Pacific Journal of Mathematics

EXPLICIT REALIZATIONS OF CERTAIN REPRESENTATIONS OF $Sp(n,\mathbb{R})$ VIA THE DOUBLE FIBRATION TRANSFORM

JODIE D. NOVAK

Volume 195 No. 2 October 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 195, No. 2, 2000

EXPLICIT REALIZATIONS OF CERTAIN REPRESENTATIONS OF $Sp(n, \mathbb{R})$ VIA THE DOUBLE FIBRATION TRANSFORM

Jodie D. Novak

We consider a family of singular infinite dimensional unitary representations of $G = Sp(n, \mathbb{R})$ which are realized as sheaf cohomology spaces on an open G-orbit D in a generalized flag variety for the complexification of G. By parametrizing an appropriate space, M_D, of maximal compact subvarieties in D, we identify a holomorphic double fibration between D and M_D which we use to define a map P, often referred to as a double fibration or Penrose transform, from the representation into sections of a corresponding sheaf on M_D. Analysis of the construction of P shows that P is injective, the image of P is the kernel of a differential operator on M_D and P is an intertwining map.

1. Introduction.

In this paper, we consider a family of singular infinite dimensional unitary representations of $G = Sp(n, \mathbb{R})$ which are realized on certain sheaf cohomology spaces of D, an open G-orbit in a generalized flag variety for the complexification of G. By parametrizing an appropriate space, M_D, of maximal compact subvarieties in D, we can identify a holomorphic double fibration between D and M_D, a well understood bounded symmetric domain. Using standard constructions from sheaf theory and the fact that M_D is Stein, we define a map P, often referred to as a double fibration or Penrose transform, from the representation into the space of sections of a corresponding sheaf on M_D. By analyzing the spectral sequences involved in the construction of P and applying the Bott-Borel-Weil theorem, we show that P is injective. Further analysis leads to the fact that the image of P is the kernel of a differential operator on M_D and that P is an intertwining map.

More generally, let G be a real semisimple Lie group and let X be a generalized flag manifold for $G_{\mathbb{C}}$, the complexification of G. If D is an open G-orbit in X, then D can be realized as G/H for some subgroup H of G. Associated to each D is a family of representations of G given by the Dolbeault cohomology spaces $H^p(D, \mathcal{L})$ where $\mathcal{L}$ is the sheaf of holomorphic sections of a homogeneous line bundle on D. Under certain conditions, these representations are non-zero, singular, irreducible, unitarizable and infinite dimensional. They provide a construction of an important and mysterious part of the unitary dual of G.

These representations can be studied using a double fibration transform whose purpose is to embed the cohomology space in a space of holomorphic sections of a vector bundle on M_D as the kernel of a differential operator. Although the technique was developed for open orbits G/H where H is compact, some results of Wolf [**Wo2**, **Wo3**] allow the possibility of extending this technique to any open G–orbit in a generalized flag manifold for $G_{\mathbb{C}}$. This technique is related to Schmid's [**S**] construction of discrete series for G associated to an orbit G/H when H is a compact Cartan contained in a maximal compact subgroup K of G.

Wells and Wolf [**WW**] studied G-orbits $D = G/H$ where H is compact. For these orbits, they showed the existence of a holomorphic double fibration where

$$\begin{array}{ccc} & Y_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & M_D. \end{array} \tag{1.1}$$

M_D is the space of $G_{\mathbb{C}}$-translates in D of the maximal compact subvariety $K/H \cap K$ and Y_D is the incidence manifold $Y_D = \{(z, Q) \in D \times M_D : z \in Q\}$. They show that M_D is Stein in this case and use the double fibration to show that $H^s(D, \mathcal{E})$ embeds in $H^0(M_D, R^s_\nu \mu^*(\mathcal{E}))$ where $\mathcal{E}$ is the sheaf of holomorphic sections of a homogeneous bundle on D. This work proves modified versions of conjectures made by Griffiths [**Gr**] while studying automorphic cohomology.

Even if H is not compact, these ideas can be used for any open orbits D if we know that M_D is a Stein manifold. Fortunately, Wolf [**Wo2**, **Wo3**] has shown that M_D is Stein for all open G-orbits D. Eastwood, Penrose, and Wells [**EPW**] used a holomorphic double fibration of this type for an open orbit of $U(2,2)$ with isotropy $U(1) \times U(1,2)$ to study the massless field equations. In this case, M_D is $U(2,2)/(U(2) \times U(2))$. Patton and Rossi [**PR1**, **PR2**], generalizing the work of Eastwood, Penrose and Wells, studied special $SU(p,q)$-orbits.

The key to using the double fibration transform is understanding the structure of M_D. There are two basic cases and, as is expected, the structure of M_D depends on the structure of D. An open orbit D is of holomorphic type if there exists a holomorphic double fibration between D and G/K. In this case M_D is G/K. An open orbit D is of nonholomorphic type if no such holomorphic double fibration exists. In this case M_D is an open submanifold of $G_{\mathbb{C}}/K_{\mathbb{C}}$ ([**WW**]). The $U(2,2)$ example studied by Eastwood, Penrose and Wells is of holomorphic type and further examples and generalizations of the holomorphic type are given in [**BE**]. In fact, open orbits of holomorphic type are well understood. Orbits of holomorphic type correspond to highest weight representations and those of nonholomorphic type correspond to representations which do not have a highest weight. The representations are discrete series if and only if H is compact.

Not as much is known in the nonholomorphic case. This case splits usefully into two subcases: When G/K is Hermitian symmetric and when it is not. When G/K is Hermitian symmetric, the structure of M_D has been computed for two families of examples: For arbitrary $U(p,q)$-orbits [**DZ**, **PR2**] and for the open $Sp(n,\mathbb{R})$-orbits in the flag variety of Lagrangian planes in $\mathbb{C}^{2n}$ [**N**]. In both families M_D is $G/K \times \overline{G/K}$ where $\overline{G/K}$ denotes G/K with the opposite complex structure. More recently, Wolf and Zierau [**WZ**] have shown that M_D is always $G/K \times \overline{G/K}$ in the nonholomorphic Hermitian symmetric case.

When G/K is not Hermitian symmetric, Wells [**We**] and Dunne and Zierau [**DZ**] determined M_D for special $SO(2m,r)$-orbits. Akheizer and Gindikin [**AG**] have also worked out a related example for this case and have suggested that M_D could be described as a particular Stein tubular neighborhood of G/K in $G_{\mathbb{C}}/K_{\mathbb{C}}$. For these examples, it is not clear whether M_D can be realized as a homogeneous space or whether these results can be generalized. No work has been done as yet on defining the transform for these cases.

1.1. Results of Paper. In this paper we will define a double fibration transform for the $Sp(n,\mathbb{R})$-representations $H^s(D_i,\mathcal{L})$. Here, D_i is one of $r-1$ open orbits in the generalized flag variety X of isotropic i-planes in $\mathbb{C}^{2n}$ where $r \leq n$ (see Section 3.2). The dimension of a maximal compact subvariety in D_i is s and $\mathcal{L}$ is the sheaf of holomorphic sections of a sufficiently negative line bundle on D_i. These orbits are in the nonholomorphic Hermitian symmetric case with noncompact H so we are studying representations which are not discrete series and which do not have a highest weight. In this paper, we will construct a double fibration transform for $H^s(D_i,\mathcal{L})$ and show that it is injective (Theorem 4.6 and 4.11). Finally, we will use the transform to realize $H^s(D_i,\mathcal{L})$ as the kernel of a differential operator on $H^0(M_{D_i}, R^s_\nu\mu^*\mathcal{L})$ (Theorem 4.11 and 5.26). Thus these representations are Frechet spaces and are continuous, facts that also follow from work by Wong [**Wg**].

Now we describe the results in more detail. Let $\mathbb{C}^{2n}$ be endowed with a symplectic form and a Hermitian form of signature (n,n). Let X be the set of r-planes in $\mathbb{C}^{2n}$ which are isotropic with respect to the symplectic form where $r \leq n$. For $1 \leq i \leq r-1$, let D_i be planes in X of signature $(i, r-i)$. Then X is a generalized flag variety for the Lie group $Sp(n,\mathbb{C})$ and D_i is the open $Sp(n,\mathbb{R})$-orbit G/H_i in X where H_i is $U(i,r-i)\times Sp(n-r,\mathbb{R})$. Let χ be a unitary character on H_i which determines a homogeneous vector bundle

$\mathbb{L}_\chi$ on D_i. Let $\mathcal{L}_\chi$ be the sheaf of holomorphic sections of $\mathbb{L}_\chi$. When the bundle satisfies a suitable negativity condition and s is the dimension of a maximal compact subvariety of D_i, then $H^s(D_i, \mathcal{L}_\chi)$ is a non-zero irreducible infinite-dimensional singular unitarizable representation of $Sp(n, \mathbb{R})$. In this paper we give another realization of this representation via a double fibration transform.

In Section 2, we outline the construction of the double fibration transform for complex manifolds D, Y and M which are related by the holomorphic double fibration (1.1). When $\mathbb{L}$ is a line bundle on D, the transform is a map from $H^p(D, \mathcal{O}(\mathbb{L}))$ to $H^0(M, R^p_\nu \mathcal{O}(\mu^*\mathbb{L}))$ which is defined using standard constructions from sheaf theory. We establish the conditions necessary for this map to be injective and for the image of $H^p(D, \mathcal{O}(\mathbb{L}))$ to be the kernel of a map from $H^0\big(M, R^p_\nu \mathcal{O}(\mu^*\mathbb{L})\big)$ to $H^0\big(M, R^p_\nu \Omega^1_\mu(\mu^*\mathbb{L})\big)$ where Ω^1_μ is the sheaf of relative holomorphic 1–forms on Y.

In Section 3, we analyze the geometry of the holomorphic double fibration used in the construction of the transform.

In Section 4, we construct the transform for $H^s(D_i, \mathcal{L}_\chi)$. This involves analyzing the sheaves and vector bundles which are in the construction. In particular, we show that each of $R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)$ and $R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of a homogeneous vector bundle. These facts, which are crucial in determining when the transform is injective, are not immediate because μ is a G-equivariant map from a $(G\times G)$-homogeneous manifold to a G-homogeneous manifold.

Next, we show that the transform is injective by analyzing the Leray spectral sequences involved in the construction of the map and by reducing the problem to an application of the Borel-Bott-Weil theorem. An abbreviated version of the main result of Section 4 is the following theorem.

Theorem 4.11. *The double fibration transform*

$$P : H^s(D_i, \mathcal{L}_\chi) \to H^0(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi))$$

is an injection and the image of P is the kernel of a map $\mathcal{D}$ from $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$.

Since $R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)$ and $R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)$ are each the sheaf of sections of a homogeneous bundle, the transform realizes $H^s\big(D_i, \mathcal{L}_\chi\big)$ as a space of functions on M_{D_i} with values in a homogeneous vector bundle.

In Section 5, we analyze the map $\mathcal{D}$ in Theorem 4.11. By construction, $\mathcal{D}$ is determined by the map from $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$ and the kernel of $\mathcal{D}$ is the image of P. The main result of Section 5 is the following theorem.

Theorem 5.26. *$\mathcal{D}$ is a G-equivariant differential operator.*

In Appendix A we consider the situation where the line bundle $\mathbb{L}_\chi$ is replaced with a finite dimensional vector bundle although it is the line bundle case that corresponds to unitarizable representations.

This paper incorporates the results of my thesis which was done at Oklahoma State University. More specifically, my thesis contains these results when $r = n$ along with the contents of [**N**]. The case when $r < n$ is not a part of my thesis. I wish to thank my advisor, Roger Zierau, and Joe Wolf and Anthony Kable for many useful conversations while I was working on these results. Thanks also to the referee for suggesting the extension to the vector bundle case.

2. The general double fibration transform.

Let D, Y, and M be complex manifolds. Then we refer to (2.1) as a holomorphic double fibration for D when μ and ν are holomorphic fibrations.

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\mu}{\swarrow} & & \overset{\nu}{\searrow} & \\ D & & & & M. \end{array} \tag{2.1}$$

Let $\mathbb{L} \to D$ be a holomorphic line bundle on D and $\mathcal{L}$ the sheaf of holomorphic sections of $\mathbb{L}$. In this setting, it is sometimes possible to define a double fibration transform from the Dolbeault cohomology space $H^s\big(D,\mathcal{L}\big)$ to $H^0\big(M, R^s_\nu\mathcal{O}(\mu^*\mathbb{L})\big)$ where $R^s_\nu\mathcal{O}(\mu^*\mathbb{L})$ is the s^{th} higher direct image of $\mathcal{O}(\mu^*\mathbb{L})$ by ν. In this paper, we will define a double fibration transform for a family of open $Sp(n,\mathbb{R})$-orbits D in the generalized flag of isotropic r-planes in $\mathbb{C}^{2n}$ when $r \leq n$.

Although the construction of the transform is described in a variety of places [see [**BE**, **EPW**, **PR2**, **WW**], for example], we include a brief discussion here, adapted to our situation, for the convenience of the reader.

The first step in the construction is to determine when $H^s(D,\mathcal{L})$ is isomorphic to $H^s(Y,\mu^{-1}\mathcal{L})$. In the setting of this paper, the fiber of μ is contractible (Proposition 3.13) and this is sufficient to guarantee, by a theorem of Buchdahl [**Bu**], that the isomorphism exists. We note, however, that the contractibility of the fiber of μ is a stronger condition than that required by Buchdahl.

The second step is to construct a resolution of $\mu^{-1}\mathcal{L}$ to which we can apply the following lemma.

Lemma 2.2. *Let*

$$0 \to \mathcal{S} \to \mathcal{S}_0 \to \mathcal{S}_1 \to \cdots \to \mathcal{S}_N \to 0$$

be an exact sequence of sheaves on a manifold Y and suppose $H^p(Y,\mathcal{S}_t) = 0$ for $p < q$ and $1 \leq t \leq N$. Then there is an injection from $H^q(Y,\mathcal{S}) \to$

$H^q(Y,\mathcal{S}_0)$. Furthermore, $H^q(Y,\mathcal{S})$ is the kernel of the induced map from $H^q(Y,\mathcal{S}_0)$ to $H^q(Y,\mathcal{S}_1)$.

To find an appropriate resolution of $\mu^{-1}\mathcal{L}$, we begin by constructing a resolution of $\mu^{-1}\mathcal{O}_D$. We denote by Ω^p_Z the sheaf of holomorphic p-forms on a complex manifold Z.

Definition 2.3.

(1) The sheaf of relative 1-forms on Y, denoted by Ω^1_μ, is defined by the exact sequence

$$\mu^*\Omega^1_D \to \Omega^1_Y \to \Omega^1_\mu \to 0$$

where $\mu^*\Omega^1_D = \mathcal{O}_Y \otimes \mu^{-1}\Omega^1_D$ and we tensor over $\mu^{-1}\mathcal{O}_D$.

