\tau}_{\mathrm{Dol}} & \longrightarrow & \mathcal{M}_{\mathrm{Dol}}(c) & \longrightarrow & \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c).
\end{array}$$

Moreover the horizontal maps are continuous injections and vertical maps are homeomorphisms. One obtains three additional commutative diagrams by respectively replacing the symbol $\mathcal{M}$ by $\mathcal{M}^s$, $\mathcal{N}$ and $\mathcal{N}^s$ in the above diagram. In the case of $\mathcal{M}^s$, the maps in the commutative diagram are smooth.

4.3. The Dual Higgs Bundles. There is a $\mathbb{Z}_2$ action on $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$. Let $(V, \Phi) \in \mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ where Φ is a holomorphic map:

$$\Phi : V \longrightarrow V \otimes \Omega.$$

This induces a map on the dual bundles

$$\Phi^* : V^* \otimes \Omega^* \longrightarrow V^*.$$

Tensoring with Ω,

$$\Phi^* \otimes 1 : V^* \longrightarrow V^* \otimes \Omega,$$

where 1 denotes the identity map on Ω. This produces the dual Higgs bundle $(V^*, \Phi^* \otimes 1)$. We shall abbreviate it as (V^*, Φ^*).

Proposition 4.3. *If $(V, \Phi) \in \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$, then $(V^*, \Phi^*) \in \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(-c)$.*

Proof. One must show that (V, Φ) is stable (semi-stable) implies (V^*, Φ^*) is stable (semi-stable). Suppose $W_1 \subset V^*$ is Φ^*-invariant. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_1 & \longrightarrow & V^* & \longrightarrow & W_2 & \longrightarrow & 0 \\
 & & \Big\downarrow \Phi^* & & \Big\downarrow \Phi^* & & \Big\downarrow \Phi^* & & \\
0 & \longrightarrow & W_1 \otimes \Omega & \longrightarrow & V^* \otimes \Omega & \longrightarrow & W_2 \otimes \Omega & \longrightarrow & 0
\end{array}$$

where $W_2 = V^*/W_1$. The proposition follows by dualizing the diagram. □

In light of Propositions 4.1 and 4.3 we have:

Corollary 4.4. *If $c_2 = \pm c_1 \bmod n$, then $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c_1)$ is homeomorphic to $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c_2)$.*

4.4. The $\mathrm{U}(1)$ and $\mathbb{C}^*$-Actions on the Complex Moduli Spaces. If $(V, \Phi) \in \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$, then for $t \in \mathbb{C}^*$, $(V, t\Phi) \in \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$. This defines an analytic action [**11**, **12**, **22**]

$$\mathbb{C}^* \times \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c) \longmapsto \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c).$$

Since $\mathrm{U}(1) \subset \mathbb{C}^*$, this also induces a $\mathrm{U}(1)$-action on $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$.

4.5. The Moment Map. The moduli space $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)^s$ is Kähler [**11**, **12**]. Denote by i, ω the corresponding complex and symplectic structures, respectively. Define the Morse function [**11**, **12**]

$$m : \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)^s \longrightarrow \mathbb{R},$$

$$m(V, \Phi) = 2i \int_X \mathrm{tr}(\Phi\Phi^\ddagger),$$

where $\Phi^\ddagger$ is the adjoint of Φ with respect to the Hermitian-Yang-Mills metric on (E, D). Denote by $\mathfrak{X}$ the vector field on $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)^s$ such that [**12**]

$$\mathrm{grad}\, m = \mathrm{i}\mathfrak{X}.$$

Theorem 4.5.

1) *The map m is proper.*
2) *The* $\mathrm{U}(1)$*-action generates $\mathfrak{X}$.*
3) *The $\mathbb{C}^*$ action is analytic with respect to* i*; therefore, the orbit of $\mathbb{C}^*$ is locally an analytic subvariety with respect to* i.

Proof. See [**11**, **12**, **22**]. □

Corollary 4.6. *Each component of $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$ contains a point that is a local minimum of m.*

Corollary 4.7. *If the $\mathbb{C}^*$ action preserves $\mathcal{M} \subset \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)^s$, then the gradient flow $\mathrm{grad}\, m$ preserves $\mathcal{M}$.*

Let m_r be the restriction of m to the subspace $\mathcal{M}^\tau_{\mathrm{Dol}} \subset \mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$.

Corollary 4.8. *Every component of $\mathcal{M}^\tau_{\mathrm{Dol}}$ contains a point that is a local minimum of m_r. If (V, Φ) is stable and is a local minimum of m_r, then (V, Φ) is a critical point of m.*

Proof. Consider

$$\mathcal{M}^\tau_B \subset \mathcal{M}_B(c) \subset \mathbb{C}\mathcal{M}_B(c).$$

Since $\mathrm{U}(p, q)$ is closed in $\mathrm{GL}(n, \mathbb{C})$, $\mathcal{M}_B(c)$ is a closed subspace of $\mathbb{C}\mathcal{M}_B(c)$. Since the obstruction map o_2 is continuous, $\mathcal{M}^\tau_B$ is a closed subspace of $\mathcal{M}_B(c)$. Hence $\mathcal{M}^\tau_B$ is closed in $\mathbb{C}\mathcal{M}_B(c)$. Hence by Theorem 4.5, m_r is proper. Thus each component of $\mathcal{M}^\tau_{\mathrm{Dol}}$ contains a local minimum of m_r.

