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In 1960, Nicolaas Kuiper showed that every surface can be tightly immersed in three-space except for the real projective plane and the Klein bottle, for which no such immersion exists, and the real projective plane with one handle, for which he could find neither a tight example nor a proof that one does not exist. It was not until more than 30 years later, in 1992, that François Haab proved that there is no smooth tight immersion into three-space of the projective plane with one handle. Haab's proof is valid only for smooth surfaces, but it, together with the fact that no polyhedral example had been found in the preceding 30 years, strongly suggested that the same would be true of polyhedral surfaces as well. Surprisingly, this is not the case. A tight polyhedral immersion of the real projective plane with one handle exists, which we demonstrate in this paper.

1. Introduction.

An immersion of a closed surface into three-space is *tight* provided it has the two-piece property, meaning that every plane cuts the surface into at most two pieces (see Section 2). Nicolaas Kuiper began a detailed study of tight immersions of surfaces in the late 1950s, and showed [**K1**], [**K2**] that: (1) every orientable surface admits a tight immersion into three-space; (2) every non-orientable surface with Euler characteristic strictly less than -1 admits a tight immersion into three-space; (3) the real projective plane, which has Euler characteristic 1, can not be tightly immersed in three-space; and (4) the Klein bottle, which has Euler characteristic 0, can not be tightly immersed in three-space.

Kuiper's original examples for (1) and (2) were of smooth surfaces, but it is not hard to modify his constructions to generate polyhedral ones, and his proof of (3) carries over to the polyhedral case as well. The proof for (4) needs some modification for polyhedral surfaces, but Kuiper provided this in [**K4**]. The smooth and polyhedral results agree for all of these surfaces.

The only surface not accounted for above is the one with characteristic -1, a non-orientable surface formed by adding a handle to the real projective plane. Kuiper could not determine whether a tight immersion of this surface exists or not. The resolution to this issue became a standard open question in the field.

An important step toward solving this problem was provided in 1992 by François Haab [**H**], who proved that there is no *smooth* tight immersion of the projective plane with one handle into space. Haab looked at projections into the plane of immersions in space, and used a counting argument concerning the number of fold curves in a projection of a tight immersion and the number of different types of saddle points on those curves to break the surface into regions, two of which contain local maxima for a height function in some direction. This contradicts tightness, since a plane just below the lower of these two maxima will cut the surface into three pieces (the two maxima each in separate components, and the remainder of the surface in a third).

Haab's proof relies heavily on the smoothness of the surface, however, so the question remained open for polyhedral surfaces. In this paper, we resolve this issue by presenting an example of a tight polyhedral immersion of the real projective plane with one handle. In light of Haab's proof, and the fact that no polyhedral example had been found in the preceding 30 years, the surface presented below is somewhat unexpected. For a discussion of the relationship of this surface to Haab's proof, see [**C2**]. For a breakdown of smooth and polyhedral tight surfaces by homotopy classes, see [**C1**].

2. Tightness for Simplicial Surfaces.

In this section, we give the essential definitions and lemmas that we will need in the following section to demonstrate that the model presented there is a tight immersion. In Section 3, we describe the polyhedral model and show that it has the desired properties.

By a *simplexwise-linear map*, we mean a map $f\colon M \to \mathbf{R}^3$ from a triangulated surface M into $\mathbf{R}^3$ such that the edges and faces of M are mapped as the convex linear combinations of their vertices. We assume that such a map is *non-degenerate*, that is, it does not reduce the dimension of any simplex of M. Two simplices are said to *intersect* if their images intersect other than at a vertex or edge common to both. If σ is a simplex of M, its *star* is the collection of all simplices containing σ. In particular, the star of a vertex v is the union of all edges and faces containing v.