(2) The relative p–forms Ω^p_μ are defined by $\wedge^p\Omega^1_\mu$.

We can think of Ω^p_μ as p-forms on Y in the direction of the fiber of μ with coefficients in $\mathcal{O}_Y$ and $d_\mu : \wedge^p\Omega^1_\mu \to \wedge^{p+1}\Omega^1_\mu$ as differentiation along the fiber.

We have the following lemma about relative p-forms.

Lemma 2.4. *Let $m = \dim Y - \dim D$.*

(1) *Then*

$$0 \to \mu^{-1}\mathcal{O}_D \to \mathcal{O}_Y \xrightarrow{d_\mu} \Omega^1_\mu \to \cdots \to \Omega^m_\mu \to 0 \tag{2.5}$$

is an exact sequence of sheaves on Y.

(2) *The sequence (2.6) is a resolution of $\mu^{-1}\mathcal{L}$.*

$$0 \to \mu^{-1}\mathcal{L} \to \mu^*\mathcal{L} \to \Omega^1_\mu(\mu^*\mathbb{L}) \to \cdots \to \Omega^m_\mu(\mu^*\mathbb{L}) \to 0. \tag{2.6}$$

The proof of (1) is the usual Poincaré lemma. To prove (2) we tensor (2.5) by $\mu^{-1}\mathcal{L}$ and observe that $\mu^*\mathcal{L} = \mathcal{O}(\mu^*\mathbb{L})$ and

$$\Omega^p_\mu \otimes_{\mu^{-1}\mathcal{O}_D} \mu^{-1}\mathcal{L} = \Omega^p_\mu \otimes_{\mathcal{O}_Y} \mathcal{O}(\mu^*\mathbb{L}).$$

To simplify notation we denote $\Omega^p_\mu \otimes_{\mu^{-1}\mathcal{O}_D} \mu^{-1}\mathbb{L}$ by let $\Omega^p_\mu(\mu^*\mathbb{L})$.

Applying Lemma 2.2 to (2.6) yields the following lemma.

Lemma 2.7. *If $H^p\big(Y,\Omega^t_\mu(\mu^*\mathbb{L})\big) = 0$ for all $p < q$ and all t, then $H^q\big(Y,\mu^{-1}\mathcal{L}\big)$ embeds in $H^q(Y,\mathcal{O}(\mu^*\mathbb{L}))$ as the kernel of the induced map from $H^q\big(Y,\mathcal{O}(\mu^*\mathbb{L})\big)$ to $H^q\big(Y,\Omega^1_\mu(\mu^*\mathbb{L})\big)$.*

For the third and final step in the construction of the transform, we must assume that M is Stein, that ν is proper, and that $\mathcal{S}$ is a coherent sheaf on Y. With these assumptions, the following theorem is the key to this final step.

Theorem 2.8. *$H^p(Y,\mathcal{S})$ is isomorphic to $H^0(M,R^p_\nu\mathcal{S})$.*

Proof. There exists a Leray spectral sequence which abuts to $H^*(Y,\mathcal{S})$ and whose E_2-term is given by $\mathrm{E}_2^{p,q} = H^p(M, R^q_\nu\mathcal{S})$. The direct image theorem [**GR**] implies that $R^q_\nu\mathcal{S}$ is coherent so $\mathrm{E}_2^{p,q} = 0$ for all nonzero p. That is, the spectral sequence collapses and the result follows. □

If $\mathcal{O}(\mu^*\mathbb{L})$ and $\Omega^1_\mu(\mu^*\mathbb{L})$ are coherent, then Theorem 2.8 implies that $H^q(Y,\mathcal{O}(\mu^*\mathbb{L}))$ is isomorphic to $H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}))$ and also that $H^q(Y,\Omega^1_\mu(\mu^*\mathbb{L}))$ is isomorphic to $H^0(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L}))$. These isomorphisms, along with the isomorphism in Lemma 2.7, determine a map $\mathcal{D}$ from $H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}))$ to $H^0(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L}))$.

In the following theorem, we combine these constructions to define the Penrose transform.

Theorem 2.9. *The Penrose tranform is the map*

$$P : H^q(D,\mathcal{L}) \to H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})).$$

The map P is an injection and the image of P is the kernel of $\mathcal{D}$ which is defined below.

More explicitly, $H^q(D,\mathcal{L})$ is isomorphic to $H^q(Y,\mu^{-1}\mathcal{L})$ by Buchdahl's theorem. Then $d_\mu : \mathcal{O}_Y \to \Omega^1_\mu$ determines a map ${d_\mu}^* : \mathcal{O}(\mu^*\mathcal{L}_X) \to \Omega^1_\mu(\mu^*\mathbb{L})$ whose kernel is $\mu^{-1}\mathcal{L}$. By Lemma 2.7, ${d_\mu}^*$ determines an injection $\mathcal{D}_\mu$ from $H^q(Y,\mathcal{O}(\mu^*\mathbb{L}))$ to $H^q(Y,\Omega^1_\mu(\mu^*\mathbb{L}))$ whose kernel is $H^q(Y,\mu^{-1}\mathcal{L})$. Then, Theorem 2.8 gives an isomorphism between $H^q(Y,\mathcal{O}(\mu^*\mathbb{L}))$ and $H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}))$ and one between $H^q(Y,\Omega^1_\mu(\mu^*\mathbb{L}))$ and $H^0(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L}))$. As a result, $\mathcal{D}_\mu$ determines a differential operator $\mathcal{D}$ such that the following diagram commutes.

$$\begin{array}{ccc} H^q(Y,\mathcal{O}(\mu^*\mathbb{L})) & \longrightarrow & H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L})) \\ {\scriptstyle\mathcal{D}_\mu}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathcal{D}} \\ H^q(Y,\Omega^1_\mu(\mu^*\mathbb{L})) & \longrightarrow & H^0(M, R^q_\nu\Omega^1_\mu(\mu^*\mathbb{L})) \end{array} .$$

In this way, the map P and $\mathcal{D}$ are defined and P embeds $H^q(D,\mathcal{L})$ is $H^0(M, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}))$ as the kernel of $\mathcal{D}$.

3. The geometry underlying the double fibration transform.

The purpose of this section is to understand the geometry of the holomorphic double fibrations (3.1) and (3.2) which we will use to define a double fibration transform for a family of $Sp(n,\mathbb{R})$-representations. Let D_i be the open $Sp(n,\mathbb{R})$-orbit of isotropic r-planes of signature $(i, r-i)$ in the generalized flag manifold X of isotropic r-planes in $\mathbb{C}^{2n}$. Then D_i is G/H_i where $H_i \simeq U(i, r-i) \times Sp(n-r,\mathbb{R})$ and $K/H_i \cap K$ is a maximal compact subvariety in D_i. Here, K is a maximal compact subgroup of G isomorphic to $U(n)$.

Let M_{X_i} be the $Sp(n,\mathbb{C})$-translates of $K/H_i \cap K$ in X. Let $\widetilde{M}_{D_i}$ be the translates contained in D_i and let M_{D_i} be the connected component of $\widetilde{M}_{D_i}$ containing $K/H_i \cap K$. Let Y_{D_i} and Y_{X_i} be the incidence spaces

$$\begin{aligned} Y_{D_i} &= \{(z,Q) \in D_i \times M_{D_i} : z \in Q\} \\ \text{and} \quad Y_{X_i} &= \{(z,Q) \in X \times M_{X_i} : z \in Q\}. \end{aligned}$$

Then we have the following holomorphic double fibrations

$$\begin{array}{ccc} & Y_{D_i} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D_i & & M_{D_i} \end{array} \tag{3.1}$$

and

$$\begin{array}{ccc} & Y_{X_i} & \\ {}^{\tilde{\mu}}\swarrow & & \searrow^{\tilde{\nu}} \\ X & & M_{X_i} \end{array} \tag{3.2}$$

with the natural projection maps.

3.1. Preliminaries. In this section, we define the bilinear forms and the Lie groups we will use to describe the manifolds in the double fibrations. In addition, we describe various Lie algebras and root systems that will be used later.

Let $\langle \cdot\,, \cdot \rangle_H$ denote the Hermitian form on $\mathbb{C}^{2n}$ corresponding to the matrix $I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and let $\omega(\cdot\,,\cdot)$ denote the symplectic form on $\mathbb{C}^{2n}$ corresponding to $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. We call a subspace y of $\mathbb{C}^{2n}$ *isotropic* if $\omega(u\,,v) = 0$ for all $u, v \in y$ and *Lagrangian* if $y = y^{\perp_\omega}$. We denote the *signature* of a subspace y by $\mathrm{sgn}(y) = (a,b,c)$ if y has a Hermitian orthogonal basis of a positive vectors, b negative vectors and c null vectors. If $c = 0$, we write $\mathrm{sgn}(y) = (a,b)$.

We will use these forms to describe certain subgroups of $GL(2n,\mathbb{C})$. The complex symplectic group $Sp(n,\mathbb{C})$ is the set of matrices that preserve the symplectic form, and $U(n,n)$ is the subgroup that preserves the Hermitian form. Then $Sp(n,\mathbb{C}) \cap U(n,n)$ is a real form of $Sp(n,\mathbb{C})$ which preserves both the symplectic and Hermitian forms. We denote $Sp(n,\mathbb{C})$ by $G_{\mathbb{C}}$ and the real form by G. We note that $G \simeq Sp(n,\mathbb{R})$.

Let $\mathfrak{g}_{\mathbb{C}}$ denote the Lie algebra of $G_{\mathbb{C}}$ and $\mathfrak{g}$ the Lie algebra of G. Fix the Cartan subalgebra

$$\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(t_1, t_2, \dots, t_n, -t_1, -t_2, \dots, -t_n) : t_j \in \mathbb{C}\}$$

of $\mathfrak{g}_{\mathbb{C}}$ where an element of $\mathfrak{t}_{\mathbb{C}}$ is a diagonal matrix with the indicated entries. Elements of $\mathfrak{t}_{\mathbb{C}}^*$ will be identified with points in $\mathbb{C}^n$ as follows. For

$\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{C}^n$, define

$$\gamma\big(\mathrm{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n)\big) = \sum \gamma_j t_j.$$

Let e_j be the element of $\mathfrak{t}_{\mathbb{C}}^*$ which corresponds to the j^{th} standard basis vector in $\mathbb{C}^{2n}$.

The element $\lambda_i = (-1, \ldots, -1 \,|\, 0, \ldots, 0 | \, 1, \ldots, 1)$ in $\mathfrak{t}_{\mathbb{C}}^*$, with i-entries before the first vertical bar, $(n-r)$-entries between the vertical bars, and $(r-i)$-entries after the last vertical bar, will be used to determine a positive system for $\mathfrak{g}_{\mathbb{C}}$. Although these objects depend on i and r, we will only indicate the dependence on i. If $\triangle(\mathfrak{g}_{\mathbb{C}})$ denotes the roots of $\mathfrak{g}_{\mathbb{C}}$, then

$$\triangle(\mathfrak{g}_{\mathbb{C}}) = \triangle(\mathfrak{h}_{i,\mathbb{C}}) \cup \triangle(\mathfrak{q}_{i,+}) \cup \triangle(\mathfrak{q}_{i,-})$$

where $\triangle(\mathfrak{h}_{i,\mathbb{C}})$ is the set of roots of $\mathfrak{g}_{\mathbb{C}}$ whose inner product with λ_i is 0 and $\triangle(\mathfrak{q}_{i,+})$ (respectively, $\triangle(\mathfrak{q}_{i,-})$) is the roots of $\mathfrak{g}_{\mathbb{C}}$ whose inner product with λ_i is positive (respectively, negative).

We fix a positive system

$$\begin{aligned}\triangle^+(\mathfrak{h}_{i,\mathbb{C}}) = &\{(e_j - e_k) : 1 \le k < j \le i \text{ or } i+n-r+1 \le k < j \le n\} \\ &\cup \{(e_j + e_k) : 1 \le j \le i, i+n-r+1 \le k \le n\} \\ &\cup \{(e_j - e_k) : i+1 \le j < k \le i+n-r\} \\ &\cup \{(e_j + e_k) : i+1 \le j \le k \le i+n-r\}.\end{aligned}$$

for $\mathfrak{h}_{i,\mathbb{C}}$ and note that the first two subsets are all the positive roots for $U(i, r-i)$ and the last two for $Sp(n-r,\mathbb{R})$. If $r = n$, the $Sp(n-r,\mathbb{R})$ piece does not appear. The corresponding simple system is

$$\begin{aligned}\Pi_i = &\{e_2 - e_1, e_3 - e_2, \ldots, e_i - e_{i-1}, e_1 + e_{i+n-r+1}\} \\ &\cup \{e_{i+n-r+2} - e_{i+n-r+1}, e_{i+n-r+3} - e_{i+n-r+2}, \ldots, e_n - e_{n-1}\} \\ &\cup \{e_{i+1} - e_{i+2}, e_{i+2} - e_{i+3}, \ldots, e_{i+n-r-1} - e_{i+n-r}, 2e_{i+n-r}\}.\end{aligned}$$

Again we note that the first two subsets are the simple roots for $U(i, r-i)$ and the last for $Sp(n-r,\mathbb{R})$. Now $\triangle^+(\mathfrak{h}_{i,\mathbb{C}}) \cup \triangle(\mathfrak{q}_{i,+})$ forms a positive system for $\triangle(\mathfrak{g}_{\mathbb{C}})$ and this is the system that we shall use throughout.

Let $\mathfrak{h}_{i,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum\limits_{\alpha \in \triangle(\mathfrak{h}_{i,\mathbb{C}})} \mathfrak{g}_\alpha$ and let $H_{i,\mathbb{C}}$ be the analytic subgroup associated to $\mathfrak{h}_{i,\mathbb{C}}$. Let $\mathfrak{h}_i = \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{g}$; then $\mathfrak{h}_i$ is isomorphic to $\mathfrak{u}(i, n-i) \oplus \mathfrak{sp}(n-r,\mathbb{R})$ and $H_i \simeq U(i, n-i) \times Sp(n-r,\mathbb{R})$ is the analytic subgroup for $\mathfrak{h}_i$. For $r = n$, H_i is the fixed points of the involution $\mathrm{Ad}\,\zeta_i$ on G where

$$\zeta_i = \begin{pmatrix} -I_i & 0 & 0 & 0 \\ 0 & I_{n-i} & 0 & 0 \\ 0 & 0 & I_i & 0 \\ 0 & 0 & 0 & -I_{n-i} \end{pmatrix}.$$

Let $\mathfrak{q}_{i,+} = \sum_{\alpha \in \triangle(\mathfrak{q}_{i,+})} \mathfrak{g}_\alpha$ and let $Q_{i,+}$ denote the analytic subgroup of $\mathfrak{q}_{i,+}$ in $G_{\mathbb{C}}$. Let $\mathfrak{q}_{i,-} = \sum_{\alpha \in \triangle(\mathfrak{q}_{i,-})} \mathfrak{g}_\alpha$ and let $Q_{i,-}$ denote the analytic subgroup of $\mathfrak{q}_{i,-}$ in $G_{\mathbb{C}}$.