The points in $(\mathcal{M}^\tau_{\mathrm{Dol}})^s$ are smooth. Suppose $(V, \Phi) \in (\mathcal{M}^\tau_{\mathrm{Dol}})^s$. Then (V, Φ) is of the form described in Proposition 3.1. Hence the $\mathbb{C}^*$ action preserves the subspace $(\mathcal{M}^\tau_{\mathrm{Dol}})^s \subset \mathbb{C}\mathcal{M}^s_{\mathrm{Dol}}$. By Corollary 4.7, the gradient flow of m preserves $(\mathcal{M}^\tau_{\mathrm{Dol}})^s$. Hence

$$\mathrm{grad}\, m_r = \mathrm{grad}\, m = \mathrm{i}\mathfrak{X}.$$

If m_r is a local minimum at (V, Φ), then

$$\mathrm{grad}\, m(V, \Phi) = \mathrm{grad}\, m_r(V, \Phi) = 0.$$

Hence (V, Φ) is a critical point of m. □

5. Bounds on Invariants.

In this section, we assume $q = 1$ and let $n = p + q = p + 1$. In light of Proposition 4.3 and Corollary 4.4, one may further assume that $\tau \geq 0$ and $0 \leq c < n$, or equivalently,

$$s(V_Q) \leq s(V) \leq s(V_P), 0 \leq c < n.$$

Proposition 5.1. *If* $(V, \Phi) = (V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in \mathcal{M}_{\mathrm{Dol}}(c)^s$ $(\mathcal{M}_{\mathrm{Dol}}(c))$, *then*

$$\begin{array}{lll} d_P & < (\leq) & \dfrac{c(n-1)}{n} + (g-1) \\ d_Q & > (\geq) & \dfrac{c}{n} - (g-1). \end{array}$$

Proof. Suppose $(V_P \oplus V_Q, \Phi) \in \mathcal{M}_{\mathrm{Dol}}(c)^s$ with $\Phi = (\Phi_1, \Phi_2)$ in the notation of Proposition 3.1. Since $s(V_P) \geq s(V)$,

$$\Phi_1 : V_P \longrightarrow V_Q \otimes \Omega$$

is non-zero.

Construct the canonical factorization for Φ_1 [**20**]: There exist holomorphic bundles V_1, V_2 and W_1, W_2 such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f_1} & V_P & \xrightarrow{f_2} & V_2 & \longrightarrow & 0 \\ & & & & \Phi_1 \big\downarrow & & \varphi \big\downarrow & & \\ 0 & \longleftarrow & W_2 & \xleftarrow{g_2} & V_Q \otimes \Omega & \xleftarrow{g_1} & W_1 & \longleftarrow & 0 \end{array}$$

commutes, and the rows are exact, $\mathrm{rank}(V_2) = \mathrm{rank}(W_1)$ and φ has full rank at a generic point of X. This implies

$$\begin{cases} \deg(V_1) + \deg(V_2) & = & d_P \\ \deg(W_1) + \deg(W_2) & = & d_Q + 2(g-1). \end{cases}$$

Since $\Phi_1 \not\equiv 0$, we have $\varphi \not\equiv 0$, $\mathrm{rank}(W_2) = 0$ and $W_1 = V_Q \otimes \Omega$.

The case of $p = 1$ has been dealt with by Hitchin [**11**], so we assume $p > 1$. Then V_1 is a Φ-invariant subbundle of positive rank. Stability implies

$$s(V_1) < s(V) = (d_P + d_Q)/n = c/n.$$

Since the map

$$V_2 \xrightarrow{\varphi} W_1 = (V_Q \otimes \Omega)$$

is not trivial,

$$\deg(V_2) \leq \deg(W_1) = \deg(V_Q \otimes \Omega).$$

So one has

$$\begin{cases} s(V_1) & < & s(V) \\ d_P & = & \deg(V_1) + \deg(V_2) \\ \deg(V_2) & \leq & d_Q + 2(g-1). \end{cases}$$

This implies

$$d_P < \frac{(n-2)c}{n} + d_Q + 2(g-1).$$

Since $d_P + d_Q = c$,

$$d_P < \frac{c(n-1)}{n} + (g-1)$$

and

$$d_Q > \frac{c}{n} - (g-1).$$

When (V, Φ) is semi-stable, one has either $\Phi \not\equiv 0$ or $\Phi \equiv 0$. In the former case, one has $s(V_1) \leq s(V)$ implying

$$\begin{aligned} d_P &\leq \frac{c(n-1)}{n} + (g-1) \\ d_Q &\geq \frac{c}{n} - (g-1). \end{aligned}$$

In the latter case, V_p is Φ-invariant. By the assumption $s(V_Q) \leq s(V_P)$, $d_P = d_Q = 0$ and $\tau = 0$. $\square$

By definition,

$$\begin{aligned} \tau &= 2\frac{d_P - pd_Q}{n} \\ &\leq \frac{2}{n}\left(\frac{c(n-1)}{n} + (g-1) - (n-1)\frac{c}{n} + (n-1)(g-1)\right) \\ &= 2(g-1). \end{aligned}$$

Equality holds only when (V, Φ) is semi-stable but not stable, in which case, the associated flat connection is reducible. This proves Theorem 1.2.

6. Reducible Higgs Bundles.

Let $p = 2$ and $q = 1$ and assume $\tau \geq 0$ and $0 \leq c < 3$. By definition, a reducible poly-stable Higgs bundle is a direct sum of stable Higgs bundles of the same slope. These Higgs bundles correspond to the reducible representations in $\mathcal{M}_B$. A direct computation shows that if (V, Φ) is reducible, then

$$\deg(V) = d_P + d_Q = 0$$

and the associated Toledo invariant τ is an even integer. Hence one has:

Proposition 6.1. *If* $c = \deg(V) \neq 0$ *and* $(V, \Phi) \in \mathcal{M}_{\mathrm{Dol}}(c)$*, then* (V, Φ) *is stable. In particular,* $\mathcal{M}_{\mathrm{Dol}}(c)$ *is smooth.*

An example of a reducible Higgs bundle is $(\mathcal{O} \oplus \Omega^{\frac{1}{2}} \oplus \Omega^{-\frac{1}{2}}, \Phi)$, where

$$\Phi : \Omega^{\frac{1}{2}} \longrightarrow \Omega^{-\frac{1}{2}} \otimes \Omega$$

is a holomorphic bundle isomorphism. That is, Φ is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Toledo invariant in this case is $2(g-1)$. All the flat $\mathrm{U}(2,1)$-connections with $\tau = 2(g-1)$ are reducible by Proposition 5.1. The fact that there is no irreducible deformation for the $\mathrm{U}(2,1)$-connections with $\tau = 2(g-1)$ was first demonstrated by Toledo [**26**]. In particular, this component is connected [**6**, **11**].

