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Let $G \subset G_{\mathbf{C}}$ be a connected reductive linear Lie group with a Cartan subgroup B which is compact modulo the center of G. Then G has discrete series representations. Further, since G is linear the characters of discrete series representations can be averaged over the Weyl group to obtain stable discrete series characters which are constant on orbits of $G_{\mathbf{C}}$ in G', and can be regarded as the restrictions of certain class functions on the regular set $G'_{\mathbf{C}}$ of $G_{\mathbf{C}}$. The main theorem of this paper expresses these class functions on $G'_{\mathbf{C}}$ as "lifts" of analogous class functions on two-structure groups for $G_{\mathbf{C}}$. These are connected reductive complex Lie groups which are not necessarily subgroups of $G_{\mathbf{C}}$, but which "share" the Cartan subgroup $B_{\mathbf{C}}$ with $G_{\mathbf{C}}$. Further, all of their simple factors have root systems of type A_1 or $B_2 \simeq C_2$.

1. Introduction.

Let $G \subset G_{\mathbf{C}}$ be a connected reductive linear Lie group with a Cartan subgroup B which is compact modulo the center of G. Then G has discrete series representations. Further, since G is linear the characters of discrete series representations can be averaged over the Weyl group to obtain stable discrete series characters which are constant on orbits of $G_{\mathbf{C}}$ in G', and can be regarded as the restrictions of certain class functions on the regular set $G'_{\mathbf{C}}$ of $G_{\mathbf{C}}$. The main theorem of this paper expresses these class functions on $G'_{\mathbf{C}}$ as "lifts" of analogous class functions on two-structure groups for $G_{\mathbf{C}}$. These are connected reductive complex Lie groups which are not necessarily subgroups of $G_{\mathbf{C}}$, but which "share" the Cartan subgroup $B_{\mathbf{C}}$ with $G_{\mathbf{C}}$. Further, all of their simple factors have root systems of type A_1 or $B_2 \simeq C_2$.

Let $G_{\mathbf{C}}$ be a connected complex reductive Lie group and fix a Cartan subgroup $B_{\mathbf{C}}$ of $G_{\mathbf{C}}$. Let G be a real form of $G_{\mathbf{C}}$ such that $B = G \cap B_{\mathbf{C}}$ is a relatively compact Cartan subgroup of G; that is B is compact modulo the center of G. Then G has (relative) discrete series representations parameterized by the set L'_B of Harish-Chandra parameters. Let Θ_λ denote the discrete series character of G parameterized by $\lambda \in L'_B$, and let Φ denote the set of roots of the Lie algebra of $G_{\mathbf{C}}$ with respect to that of $B_{\mathbf{C}}$. The Weyl group $W(\Phi)$ corresponding to Φ acts on L'_B, and there is a stable discrete series character $\overline{\Theta}_\lambda$ of G parameterized by λ which is given, up to a constant, by $\sum_{w \in W(\Phi)} \Theta_{w\lambda}$. These stable characters can be regarded as the restrictions of certain class functions on $G_{\mathbf{C}}$. That is, given a discrete series parameter $\lambda \in L'_B$, there is a class function T_λ on $G'_{\mathbf{C}}$, the set of regular semisimple elements of $G_{\mathbf{C}}$, so that if G is any real form of $G_{\mathbf{C}}$ with $G \cap B_{\mathbf{C}} = B$, then the restriction of T_λ to $G' = G \cap G'_{\mathbf{C}}$ is equal to $\overline{\Theta}_\lambda$ up to a sign which depends on the real form. We call T_λ the stable discrete series class function on $G_{\mathbf{C}}$ parameterized by $\lambda \in L'_B$.

Two-structures were first defined in [**H1**] and were used to prove an identity for the constants occuring in stable discrete series character formulas. In this paper we use this identity to prove a formula expressing the stable discrete series class functions $T_\lambda, \lambda \in L'_B$, on $G'_{\mathbf{C}}$ as "lifts" of the analogous class functions on groups corresponding to two-structures. The set of two-structures for Φ and two-structure groups are defined as follows.

A root subsystem $\varphi \subset \Phi$ is called a two-structure for Φ if it satisfies the following two properties.

(i) Every irreducible factor of φ is of type A_1 or $B_2 \simeq C_2$.

(ii) Let φ^+ be any choice of positive roots for φ. Then if $w \in W(\Phi)$ with $w\varphi^+ = \varphi^+$ we have $\det w = 1$.

Two-structures exist for any root system Φ, and are all conjugate via $W(\Phi)$. We let $\mathcal{T}(\Phi)$ denote the set of all two-structures for Φ.