Let Θ be the Cartan involution on $\mathfrak{g}_{\mathbb{C}}$ given by $\Theta(X) = -{}^{t}\overline{X}$. Denote by $K_{\mathbb{C}}$ the analytic subgroup of $G_{\mathbb{C}}$ corresponding to the $(+1)$-eigenspace of Θ. The (-1)-eigenspace $\mathfrak{p}_{\mathbb{C}}$ of Θ decomposes into the $K_{\mathbb{C}}$-invariant subspaces $\mathfrak{p}_+$ and $\mathfrak{p}_-$. Let $P_+ = \exp(\mathfrak{p}_+)$ and $P_- = \exp(\mathfrak{p}_-)$. In this case K, the real form of $K_{\mathbb{C}}$, is isomorphic to $U(n)$.

3.2. The Double Fibration for $Sp(n, \mathbb{R})$. The geometry of $\mathbb{C}^{2n}$ induced by the Hermitian and symplectic forms provides a useful tool for describing the spaces D_i, Y_{D_i}, and M_{D_i} in the double fibration (3.1), for realizing them as homogeneous manifolds, and for examining the relationship between the double fibrations (3.1) and (3.2).

We begin by observing that $G_{\mathbb{C}}$ acts transitively on X, the set of isotropic r-planes in $\mathbb{C}^{2n}$, by Witt's theorem (see, for example, [**A**]). If we choose $x_i = \mathrm{span}\{e_1, \ldots, e_i, e_{2n-r+i+1}, \ldots, e_{2n}\}$ as a basepoint in X, then $G_{\mathbb{C}}$ acts with isotropy subgroup $H_{i,\mathbb{C}}Q_{i,-}$. Then X, as a generalized flag manifold for $G_{\mathbb{C}}$, can be realized in several ways; if it is important to specify a realization we will use the convention $X_i = G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}$. We also note that if $r = n$ and $i = 0$ or n, then the isotropy subgroup is $K_{\mathbb{C}}P_+$ or $K_{\mathbb{C}}P_-$, respectively.

The relationship between X and D_i, the set of isotropic $(i, r-i)$-planes, is given in the following proposition.

Proposition 3.3. *D_i is an open G-orbit in X.*

This can be seen in two ways. First, for a fixed r, the open G-orbits in X are parametrized by the signatures $(i, r-i)$ [**Wo1**]. Second, a generalization of Witt's theorem (see [**A**], for example) implies that G acts transitively on D_i. For the basepoint x_i, the stabilizer of this action is H_i and a dimension count shows that D_i is open.

Thus D_i is a complex manifold. If $r = n$ and $i = 0$ or n, then D_i is the Hermitian symmetric space G/K and is of holomorphic type. If $r = n$ and $i \neq 0$ or n, then D_i is the indefinite Kähler symmetric space G/H_i and is of non-holomorphic type. If $r < n$, then D_i is G/H_i which is not a symmetric space. In this case, if $i = 0$ or r, then D_i is of holomorphic type and if $i \neq 0$ or r then D_i is of nonholomorphic type as described in Section 1.

We now define two other members of the double fibrations: M_{D_i} and M_{X_i}.

Definition 3.4. The space M_{X_i} is the set of $G_{\mathbb{C}}$-translates of Kx_i. Let $\widetilde{M}_{D_i}$ be the $G_{\mathbb{C}}$-translates of Kx_i contained in D_i and let M_{D_i} be the connected component of $\widetilde{M}_{D_i}$ containing Kx_i.

To analyze the structure of M_{D_i} and M_{X_i}, we need to understand the structure of the K-orbit of x_i in D_i. First, work of Schmid and Wolf [**SW**] implies that Kx_i is a maximal compact subvariety of D_i.

If $r = n$ and $i = 0$ or n, then $Kx_i = x_i$ and M_{D_i} is D_i. If $r = n$ with $i \neq 0$ or n and if $r < n$ with $i = 0$ or r, then Kx_i is biholomorphic to the Grassmanian of i-planes in $\mathbb{C}^n$ in the first case and to the Grassmanian of r-planes in $\mathbb{C}^n$ in the second. In all cases, Kx_i is realized as the homogeneous space $K/H_i \cap K$. The parametrization of M_{D_i} is given in the following theorem for $r = n$ with $i \neq 0$ or n and for all $r < n$ with $i \neq 0$ or r.

Theorem 3.5. *The manifold M_{D_i} is biholomorphic to $G/K \times \overline{G/K}$, where $\overline{G/K}$ denotes G/K with the opposite complex structure.*

For $r = n$ and $i \neq 0$ or n, the proof of this theorem is the main result of [**N**]. More recently, Wolf and Zierau [**WZ**] have proven this theorem for all open orbits of nonholomorphic type when G/K is a Hermitian symmetric space and G is a classical group. We will outline the idea of the proof in [**N**] so that we can use the explicit description of M_D given there in the proof of the contractibility of the fiber.

First, we describe how to associate a pair of transverse Lagrangian planes in $\mathbb{C}^{2n}$ to a $G_{\mathbb{C}}$-translate of Kx_i. The difficulty here is showing that the association is unique. Once this is complete, we have the parametrization of M_{X_i} given below.

Lemma 3.6. *M_{X_i} is the manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ when $2i \neq r$ and $G_{\mathbb{C}}/L$ when $2i = r$.*

This is another result of [**N**] where L is the subgroup of $G_{\mathbb{C}}$ generated by $K_{\mathbb{C}}$ and the matrix J which defines the symplectic form.

To describe the association, we observe that Kx_i is the set of isotropic r-planes of signature $(i, r-i)$ which meet $y_0 = \text{span}\{e_1, \ldots, e_n\}$ in an i-plane and $w_0 = \text{span}\{e_{n+1}, \ldots, e_{2n}\}$ in an $(r-i)$-plane. More specifically, each i-plane u in y_0 together with any $(r-i)$-plane u' in $u^{\perp_\omega} \cap w_0$ forms an isotropic $(i, r-i)$-plane in Kx_i and each element of Kx_i can be described in this fashion. We make two observations. First, when $r = n$, the dimension of $u^{\perp_\omega} \cap w_0$ is $n-i$ so for each i-plane in y_0 there exists exactly one $(n-i)$-plane u' in $u^{\perp_\omega} \cap w_0$ such that $u \oplus u'$ is an element of Kx_i. Second, the above description of Kx_i does not depend on the signature of the planes y_0 and w_0, only that the planes are transverse and Lagrangian. In light of this, translating Kx_i by $g \in G_{\mathbb{C}}$ element by element is equivalent to translating y_0 and w_0 by g and creating gKx_i from the translated planes.

To reflect the relationship between Kx_i and the two transverse Lagrangian planes y_0 and w_0, we denote Kx_i by $V^i(y_0,w_0)$. Then gKx_i will be denoted by $V^i(gy_0,gw_0)$ and M_{X_i} is the set of maximal compact subvarieties $V^i(y,w)$ for any pair of transverse Lagrangian planes y and w. The main difficulty in parametrizing M_{X_i} is determining the stabilizer of the action of $G_{\mathbb{C}}$ on M_{X_i}. That is, showing the level of uniqueness of the representation of a maximal compact subvariety by $V^i(y,w)$. When $2i \neq r$, $V^i(y,w) = V^i(y',w')$ if and only if $y = y'$ and $w = w'$. When $2i = r$, it is also the case that $V^i(y,w) = V^i(y',w')$ when $y = w'$ and $w = y'$. This happens because switching the position of y and w in $V^i(y,w)$ does not change the maximal compact subvariety.

To parametrize M_{D_i}, we must identify which pairs of transverse Lagrangian planes are asociated to elements of M_{D_i}. Clearly, if y is positive and w is negative, then $V^i(y,w)$ is in D_i and hence in $\widetilde{M}_{D_i}$. The difficulty lies in showing that such $V^i(y,w)$ are in M_{D_i} and that only $V^i(y,w)$ of this type are in M_{D_i}. See [**N**] for details.

The descriptions of M_{D_i} and M_{X_i} are useful for determining the structure of

$$Y_{D_i} = \left\{(z, V^i(y,w)) \in D_i \times M_{D_i} :\ z \in V^i(y,w)\right\} \quad (3.7)$$

$$\text{and} \quad Y_{X_i} = \left\{(z, V^i(y,w)) \in X \times M_{X_i} :\ z \in V^i(y,w)\right\}.$$

It is not too difficult to show that $G_{\mathbb{C}}$ acts transitively on Y_{X_i} by $g \cdot (z, V^i(y,w)) = (gz, V^i(gy,gw))$. Making use of the parametrization of M_{X_i} and X_i, we have the following theorem.

Theorem 3.8. *When $2i \neq r$ the manifold Y_{X_i} is ${G_{\mathbb{C}}}/{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}}$ and when $2i = r$ the manifold Y_{X_i} is ${G_{\mathbb{C}}}/{H_{i,\mathbb{C}}Q_{i,-} \cap L}$*

We now turn our attention to analyzing the structure of Y_{D_i}. Although G acts on Y_{D_i} as $G_{\mathbb{C}}$ acts on Y_{X_i}, this action is not transitive. Fortunately, $G \times G$ acts transitively on Y_{D_i}. First, we define the action of $G \times G$ on the basepoint $(x_i, V^i(y_0,w_0))$ by

$$\begin{aligned}(g_1, g_2) \cdot &\left(x_i, V^i(y_0,w_0)\right) \\ &= \left((g_1 \exp(X_+)x_i, V^i(g_1 \exp(X_+)y_0, g_1 \exp(X_+)w_0))\right).\end{aligned}$$

where $\exp(X_+)k\exp(X_-)$ is the Harish-Chandra decomposition (see [**K**], for example) of $g_1^{-1}g_2$. We note that the action in the second factor simplifies to $V^i(g_1y_0, g_2w_0)$. This action of $G \times G$ maps $(x_i, V^i(y_0,w_0))$ onto Y_{D_i} as follows. Since $G \times G$ acts transitively on M_{D_i}, there exists $g_1, g_2 \in G$ such that $(g_1, g_2)V^i(y_0,w_0) = V^i(y,w)$ for any $V^i(y,w)$ in M_{D_i}. Since $K \times K$ fixes $V^i(y_0,w_0)$, as k_1 and k_2 run through K, (g_1k_1, g_2k_2) acting on x_i run through every element of $V^i(g_1y_0,g_2w_0)$. Thus each $(z, V^i(y,w))$ is a translate of $(x_i, V^i(y_0,w_0))$. Then $G\times G$ acts on $(z, V^i(y,w))$ by first writing $(z, V^i(y,w))$ as $(g_3, g_4)(x_i, V^i(y_0,w_0))$ and letting (g_1g_3, g_2g_4) act on $(x_i, V^i(y_0,w_0))$.

We have the following theorem.

Theorem 3.9. *Y_{D_i} is biholomorphic to* $\left. G \middle/ H_i \cap K \right. \times \overline{\left. G \middle/ K \right.}$.

Proof. As shown above, $G\times G$ acts transitively on Y_{D_i}. Then the stabilizer of $(x_i, V^i(y_0,w_0))$ is $(H_i \cap K)\times K$ so there is an isomorphism between Y_{D_i} and $G/H_i \cap K \times \overline{G/K}$ which endows Y_{D_i} with a differentiable structure. The complex structure comes from using the Harish-Chandra decomposition to embed $G/H_i \cap K$ into $G_{\mathbb{C}}/(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})P_+$ and G/K in $G_{\mathbb{C}}/K_{\mathbb{C}}P_-$. The opposite complex structure is needed in the second factor since P_+ is replaced with P_-. □

We have the following two observations about the action of $G\times G$ on Y_{D_i}. First, if we had decomposed $g_2^{-1}g_1$ as $\exp(X_-)k\exp(X_+)$ instead of decomposing $g_1^{-1}g_2$, then $(g_1, g_2)(x_i, V^i(y_0,w_0)) = (g_2\exp(X_-)x_i, V^i(g_1y_0,g_2w_0))$ determines another action of $G \times G$ on Y_{D_i}. In this case, the space Y_{D_i} would have been realized as $G/K \times \overline{G/H_i \cap K}$. If this action were chosen, the factors would be switched throughout the construction.

Second, when $r = n$ we can describe the action of $G \times G$ on the first component of $(x_i, V^i(y_0,w_0))$ geometrically. This is possible because, as described after Lemma 3.6, each element of Kx_i is of the form $u \oplus u'$ with u an i-plane in y_0 and u' an $(n-i)$-plane in $u^{\perp_\omega} \cap w_0$. When $r = n$, we have $u' = u^{\perp_\omega} \cap w_0$. That is, each element of Kx_i is completely determined by its intersection with y_0. So, if we move $x_i \cap y_0$ by g_1 to $g_1(x_i \cap y_0)$, then $g_1(x_i \cap y_0)$ meets g_1y_0 in an i-plane and the image of x_i under (g_1, g_2) is $z' = g_1(x_i \cap y_o) \oplus [\{g_1(x_i \cap y_o)\}^{\perp_\omega} \cap (g_2w_o)]$ which is an element of $V^i(g_1y_0,g_2w_0)$. Using the Harish-Chandra decompostition of $g_1^{-1}g_2$, we have $z' = g_1\exp(X_+)x_i$. Thus, the action of $G\times G$ on Y_{D_i} can be interpreted in terms of planes.

3.3. Relating the two double fibrations. A good understanding of the relationship between the double fibrations (3.1) and (3.2) is crucial for giving an explicit realization of the differential operator in Theorem 2.9. We have already discussed the relationship between D_i and X_i in Proposition 3.3. In this section, we consider the relationship between the other pairs.

From the descriptions of Y_{D_i} and Y_{X_i} in (3.7) as certain pairs of isotropic r-planes and maximal compact subvarieties, it is clear that Y_{D_i} is contained in Y_{X_i}. When these spaces are realized as homogeneous manifolds, the embedding of Y_{D_i} in Y_{X_i} is given by the following theorem.

Theorem 3.10. *The map*

$$\varphi : \left. G \middle/ H_i \cap K \right. \times \overline{\left. G \middle/ K \right.} \to \left. G_{\mathbb{C}} \middle/ H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}} \right.$$

defined by $\varphi(\overline{g_1}, \overline{g_2}) = \overline{g_1\exp(X_+)}$ *is a holomorphic injection where the Harish-Chandra decomposition of* $g_1^{-1}g_2$ *is* $\exp(X_+)k\exp(X_-)$ *with* $X_+ \in \mathfrak{p}_+$, $k \in K_{\mathbb{C}}$ *and* $X_- \in \mathfrak{p}_-$.