7. Hodge Bundles and Deformation.

Let $p = 2$ and $q = 1$ and assume $\tau \geq 0$ and $0 \leq c < 3$. A Hodge bundle on X is a direct sum of holomorphic bundles [**22**]

$$V = \bigoplus_{s,t} V^{s,t}$$

together with holomorphic maps (Higgs field)

$$\Phi_i : V^{s,t} \longrightarrow V^{s-1,t+1} \otimes \Omega.$$

An immediate consequence of Proposition 3.1 is:

Corollary 7.1. *Suppose* $(V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in \mathcal{M}_{\mathrm{Dol}}(c)$ *(in the notations of Proposition* 3.1*). Then* $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ *is a Hodge bundle if and only if* $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ *is either binary or ternary in the following sense:*

1) *Binary:* $\Phi_2 \equiv 0$.
2) *Ternary:* $V_P = V_1 \oplus V_2$ *and the Higgs field consists of two maps:*

$$\Phi_1 : V_2 \longrightarrow V_Q \otimes \Omega,$$
$$\Phi_2 : V_Q \longrightarrow V_1 \otimes \Omega.$$

Denote by $B(d_P, d_Q)$ the space of all poly-stable (or S-equivalence classes of semi-stable) binary Hodge bundles $(V_P \oplus V_Q, (\Phi_1, 0))$ with $\deg(V_P) = d_P$ and $\deg(V_Q) = d_Q$. Denote by $T(d_1, d_2, d_Q)$ the space of all poly-stable (or S-equivalence classes of semi-stable) ternary Hodge bundles $(V_1 \oplus V_2 \oplus V_Q, (\Phi_1, \Phi_2))$ with $\deg(V_1) = d_1$, $\deg(V_2) = d_2$ and $\deg(V_Q) = d_Q$. Denote the subspaces of stable Hodge bundles by $B(d_P, d_Q)^s, T(d_1, d_2, d_Q)^s$. When τ is not an integer, these are the type (2,1) and (1,1,1) spaces in [**8**]. Note the (1,2) types give $\tau < 0$ and therefore need not be considered here.

Proposition 7.2. *Every stable binary Hodge bundle in* $(\mathcal{M}^{\tau}_{\mathrm{Dol}})^s$ *may be deformed to a stable ternary Hodge bundle within* $\mathcal{M}^{\tau}_{\mathrm{Dol}}$.

A family (or flat family) of Higgs pairs (V_Y, Φ_Y) is a variety Y such that there is a vector bundle V_Y on $X \times Y$ together with a section $\Phi_Y \in \Gamma(Y, (\pi_Y)_*(\pi_X^*\Omega \otimes End(V_Y)))$ [18]. $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ being a moduli space implies that if Y is a family of stable (poly-stable or S-equivalence classes of semi-stable) Higgs bundles, then there is a natural morphism [15, 17]

$$t : Y \longrightarrow \mathbb{C}\mathcal{M}_{\mathrm{Dol}}.$$

Moreover t takes every point $y \in Y$ to the point of $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}$ that corresponds to the Higgs bundle in the family over y [15, 17, 18].

The space $\mathcal{M}_{\mathrm{Dol}}(c)$ is a subvariety of $\mathbb{C}\mathcal{M}_{\mathrm{Dol}}(c)$; hence, to show that two stable (poly-stable or S-equivalence classes of semi-stable) Higgs bundles (V_1, Φ_1) and (V_2, Φ_2) belong to the same component of $\mathcal{M}_{\mathrm{Dol}}(c)$, it suffices to exhibit a connected family Y (within $\mathcal{M}_{\mathrm{Dol}}(c)$) of stable (poly-stable or S-equivalence classes of semi-stable) Higgs bundles containing both (V_1, Φ_1) and (V_2, Φ_2).

Proof. Suppose $(V, \Phi) = (V_P \oplus V_Q, (\Phi_1, 0)) \in B(d_P, d_Q)^s \subset (\mathcal{M}^{\tau}_{\mathrm{Dol}})^s$. Since $s(V_P) \geq s(V)$ (This is due to the assumption $\tau \geq 0$, and $0 \leq c < 3$), $\Phi_1 \not\equiv 0$. Construct the canonical factorization for Φ_1:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1 & \xrightarrow{f_1} & V_P & \xrightarrow{f_2} & V_2 & \longrightarrow & 0 \\
 & & & & {\scriptstyle \Phi_1}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & \\
0 & \longleftarrow & W_2 & \xleftarrow{g_2} & V_Q \otimes \Omega & \xleftarrow{g_1} & W_1 & \longleftarrow & 0\,.
\end{array}$$

V_1 being Φ_1 invariant implies

$$\deg(V_1) = s(V_1) < s(V) \leq s(V_P) \leq s(V_2) = \deg(V_2).$$

The space $Pic^0(X)$ of line bundles of degree 0 over X is identified with the Jacobi variety $J_0(X)$. Construct the universal bundle [2, 19]

$$U \longrightarrow X \times J_0(X)$$

such that U restricts to the bundle $L \otimes V_1 \otimes V_2^{-1}$ on (X, L). Let π be the projection

$$\pi : X \times J_0(X) \longrightarrow J_0(X).$$

Applying the right derived functor R^1 to π gives the sheaf $\mathcal{F} = R^1\pi_*(U)$ [10] such that

$$\mathcal{F}|_L = \mathrm{H}^1(X, L \otimes V_1 \otimes V_2^{-1}).$$

Since

$$\deg(L \otimes V_1 \otimes V_2^{-1}) = \deg(V_1) - \deg(V_2) < 0,$$

by Riemann-Roch,

$$\begin{aligned} h^1(L\otimes V_1\otimes V_2^{-1}) &= h^0(L\otimes V_1\otimes V_2^{-1}) - \deg(L\otimes V_1\otimes V_2^{-1}) + (g-1) \\ &= \deg(V_2) - \deg(V_1) + (g-1) \end{aligned}$$

is a constant. By Grauert's theorem, $\mathcal{F}$ is locally free, hence, is associated with a vector bundle

$$F \longmapsto J_0(X)$$

of rank $\deg(V_2)-\deg(V_1)+(g-1)$. In particular the total space F is smooth and parameterizes extensions [9, 10]:

$$0 \longrightarrow L\otimes V_1 \xrightarrow{f_3} W_P \xrightarrow{f_4} V_2 \longrightarrow 0$$

for fixed V_1, V_2. Tensoring the above sequence with Ω gives:

$$0 \longrightarrow L\otimes V_1\otimes\Omega \xrightarrow{g_3} W_P \ \otimes\Omega \xrightarrow{g_4} V_2\otimes\Omega \longrightarrow 0.$$

Fix φ. Then F also parameterizes a family of Higgs bundles (W_P, Φ_1') that fit into the factorization

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L\otimes V_1 & \xrightarrow{f_3} & W_P & \xrightarrow{f_4} & V_2 & \longrightarrow & 0 \\ & & & & \Phi_1'\big\downarrow & & \varphi\big\downarrow & & \\ 0 & \longleftarrow & W_2 & \xleftarrow{g_2} & V_Q\otimes\Omega & \xleftarrow{g_1} & W_1 & \longleftarrow & 0\,. \end{array}$$

Let $\mathcal{V}\subset F$ be the subset of stable extensions (i.e., $W_P\in\mathcal{V}$ implies W_P is a stable holomorphic bundle [19]).