Fix $\varphi \in \mathcal{T}(\Phi)$, and write $\varphi = \varphi_1 \cup \ldots \cup \varphi_k$ for its decomposition into irreducible factors. Each $\varphi_i, 1 \leq i \leq k$, is an irreducible subroot system of Φ and so corresponds naturally to a connected simple subgroup $G_{i,\mathbf{C}}$ of $G_{\mathbf{C}}$ with Cartan subgroup $B_{i,\mathbf{C}} = B_{\mathbf{C}} \cap G_{i,\mathbf{C}}$. Let $\underline{b}_{\mathbf{C}}$ denote the Lie algebra of $B_{\mathbf{C}}$. We also define $G_{0,\mathbf{C}} = B_{0,\mathbf{C}} = \exp(\underline{b}_{0,\mathbf{C}})$ where $\underline{b}_{0,\mathbf{C}} = \{H \in \underline{b}_{\mathbf{C}} : \alpha(H) = 0 \ \forall \ \alpha \in \varphi\}$. Since the irreducible factors of φ are of type A_1 or B_2, each of the groups $G_{i,\mathbf{C}}, 1 \leq i \leq k$, is locally isomorphic to either $SL(2,\mathbf{C})$ or $SO(5,\mathbf{C})$. The group $G_{0,\mathbf{C}}$ is abelian.

Let $G_{0,\mathbf{C}} \times G_{1,\mathbf{C}} \times \cdots \times G_{k,\mathbf{C}}$ denote the abstract direct product of the groups $G_{i,\mathbf{C}}, 0 \leq i \leq k$. Since $B_{i,\mathbf{C}} \subset B_{\mathbf{C}}, 0 \leq i \leq k$, and $B_{\mathbf{C}}$ is abelian, the mapping

$$f : B_{0,\mathbf{C}} \times \cdots \times B_{k,\mathbf{C}} \to B_{\mathbf{C}} \text{ given by } f(b_0, \ldots, b_k) = b_0 \cdots b_k, \quad b_i \in B_{i,\mathbf{C}}, \ 0 \leq i \leq k,$$

is a group homomorphism. Let Z denote the kernel of this homomorphism. It is a central subgroup of $G_{0,\mathbf{C}} \times \cdots \times G_{k,\mathbf{C}}$. Define

$$G_{\varphi,\mathbf{C}} = (G_{0,\mathbf{C}} \times \cdots \times G_{k,\mathbf{C}})/Z, \qquad B_{\varphi,\mathbf{C}} = (B_{0,\mathbf{C}} \times \cdots \times B_{k,\mathbf{C}})/Z.$$

Then $G_{\varphi,\mathbf{C}}$ is a connected complex reductive Lie group with Cartan subgroup $B_{\varphi,\mathbf{C}}$, and the mapping $f_B : B_{\varphi,\mathbf{C}} \to B_{\mathbf{C}}$ induced by f is an isomorphism onto $B_{\mathbf{C}}$. We will use the isomorphism f_B to identify $B_{\varphi,\mathbf{C}}$ and $B_{\mathbf{C}}$. Thus we will think of $B_{\mathbf{C}}$ as a Cartan subgroup of both $G_{\mathbf{C}}$ and $G_{\varphi,\mathbf{C}}$.

Note that the different subgroups $G_{i,\mathbf{C}}$ do not necessarily commute with each other inside $G_{\mathbf{C}}$. This is because, although roots in different irreducible factors of φ are orthogonal to each other, they need not be strongly orthogonal as elements of Φ. Thus $G_{\varphi,\mathbf{C}}$ can not necessarily be embedded as a subgroup of $G_{\mathbf{C}}$. However, we can define an orbit mapping from $G_{\varphi,\mathbf{C}}$ to $G_{\mathbf{C}}$ and a lifting of class functions from $G_{\varphi,\mathbf{C}}$ to $G_{\mathbf{C}}$ as follows.

For any $g \in G_{\mathbf{C}}$, let $\mathcal{O}_{\mathbf{C}}(g)$ denote the orbit of g in $G_{\mathbf{C}}$. Similarly, let $\mathcal{O}_{\varphi,\mathbf{C}}(x), x \in G_{\varphi,\mathbf{C}}$, denote the orbit of x in $G_{\varphi,\mathbf{C}}$. Let $x \in G'_{\varphi,\mathbf{C}}$, the set of regular semisimple elements of $G_{\varphi,\mathbf{C}}$. Then there exists $b \in B_{\mathbf{C}}$ (not unique) such that $b \in \mathcal{O}_{\varphi,\mathbf{C}}(x)$. We define

$$F_{\varphi,\mathbf{C}}(\mathcal{O}_{\varphi,\mathbf{C}}(x)) = \mathcal{O}_{\mathbf{C}}(b),$$

and prove that the orbit $\mathcal{O}_{\mathbf{C}}(b)$ is independent of the choice of $b \in B_{\mathbf{C}} \cap \mathcal{O}_{\varphi,\mathbf{C}}(x)$.