Proof. In the following diagram

$$\begin{array}{ccc} {}^{G}\!\big/_{H_i \cap K} \times \overline{{}^{G}\!\big/_{K}} & \xrightarrow{\varphi} & {}^{G_{\mathbb{C}}}\!\big/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}} \\ \alpha\big\downarrow & & \big\downarrow\beta \\ {}^{G_{\mathbb{C}}}\!\big/_{(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})P_+} \times {}^{G_{\mathbb{C}}}\!\big/_{K_{\mathbb{C}}P_-} & \xrightarrow{i} & {}^{G_{\mathbb{C}}}\!\big/_{(H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})P_+} \times {}^{G_{\mathbb{C}}}\!\big/_{K_{\mathbb{C}}P_-} \end{array}$$

the embeddings α and β are given by $\alpha(\overline{g_1}, \overline{g_2}) = (\overline{g_1}, \overline{g_2})$ and $\beta(\overline{g}) = (\overline{g}, \overline{g})$ and i is the identity map. The following calculation shows that the image of α is contained in the image of β:

$$\begin{aligned} \alpha(\overline{g_1}, \overline{g_2}) &= (\overline{g_1}, \overline{g_2}) \\ &= g_1 \cdot (\overline{e}, \overline{g_1^{-1} g_2}) \\ &= g_1 \cdot (\overline{e}, \overline{\exp(X_+) k \exp(X_-)}) \\ &= g_1 \cdot (\overline{\exp(X_+)}, \overline{\exp(X_+)}) \\ &= \beta(\overline{g_1 \exp(X_+)}). \end{aligned}$$

Thus this a commutative diagram and the result follows. □

For $2i \neq r$, the map φ embeds Y_{D_i} in Y_{X_i}. For $2i = r$, the realization of Y_{X_i} accounts for the fact that, in this case, $V^i(y,w)$ and $V^i(w, y)$ are the same maximal compact subvariety. The natural projection map

$$\pi : {}^{G_{\mathbb{C}}}\!\big/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}} \to {}^{G_{\mathbb{C}}}\!\big/_{H_{i,\mathbb{C}}Q_{i,-} \cap L}$$

reflects this identification. Since only one of these realizations occurs in the parametrization of Y_{D_i}, the map $\pi \circ \varphi$ is an injection and gives the embedding of Y_{D_i} in Y_{X_i} in this case.

The situation for M_{D_i} and M_{X_i} is similar and we use the following theorem to embed M_{D_i} in M_{X_i}.

Theorem 3.11. *The map*

$$\psi : {}^{G}\!\big/_{K} \times \overline{{}^{G}\!\big/_{K}} \to {}^{G_{\mathbb{C}}}\!\big/_{K_{\mathbb{C}}}$$

defined by $\psi(\overline{g_1}, \overline{g_2}) = \overline{g_1 \exp(X_+)}$ *is a holomorphic injection where* $\exp(X_+) k \exp(X_-)$ *is the Harish-Chandra decomposition of* $g_1^{-1} g_2$.

Proof. We embed $G/K \times \overline{G/K}$ and $G_{\mathbb{C}}/K_{\mathbb{C}}$ in $G_{\mathbb{C}}/K_{\mathbb{C}}P_+ \times G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ and proceed as in Theorem 3.10. □

For $2i \neq r$, the map ψ embeds M_{D_i} in M_{X_i}. For $2i = r$, let $\pi : G_{\mathbb{C}}/K_{\mathbb{C}} \to G_{\mathbb{C}}/L$ be the natural projection map. Then, as before, $\pi \circ \psi$ embeds M_{D_i} in M_{X_i}.

3.4. The fiber of μ. The geometry of the fiber of μ plays an important role in the first step of the construction of the Penrose transform. In particular, we need the fiber of μ to be contractible to apply Buchdahl's condition [**Bu**] to conclude that $H^s(D_i, \mathcal{L}_\chi)$ is isomorphic to $H^s(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi)$. Since μ is a G-equivariant map, it suffices to consider the geometry of $\mu^{-1}(x_i)$ where, as before, we have $x_i = \text{span}\{e_1, \dots, e_i, e_{2n-r+i+1}, \dots, e_{2n}\}$. We will show that $\mu^{-1}(x_i)$ is contractible by showing that it fibers over a contractible space with contractible fiber. Let $G(j) = Sp(j,\mathbb{C}) \cap U(j,j)$.

Theorem 3.12. *Let*

$$\pi : \mu^{-1}(x_i) \to$$
$$H_i/U(i) \times G(n-r) \times U(r-i) \times H_i/U(i) \times G(n-r) \times U(r-i)$$

be the map defined by $\pi(x_i, V^i(y,w)) = (\overline{h_1}, \overline{h_2})$ *where* $h_1(x_i \cap y_0) = x_i \cap y$ *and* $h_2(x_i \cap w_0) = x_i \cap w$. *Then* $\mu^{-1}(x_i)$ *fibers over*

$$H_i/U(i) \times G(n-r) \times U(r-i) \times H_i/U(i) \times G(n-r) \times U(r-i)$$

with fiber

$$G(n-i)/U(n-i) \times G(i+n-r)/U(i+n-r).$$

This theorem together with the observation that both the base space and the fiber of π are contractible give us the following proposition.

Proposition 3.13. $\mu^{-1}(x_i)$ *is contractible.*

Proof of Theorem 3.12. To understand the geometry of $\mu^{-1}(x_i)$, we must identify all maximal compact subvarieties $V^i(y,w)$ in M_{D_i} containing x_i. Given the parametrization of M_{D_i}, this is equivalent to finding all positive Lagrangain planes y and all negative Lagrangian planes w such that y meets x_i is an i-plane and w meets x_i is an $(r-i)$-plane.

We begin by looking at a special case: The positive i-plane $u_i = \text{span}\{e_1, \dots, e_i\}$ in x_i. We can extend u_i to a positive Lagrangian plane by any positive $(n-i)$-plane in $u_i^{\perp_\omega} \cap u_i^{\perp_H} = \text{span}\{e_{i+1}, \dots, e_n, e_{n+i+1}, \dots, e_{2n}\}$. One such plane is $v_i = \text{span}\{e_{i+1}, \dots, e_n\}$. To find the others we observe that, in G, the plane $u_i^{\perp_\omega} \cap u_i^{\perp_H}$ is fixed by $G(i) \times G(n-i)$ and the stabilizer of v_i in $G(i) \times G(n-i)$ is $G(i) \times U(n-i)$. Thus, all positive Lagrangian planes containing u_i are of the form $u_i \oplus g v_i$ where $\overline{g} \in G(n-i)/U(n-i)$.

More generally, any positive i-plane u in x_i is an H_i-translate of u_i and the stabilizer of u_i in H_i is $U(i) \times G(n-r) \times U(r-i)$. Thus the positive i-planes in x_i are parametrized by $H_i/(U(i) \times G(n-r) \times U(r-i))$ and for each positive i-plane in x_i the set of positive Lagrangian planes containing it is parametrized by $G(n-i)/U(n-i)$.

In a similar fashion, one can show that the negative $(r-i)$-planes in x_i are parametrized by $H_i/U(i) \times G(n-r) \times U(r-i)$ and for each negative

$(r-i)$-plane in x_i, the set of negative Lagrangain planes containing it is parametrized by $G(i+n-r)/U(i+n-r)$. □

4. Constructing the double fibration transform for $H^s(D_i, \mathcal{L}_\chi)$.

In this section we will define a double fibration transform for the $Sp(n,\mathbb{R})$-representations $H^s(D_i, \mathcal{L}_\chi)$ where s is the dimension of the maximal compact subvariety $K/H_i \cap K$ in D_i and χ is the character on H_i whose differential is given by $\chi = (-a, \dots, -a \mid 0, \dots, 0 \mid a, \dots, a)$. That is, $\chi = \sum_{j=1}^{i} -ae_j + \sum_{p=i+n-r+1}^{n} ae_p$ in $\mathfrak{h}^*_{i,\mathbb{C}}$. In this case $H^s(D_i, \mathcal{L}_\chi)$ is an irreducible, unitarizable nonzero infinite dimensional representation of $Sp(n,\mathbb{R})$ if $a < -2n+r$ [**Wg**]. In the process of defining the transform, it will be necessary to impose additional restrictions on χ so that the transform will be injective.

4.1. Pulling up $H^s(D_i, \mathcal{L}_\chi)$ by μ to Y_{D_i}. The first step in defining the transform is transferring $H^s(D_i, \mathcal{L}_\chi)$ to Y_{D_i}. Since the fiber of μ is contractible (Proposition 3.13), a theorem of Buchdahl [**Bu**] implies the following theorem.

Theorem 4.1. *$H^s\big(D_i, \mathcal{L}_\chi\big)$ is isomorphic to $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi\big)$.*

Now Lemma 2.4 implies that

$$0 \to \mu^{-1}\mathcal{L}_\chi \to \mathcal{O}(\mu^*\mathbb{L}_\chi) \to \Omega^1_\mu(\mu^*\mathbb{L}_\chi) \to \cdots \to \Omega^m_\mu(\mu^*\mathbb{L}_\chi) \to 0 \tag{4.2}$$

is a resolution of $\mu^{-1}\mathcal{L}_\chi$ where $\Omega^p_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of relative p-forms on Y_{D_i} with values in the bundle $\mu^*\mathbb{L}_\chi$ and $m = \dim Y_{D_i} - \dim D_i$.

Upon first inspection, the sheaves in the resolution of $\mu^{-1}\mathcal{L}_\chi$ do not appear to be sheaves of holomorphic sections of homogeneous vector bundles due to the fact that μ is a G–equivariant map, not $G\times G$–equivariant, from the $G\times G$-homogeneous space Y_{D_i} to the G-homogeneous space D_i. We will show, using the natural projection map $\tilde{\mu} : Y_{X_i} \to X_i$, that these sheaves are holomorphic sections of a homogeneous vector bundle. We begin with the sheaf $\mathcal{O}(\mu^*\mathbb{L}_\chi)$.

Theorem 4.3. *The bundle $\mu^*\mathbb{L}_\chi$ on Y_{D_i} is a homogeneous bundle with fiber $\mathbb{C}_\chi$ where $(H_i \cap K) \times K$ acts by $\chi \otimes 1$.*

Proof. Let $\tilde{\chi}$ be the extension of χ to $H_{i,\mathbb{C}}Q_{i,-}$ with $\tilde{\chi}$ trivial on $Q_{i,-}$. Then $\tilde{\mu}^\star\mathbb{L}_{\tilde{\chi}}$ is the homogeneous line bundle on Y_{X_i} with fiber $\mathbb{C}_{\tilde{\chi}}$ and its restriction to $\varphi(Y_{D_i})$ is isomorphic to $\mu^*\mathbb{L}_\chi$ on Y_{D_i} where φ is the embedding of Y_{D_i} in Y_{X_i} in Theorem 3.10. This isomorphism allows us to show that $\mu^*\mathbb{L}_\chi$ is a $G\times G$-homogeneous bundle once we have an explicit expression for the action of $G\times G$ on $\varphi(Y_{D_i})$.

Let $\overline{g} \in \varphi(Y_{D_i})$ and $g_1, g_2 \in G$. Assume, for the moment, that the Harish-Chandra decomposition of $(g_1g)^{-1}g_2g$ as $\exp(X_+)k\exp(X_-)$ exists. The key to seeing that $(g_1, g_2)\cdot\overline{g} = \overline{g_1g\exp(X_+)}$ defines an action of $G{\times}G$ on $\varphi(Y_{D_i})$ is the following computation. Using the identification of $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}}$ with Y_{X_i}, we have

$$\begin{aligned}(g_1, g_2)\cdot\overline{g} &= (g_1, g_2)\cdot\big(gx_i, V^i(gy_0,gw_0)\big)\\ &= (g_1g, g_2g)(x_i, V^i(y_0,w_0))\\ &= \big(g_1g\exp(X_+)x_i, V^i(g_1g\exp(X_+)y_0, g_1g\exp(X_+)w_0)\big)\\ &= g_1g\exp X_+(x_i, V^i(y_0,w_0))\\ &= \overline{g_1g\exp X_+}.\end{aligned}$$

Now we address the Harish-Chandra decomposition of $(g_1g)^{-1}g_2g$. Since $\overline{g} \in \varphi(Y_{D_i})$, there exist $g_3, g_4 \in G$ such that $\varphi(\overline{g_3}, \overline{g_4}) = \overline{g}$. That is, there exist $X_+' \in \mathfrak{p}_+$ and $h \in H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}}$ such that $g = g_3\exp(X_+')h$. Using this expression for g in $(g_1g)^{-1}g_2g$ and the Harish-Chandra decomposition of $(g_1g_3)^{-1}g_2g_4$ yields the decomposition of $(g_1g)^{-1}g_2g$.

Let $\mathbb{W}$ denote the restriction of $\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}}$ to $\varphi(Y_{D_i})$ and let $[g, w]$ be in $\mathbb{W}_{\overline{g}}$. For $g_1, g_2 \in G$,

$$(g_1, g_2)\cdot[g, w] = \big[g_1g\exp(X_+), w\big]$$

defines an action of $G\times G$ on $\mathbb{W}$. Then $\mathbb{W}$ is the $G\times G$-homogeneous line bundle on Y_{D_i} with fiber $\mathbb{C}_\chi\otimes 1$. □

We note that the action of G on $\mathbb{W}$ as a subgroup of $G_{\mathbb{C}}$ is equivalent to the action of G as the diagonal subgroup of $G{\times}G$.

In the remainder of this section, we will show that the sheaves $\Omega^p_\mu(\mu^*\mathbb{L}_\chi)$ in (4.2) are sheaves of sections of homogeneous bundles on Y_{D_i}. Since $\Omega^p_\mu(\mu^*\mathbb{L}_\chi) = \Omega^p_\mu\otimes\mathcal{O}(\mu^*\mathbb{L}_\chi)$, it suffices to show that Ω^p_μ is homogeneous.

First we describe the sheaf of relative differential 1-forms for a general fibration between differentiable manifolds. Let $f : Y \to X$ be a C^∞ fibration. Then $\ker df$, the relative tangent bundle, is a subbundle of the tangent bundle of Y whose stalk at y is the kernel of df_y and $(\ker df)^*$ is the relative cotangent bundle. Let $\mathcal{E}^1_M$ denote the sheaf of smooth differential 1-forms on a manifold M and let $\mathcal{E}_M$ be the sheaf of C^∞ functions on M. Then $f^*\mathcal{E}^1_X = f^{-1}\mathcal{E}^1_X\otimes_{f^{-1}\mathcal{E}_X}\mathcal{E}_Y$ and the sheaf of relative differential 1-forms is $\mathcal{E}^1_f = \mathcal{E}^1_Y/f^*\mathcal{E}^1_X$.