The map $f\colon M \to \mathbf{R}^3$ of a simplicial surface M is an *embedding* if it is a one-to-one mapping of M into $\mathbf{R}^3$, and it is an *immersion* if it is locally one-to-one (for smooth surfaces one usually requires additional properties that guarantee a unique, well-defined tangent plane at every point). A simplicial surface can fail to be immersed in essentially only two ways: Either two faces with a common vertex intersect (so that no neighborhood of the vertex will be mapped one-to-one) or two triangles with a common edge overlap (so no neighborhood of any point of the edge is mapped one-to-one). The latter, however, implies that the former occurs at the vertices of the edge, which gives us the following characterization of immersions:

Lemma 2.1. *A simplexwise-linear map $f\colon M \to \mathbf{R}^3$ is an immersion if, and only if, the star of every vertex of M is embedded by f.*

An immersion $f\colon M \to \mathbf{R}^3$ is *tight* if it has the *two-piece property*, namely that the pre-image of every half-space is connected in M [**B1**], or in other words, every plane in $\mathbf{R}^3$ cuts the surface into at most two parts. (For some equivalent definitions of tightness, see [**BK**], [**CR1**], [**K3**], [**Ku1**], and [**K1**].) Tightness is closely related to convexity; and, in fact, it is identical to convexity for topological spheres. A tight surface need not be convex, however; for example, a torus of revolution is tight.

Closed convex surfaces have the property that, for almost every direction, the height function induced on the surface in this direction has a single maximum and a single minimum; that is, local extrema are also global extrema. The same is true for closed tight surfaces as well, for if there were two local maxima for a particular direction, then a plane slightly below the lower of the two would cut off both maxima, breaking the surface into at least three parts. For smooth tight surfaces, this means that all the positive curvature must be on the convex hull of the surface, while all the points inside the convex hull have negative (or zero) curvature.

A similar geometric interpretation is possible in the polyhedral case as well. A vertex v of M is called a *local extreme vertex* if $f(v)$ is a vertex of the convex hull of the image of the star of v (that is, it is an isolated local maximum for the height function on $f(M)$ in some direction), and it is a (*global*) *extreme vertex* if $f(v)$ is a vertex of the convex hull of $f(M)$. A local extreme vertex corresponds to a point of positive curvature in a smooth surface, while a vertex that lies in the interior of the convex hull of its neighbors corresponds to a point of negative curvature. Note that v will not be an extreme vertex (local or global) if it lies in the interior of the convex hull of some subset of its adjacent vertices; for example, if v lies on the line segment between two of its neighbors, then v can not be locally or globally extreme.

With these definitions, we can characterize tightness for simplicial surfaces as follows:

Lemma 2.2. *A simplexwise-linear map $f\colon M \to \mathbf{R}^3$ of a closed, compact, connected surface M is tight if, and only if,*

1. *every local extreme vertex is a global extreme vertex,*
2. *every edge of the convex hull of $f(M)$ is contained in $f(M)$, and*
3. *every vertex of the convex hull of $f(M)$ is the image of a single vertex of M.*

This lemma can be found in the literature ([**BK**] or [**Ku1**], for example) as a result for embedded surfaces, without the third condition. This condition is required for immersions, however, as shown by Figure 1, which is a sphere with two points touching. It satisfies (1) and (2), but is not tight since a horizontal plane just below the upper vertex cuts off two pieces of M at the top.

This example is a bit contrived, however, since each of the vertices at the top of Figure 1 has more directions for which it is a local maximum than a global one. In a sense, then, these are not "true" extreme vertices, since the local and global convex hulls don't agree, so we might not be surprised that this example is not tight.

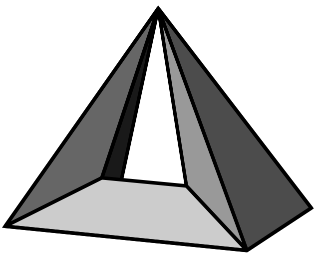

Figure 1. A sphere with two points touching that satisfies the first two conditions of Lemma 2.2 but not the third. It is not tight since a horizontal plane just below the upper vertex cuts M into three pieces.

A more serious example of the need for the third condition is a doubly covered torus of revolution (that is, a tube that wraps around the torus twice), or rather its polyhedral equivalent. Although the image of this surface is the same as the corresponding embedded torus, which *is* tight, the doubly-covered torus is *not* tight, since cutting off any patch of the smooth version or any vertex of the polyhedral one actually cuts off two patches, separating the surface into at least three parts. It is important to realize that tightness is a property of the mapping, f, not of its image in space.