Lemma 7.3. *$\mathcal{V}\cap \mathrm{H}^1(L\otimes V_1\otimes V_2^{-1})$ and $\mathcal{V}$ are open and dense in $\mathrm{H}^1(L\otimes V_1\otimes V_2^{-1})$ and F, respectively. Moreover if $W_P\in\mathcal{V}$, then $(W_P\oplus V_Q, (\Phi_1', 0))$ is stable.*

Proof. Since $\deg(L\otimes V_1) < \deg(V_2)$ for each $L\in J_0(X)$, by a theorem of Lange and Narasimhan [13], there always exists a stable extension $W_P\in \mathrm{H}^1(L\otimes V_1\otimes V_2^{-1})$. In addition, a theorem of Maruyama states that being stable is an open property [14]. The open dense property follows from the smoothness of F and $\mathrm{H}^1(L\otimes V_1\otimes V_2^{-1})$.

Let p_P, p_Q be the projections of $W_P\oplus V_Q$ onto its W_P and V_Q factors, respectively. Suppose W is $(\Phi_1', 0)$-invariant. Suppose W has rank 1. If $P_Q(W)=0$, then $W = L\otimes V_1$; otherwise, $\deg(W)\le\deg(V_Q)$. In either case, $s(W)<s(V)$. Suppose W has rank 2. If $p_Q(W)=0$, then $W=W_P$ and $s(W)<s(V)$. Suppose $P_Q(W)\neq 0$. Then there exists a line bundle L_1 such that

$$0 \longrightarrow L_1 \longrightarrow W \xrightarrow{p_Q} p_Q(W) \longrightarrow 0.$$

Now let $L_P = p_P(L_1)\subset W_P$. Then

$$\deg(W) = \deg(L_1) + \deg(p_Q(W)) \le \deg(L_P) + \deg(V_Q).$$

Since W_P is stable, $s(L_P) < s(W_P)$. By the assumptions $\tau \geq 0$ and $0 \leq c < 3$, one has $s(V_Q) \leq 0$ and $s(W_P) \geq 0$. Therefore,

$$s(W) \leq s(L_P \oplus V_Q) = \frac{s(L_P) + s(V_Q)}{2} < \frac{s(W_P) + s(V_Q)}{2}$$
$$= \frac{\deg(W_P)}{4} + \frac{\deg(V_Q)}{2} \leq \frac{\deg(W_P) + \deg(V_Q)}{3} = s(V).$$

Thus $(W_P \oplus V_Q, (\Phi_1', 0))$ is stable. □

Since $\Phi_1 \not\equiv 0$, $\deg(V_2) \leq d_Q + 2(g-1)$ and

$$\deg(V_1) = d_P - \deg(V_2) \geq d_P - d_Q - 2(g-1).$$

Hence

$$\deg(V_Q^{-1} \otimes V_1 \otimes \Omega) = -d_Q + \deg(V_1) + 2(g-1) \geq d_P - 2d_Q > 0.$$

Hence there exists $L' \in J(X)$ such that

$$h^0(V_Q^{-1} \otimes L' \otimes V_1 \otimes \Omega) > 0$$

implying there exists a non-trivial holomorphic map

$$\phi : V_Q \longrightarrow L' \otimes V_1 \otimes \Omega.$$

Fix $\phi \not\equiv 0$. By Lemma 7.3, the family parameterized by $\mathcal{V}$ contains both $(V_P \oplus V_Q, (\Phi_1, 0))$ and $(W_P \oplus V_Q, (\Phi_1', 0))$ implying there is deformation between the two.

Set $L = L'$ and $\Phi_2' = g_3 \circ \phi$. Then the family of stable Higgs bundles parameterized by $\mathrm{H}^0(X, V_Q^{-1} \otimes L' \otimes V_1 \otimes \Omega)$ contains $(W_P \oplus V_Q, (\Phi_1', 0))$ and $(W_P \oplus V_Q, (\Phi_1', \Phi_2'))$.

Now the family of bundle extensions of V_2 by $L' \otimes V_1$ is $\mathrm{H}^1(L' \otimes V_1 \otimes V_2^{-1})$. With a fixed ϕ and the canonical factorization with φ fixed, $\mathrm{H}^1(L' \otimes V_1 \otimes V_2^{-1})$ parameterizes a family of Higgs bundles. This family contains $(W_P \oplus V_Q, (\Phi_1', \Phi_2'))$. The zero element in $\mathrm{H}^1(L' \otimes V_1 \otimes V_2^{-1})$ corresponds to the bundle extension

$$0 \longrightarrow L' \otimes V_1 \xrightarrow{f_5} (L' \otimes V_1) \oplus V_2 \xrightarrow{f_6} V_2 \longrightarrow 0.$$

Tensoring with Ω gives

$$0 \longrightarrow L' \otimes V_1 \otimes \Omega \xrightarrow{g_5} ((L' \otimes V_1) \oplus V_2) \otimes \Omega \xrightarrow{g_6} V_2 \otimes \Omega \longrightarrow 0.$$

Lemma 7.4. *If $(W_P \oplus V_Q, (\Phi_1', \Phi_2'))$ is stable (semi-stable), then $\mathrm{H}^1(L' \otimes V_1 \otimes V_2^{-1})$ parameterizes a stable (semi-stable) family.*

Proof. Suppose $(U_p \oplus V_Q, (\Psi_1, \Psi_2)) \in \mathrm{H}^1(L' \otimes V_1 \otimes V_2^{-1})$ and $W \subset U_P \oplus V_Q$ is (Ψ_1, Ψ_2)-invariant. Since $\varphi, \phi \not\equiv 0$, one has $W = V_1$ or $W = V_Q \oplus V_1$. A direct computation shows $s(W) < s(U_P \oplus V_Q)$ $(s(W) \leq s(U_P \oplus V_Q))$. □

Proposition 7.2 follows from Lemma 7.4 because the family of Higgs bundles parameterized by $\mathrm{H}^1(L' \otimes V_1 \otimes V_2^{-1})$ contains $(W_P \oplus V_Q, (\Phi_1', \Phi_2'))$ and $((L' \otimes V_1) \oplus V_2 \oplus V_Q, (g_1 \circ \varphi \circ f_6, g_5 \circ \phi))$.