Theorem 4.4. *$\mathcal{E}^1_f$ and $\mathcal{E}((\ker df)^*)$ are isomorphic as sheaves.*

Sketch of Proof. Since it suffices to check this on sufficiently small open sets, we may assume that U, an open subset of Y, is isomorphic to $\mathbb{R}^n\times\mathbb{R}^k$. The map $\gamma_U : \mathcal{E}^1_f(U) \to \mathcal{E}((\ker df)^*)(U)$ defined by $\gamma_U([w]) = \gamma_w$ where $\gamma_w(y) = w(y)|_{\ker df}$ and $y\in Y$ gives the isomorphism. □

We will use this theorem to describe the relative holomorphic (1,0)-forms for the holomorphic fibration from $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}$ to $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}$ given by the natural projection map $\tilde{\mu}$. As is customary, we identify the holomorphic tangent space $T^{1,0}(G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})$ with $T(G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}})$; we do likewise for $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-}$. Under these identifications, Theorem 4.4 implies that the sheaf of relative holomorphic 1-forms $\Omega^1_{\tilde{\mu}}$ is isomorphic to the sheaf $\mathcal{O}((\ker d\tilde{\mu})^*)$. Now $\ker d\tilde{\mu}$ is the $G_{\mathbb{C}}$–homogeneous bundle with fiber $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}$.

Theorem 4.4 again implies that Ω^1_{μ} is the sheaf $\mathcal{O}((\ker d\mu)^*)$ where here d_μ is the map from $T^{1,0}(Y_{D_i})$ to $T^{1,0}(D_i)$. Since the map μ from $G/H_i \cap K \times \overline{G/K}$ to G/H_i is given in terms of isotropic planes and maximal compact subvarieties and not as a map of homogeneous spaces, we are unable to use the definition of μ to determine $\ker d\mu$. However, we can give an explicit description of Ω^1_{μ} by understanding the relationship between Ω^1_{μ} and $\Omega^1_{\tilde{\mu}}$.

Theorem 4.5.

(1) *The sheaf* Ω^1_{μ} *is isomorphic to* $\mathcal{O}((\ker d\mu)^*)$.
(2) *The vector bundle* $\ker d\mu$ *is* $(G \times G)$*-homogeneous with fiber* $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$ *where* $(H_i \cap K) \times K$ *acts by* $Ad \otimes 1$.
(3) *The vector bundle* $(\ker d\mu)^*$ *is* $(G \times G)$*-homogeneous with fiber* $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}$ *where* $(H_i \cap K) \times K$ *acts by* $Ad \otimes 1$.

Proof of (1). This follows from the discussion before the statement of Theorem 4.5. □

Proof of (2). Recall the map

$$\varphi : {}^{G}\!/_{H_i \cap K} \times \overline{{}^{G}\!/_{K}} \to {}^{G_{\mathbb{C}}}\!/_{H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}}$$

from Section 3.3. Since the image of φ is open in $G_{\mathbb{C}}/H_{i,\mathbb{C}}Q_{i,-} \cap K_{\mathbb{C}}$, the fiber of μ is open in the fiber of $\tilde{\mu}$. Thus, we have $\ker d\mu = (\ker d\tilde{\mu})|_{Im(\varphi)}$. Then, as in Theorem 4.3, we can define an action of $G \times G$ on $\ker d\mu$ and the action of $(H_i \cap K) \times K$ on $(\ker d\mu)_{\overline{e}}$ is determined by its action on $(\ker d\tilde{\mu})|_{Im(\varphi)}$. Thus $(H_i \cap K) \times K$ acts on $\big((\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}\big)$ via $\mathrm{Ad} \otimes 1$. Since $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$ is an $\big[(H_i \cap K) \times K\big]$-invariant complement to $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_{\mathbb{C}}$ in $\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}$, the bundle $\ker d\mu$ is $(G \times G)$–homogeneous with fiber $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$. □

Proof of (3). The Killing form can be used to identify $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}$ as the dual of $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{p}$ in $\mathfrak{g}_{\mathbb{C}}$. □

We are now ready to apply Lemma 2.2.

4.2. The Vanishing Condition. We will show in this section that $H^p(Y_{D_i}, \Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ vanishes for all $p < s$ and $1 \le q \le m$ if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$. That is, we obtain the hypothesis of Lemma 2.2 for the resolution of $\mu^{-1}\mathcal{L}_\chi$ given in (4.2). Once this is accomplished, we have:

Theorem 4.6. *If $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$, then there is an injection from $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi\big)$ into $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ whose image is the kernel of the induced map from $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$.*

To obtain the vanishing condition, we make the following observations. First, the manifold M_{D_i} is Stein [**Wo2**, **Wo3**]. Second, since the map ν is a fibration, it is proper because the inverse image of a point in M_{D_i} under ν is isomorphic to the compact submanifold $K/H_i \cap K$. Third, the sheaves $\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ are coherent since each is the sheaf of sections of a homogeneous vector bundle. (See Theorem 4.3 and 4.5.)

Now we can apply Theorem 2.8 to $H^p(Y_{D_i}, \Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ to obtain the following theorem.

Theorem 4.7. *$H^p(Y_{D_i}, \Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ is isomorphic to $H^0(M_{D_i}, R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ for all p and q.*

Now we will show that $H^0(M_{D_i}, R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi))$ vanishes for all $p < s$ and $1 \le q \le m$ if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$. Recall that $\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous bundle $\mathbb{V}^q_\chi = \wedge^q(\ker d\mu)^* \otimes \mu^*\mathbb{L}_\chi$ on Y_{D_i} whose fiber is $\wedge^q\big[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}\big] \otimes \mathbb{C}_\chi$.

Before we look at the structure of $R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ we identify the fiber of ν with $K/H_i \cap K$. Under this identification, the restriction of $\mu^*\mathbb{L}_\chi$ to the fiber of ν is the bundle $K \times_{(H_i \cap K)} \mathbb{C}_\chi$ and the restriction of $(\ker d\mu)^*$ is the K-homogeneous bundle with fiber $\wedge^q\big[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}\big]$ (see Theorem 4.5).

With these identifications, the restriction of $\mathbb{V}^q_\chi$ to the fiber of ν is the bundle

$$K \times_{(H_i \cap K)} [\wedge^q[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}] \otimes \mathbb{C}_\chi]$$

which we also denote by $\mathbb{V}^q_\chi$. With this in mind, a theorem of Bott [**B**] implies that $R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of the $(G\times G)$–homogeneous vector bundle $\mathbb{H}^p\,(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$ on M_{D_i} whose fiber is $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$.

We summarize this discussion with the following lemma.

Lemma 4.8. *The sheaf $R^p_\nu\Omega^q_\mu(\mu^*\mathbb{L}_\chi)$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous vector bundle on M_{D_i} whose fiber is $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$ and the action of $K \times K$ on the fiber is given by $(k_1, k_2)\cdot\omega = \ell^*_{k_1^{-1}}\omega$*

where ℓ_k is the map from $K/H_i \cap K$ to itself given by left translation. We denote $R^p_\nu \Omega^q_\mu(\mu^* \mathbb{L}_\chi)$ by $\mathcal{O}\big[\mathbb{H}^p\big(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi)\big)\big]$.

We now state the vanishing condition.

Theorem 4.9. *$H^p\left({}^{K}\!/_{H_i \cap K}, \mathcal{O}(\mathbb{V}^q_\chi)\right)$ vanishes for $p < s$ and $1 \le q \le m$ if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$.*

The proof of this theorem is an application of Bott-Borel-Weil along with the following observations. Since the fiber V^q_χ of $\mathbb{V}^q_\chi$ is reducible, we decompose V^q_χ into irreducible subrepresentations $V_1, \ldots, V_j$. Then Bott-Borel-Weil determines a condition on χ such that $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}_i))$ vanishes for all $p \neq s$. Thus $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q_\chi))$ also vanishes. Since we do not know the V_i's or their highest weights, we choose χ such that $\langle \chi + \gamma + \rho_K, \alpha \rangle < 0$ for all $\alpha \in \triangle(\mathfrak{q}_{i,+} \cap \mathfrak{k}_\mathbb{C})$ and for all weights γ of V^q_χ which guarantees the vanishing of $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}_i))$ for all i and for all $p \neq s$. If χ is chosen such that $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and if $a < -3n + r$ when $r < n$, then the vanishing is guaranteed. The calculations for this theorem were done with $\triangle^+(\mathfrak{k}_\mathbb{C}) = \{e_j - e_k : 1 \le k < i \text{ and } k \le j \le n, i+1 \le j < k \le i+n-r, i+1 \le k \le i+n-r < j \le n, \text{ or } i+n-r+1 \le k < j \le n\}$. Thus we have obtained the hypothesis for Lemma 2.2 and have proved Theorem 4.6.

4.3. Pushing Down to M_{D_i} by ν. Now we will push $H^s(Y_{D_i}, \mathcal{O}(\mu^* \mathbb{L}_\chi))$ down to M_{D_i} and construct the double fibration transform.

Theorem 4.10. *$H^s\big(Y_{D_i}, \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ is isomorphic to $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ which is isomorphic to $H^0\big(M_{D_i}, \mathcal{O}\big[\mathbb{H}^s(K/H_i \cap K, \mathcal{L}_\chi)\big]\big)$.*

Proof. This is Theorem 4.3 and Lemma 4.8 applied to the sheaf $\mathcal{O}(\mu^* \mathbb{L}_\chi)$. □

Now we can define the double fibration transform.

Theorem 4.11. *Define the map*

$$P : H^s\big(D_i, \mathcal{L}_\chi\big) \to H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$$

by the composition of the maps in Theorem 4.1, Theorem 4.6, and Theorem 4.10. Then P is the double fibration transform and it is an injection if $a < \frac{1}{2} - \frac{3}{2}n$ when $r = n$ and $a < -3n + r$ when $r < n$. Also, the image of P is isomorphic to the kernel of a map $\mathcal{D}$ from $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ to $H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^* \mathbb{L}_\chi)\big)$ where $\mathcal{D}$ is defined in Theorem 2.9.

4.4. Bott-Borel-Weil applied to $H^s(K/H_i \cap K, \mathcal{O}(\mathcal{L}_\chi))$. Before we investigate the map $\mathcal{D}$ in the next section, we will further our understanding of $H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^* \mathbb{L}_\chi)\big)$ As before, a theorem of Bott [**B**] implies that the fiber of $R^s_\nu(\mu^*\mathbb{L}_\chi)$ is $H^s(K/H_i \cap K, \mathcal{L}_\chi)$.

We have the following lemma.

Lemma 4.12. *For $r = n$, if $a < \frac{1}{2}(1-n)$, the cohomology space $H^p(K/H_i \cap K, \mathcal{L}_\chi)$ vanishes whenever $p < s$ and whenever $p = s$ it is the nonzero irreducible K-representation of highest weight $(a+i, \dots, a+i\ ;\ -a-n+i, \dots, -a-n+i)$ where there are $(n-i)$ entries before the semicolon.*

For $r < n$ if $a < -n+1$ then $H^p(K/H_i \cap K, \mathcal{L}_\chi)$ vanishes whenever $p < s$ and whenever $p = s$ it is a nonzero irreducible K-representation. If $r - i \leq i$, the highest weight of the representation is

$$\begin{aligned}(a+i+n-r, \dots, a+i+n-r;\\ 2i-r, \dots 2i-r| - a-n+i, \cdots -a-n+i;\\ 2i-r, \dots, 2i-r| - a-n+i, \dots, -a-n+i)\end{aligned}$$

where there are $(r-i)$-entries before the first semicolon, a total of i-entries before the first vertical bar, $(2i-r)$-entries between the first vertical bar and the second semicolon, a total of $(n-r)$-entries between the vertical bars, and $(r-i)$-entries after the second vertical bar.

If $r - i > i$, the highest weight of the representation is

$$\begin{aligned}(a+i+n-r, \dots, a+i+n-r|2i-r, \dots 2i-r;\\ a+i+n-r, \dots a+i+n-r|2i-r, \dots, 2i-r;\\ -a-n+i, \dots, -a-n+i)\end{aligned}$$

where there are i-entries before the first vertical bar, $(n+2i-2r)$-entries between the first vertical bar and the first semicolon, a total of $(n-r)$-entries between the two vertical bars, $(r-2i)$-entries between the second vertical bar and the second semicolon, a total of $(r-i)$-entries after the second vertical bar.

The proof is an application of Bott-Borel-Weil.

5. The differential operator.

The double fibration transform realizes the representation $H^s\big(D_i, \mathcal{L}_\chi\big)$ as the kernel of the map

$$\mathcal{D} : H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\left(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\right) \tag{5.1}$$

defined in Theorem 4.11 and Theorem 2.9. In this section, we will describe $\mathcal{D}$ more explicitly and show that it is a G-invariant differential operator.

Recall that $H^s\big(D_i, \mathcal{L}_\chi\big)$ is isomorphic to $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi\big)$ and that $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{L}_\chi\big)$ is the kernel of the map $\partial_\mu^* : H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$ where ∂_μ^* is induced from the map $\partial_\mu : \mathcal{O}_{Y_{D_i}} \to \Omega^1_\mu$. Now a Leray spectral sequence argument shows that $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ is isomorphic to $H^0\big(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ and that $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$ is isomorphic to $H^0\big(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$. The map ∂_μ determines a map between the spectral sequences and induces the map $\mathcal{D}$ in (5.1).

To understand $\mathcal{D}$, we need a better understanding of the map $\partial_\mu : \mathcal{O}_{Y_{D_i}} \to \Omega^1_\mu$. By definition $\partial_\mu = \pi \circ \partial$ where $\partial : \mathcal{O}_{Y_{D_i}} \to \Omega^1_{Y_{D_i}}$ is the standard holomorphic deRahm operator on Y_{D_i} and π is the quotient map from $\Omega^1_{Y_{D_i}}$ to $\Omega^1_\mu = \Omega^1_{Y_{D_i}}/\mu^*\Omega^1_{D_i}$. We note that, in this case, $d = \partial$ since $\overline{\partial} = 0$ on the sheaves of interest. Although we can realize both $\Omega^1_{Y_{D_i}}$ and Ω^1_μ as sheaves of holomorphic sections of a $(G\times G)$–homogeneous bundles, the map π is not determined by a $(G\times G)$-equivariant bundle map. To understand π we will decompose it into $\pi_1 \circ \pi_2$ where $\pi_2 : \Omega^1_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ and $\pi_1 : \nu^*\Omega^1_{M_{D_i}} \to \Omega^1_\mu$. Once we show that π_2 is a $(G\times G)$-equivariant map and π_1 is equivariant for the diagonal embedding of G in $G\times G$, then π will be a G–equivariant map.