3. The Polyhedral Tight Immersion.

With these definitions and lemmas, we can now present the tight polyhedral model of the real projective plane with one handle. The model has 13 vertices; their mapping into three-space is given in Figure 2, together with the 28 faces that make up the surface.

Figure 3 provides a view of this surface; it is broken into two parts, the central core (a projective plane minus two disks) and the outer handle (that forms the intersection of the surface with its convex envelope). In these pictures, the x-axis is parallel to the edge ef while the y-axis is parallel to fg. The viewpoint is approximately $(-30, -10, 60)$. The surface has self-intersection, shown in the diagram where faces meet without a heavy black line. A triple point is visible at the center of the core projective plane.

To verify that this is really the projective plane with one handle, we compute its Euler characteristic: Since there are 13 vertices, 42 edges, and

Vertices:

$a \to (-2,0,0)$	$h \to (0,3,2)$
$b \to (0,0,0)$	$i \to (-3/8,0,1/2)$
$c \to (1,0,0)$	$j \to (1/2,1/4,1)$
$d \to (0,1,0)$	$k \to (-1/4,7/12,7/6)$
$e \to (-2,-1,2)$	$l \to (0,3/4,7/6)$
$f \to (1,-1,2)$	$m \to (1/4,0,1/2)$
$g \to (1,1,2)$	

Faces:

abk	bgk	bgj	bij	gfj
adl	akl	cdl	flm	clm
hij	ehi	bci	cim	fim
efi	hkl	hjl	fjl	ghk
abe	bef	bcf	cfg	
cgh	cdh	adh	aeh	

Figure 2. The positions of the vertices and the triangles that make up the tight polyhedral immersion of the real projective plane with one handle.

28 faces, the Euler characteristic is $V - E + F = -1$, as desired. Another way to see this is from the triangulation given in Figure 4, which shows a real projective plane (left) with two disks removed (grey) together with a tube (right) that connects the two holes. The dotted line represents the pre-image of the self-intersection, called the *double set* or *double locus*.

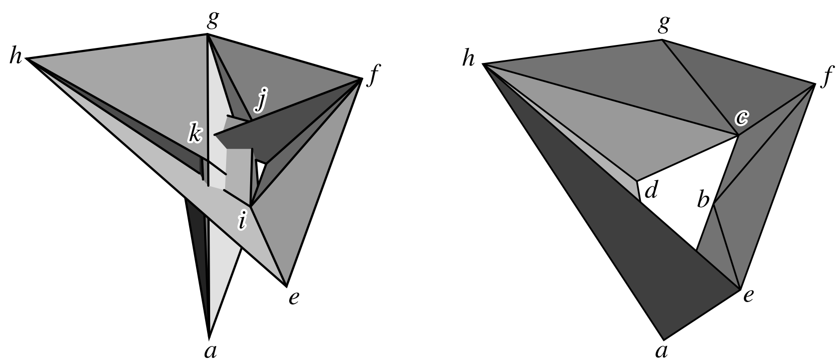

Figure 3. The tight polyhedral model of the real projective plane with one handle broken into two parts: The central projective plane with two disks removed (left) and the tube forming the handle that makes the surface tight (right).

To see that it is an immersion, we need to check that the star of each vertex is embedded (Lemma 2.1). This can be seen in the triangulation in Figure 4, since the double locus does not contain any vertex. Any immersion of a surface of odd Euler characteristic is required to have at least one triple point [**B4**]; our model has exactly one. A triple point has six arcs of double points emanating from it, and in an immersion with only one triple point, we expect the double curve to connect these arcs in pairs. This is exactly the behavior we see in the model presented here, where the image of the double set forms three loops with a single point in common (the triple point visible in Figure 3).