To summarize, a stable binary Hodge bundle $(V_P \oplus V_Q, (\Phi_1, 0))$ is first deformed to $(W_P \oplus V_Q, (\Phi_1', 0))$ such that non-trivial holomorphic maps exist between V_Q and $(L' \otimes V_1) \otimes \Omega \subset W_P \otimes \Omega$. Such a non-trivial map Φ_2' is then chosen and attached to the existing Higgs field Φ_1'. Finally W_P is deformed to a direct sum making the resulting stable Higgs bundle a ternary Hodge bundle. □

Let $B = B(0,0) \backslash (B(0,0)^s \cup T(0,0,0))$.

Proposition 7.5. *B is connected and can be deformed to a stable ternary Hodge bundle in $\mathcal{M}^0_{\mathrm{Dol}}$.*

Proof. Consider the space $U \times J_0(X)$, where $J_0(X)$ is the Jacobi variety identified with the set of holomorphic line bundles of degree zero on X and U is the moduli space of rank-2 poly-stable holomorphic bundles of degree 0 on X. The space U is connected [**2**, **19**]. Hence $U \times J_0(X)$ is connected. Each poly-stable Higgs bundle in B is contained in the family of Higgs bundles parameterized by $U \times J_0(X)$. Hence the natural morphism

$$t : U \times J_0(X) \longrightarrow B$$

is surjective. This proves that the set B is connected.

Choose holomorphic line bundles V_1, V_2, V_Q of degrees $-1, 1, 0$, respectively such that

$$h^0(X, V_2^{-1} \otimes V_Q \otimes \Omega) > 0,$$
$$h^0(X, V_Q^{-1} \otimes V_1 \otimes \Omega) > 0.$$

Choose

$$0 \not\equiv \psi_1 \in \mathrm{H}^0(X, V_2^{-1} \otimes V_Q \otimes \Omega)$$
$$0 \not\equiv \psi_2 \in \mathrm{H}^0(X, V_Q^{-1} \otimes V_1 \otimes \Omega).$$

The space of extension of V_2 by V_1,

$$0 \longrightarrow V_1 \xrightarrow{f_1} V_P \xrightarrow{f_2} V_2 \longrightarrow 0,$$

is $\mathrm{H}^1(X, V_1 \otimes V_2^{-1})$. Tensoring the exact sequence with Ω gives

$$0 \longrightarrow V_1 \otimes \Omega \xrightarrow{g_1} V_P \otimes \Omega \xrightarrow{g_2} V_2 \otimes \Omega \longrightarrow 0.$$

Since $\deg(V_1) < \deg(V_2)$, by the theorem of Lange and Narasimhan [**13**], stable extensions always exist. Fix a stable extension V_P and set

$$\Phi_1 = \psi_1 \circ f_2,$$
$$\Phi_2 = g_1 \circ \psi_2.$$

Note $(V_P \oplus V_Q, 0) \in B$. The connected family

$$FC = \mathrm{H}^0(X, V_2^{-1} \otimes V_Q \otimes \Omega) \times \mathrm{H}^0(X, V_Q^{-1} \otimes V_1 \otimes \Omega)$$

of Higgs bundles contains $(V_P \oplus V_Q, 0)$ and $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$. Note the family FC contains semi-stable Higgs bundles. This is allowed since the points in the moduli space $\mathcal{M}_{\mathrm{Dol}}$ are also interpreted as S-equivalence classes of semi-stable Higgs bundles. However one may choose FC to be a strictly poly-stable family:

$$\begin{aligned} FC &= (\mathrm{H}^0(X, V_2^{-1} \otimes V_Q \otimes \Omega) \times \mathrm{H}^0(X, V_Q^{-1} \otimes V_1 \otimes \Omega)) \setminus \\ &\quad ((\{0\} \times \mathrm{H}^0(X, V_Q^{-1} \otimes V_1 \otimes \Omega)) \cup (\mathrm{H}^0(X, V_2^{-1} \otimes V_Q \otimes \Omega) \times \{0\})). \end{aligned}$$

Since V_P is stable, by Lemma 7.3, any element in FC is semi-stable. Hence the family FC provides a deformation between $(V_P \oplus V_Q, 0)$ and $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$. The cohomology $\mathrm{H}^1(X, V_1 \otimes V_2^{-1})$ parameterizes bundle extensions of V_2 by V_1 and also parameterizes a family of Higgs bundles with fixed ψ_1, ψ_2. By Lemma 7.4, this is a stable family which contains $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ and $(V_1 \oplus V_2 \oplus V_Q, (\psi_1 \circ f_4, g_3 \circ \psi_2))$ where f_3, f_4, g_3, g_4 come from the trivial extensions

$$0 \longrightarrow V_1 \xrightarrow{f_3} V_1 \oplus V_2 \xrightarrow{f_4} V_2 \longrightarrow 0,$$

$$0 \longrightarrow V_1 \otimes \Omega \xrightarrow{g_3} (V_1 \oplus V_2) \otimes \Omega \xrightarrow{g_4} V_2 \otimes \Omega \longrightarrow 0.$$

Hence $\mathrm{H}^1(X, V_1 \otimes V_2^{-1})$ provides a deformation between $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ and $(V_1 \oplus V_2 \oplus V_Q, (\psi_1 \circ f_4, g_3 \circ \psi_2)) \in T(-1, 1, 0)$.