Let $\partial_2 = \pi_2 \circ \partial$ and let ∂_2^* be the induced map from the cohomology space $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ to $H^s\big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big)$. We will see in Section 5.1 that the corresponding map

$$\mathcal{D}_2 : H^0\big(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\big(M_{D_i}, R^s_\nu\big[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big]\big)$$

is the standard holomorphic d operator on M_{D_i} and that $\mathcal{D}_2$ is a $(G\times G)$-invariant first-order differential operator.

Now π_1 induces a map

$$\pi_1 : H^s\Big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\Big) \to H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big).$$

In Section 5.2, we will show that the corresponding map

$$\mathcal{D}_1 : H^0\Big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\Big) \to H^0\big(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$$

is a G-invariant zeroth-order differential operator. As $\mathcal{D}_1$ is G-invariant and $\mathcal{D}_2$ is $(G\times G)$-invariant, the map $\mathcal{D} = \mathcal{D}_1 \circ \mathcal{D}_2$ is a G-invariant first-order differential operator for the diagonal embedding of G in $G\times G$.

5.1. The operator $\mathcal{D}_2$. In this section we will define $\pi_2 : \Omega^1_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ and give an explicit realization of the map $\partial_2 : \mathcal{O}_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ which induces the map

$$\partial_2^* : H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^s\big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big).$$

We will then see how ∂_2^* determines the $(G{\times}G)$-invariant differential operator

$$\mathcal{D}_2 : H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big)$$

by examining the maps between the Leray spectral sequences for $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ and $H^s\big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big)$.

The Leray spectral sequence which defines the isomorphism between the cohomology spaces $H^s\big(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ and $H^0\big(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\big)$ is realized from a filtration of the resolution

$$0 \to \mathcal{O}(\mu^*\mathbb{L}_\chi) \to \mathcal{E}^{0,0}(\mu^*\mathbb{L}_\chi) \to \mathcal{E}^{0,1}(\mu^*\mathbb{L}_\chi) \to \cdots \to \mathcal{E}^{0,a}(\mu^*\mathbb{L}_\chi) \to 0$$

with respect to the fiber of ν. (See, for example, [**G**].) Using the homogeneous structure of $\mathcal{E}^{0,c}$ and $\mu^*\mathbb{L}_\chi$, the E_0-term is given by

$$E_0^{p,q} = C^\infty\big(G{\times}G, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes \wedge^p \mathfrak{d}\big)^{(H_i \cap K) \times K} \tag{5.2}$$

where $\mathbb{C}_\chi$ is an $(H_i \cap K)$-representation and where $\mathfrak{c}_i$ and $\mathfrak{d}$ represent the fiber of the antiholomorphic cotangent space of $K/H_i \cap K$ and $G/K \times \overline{G/K}$ respectively. Since $\wedge^p\mathfrak{d}$ is a $(K \times K)$-representation, the following lemma gives another realization of $E_0^{p,q}$.

Lemma 5.3.

$$E_0^{p,q} = C^\infty\Big(G{\times}G, C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes \wedge^p\mathfrak{d}\Big)^{K \times K}. \tag{5.4}$$

Proof. The isomorphism is given by sending φ to $\tilde\varphi$ where $\tilde\varphi(g_1, g_2)(k) = \varphi(g_1 k, g_2)$. A straightforward computation shows that $\tilde\varphi$ has the correct invariance property. □

Then $E_1^{p,q} = C^\infty\big(G{\times}G, H^q(K/H_i \cap K, \mathcal{O}(\mathbb{C}_\chi)) \otimes \wedge^p\mathfrak{d}\big)^{K\times K}$ and

$$\begin{aligned} E_2^{p,q} &= H^p\Big(G/K \times \overline{G/K}, \mathcal{O}\big[\mathbb{H}^q(K/H_i \cap K, \mathcal{O}(\mathbb{C}_\chi))\big]\Big) \\ &= H^p\Big(G/K \times \overline{G/K}, R^q_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)\Big). \end{aligned}$$

Now we turn our attention to the Leray spectral sequence for the cohomology space $H^s\Big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\Big)$. The E_0-term of the Leray spectral sequence which defines the isomorphism between the spaces $H^s\Big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\Big)$ and $H^0\Big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\Big)$ is given by

$$E_{0,M}^{p,q} = C^\infty\big(G{\times}G, \mathbb{C}_\chi \otimes \wedge^q\mathfrak{c}_i \otimes \wedge^p\mathfrak{d} \otimes (\mathfrak{p}_+ \oplus \mathfrak{p}_-)\big)^{(H_i\cap K)\times K} \tag{5.5}$$

where here we are identifying $(\mathfrak{g}_\mathbb{C}/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\pm))^*$ with $\mathfrak{p}_\mp$.

Lemma 5.6.

$$E_{0,M}^{p,q} = C^\infty\Big(G\times G, \big[C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes \mathfrak{p}_+\big)^{H_i\cap K} \otimes \wedge^p \mathfrak{d}\big] \oplus \big[C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes \big(\mathfrak{p}_- \otimes \wedge^p \mathfrak{d}\big)\big]\Big)^{K\times K}. \tag{5.7}$$

Proof. As in Lemma 5.3, we see that (5.5) is isomorphic to

$$C^\infty\big(G\times G, C^\infty\big(K\times K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes (\mathfrak{p}_+ \oplus \mathfrak{p}_-)\big)^{(H_i\cap K)\times K} \otimes \wedge^p \mathfrak{d}\big)^{K\times K}. \tag{5.8}$$

Since $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are K-representations, the inside of (5.8) is isomorphic to

$$C^\infty\big(K\times K, \big(\mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big) \otimes 1\big)^{(H_i\cap K)\times K} \otimes \big((\mathfrak{p}_+ \otimes 1) \oplus (1 \otimes \mathfrak{p}_-)\big)$$

which is isomorphic to

$$C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes \big((\mathfrak{p}_+ \otimes 1) \oplus (1 \otimes \mathfrak{p}_-)\big).$$

The lemma follows from splitting up the direct sum and identifying

$$C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i\big)^{H_i\cap K} \otimes (\mathfrak{p}_+ \otimes 1)$$

with

$$C^\infty\big(K, \mathbb{C}_\chi \otimes \wedge^q \mathfrak{c}_i \otimes \mathfrak{p}_+\big)^{H_i\cap K}.$$

□

Now that we have explicit descriptions of the Leray spectral sequences, we look at the map π_2 and $\pi_2 \circ \partial$ in more detail so we can define a map between the spectral sequences. To define the map π_2, we observe that $\nu^*\Omega^1_{M_{D_i}}$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous bundle on Y_{D_i} with fiber

$$\Big(\mathfrak{g}_\mathbb{C}/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_+) \quad \oplus \quad \mathfrak{g}_\mathbb{C}/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_-)\Big)^* \tag{5.9}$$

and $\Omega^1_{Y_{D_i}}$ is the sheaf of holomorphic sections of the $(G\times G)$-homogeneous bundle with fiber

$$\Big(\mathfrak{g}_\mathbb{C}/[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_\mathbb{C}] \oplus \mathfrak{p}_+ \quad \oplus \quad \mathfrak{g}_\mathbb{C}/(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_-)\Big)^*. \tag{5.10}$$

The natural map from (5.9) to (5.10) induces the $(G\times G)$–equivariant map $\pi_2 : \Omega^1_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$. Then $\partial_2 = \pi_2 \circ \partial$ is a map from $\mathcal{O}_{Y_{D_i}}$ to $\nu^*\Omega^1_{M_{D_i}}$.

In the following lemma, we give an explicit formula for $\partial_2 : \mathcal{O}_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ which will lead to a formula for $\partial_{2,0} : E_0^{p,q} \to E_{0,M}^{p,q}$.

Lemma 5.11. *The map* $\partial_2 : \mathcal{O}_{Y_{D_i}} \to \nu^*\Omega^1_{M_{D_i}}$ *is given by*

$$\partial_2(\psi) = \sum_{\alpha\in\mathfrak{p}_-} r_1(X_\alpha)\psi \otimes X_{-\alpha} + \sum_{\beta\in\mathfrak{p}_+} r_2(X_\beta)\psi \otimes X_{-\beta} \tag{5.12}$$

where ψ represents the corresponding element of $\mathcal{O}(G\times G)^{(H_i\cap K)\times K}$. Here

$$\big(r_1(X_\alpha)\psi\big)(g_1,g_2)=\frac{d}{dt}\Big|_{t=0}\psi(g_1\exp tX_\alpha,g_2)$$

and

$$\big(r_2(X_\beta)\psi\big)(g_1,g_2)=\frac{d}{dt}\Big|_{t=0}\psi(g_1,g_2\exp tX_\beta).$$

Proof. The manifolds $G/H_i\cap K$ and $\overline{G/K}$ are open orbits in the generalized flags $G_{\mathbb{C}}/(H_{i,\mathbb{C}}Q_{i,-}\cap K_{\mathbb{C}})P_+$ and $G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ respectively. Griffiths and Schmid's [**GS**] formula for the standard $\overline{\partial}$ operator implies that the standard ∂ operator from $\mathcal{O}_{Y_{D_i}}$ to $\Omega^1_{Y_{D_i}}$ is given by

$$\partial(\psi)=\sum_{\alpha\in(\mathfrak{q}_{i,+}\cap\mathfrak{k}_{\mathbb{C}})\oplus\mathfrak{p}_-}r_1(X_\alpha)\psi\otimes X_{-\alpha}+\sum_{\beta\in\mathfrak{p}_+}r_2(X_\beta)\psi\otimes X_{-\beta}.$$

The lemma follows from the fact that $\partial_2=\pi_2\circ\partial$. □

The map $\partial_2:\mathcal{O}_{Y_{D_i}}\to\nu^*\Omega^1_{M_{D_i}}$ determines a map between the resolutions $\mathcal{E}^{0,\bullet}\otimes\mathcal{O}_{Y_{D_i}}(\mu^*\mathcal{L}_\chi)$ and $\mathcal{E}^{0,\bullet}\otimes\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathcal{L}_\chi)$ which respects the filtration along the fibers of ν. This map of resolutions induces a map between the associated Leray spectral sequences (see, for example, [**G**]). Let $\partial_{2,0}$ be the induced map from $E_0^{p,q}$ to $E_{0,M}^{p,q}$ (i.e., from (5.2) to (5.5)). Thus the formula for $\partial_{2,0}$ is the same as the formula for ∂_2.

Lemma 5.13. *The map $\tilde{\partial}_{2,0}$ from* (5.4) *to* (5.7) *is given by* (5.12).

Proof. The isomorphism between (5.2) and (5.4) and the one between (5.5) and (5.7) as defined in Lemma 5.3 and 5.6 respectively imply that

$$(\tilde{\partial}_{2,0}\psi)(g_1,g_2)(k)=\sum_{\alpha\in\mathfrak{p}_-}\big(r_1(X_\alpha)\psi\big)(g_1,g_2)(k)\otimes\operatorname{Ad}(k^{-1})X_{-\alpha}$$
$$+\sum_{\beta\in\mathfrak{p}_+}\big(r_2(X_\beta)\psi\big)(g_1,g_2)(k)\otimes X_{-\beta}.$$

Since $C^\infty\big(K,\mathbb{C}_\chi\otimes\wedge^q\mathfrak{c}_i\otimes\mathfrak{p}_+\big)^{H_i\cap K}$ is isomorphic to $C^\infty\big(K,\mathbb{C}_\chi\otimes\wedge^q\mathfrak{c}_i\big)^{H_i\cap K}\otimes\mathfrak{p}_+$, the lemma follows. □

Thus the maps $\tilde{\partial}_{2,0}$ and $\partial_{2,1}:E_1^{p,q}\to E_{1,M}^{p,q}$ where

$$E_{1,\mu}^{p,q}=C^\infty\big(G\times G,\big[\big(H^q(K/H_i\cap K,\mathcal{O}(\mathcal{L}_\chi))\otimes\mathfrak{p}_+\big)\otimes\wedge^p\mathfrak{d}\big]\oplus$$
$$\big[H^q(K/H_i\cap K,\mathcal{O}(\mathcal{L}_\chi))\otimes(\mathfrak{p}_-\otimes\wedge^p\mathfrak{d})\big]\big)^{K\times K}$$

are each the standard holomorphic d operator on M_{D_i}. Since both spectral sequences collapse at the E_2–term, the map $\partial_{2,2}:E_2^{p,q}\to E_{2,M}^{p,q}$ is the zero map except when $p = 0$ and $q = s$. In that case, it is the standard holomorphic d operator on M_{D_i}

$$\mathcal{D}_2 : H^0\big(M_{D_i}, R^s_\nu \mathcal{O}(\mu^*\mathbb{L}_\chi)\big) \to H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big)$$

and it is a $(G\times G)$-equivariant map.

5.2. The operator $\mathcal{D}_1$. In this section, we define $\pi_1 : \nu^*\Omega^1_{M_{D_i}} \to \Omega^1_\mu$ and then see how this determines the operator

$$\mathcal{D}_1 : H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big) \to H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big).$$

To define the map π_1 we observe that the restriction of $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ and $\Omega^1_{\tilde{\mu}}$ to Y_{D_i} is isomorphic to $\nu^*\Omega^1_{M_{D_i}}$ and Ω^1_μ respectively. Because the embedding of Y_{D_i} in Y_{X_i} is G-equivariant, these two isomorphisms are G–equivariant. Now $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ is the sheaf of holomorphic sections of the $G_\mathbb{C}$-homogeneous bundle with fiber

$$\left({}^{\mathfrak{g}_\mathbb{C}}\big/_{\mathfrak{k}_\mathbb{C}}\right)^* \tag{5.14}$$

and $\Omega^1_{\tilde{\mu}}$ is the sheaf of holomorphic sections of the $G_\mathbb{C}$-homogeneous bundle with fiber

$$\left({}^{(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+})}\big/_{(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{k}_\mathbb{C}}\right)^*. \tag{5.15}$$

Then $\tilde{\pi}_1$ is the $G_\mathbb{C}$–equivariant map induced by the natural restriction map from (5.14) to (5.15) and $\pi_1 = \tilde{\pi}_1|_{\nu^*\Omega^1_{M_{D_i}}}$ is the G-equivariant map from $\nu^*\Omega^1_{M_{D_i}}$ to Ω^1_μ.