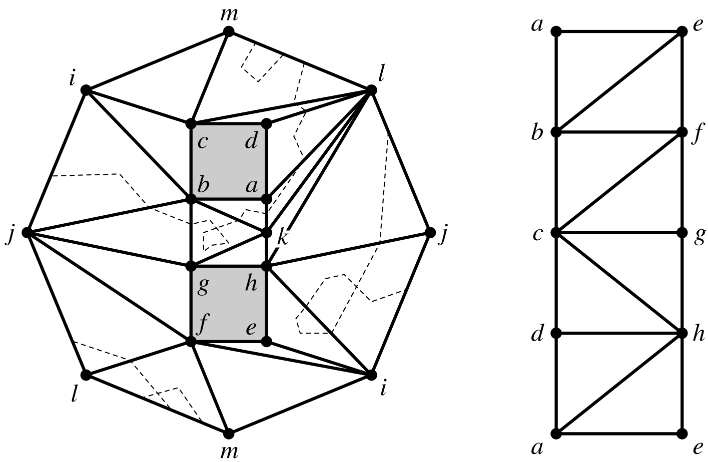

Figure 4. The triangulation of the projective plane with one handle; the projective plane (left) has two disks removed (grey) and the strip (right) has its top and bottom identified to form a tube whose ends are attached where the disks where removed.

To check that the model is tight, we must show that all the edges of the convex hull are contained in the surface itself, and that any vertex that is not a vertex of the convex hull of the surface lies in the convex hull of its neighbors (Lemma 2.2). First, note that the convex envelope has seven vertices: a, c, d, e, f, g, and h (although b is on the convex envelope, it is not a vertex of it) and that the last eight faces listed contain all the edges of the convex envelope. Of the remaining six vertices, two lie on the straight line segment between two neighbors (b lies on the segment ac, and k on the segment ag), two lie within a triangle formed by three neighbors (i lies within triangle beh, and j within bfg), and two lie inside tetrahedra formed by four neighbors (l lies within $acfh$, and m within $cfil$). Thus the surface is tight, as claimed.

4. Generating Tight Polyhedral Projective Planes with Handles.

The breakdown of the surface presented in Section 3 into an inner core and an outer shell (Figure 3) is typical of tight surfaces, and corresponds to having all the positive curvature (i.e., the local extreme vertices) on the convex hull and all the negative curvature (the non-locally extreme vertices) inside the convex hull. A standard approach to generating tight surfaces is the following: Take some core surface and make as much of it as possible have negative curvature, then cut off the areas of positive curvature. Place the result inside a large sphere with a corresponding number of disks removed and attach negatively curved tubes between these holes and the removed areas of positive curvature on the original surface. The result is a closed surface that is the base surface with some handles attached; if carefully done, it will be tight.

Kuiper used this approach in his search for a tight real projective plane with one handle. He began by describing the level sets of an immersed projective plane that has exactly three critical points: One maximum, one minimum, and one saddle (see Figure 5). There must be positive curvature at the maximum and minimum, but Kuiper hoped to fill in the space between the remaining levels with pieces of ruled surface (which have zero curvature) and paste these pieces together by strips having negative curvature. Had he been able to do this, he would have cut off the top and bottom, and placed the negatively curved core inside a sphere with two holes removed (i.e., a tube) to obtain a tight smooth immersion of the projective plane with one handle.

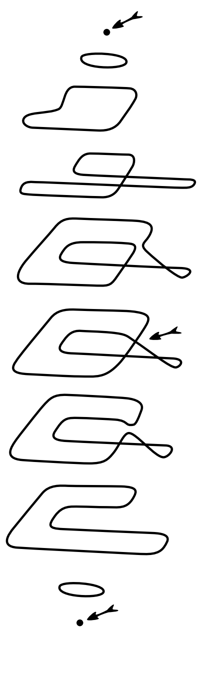

Figure 5. (After Kuiper [**K2**]) The level sets for an immersion of the real projective plane with only three critical points (arrows) originally described by Kuiper. The polyhedral model of Section 3 is based on these.

Unfortunately, he could not patch the level sets together without introducing at least one small region of positive curvature (near the tip of the "tongue" in the 3rd level down from the top of Figure 5). By cutting off this region and adding a handle out to the enclosing tube, Kuiper did produce a tight smooth immersion of the real projective plane with *two* handles [**K2**].