To summarize, one first shows that the space B is connected. Then choose a specific element $(V_P \oplus V_Q, 0) \in B$ with V_P a stable extension of V_2 by V_1 and that there exists non-trivial holomorphic maps

$$\begin{aligned} \psi_1 &: V_2 \longrightarrow V_Q \otimes \Omega \\ \psi_2 &: V_Q \longrightarrow V_1 \otimes \Omega. \end{aligned}$$

This provides a deformation from $(V_P \oplus V_Q, 0)$ to $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$. Finally, since V_P is an extension of V_2 by V_1, $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is deformed to $(V_1 \oplus V_2 \oplus V_Q, (\psi_1 \circ f_4, g_3 \circ \psi_2))$ in $\mathrm{H}^1(X, V_1 \otimes V_2^{-1})$. □

Corollary 7.6. *Every Binary Hodge bundle can be deformed to a ternary Hodge bundle.*

Proof. Every poly-stable reducible Hodge bundle is either ternary or in B. The result then follows from Proposition 7.2 and 7.5. □

Lemma 7.7. *For fixed integers d_1, d_2, d_3, $T(d_1, d_2, d_3)$ is connected.*

Proof. We first consider the stable bundles. Stability implies the Higgs fields Φ_1, Φ_2 are not identically zero. Denote by $J_d(X)$ the Jacobi variety identified with the set of holomorphic line bundles of degree d. For each $L_1 \in J_{d_1}(X)$, the set of all (L_3, Φ_2) such that $L_3 \in J_{d_3}(X)$ and

$$0 \not\equiv \Phi_2 \in \mathrm{H}^0(X, L_3^{-1} \otimes L_1 \otimes \Omega)$$

is $\mathbb{C}^* \times Sym^{d_1+2(g-1)-d_3}X$, where $Sym^d X$ is the d-th symmetric product of X. Hence the set of all triples (L_3, L_1, Φ_2) such that

$$L_3 \xmapsto{\Phi_2} L_1 \otimes \Omega$$

with $\Phi_2 \not\equiv 0$ is the space $(\mathbb{C}^* \times Sym^{d_1+2(g-1)-d_3}X) \times J_{d_1}(X)$.

Similarly, for each $L_3 \in J_{d_3}(X)$, the space of all triples (L_2, L_3, Φ_1) such that

$$L_2 \xmapsto{\Phi_1} L_3 \otimes \Omega$$

with $\Phi_1 \not\equiv 0$ is $\mathbb{C}^* \times Sym^{d_3+2(g-1)-d_2}X$. The set of Higgs bundles parameterized by the total space

$$S = (\mathbb{C}^* \times Sym^{d_3+2(g-1)-d_2}X) \times (\mathbb{C}^* \times Sym^{d_1+2(g-1)-d_3}X) \times J_{d_3}(X)$$

contains every Higgs bundle in $T(d_1, d_2, d_3)$. Hence the natural morphism

$$t : S \longrightarrow T(d_1, d_2, d_3)$$

is surjective. Since S is connected, $T(d_1, d_2, d_3)$ is connected.

The reducible bundles consist of $T(0,0,0)$ and $T(0, d_2, -d_2)$. All polystable Higgs bundles associated with the points in $T(0,0,0)$ and $T(0, d_2, -d_2)$ are contained in the families parameterized by

$$S_1 = J_0(X) \times J_0(X) \times J_0(X)$$

and

$$S_2 = (\mathbb{C}^* \times Sym^{2(g-1)-2d_2}X) \times J_{-d_2}(X) \times J_0(X),$$

respectively. Both S_1, S_2 are connected. Since the natural morphisms

$$t_1 : S_1 \longrightarrow T(0,0,0)$$

$$t_2 : S_2 \longrightarrow T(0, d_2, -d_2)$$

are surjective, both $T(0,0,0)$ and $T(0, d_2, -d_2)$ are connected. □

Proposition 7.8. *Every component of $\mathcal{M}^{\tau}_{\mathrm{Dol}}$ contains a Hodge bundle.*

Proof. By Corollary 4.8, every component of $\mathcal{M}^{\tau}_{\mathrm{Dol}}$ contains a local minimum (V, Φ) of m_r. If (V, Φ) is a smooth point, then (V, Φ) is a critical point of m. A theorem of Hitchin and Simpson implies that (V, Φ) is a Hodge bundle [**12**, **22**]. Singular points of $\mathcal{M}^{\tau}_{\mathrm{Dol}}$ correspond to reducible Higgs bundles. The space of all reducible Higgs bundles correspond to either the space of $\mathrm{U}(2) \times \mathrm{U}(1)$ representations or the space of $\mathrm{U}(1) \times \mathrm{U}(1,1)$ representations. Each component of $\mathrm{U}(2) \times \mathrm{U}(1)$ and $\mathrm{U}(1) \times \mathrm{U}(1,1)$ representations contains points that correspond to Hodge bundles [**11**]. In fact, these points are exactly the ones corresponding to the points in B and $T(0, d_2, -d_2)$. □

Let K be a divisor of Ω and let

$$w : X \longrightarrow |K| \cong \mathbb{CP}^{g-1}$$

be the canonical map [**10**].

Lemma 7.9. *Ω has a section with simple zeros.*

Proof. The linear system $|K|$ is base point free [**10**]. If X is hyperelliptic, then the map w is a 2-1 branch map into $\mathbb{CP}^{g-1}$ and an embedding otherwise. In both cases, by Bertini's theorem, there exists a hyperplane $\mathrm{H} \in \mathbb{CP}^{g-1}$ such that $\mathrm{H} \cap X$ is regular. Then $w^{-1}(\mathrm{H})$ is an effective divisor equivalent to K and with simple zeros. □

Choose

$$K = \{x_1, x_2, \dots, x_{2(g-1)}\},$$

such that the x_i's are all distinct.

Proposition 7.10. *Let $0 \le \tau < 2(g-1)$. Suppose*

$$T(d_1 - 1, d_2 + 1, d_Q), T(d_1, d_2, d_Q) \subset \mathcal{M}^{\tau}_{\mathrm{Dol}}.$$

Then there is deformation between $T(d_1, d_2, d_Q)$ and $T(d_1 - 1, d_2 + 1, d_Q)$ within $\mathcal{M}^{\tau}_{\mathrm{Dol}}$.