As in Section 5.1, the Leray spectral sequences for $H^s\big(Y_{D_i}, \nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big)$ and $H^s\big(Y_{D_i}, \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$ together with the map $\pi_1 : \nu^*\Omega^1_{M_{D_i}} \to \Omega^1_\mu$ determine a map

$$\mathcal{D}_1 : H^0\big(M_{D_i}, R^s_\nu[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)]\big) \to H^0\big(M_{D_i}, R^s_\nu \Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big).$$

In this case, this process does not yield an explicit description of $\mathcal{D}_1$ because we do not have an explicit description of π_1 in terms of the homogeneous vector bundles for $\nu^*\Omega^1_{M_{D_i}}$ and Ω^1_μ. To resolve this difficulty we use the fact that π_1 is the restriction of the map $\tilde{\pi}_1$ to $\nu^*\Omega^1_{M_{D_i}}$. Since $\tilde{\pi}_1$ can be described explicitly we use the Leray spectral sequences for $H^s\big(Y_{X_i}, \tilde{\nu}^*\Omega^1_{M_{X_i}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\big)$ and $H^s\big(Y_{X_i}, \Omega^1_{\tilde{\mu}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\big)$ to determine the map $\widetilde{\mathcal{D}}_1$ from $H^0\big(M_{X_i}, R^s_{\tilde{\nu}}[\tilde{\nu}^\star\Omega^1_{M_{X_i}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})]\big)$ to $H^0\big(M_{X_i}, R^s_{\tilde{\nu}}\Omega^1_{\tilde{\mu}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\big)$. Then we will show that $\widetilde{\mathcal{D}}_1$ restricts to $\mathcal{D}_1$.

Using the homogeneous structure of $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ we see that the E_0-term of the Leray spectral sequence for $H^s\Big(Y_{X_i}, \tilde{\nu}^*\Omega^1_{M_{X_i}}(\tilde{\mu}^*\mathbb{L}_{\tilde{\chi}})\Big)$ is given by

$$\widetilde{E}^{p,q}_{0,M} = C^\infty\big(G_\mathbb{C}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes \wedge^p \mathfrak{d} \otimes F_2\big)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_\mathbb{C}}$$

where $F_2 = (\mathfrak{g}_\mathbb{C}/\mathfrak{k}_\mathbb{C})^*$. Now $\widetilde{E}^{p,q}_{0,M}$ is isomorphic to

$$C^\infty\big(G_\mathbb{C}, C^\infty(K_\mathbb{C}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes F_2)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_\mathbb{C}} \otimes \wedge^p \mathfrak{d}\big)^{K_\mathbb{C}}$$

by sending φ to $\tilde{\varphi}$ where $\tilde{\varphi}(g)(k) = \varphi(gk)$. Likewise the E_0–term of the Leray spectral sequence for $H^s\Big(Y_{X_i}, \Omega^1_{\tilde{\mu}}(\mu^*\mathbb{L}_{\tilde{\chi}})\Big)$ is given by

$$\widetilde{E}^{p,q}_{0,\mu} = C^\infty\big(G_\mathbb{C}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes \wedge^p \mathfrak{d} \otimes F_3\big)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_\mathbb{C}}$$

where $F_3 = \big((\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_\mathbb{C}\big)^*$ and $\widetilde{E}^{p,q}_{0,\mu}$ is isomorphic to

$$C^\infty\big(G_\mathbb{C}, C^\infty(K_\mathbb{C}, \mathbb{C}_{\tilde{\chi}} \otimes \wedge^q \mathfrak{c}_i \otimes F_3)^{H_{i,\mathbb{C}}Q_{i,-}\cap K_\mathbb{C}} \otimes \wedge^p \mathfrak{d}\big)^{K_\mathbb{C}}.$$

Since the homogeneous structures of $\tilde{\nu}^*\Omega^1_{M_{X_i}}$ and $\Omega^1_{\tilde{\mu}}$ are compatible, we can give an explicit realization of the map $\tilde{\pi}_{1,0} : \widetilde{E}^{p,q}_{0,M} \to \widetilde{E}^{p,q}_{0,\mu}$ induced by π_1. Let r be the map from F_2 to F_3 given by restriction. Then $(\tilde{\pi}_{1,0}(\varphi))\,(g)(k) = r(\varphi(g)(k))$.

For the E_1-terms we have that

$$\widetilde{E}^{p,q}_{1,M} = C^\infty\big(G_\mathbb{C}, H^q(K_\mathbb{C}/H_{i,\mathbb{C}}Q_{i,-} \cap K_\mathbb{C}, \mathcal{O}(\mathbb{F}_{2,\tilde{\chi}})) \otimes \wedge^p \mathfrak{d}\big)^{K_\mathbb{C}} \tag{5.16}$$

and

$$\widetilde{E}^{p,q}_{1,\mu} = C^\infty\big(G_\mathbb{C}, H^q(K_\mathbb{C}/H_{i,\mathbb{C}}Q_{i,-} \cap K_\mathbb{C}, \mathcal{O}(\mathbb{F}_{3,\tilde{\chi}})) \otimes \wedge^p \mathfrak{d}\big)^{K_\mathbb{C}} \tag{5.17}$$

where $\mathbb{F}_{j,\tilde{\chi}}$ is the homogeneous bundle on $K_\mathbb{C}/H_{i,\mathbb{C}}Q_{i,-} \cap K_\mathbb{C}$ with fiber $F_j \otimes \mathbb{C}_{\tilde{\chi}}$.

Since $K/H_i \cap K = K_\mathbb{C}/(H_{i,\mathbb{C}}Q_{i,-} \cap K_\mathbb{C})$ we can identify the cohomology space in (5.16) with

$$H^q\big(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})\big) \tag{5.18}$$

and the cohomology space in (5.17) with

$$H^q\big(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})\big) \tag{5.19}$$

where $\mathbb{F}_{j,\chi}$ is the homogeneous bundle on $K/H_i \cap K$ with fiber $F_j \otimes \mathbb{C}_\chi$.

The map $\tilde{\pi}_{1,0}$ induces a map from $\tilde{E}^{p,q}_{1,M}$ to $\tilde{E}^{p,q}_{1,\mu}$ which is determined by the map from (5.18) to (5.19). To determine this map for $q = s$, we let $F_1 = (\mathfrak{g}_\mathbb{C}/(\mathfrak{h}_{i,\mathbb{C}} + \mathfrak{q}_{i,-} + \mathfrak{k}_\mathbb{C}))^*$.

Lemma 5.20.

$$0 \to F_1 \xrightarrow{j} F_2 \xrightarrow{r} F_3 \to 0 \tag{5.21}$$

is a short exact sequence where j is the natural inclusion map.

Proof. For vector spaces $W \subset V$ the dual space $(V/W)^*$ can be identified with the set of $\lambda \in V^*$ such that $\lambda|_W = 0$. Since F_3 can also be written as $\big((\mathfrak{h}_{i,\mathbb{C}} + \mathfrak{q}_{i,-} + \mathfrak{k}_{\mathbb{C}})/\mathfrak{k}_{\mathbb{C}}\big)^*$ the lemma follows. □

Now (5.21) induces a short exact sequence in cohomology.

Lemma 5.22.

$$0 \to H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{1,\chi})) \xrightarrow{j^*} H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) \xrightarrow{r^*} H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) \to 0$$

is an exact sequence for $r = n$ if $a < \frac{1}{2} - \frac{3}{2}n$ and for $r < n$ if $a < -3n + r$.

Proof. The short exact sequence (5.21) induces the following short exact sequence of sheaves

$$0 \to \mathcal{O}(\mathbb{F}_{1,\chi}) \xrightarrow{\tilde{j}} \mathcal{O}(\mathbb{F}_{2,\chi}) \xrightarrow{\tilde{r}} \mathcal{O}(\mathbb{F}_{3,\chi}) \to 0 \tag{5.23}$$

since j and r are equivariant for $H_i \cap K$ and $\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}}$. The sequence (5.23) induces a long exact sequence in cohomology. Since the dimension of $K/H_i \cap K$ is s, cohomology vanishes in degree greater than s. The space $H^{s-1}(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) = 0$ by Theorem 4.9 and the lemma follows. □

We will now determine r^* explicitly.

Lemma 5.24. *r^* is a linear projection map.*

Proof. Since K is compact, representations of K are semisimple so the short exact sequence splits. Thus, $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{1,\chi}))$ has a complement in $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ which must map isomorphically onto $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$. Thus, r^* is a linear projection map. □

In Appendix B, we decompose the K-representations $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ and determine r^* explicitly for the case when $r = n$.

The map from $\widetilde{E}^{p,s}_{1,M} \to \widetilde{E}^{p,s}_{1,\mu}$ is given by sending φ to $r^* \circ \varphi$ where $(r^* \circ \varphi)(g)(k) = r^*(\varphi(g)(k))$. Thus the map $\widetilde{\mathcal{D}}_1$ from $\widetilde{E}^{0,s}_{2,M} \to \widetilde{E}^{0,s}_{2,\mu}$ is the restriction of the map from $\widetilde{E}^{0,s}_{1,M} \to \widetilde{E}^{0,s}_{1,\mu}$ to holomorphic sections of the bundle $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $\widetilde{\mathcal{D}}_1$ is a $G_{\mathbb{C}}$–invariant zeroth-order differential operator. Since M_{D_i} is open in M_{X_i}, the map $\widetilde{\mathcal{D}}_1$ restricts to a differential operator on M_{D_i}. Now

$$R^s_{\tilde{\nu}}[\tilde{\nu}^* \Omega^1_{M_{X_i}}(\tilde{\mu}^* \mathbb{L}_{\tilde{\chi}})] \simeq R^s_{\nu}[\nu^* \Omega^1_{M_{D_i}}(\mu^* \mathbb{L}_{\chi})] \tag{5.25}$$

so $\widetilde{\mathcal{D}}_1$ restricted to M_{D_i} is the map from $H^0\Big(M_{D_i}, R^s_{\nu}[\nu^* \Omega^1_{M_{D_i}}(\mu^* \mathbb{L}_{\chi})]\Big)$ to $H^0\big(M_{D_i}, R^s_{\nu} \Omega^1_{\mu}(\mu^* \mathbb{L}_{\chi})\big)$ which is given by sending φ to $r^* \circ \varphi$. Since

$\tilde{\pi}_1|_{\nu^*\Omega^1_{M_{D_i}}} = \pi_1$ this map is $\mathcal{D}_1$. The G-equivariance of the map π_1 implies that $\mathcal{D}_1$ is also G-invariant. Thus we have proven the following theorem.

Theorem 5.26.

(1) *The map*

$$\mathcal{D}_1 : H^0\Big(M_{D_i}, R^s_\nu\big[\nu^*\Omega^1_{M_{D_i}}(\mu^*\mathbb{L}_\chi)\big]\Big) \to H^0\big(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$$

is a G-equivariant zeroth–order differential operator.

(2) *The map*

$$\mathcal{D} : H^0(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{L}_\chi)) \to H^0\big(M_{D_i}, R^s_\nu\Omega^1_\mu(\mu^*\mathbb{L}_\chi)\big)$$

is given by $\mathcal{D} = \mathcal{D}_1 \circ \mathcal{D}_2$ and $\mathcal{D}$ is a G-equivariant first–order differential operator.

Appendix A.

The main body of the paper considers the double fibration transform for a family of representations of $Sp(n,\mathbb{R})$ which are realized in cohomology with values in a line bundle. The outline for constructing the double fibration transform, as given in Section 2, is valid if we replace the line bundle $\mathbb{L}$ with a finite dimensional vector bundle $\mathbb{V}$. In this appendix, we consider the details of the construction when the line bundle is replaced with a vector bundle.

Let F_λ be a finite-dimensional, irreducible representation of H_i with highest weight λ and $\mathbb{F}_\lambda$ the corresponding homogeneous vector bundle on D_i. When $r = n$ and $\lambda = (a_1, \cdots, a_i \mid a_{i+1}, \cdots, a_n)$, then λ is a highest weight if

$$a_i \geq \cdots \geq a_1 \geq -a_{i+1} \geq \cdots \geq -a_n. \tag{A.1}$$

When $r < n$ and $\lambda = (a_1, \cdots, a_i \mid a_{i+1}, \cdots, a_{i+n-r} \mid a_{i+n-r+1}, \cdots, a_n)$, then λ is a highest weight if

$$a_i \geq \cdots \geq a_1 \geq -a_{i+n-r+1} \geq \cdots \geq -a_n \tag{A.2}$$

and

$$a_{i+1} \geq \cdots \geq a_{i+n-r} \geq 0. \tag{A.3}$$

The representation $H^s(D_i, \mathcal{O}(\mathbb{F}_\lambda))$ is infinite-dimensional, non-zero, and irreducible [**Wg**] under the following circumstances: When $r = n$, in addition to A.1, we require that $-a_n > n$ and when $r < n$, in addition to A.2 and A.3, we require that $-a_n > n$ and $a_{i+1} + a_n < -2n + r$. Unlike the line bundle case, these representations are not unitarizable.

Now we consider the construction of the double fibration transform. The first step, using Buchdahl's theorem [**Bu**] to identify $H^s(D_i, \mathcal{O}(\mathbb{F}_\lambda))$ with

$H^s\big(Y_{D_i}, \mu^{-1}\mathcal{O}(\mathbb{F}_\lambda)\big)$, remains valid because Buchdahl's theorem, which applies to vector bundles, requires only that the fiber of μ be contractible, which we already have.

The second step, embedding $H^s\big(Y_{D_i}, \mu^{-1}\mathcal{O}(\mathbb{F}_\lambda)\big)$ in $H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{F}_\lambda))$, is more complicated. As in Theorem 4.9, this requires a condition on λ which guarantees that $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{V}^q \otimes \mathbb{F}_\lambda))$ vanish for all $p < s$ and all $1 \leq q \leq m$. Recall that $\mathbb{V}^q$ is the bundle $\wedge^q(\ker d_\mu)^*$ whose fiber is $V^q = \wedge^q[(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}]$. When we consider $V^q \otimes F_\lambda$ as an $K/H_i \cap K$ representation, we see the main difference from the line bundle case. In the line bundle case, when the representation $\mathbb{C}_\chi$ is restricted to $H_i \cap K$, it remains irreducible. This allows us to know explicitly the form of the highest weights of the irreducible components of $V^q \otimes \mathbb{C}_\chi$ (see the discussion after Theorem 4.9) and to compute a specific condition on χ to guarantee vanishing.