Kuiper's presentation of this surface gives a descriptive rather than a parametric construction, and it would be difficult to carry out in practice. In [**Ku2**], however, Kühnel and Pinkall give an explicit polyhedral version of a tight immersion of the projective plane with two handles. They construct a polyhedral model of the Boy surface that has all its positive curvature isolated at three symmetric locations, then cut these off and place the result inside a large tetrahedron. The result is a tight projective plane with two handles having an axis of 3-fold rotational symmetry. They also provide a smoothing algorithm that will produce a tight smooth example of this surface also having the same symmetry. Their smoothing algorithm does not apply to all polyhedral models, however, and in particular, the polyhedral surface presented here does not satisfy the conditions that it requires. (A more detailed analysis of specific configurations within this surface, and their potential smoothability, can be found in [**C2**].)

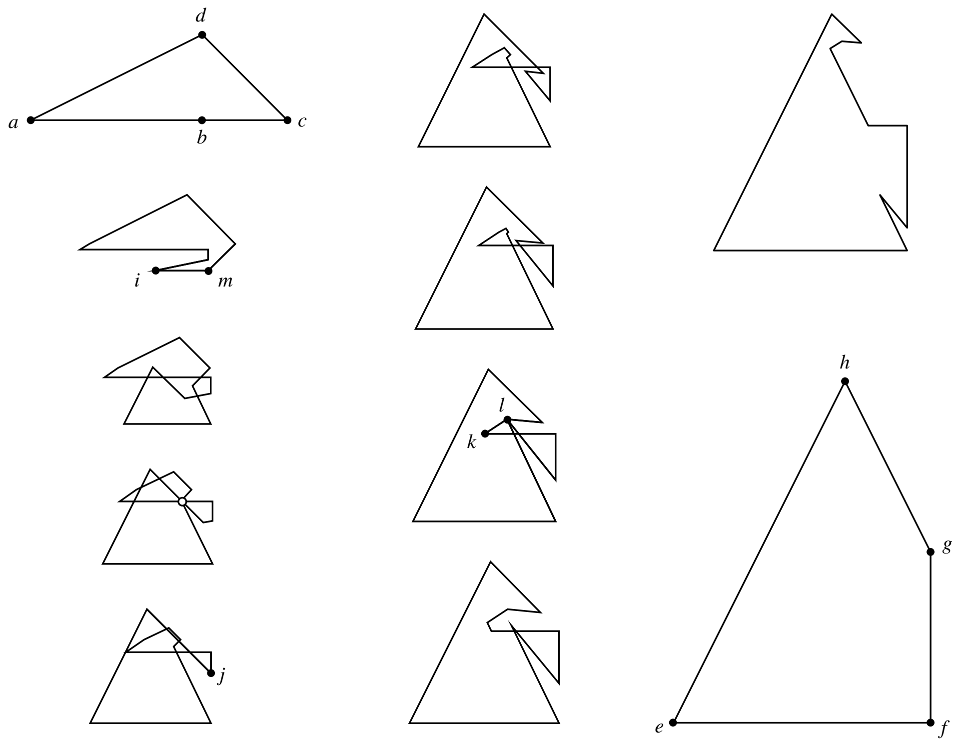

Figure 6. Selected level sets for the central projective plane of the tight immersion. The vertices (black dots) are labeled; the triple point (white dot) is marked.

The surface in Section 3 was found using the plan mapped out by Kuiper: Start with the level sets of Figure 5 and fill in, only this time use flat triangular faces rather than smooth patches. The level sets for the polyhedral core are shown in Figure 6; they bear a striking resemblance to Kuiper's smooth ones. Note how the level sets begin large, get small toward the middle and then get large again at the end, giving the core a "negatively curved" basic shape (this is reflected in the fact that none of the interior vertices is locally extreme). Kuiper's area of positive curvature appeared near our vertex j, but no positive curvature is introduced in the polyhedral case since j is not locally extreme. Once the core projective plane with only two areas of positive curvature was developed, the top and bottom were cut off, and the remainder of the convex hull was added, as shown in Figure 3, thus completing the surface as a tight polyhedral immersion of the projective plane with one handle.

5. Conclusion.

The polyhedral example described here is significant in that it represents one of only a handful of low-dimensional examples where the smooth and polyhedral theories differ in a significant way. The circumstances that provide for the difference in this specific case still deserve investigation. In particular, a close study of how this model fails to meet the conditions of Haab's paper should provide valuable insight into both the smooth and the polyhedral situations. A hypertext description of the result presented in this paper is available at [**C3**].

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