For $x \in G_{\mathbf{C}}$, write $\det(t - 1 + Ad(x)) = D(x)t^n +$ terms of higher degree, where t is an indeterminate. Then D is a class function on $G_{\mathbf{C}}$, and x is regular just in case $D(x) \neq 0$. We also write $D_\varphi(x), x \in G_{\varphi,\mathbf{C}}$, for the corresponding function on $G_{\varphi,\mathbf{C}}$. Let $x \in G'_{\varphi,\mathbf{C}}, g \in G'_{\mathbf{C}}$ such that $F_{\varphi,\mathbf{C}}(\mathcal{O}_{\varphi,\mathbf{C}}(x)) = \mathcal{O}_{\mathbf{C}}(g)$. Then we define the transfer factor

$$D^\Phi_\varphi(x) = |D(g)|^{-\frac{1}{2}} |D_\varphi(x)|^{\frac{1}{2}}.$$

For $g \in G'_{\mathbf{C}}$, we let $X_{\varphi,\mathbf{C}}(g)$ denote a complete set of representatives for the $G_{\varphi,\mathbf{C}}$ orbits which map to $\mathcal{O}_{\mathbf{C}}(g)$ under the orbit correspondence $F_{\varphi,\mathbf{C}}$.

Let Θ be a class function defined on $G'_{\varphi,\mathbf{C}}$. Now for $g \in G'_{\mathbf{C}}$, we define

$$(\mathrm{Lift}^\Phi_\varphi \Theta)(g) = \sum_{x \in X_{\varphi,\mathbf{C}}(g)} D^\Phi_\varphi(x)\Theta(x).$$

Then $\mathrm{Lift}^\Phi_\varphi\Theta$ is a class function on $G'_{\mathbf{C}}$.

Let Φ^+ denote a choice of positive roots for Φ and let $\varphi^+ = \Phi^+ \cap \varphi$. Then we have

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad \rho_\varphi = \frac{1}{2} \sum_{\alpha \in \varphi^+} \alpha.$$

Then L'_B, the set of discrete series parameters for real forms G of $G_{\mathbf{C}}$ with $G \cap B_{\mathbf{C}} = B$, is the set of all $\lambda \in i\underline{b}^*$ such that $e^{\lambda-\rho}$ is well-defined on B and $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Phi$. For each $\lambda \in L'_B$, let T_λ be the corresponding stable discrete series class function on $G'_{\mathbf{C}}$.

Assume that Φ contains no irreducible factors of type $A_{2k}, k \geq 1$. Then by [**H4**, Theorem 5.7], $\rho - \rho_\varphi$ is in the root lattice of Φ, so that $e^{\rho-\rho_\varphi}$ is well-defined on $B_{\mathbf{C}}$. Thus for any $\lambda \in L'_B$, $e^{\lambda-\rho_\varphi} = e^{\lambda-\rho}\ e^{\rho-\rho_\varphi}$ is well-defined on B and $< \alpha, \lambda > \neq 0$ for all $\alpha \in \varphi$. Thus every $\lambda \in L'_B$ is also a discrete series parameter for real forms G_φ of $G_{\varphi,\mathbf{C}}$ such that $G_\varphi \cap B_{\mathbf{C}} = B$. Now for each $\lambda \in L'_B$ we have the stable discrete series class function T^φ_λ on $G'_{\varphi,\mathbf{C}}$. In §4 we will define a sign $\epsilon^\Phi_\varphi(\lambda) = \pm 1$ corresponding to each $\varphi \in \mathcal{T}(\Phi), \lambda \in L'_B$. Since L'_B is stable under $W(\Phi)$, we can also define

$$S^\varphi_\lambda = [W(\Phi,\varphi)]^{-1} \sum_{w \in W(\Phi)} \epsilon^\Phi_\varphi(w\lambda)\ T^\varphi_{w\lambda}$$

where $W(\Phi,\varphi) = \{w \in W(\Phi) : w\varphi = \varphi\}$.