Such is not the case for vector bundles. The representation F_λ, when restricted to $H_i \cap K$, may be reducible. If we decompose F_λ as $F_\lambda = \bigoplus_{j=1}^{k} F_{\lambda_j}$ with F_{λ_j} an irreducible representation of $H_i \cap K$ with highest weight λ_j, we can say something about the λ_j's. Since $H_i \cap K = U(i) \times U(r-i) \times Sp(n-r, \mathbb{R})$ is reductive, the highest weight λ_j splits into two pieces: A highest weight λ'_j for the semisimple piece and a character χ_j on the center. Similarly, λ itself is of the form $\lambda = \chi + \lambda'$ when λ is a highest weight of $H_i = U(i, r-i) \times Sp(n-r, \mathbb{R})$. Since the one-dimensional representation remains irreducible under restriction, we have that $\chi_j = \chi$ for all j where $\chi = (-a, \cdots, -a \mid 0, \cdots, 0 \mid a, \cdots, a)$. So, each λ_j is of the form $\chi + \lambda'_j$. Now we replace χ with $\chi + \lambda'_j$ in the proof of Theorem 4.9. Then let

$$C = \max_j \left\{ \left\langle \lambda'_j, e_n - e_1 \right\rangle \right\}$$

and

$$D = \max_{j,t} \left\{ \left\langle \lambda'_j, \alpha_t \right\rangle \right\}$$

with $\alpha_1 = e_{i+1} - e_1, \alpha_2 = e_n - e_i, \alpha_3 = e_n - e_{i+n-r}$. Then the vanishing condition holds for $r = n$ when $a < -3n + 1 - C$ and for $r < n$ when $a < -3n + r - D$.

The third step, pushing $H^s(Y_{D_i}, \mathcal{O}(\mu^*\mathbb{F}_\lambda))$ down to $H^0(M_{D_i}, R^s_\nu\mathcal{O}(\mu^*\mathbb{F}_\lambda))$, is unaffected by changing from a line bundle to a vector bundle.

Likewise, the differential operator is not affected by changing from a line bundle to a vector bundle. Although, as in the line bundle case, when $r < n$, it is difficult to decompose the representations in Lemma 5.22 to give an explicit description of the projection operator r^* as was done when $r = n$ in Appendix B.

Appendix B.

For the case when $r = n$, we will decompose the spaces $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ and determine the map $r^* : H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) \to H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ in Lemma 5.22 explicitly. Since K is compact, each of $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ can be decomposed into a direct sum of irreducible K-representations.

First, we decompose $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$. We cannot apply Bott-Borel-Weil directly to $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ since $\big((\mathfrak{h}_\mathbb{C} \oplus \mathfrak{q}_{i,-})/(\mathfrak{h}_\mathbb{C} \oplus \mathfrak{q}_{i,-}) \cap \mathfrak{k}_\mathbb{C}\big)^*$ $= F_3$ is not an irreducible $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C})$-representation. Although we can decompose F_3 into a direct sum of irreducible $(H_i \cap K)$-representations, in order to decompose $\mathcal{O}(\mathbb{F}_{3,\chi})$ accordingly the decomposition of V must also be as $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C})$-modules (see [**TW**]). If we use the killing form to identify F_3 with $(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}$, then on $\mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}$ the action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ is trivial and on $\mathfrak{q}_{i,+} \cap \mathfrak{p}$,the action is by ad. Since we cannot find a decomposition of F_3 which respects the action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$, we will use a composition series for F_3 to determine $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi}))$ as indicated in the following theorem.

Theorem B.1. *Let V be a representation of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ and let $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = V$ be a composition series for V. Let W_j denote V_j/V_{j-1} and let $\mathbb{V}$ and $\mathbb{W}_j$ be the associated homogeneous vector bundles on $K/H_i \cap K$. Then there exists a spectral sequence with $\mathrm{E}_1^{p,q} = H^{p+q}\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_{N-p})\big)$ which abuts to $H^*\big(K/H_i \cap K, \mathcal{O}(\mathbb{V})\big)$.*

Proof. Since the representations are stable under the action of the antiholomorphic tangent space $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$, the filtration of V induces a filtration in the Dolbeault complex. By the proposition on page 440 of [**GH**] it follows that there exists a spectral sequence with $\mathrm{E}_1^{p,q} = H^{p+q}\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_{N-p})\big)$ which abuts to $H^*\big(K/H_i \cap K, \mathcal{O}(\mathbb{V})\big)$. □

Corollary B.2. *If $H^p\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j)\big) = 0$ for all $p \neq p_0$ and all j, then* $H^{p_0}\big(K/H_i \cap K, \mathcal{O}(\mathbb{V})\big) = \sum_{j=1}^{N} H^{p_0}\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j)\big)$.

Proof. The spectral sequence collapses in Theorem B.1 giving the conclusion. □

Once we find an appropriate decomposition series of F_3 we can use Bott-Borel-Weil to determine when $H^p\big(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j \otimes \mathbb{L}_\chi)\big)$ vanishes for all j and for all $p \neq s$. Choose the following elements for the composition series:

$$F_3 = V_4 = (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p} \qquad V_2 = (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}_+$$
$$V_3 = [(\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,+}) \cap \mathfrak{p}_+] \oplus (\mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_-) \quad V_1 = \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_+.$$

Then each V_j is a representation for $H_i \cap K$ and $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ where the action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ on V_j is the restriction of its action on V. Let $W_j = V_j/V_{j-1}$.

Then the successive quotients are

$$W_4 \simeq \mathfrak{q}_{i,+} \cap \mathfrak{p}_- \qquad W_2 \simeq \mathfrak{q}_{i,+} \cap \mathfrak{p}_+$$
$$W_3 \simeq \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_- \qquad W_1 \simeq \mathfrak{h}_{i,\mathbb{C}} \cap \mathfrak{p}_+.$$

Each W_j is an irreducible $(H_i \cap K)$–representation (since each is the realization of the holomorphic or anti-holomorphic tangent space of some symmetric space) and each W_j is a $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C})$–representation. The induced action of $\mathfrak{q}_{i,-} \cap \mathfrak{k}_\mathbb{C}$ on W_j is trivial. Let λ_j denote the highest weight of W_j. Then

$$\lambda_4 = -2e_1 \qquad \lambda_2 = 2e_n$$
$$\lambda_3 = -e_1 - e_{i+1} \qquad \lambda_1 = e_i + e_n.$$

Lemma B.3. *If $a < -\frac{1}{2}(n+1)$ then $H^p(K/H_i \cap K, \mathcal{O}(\mathbb{W}_j \otimes \mathbb{L}_\chi)) = 0$ for all j whenever $p < s$ and whenever $p = s$ it is an irreducible K-representation with highest weight $\xi + \lambda'_j$. Here $\xi = (a+i, \ldots, a+i\ ;\ -a-n+i, \ldots, -a-n+i)$ with $(n-i)$ entries before the semicolon and*

$$\lambda'_4 = -2e_{n-i+1} \qquad \lambda'_2 = 2e_{n-i}$$
$$\lambda'_3 = -e_1 - e_{n-i+1} \qquad \lambda'_1 = e_{n-i} + e_n.$$

The proof is an application of Bott-Borel-Weil.

Thus we have proven the following theorem.

Theorem B.4. *If $a < -\frac{1}{2}(n+1)$, then*

$$H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) = \bigoplus_{j=1}^{4} E_{\tau_j}$$

where E_{τ_j} is the irreducible K-representation with highest weight $\tau_j = \xi + \lambda'_j$ where ξ and λ'_j are given in Lemma B.3.

Now we will decompose $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$.

Theorem B.5. *If $a < -\frac{1}{2}(n+1)$, then*

$$H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) = \bigoplus_{j=1}^{6} E_{\tau_j}$$

where E_{τ_j} is the irreducible K–representation of highest weight $\tau_j = \xi + \lambda'_j$. Here ξ and λ'_j are given in Lemma B.3 *for $j = 1, \ldots, 4$. Let $\lambda'_5 = 2e_n$ and $\lambda'_6 = -2e_1$.*

Proof. First we identify $F_2 = (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^*$ with $\mathfrak{p}$. The decomposition $\mathfrak{p}_+ \oplus \mathfrak{p}_-$ of $\mathfrak{p}$ respects the action of $H_i \cap K$ and $\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}}$. Thus $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi}))$ decomposes into the direct sum of the K-representations

$$H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i\cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_+)\big]\big) \qquad \text{and}$$

$$H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i\cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_-)\big]\big).$$

Since $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are indecomposable as $(\mathfrak{q}_{i,-} \cap \mathfrak{k}_{\mathbb{C}})$-representations, the cohomology spaces will be computed using a composition series.

Now $U_3 = \mathfrak{p}_+, U_2 = \mathfrak{p}_+ \cap (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})$ and $U_1 = \mathfrak{p}_+ \cap \mathfrak{q}_{i,-}$ is a composition series for $\mathfrak{p}_+$ and $Z_3 = \mathfrak{p}_-, Z_2 = \mathfrak{p}_- \cap (\mathfrak{h}_{i,\mathbb{C}} \oplus \mathfrak{q}_{i,-})$ and $Z_1 = \mathfrak{p}_- \cap \mathfrak{q}_{i,-}$ is a composition series for $\mathfrak{p}_-$. Lemma B.3 implies that

$$H^p\big(K/H_i \cap K, \mathcal{O}(K \times_{H_i\cap K} (\mathbb{C}_\chi \otimes W))\big) = 0 \tag{B.6}$$

for $p < s$ where $W = U_3/U_2, U_2/U_1, Z_3/Z_2$, and Z_2/Z_1. We will determine the condition necessary for (B.6) to hold when W is U_1 or Z_1.

Let $\lambda_5 = 2e_i$ and $\lambda_6 = -2e_{i+1}$. Then λ_5 (respectively λ_6) is the highest weight of the irreducible K-representation U_1 (respectively Z_1). As in the proof of Lemma B.3, to show (B.6) it suffices to show that $\langle \chi + p_k + \lambda_j, e_n - e_1\rangle < 0$ for $j = 5, 6$. Since $\langle \chi + p_k + \lambda_j, e_n - e_1\rangle = 2a + n - 1$ we see that (B.6) is true when $a < -\frac{1}{2}(n+1)$. Thus Theorem B.1 and Corollary B.2 together imply that $H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i\cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_+)\big]\big) = E_{\tau_1} \oplus E_{\tau_2} \oplus E_{\tau_5}$ and that $H^s\big(K/H_i \cap K, \mathcal{O}\big[K \times_{H_i\cap K} (\mathbb{C}_\chi \otimes \mathfrak{p}_-)\big]\big) = E_{\tau_3} \oplus E_{\tau_4} \oplus E_{\tau_6}$ □

We will now determine r^* explicitly.

Lemma B.7. *r^* is a linear projection map.*

Proof. The map r^* is onto by Lemma 5.22. Since $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{3,\chi})) = \bigoplus_{j=1}^{4} E_{\tau_j}$ and $H^s(K/H_i \cap K, \mathcal{O}(\mathbb{F}_{2,\chi})) = \bigoplus_{j=1}^{6} E_{\tau_j}$ and each E_{τ_j} is an irreducible K-representation, r^* is the natural projection map from $\bigoplus_{j=1}^{6} E_{\tau_j}$ to $\bigoplus_{j=1}^{4} E_{\tau_j}$. □

References

[AG] D. Akheizer and S. Gindikin, *On the Stein extensions of real symmetric spaces*, Math. Ann., **286** (1990), 1-12.

[A] E. Artin, *Geometric Algebra*, Interscience Publishers, Inc., New York, 1988.

[BE] R.J. Baston and M.G. Eastwood, *The Penrose Transform: Its Interaction with Representation Theory*, Clarendon Press, Oxford, 1989.

[B] R. Bott, *Homogeneous vector bundles*, Ann. of Math., **66**(**2**) (1957), 203-248.

[Bu] N. Buchdahl, *On the relative DeRham sequence*, Proc. Amer. Math. Soc., **87** (1983), 363-366.

[DZ] E.G. Dunne and R. Zierau, *Twistor theory for indefinite Kähler symmetric spaces*, Contem. Math., **154** (1993), 117-132.

[EPW] M.G. Eastwood, R. Penrose and R.O. Wells, *Cohomology and massless fields*, Comm. Math. Phys., **78** (1981), 305-351.

[G] R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann and Cie, Paris, 1964.

[GR] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Springer-Verlag, Berlin, 1984.

[Gr] P.A. Griffiths, *Periods of integrals on algebraic manifolds; summary of main results and discussion of open problems*, Bull. Amer. Math. Soc., **76** (1970), 228-296.

[GH] P.A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[GS] P.A. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta. Math., **123** (1969), 253-302.

[K] A.W. Knapp, *Representation Theory of Semisimple Groups*, Princeton Mathematical Series # 36, Princeton University Press, Princeton, 1986.

[N] J.D. Novak, *Parametrizing maximal compact subvarieties*, Proc. Amer. Math. Soc., **124** (1996), 969-975.

[PR1] C.M. Patton and H. Rossi, *Unitary structures on cohomology*, Trans. Amer. Math. Soc., **290** (1985), 235-258.

[PR2] ———, *Cohomology on complex homogeneous manifolds with compact subvarieties*, Contem. Math., **58** (1986), 199-211.

[S] W. Schmid, *Homogeneous complex manifolds and representations of semi-simple Lie groups*, Thesis, Univ. of Calif. at Berkeley, 1967.

[SW] W. Schmid and J.A. Wolf, *A vanishing theorem for open orbits on complex flag manifolds*, Proc. Amer. Math. Soc., **92** (1984), 461-464.

[TW] J.A. Tirao and J.A. Wolf, *Homogeneous holomorphic vector bundles*, Indiana Univ. Math. J., **20**(**1**) (1970), 15-31.

[We] R.O. Wells, *Parametrizing the compact submanifolds of a period matrix domain by a Stein manifold*, in 'Symposium on Several Complex Variables', Lecture Notes in Mathematics (Park City, Utah, 1970) (R.M. Brooks, ed.) Vol. 184, Springer-Verlag, New York, (1971), 121-150.

[WW] R.O. Wells and J.A. Wolf, *Poincaré series and automorphic cohomology on flag domains*, Ann. of Math., **105** (1977), 397-448.

[Wo1] J.A. Wolf, *Fine structure of Hermitian symmetric spaces*, Pure and Applied Math., **8** (1972), 271-357.

[Wo2] ———, *The Stein condition for cycle spaces of open orbits on complex flag manifolds*, Ann. of Math., **136**(**2**) (1992), 541-555.

[Wo3] ———, *Exhaustion functions and cohomology vanishing theorems for open orbits on complex flag manifolds*, Mathematical Research Letters, **2** (1995), 179-191.

[WZ] J.A. Wolf and R. Zierau, *Linear cycle spaces in flag domains,* I: *Flags corresponding to the classical groups*, Erwin Schrodinger Institute Preprint Series, **357** (1996).

[Wg] H.W. Wong, *Dolbeault cohomologies and Zuckerman modules associated with finite rank representations*, Thesis, Harvard University, 1992.

Received December 4, 1998 and revised August 23, 1999.

Univeristy of Northern Colorado
Greeley, Colorado 80639
E-mail address: jdnovak@unco.edu