Proof. Suppose

$$(V_1 \oplus V_2 \oplus V_Q, (\Phi_1, \Phi_2)) \in T(d_1 - 1, d_2 + 1, d_Q),$$

$$(U_1 \oplus U_2 \oplus U_Q, (\Psi_1, \Psi_2)) \in T(d_1, d_2, d_Q).$$

By the semi-stability of $(U_1 \oplus U_2 \oplus U_Q, (\Psi_1, \Psi_2))$ and the assumptions $\tau \ge 0, 0 \le c < 3$, one has $d_Q \le 0$ and

$$d_1 - 1 < d_1 \le \frac{d_P + d_Q}{3} < 1;$$

hence,

$$d_1 - 1 < d_1 \le 0 \text{ and } d_2 + 1 > 0.$$

This implies $(V_1 \oplus V_2 \oplus V_Q, (\Phi_1, \Phi_2))$ is stable. Hence $\Phi_1 \not\equiv 0$ and

$$-\deg(V_2) + d_Q + 2(g-1) \ge 0.$$

On the other hand, $\deg(V_1) + \deg(V_2) = d_P$, so

$$d_P - \deg(V_1) - d_Q \le 2(g-1),$$

$$-d_1 < 1 - d_1 = -\deg(V_1) \le -d_P + d_Q + 2(g-1) \le 2(g-1).$$

In light of Lemma 7.7, it suffices to demonstrate the existence of $(U_1 \oplus U_2 \oplus U_Q, (\Psi_1, \Psi_2)) \in T(d_1, d_2, d_Q)$ and $(V_1 \oplus V_2 \oplus U_Q, (\Phi_1, \Phi_2)) \in T(d_1 - 1, d_2 + 1, d_Q)$ and a deformation between the two.

Since $|K|$ is base point free, there exists $K' \in |K|$ such that

$$K' = \{y_1, y_2, \dots, y_{2(g-1)}\}$$

with $y_i \neq x_{2(g-1)}$ for all $1 \leq i \leq 2g$. The bounds on the degrees of the various bundles allow us to construct the following divisors:

$$\begin{cases} D_1 & = & \{-x_1, \dots, -x_{-\deg(U_1)}\} \\ D_2 & = & \{y_1, \dots, y_{d_P - \deg(V_1)}, -x_{2(g-1)}\} \\ D_Q & = & \{-y_{d_P-\deg(V_1)+1}, \dots, -y_{d_P-\deg(V_1)-d_Q}\}. \end{cases}$$

Let u be the basic epimorphism [1]

$$u : \mathrm{Div}\,(X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}^*)$$

and set

$$\begin{cases} U_1 & = & u(D_1) \\ U_2 & = & u(D_2) \\ U_Q & = & u(D_Q) \\ U_P & = & U_1 \oplus U_2. \end{cases}$$

Let ψ_1, ψ_2 be meromorphic sections associated with the divisors D_1, D_2. Then the meromorphic section $\psi_1 \oplus \psi_2$ of U_P is associated with the divisor

$$D_1' = \{-x_1, \dots, -x_{-\deg(U_1)}, -x_{2(g-1)}\}.$$

Hence there exists $V_1 \subset U_P$ [9] such that

$$V_1 = u(D_1').$$

Let

$$V_2 = U_P/V_1.$$

Since

$$V_1 \otimes V_2 = \det(U_P) = U_1 \otimes U_2,$$
$$V_2 = u(D_2'),$$

where

$$D_2' = \{y_1, \dots, y_{d_P-\deg(V_1)}\}.$$

In short, the bundle U_P is constructed in such a way that it is the trivial extension of U_2 by U_1, and is also an extension of V_2 by V_1:

$$0 \longrightarrow U_1 \xrightarrow{f_1} U_P \xrightarrow{f_2} U_2 \longrightarrow 0$$

$$0 \longrightarrow V_1 \xrightarrow{f_3} U_P \xrightarrow{f_4} V_2 \longrightarrow 0.$$

Tensoring with Ω gives

$$0 \longrightarrow U_1 \otimes \Omega \xrightarrow{g_1} U_P \otimes \Omega \xrightarrow{g_2} U_2 \otimes \Omega \longrightarrow 0$$

$$0 \longrightarrow V_1 \otimes \Omega \xrightarrow{g_3} U_P \otimes \Omega \xrightarrow{g_4} V_2 \otimes \Omega \longrightarrow 0.$$

Since

$$\begin{cases} -D_2 + D_Q + K' &= \Big\{x_{2(g-1)}, y_{d_P-\deg(V_1)-d_Q+1}, \cdots, y_{2(g-1)}\Big\} \\ -D_Q + D_1 + K &= \Big\{y_{d_P-\deg(V_1)+1}, \cdots, y_{d_P-\deg(V_1)-d_Q}, \\ & \qquad\qquad x_{-\deg(U_1)+1}, \cdots, x_{2(g-1)}\Big\} \end{cases}$$

are effective divisors, there exists

$$0 \not\equiv \psi_1 \in \mathrm{H}^0(X, U_2^{-1} \otimes U_Q \otimes \Omega)$$
$$0 \not\equiv \psi_2 \in \mathrm{H}^0(X, U_Q^{-1} \otimes U_1 \otimes \Omega).$$

Set

$$\Psi_1 = \psi_1 \circ f_2 \text{ and } \Psi_2 = g_1 \circ \psi_2.$$

Then $(U_1 \oplus U_2 \oplus U_Q, (\Psi_1, \Psi_2))$ is a semi-stable ternary Hodge bundle.

The divisors

$$\begin{cases} -D_2' + D_Q + K' &= \Big\{y_{d_P-\deg(V_1)-d_Q+1}, \cdots, y_{2(g-1)}\Big\} \\ -D_Q + D_1' + K &= \Big\{x_{-\deg(U_1)+1}, \cdots, x_{2(g-1)-1}, \\ & \qquad\qquad y_{d_P-\deg(V_1)+1}, \cdots, y_{d_P-\deg(V_1)-d_Q}\Big\} \end{cases}$$

are effective. Hence there exist

$$0 \not\equiv \phi_1 \in \mathrm{H}^0(X, V_2^{-1} \otimes U_Q \otimes \Omega)$$
$$0 \not\equiv \phi_2 \in \mathrm{H}^0(X, U_Q^{-1} \otimes V_1 \otimes \Omega).$$

Remark 2. This is the critical step where the assumption $\tau < 2(g-1)$ is needed. In the case of $\tau = 2(g-1)$, the degree of $V_2^{-1} \otimes U_Q \otimes \Omega$ equals -1 thus rendering it impossible to find a non-zero global section ϕ_1. This reflects the fact that every representation with $\tau = 2(g-1)$ is reducible. (See Section 6.)

Set

$$\Psi_1' = \phi_1 \circ f_4 \text{ and } \Psi_2' = g_3 \circ \phi_2.$$

Then $(U_P \oplus U_Q, (\Psi_1', \Psi_2'))$ is a semi-stable Higgs bundle. Since

$$h^0(X, U_2^{-1} \otimes U_Q \otimes \Omega) > 0$$
$$h^0(X, U_Q^{-1} \otimes U_1 \otimes \Omega) > 0,$$

$\mathrm{H}^0(X, U_1^{-1} \otimes U_Q \otimes \Omega)$ and $\mathrm{H}^0(X, U_Q^{-1} \otimes U_2 \otimes \Omega)$ are proper subspaces of $\mathrm{H}^0(X, U_P^{-1} \otimes U_Q \otimes \Omega)$ and $\mathrm{H}^0(X, U_Q^{-1} \otimes U_P \otimes \Omega)$, respectively. Hence

$$\begin{aligned} FC &= (\mathrm{H}^0(X, U_P^{-1} \otimes U_Q \otimes \Omega) \setminus \mathrm{H}^0(X, U_1^{-1} \otimes U_Q \otimes \Omega)) \times \\ &\quad (\mathrm{H}^0(X, U_Q^{-1} \otimes U_P \otimes \Omega) \setminus \mathrm{H}^0(X, U_Q^{-1} \otimes U_2 \otimes \Omega)) \end{aligned}$$

is connected and parameterizes a family of semi-stable Higgs bundles that contains both $(U_P \oplus U_Q, (\Psi_1, \Psi_2))$ and $(U_P \oplus U_Q, (\Psi_1', \Psi_2'))$. Hence there is deformation between the two.

The space of bundle extensions of V_2 by V_1,

$$0 \longrightarrow V_1 \xrightarrow{f_5} V \xrightarrow{f_6} V_2 \longrightarrow 0,$$

is parameterized by the vector space $\mathrm{H}^1(V_1 \otimes V_2^{-1})$ containing both U_P and $V_1 \oplus V_2$ (the zero element in $\mathrm{H}^1(V_1 \otimes V_2^{-1})$). Again tensoring with Ω gives

$$0 \longrightarrow V_1 \otimes \Omega \xrightarrow{g_5} V \otimes \Omega \xrightarrow{g_6} V_2 \otimes \Omega \longrightarrow 0.$$

Let

$$\Phi_1 = \phi_1 \circ f_6' \text{ and } \Phi_2 = g_5' \circ \phi_2,$$

where

$$0 \longrightarrow V_1 \xrightarrow{f_5'} V_1 \oplus V_2 \xrightarrow{f_6'} V_2 \longrightarrow 0$$

$$0 \longrightarrow V_1 \otimes \Omega \xrightarrow{g_5'} (V_1 \oplus V_2) \otimes \Omega \xrightarrow{g_6'} V_2 \otimes \Omega \longrightarrow 0$$

correspond to the trivial extensions. By Lemma 7.4, $\mathrm{H}^1(V_1 \otimes V_2^{-1})$ parameterizes a family of semi-stable Higgs bundles that contains both $(U_P \oplus U_Q, (\Psi_1', \Psi_2'))$ and $(V_1 \oplus V_2 \oplus U_Q, (\Phi_1, \Phi_2))$.

To summarize, the first step consists of fixing $U_P = U_1 \oplus U_2$ and deform the Higgs field (Ψ_1, Ψ_2) to (Ψ_1', Ψ_2'). In the second step, fix ϕ_1, ϕ_2 and deform U_P to $V_1 \oplus V_2$.

□

Consider the space $T(0, d_2, -d_2)$. By Proposition 7.5, one may assume $d_2 > 0$. To deform points in $T(0, d_2, -d_2)$, the family FC constructed in the above proof contains semi-stable Higgs bundles. However, one may also opt to construct the deformation family of poly-stable Higgs bundles by setting:

$$\begin{aligned} FC \;=\; & (\mathrm{H}^0(X, U_P^{-1} \otimes U_Q \otimes \Omega) \setminus \\ & (\mathrm{H}^0(X, U_1^{-1} \otimes U_Q \otimes \Omega) \cup \mathrm{H}^0(X, U_2^{-1} \otimes U_Q \otimes \Omega))) \times \\ & (\mathrm{H}^0(X, U_Q^{-1} \otimes U_P \otimes \Omega) \setminus \\ & (\mathrm{H}^0(X, U_Q^{-1} \otimes U_2 \otimes \Omega) \cup \mathrm{H}^0(X, U_Q^{-1} \otimes U_1 \otimes \Omega))) \\ & \cup (\mathrm{H}^0(X, U_2^{-1} \otimes U_Q \otimes \Omega) \times \{0\}). \end{aligned}$$

The case with $\tau = 2(g-1)$ has been covered in Section 6 and $\mathcal{M}_{\mathrm{Dol}}^{2(g-1)}$ is connected. Suppose $\tau < 2(g-1)$. By Proposition 7.8, every component of $\mathcal{M}_{\mathrm{Dol}}^{\tau}$ contains a Hodge bundle. By Corollary 7.6, every component of $\mathcal{M}_{\mathrm{Dol}}^{\tau}$ contains a ternary Hodge bundle. It follows from Proposition 7.10 and induction that $\mathcal{M}_{\mathrm{Dol}}^{\tau}$ is connected. Since

$$\mathcal{N}_{\mathrm{Dol}}^{\tau} = \mathcal{M}_{\mathrm{Dol}}^{\tau} / \mathrm{H}^1(X, \mathcal{O}^*),$$

Theorem 1.1 then follows from Corollary 4.2.

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