Let $G''_{\mathbf{C}}$ denote the set of all strongly regular elements of $G_{\mathbf{C}}$. It is the set of all elements $g \in G_{\mathbf{C}}$ such that the centralizer of g in $G_{\mathbf{C}}$ is a Cartan subgroup, and is a dense open subset of $G'_{\mathbf{C}}$. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that Φ has no irreducible factors of type $A_{2k}, k \geq 1$, and let $\lambda \in L'_B, g \in G''_{\mathbf{C}}$. Then*

$$T_\lambda(g) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^\Phi_\varphi(\lambda)\ (\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda)(g).$$

Equivalently, for any $\varphi \in \mathcal{T}(\Phi)$,

$$T_\lambda(g) = (\mathrm{Lift}^\Phi_\varphi S^\varphi_\lambda)(g).$$

The formulas in Theorem 1.1 can be interpreted as follows. The functions $T_\lambda, \lambda \in L'_B$, have two invariance properties. First, they are class functions on $G'_{\mathbf{C}}$. Second, $T_{w\lambda} = T_\lambda$ for all $w \in W(\Phi)$. For each $\varphi \in \mathcal{T}(\Phi), \lambda \in L'_B$, the functions S^φ_λ have the second invariance property. When we lift from $G_{\varphi,\mathbf{C}}$ to $G_{\mathbf{C}}$, they become class functions on $G_{\mathbf{C}}$. Thus $\mathrm{Lift}^\Phi_\varphi S^\varphi_\lambda$ has the same two invariance properties as T_λ. The lifts $\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda$ will also be class functions on $G'_{\mathbf{C}}$. However, they will not have the second invariance property of T_λ. We will see in §5 that for any $w \in W(\Phi)$,

$$\epsilon^\Phi_\varphi(w\lambda)\ \mathrm{Lift}^\Phi_\varphi T^\varphi_{w\lambda} = \epsilon^\Phi_{w^{-1}\varphi}(\lambda)\ \mathrm{Lift}^\Phi_{w^{-1}\varphi} T^{w^{-1}\varphi}_\lambda.$$

Thus

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^\Phi_\varphi(\lambda)\ \mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda$$

is also invariant under $\lambda \mapsto w\lambda, w \in W(\Phi)$.

Suppose that Φ contains an irreducible factor of type $A_{2k}, k \geq 1$. Then there is an invariant neighborhood Ω of the identity in $G_{\mathbf{C}}$ with the following properties. Let $\varphi \in \mathcal{T}(\Phi)$ and let Ω'_φ denote the union of all orbits in $G'_{\varphi,\mathbf{C}}$ which map into $\Omega \cap G'_{\mathbf{C}}$ via the orbit correspondence $F_{\varphi,\mathbf{C}}$. Then for any $\lambda \in L'_B$ we can define a class function T^φ_λ on Ω'_φ which is related to stable discrete series characters for real forms of a two-fold cover of $G_{\varphi,\mathbf{C}}$. Further, we can define $\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda$ in $\Omega \cap G'_{\mathbf{C}}$, and the formulas of Theorem 1.1 are valid for $g \in \Omega \cap G''_{\mathbf{C}}$.

In [**H4**] we proved a formula similar to Theorem 1.1 for discrete series characters. In this case we started with a connected reductive Lie group G (not necessarily linear) with a relatively compact Cartan subgroup B, roots $\Phi = \Phi(\underline{g}_{\mathbf{C}}, \underline{b}_{\mathbf{C}})$, and discrete series parameters L'_B. Corresponding to each $\varphi \in \mathcal{T}(\Phi)$ we defined a connected reductive group G_φ with a relatively compact Cartan subgroup $B_\varphi \simeq B$, an orbit mapping from certain good regular semisimple orbits of G_φ to orbits of G, and a lifting of class functions from G'_φ to G'. When Φ has no irreducible factors of type A_{2k}, every $\lambda \in L'_B$ is a discrete series parameter for G_φ, so that we had discrete series characters Θ_λ of G and Θ^φ_λ of G_φ corresponding to λ. Theorem 6.5 of [**H4**] said that

$$\Theta_\lambda(g) = c(g) \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^G_\varphi(\lambda)\ (\mathrm{Lift}^G_\varphi \Theta^\varphi_\lambda)(g), g \in G''.$$

Here as in Theorem 1.1, G'' is a dense open subset of G' and $\epsilon^G_\varphi(\lambda) = \pm 1$. The $c(g), g \in G''$, are integers which are constant on G-orbits, and on connected components of Cartan subgroups.

When G is linear, we can use this theorem to write

$$\sum_{w \in W(\Phi)} \Theta_{w\lambda}(g) = c(g) \sum_{\varphi \in \mathcal{T}(\Phi)} \sum_{w \in W(\Phi)} \epsilon^G_\varphi(w\lambda)\ (\mathrm{Lift}^G_\varphi \Theta^\varphi_{w\lambda})(g)$$

$$= c(g) \sum_{\varphi \in \mathcal{T}(\Phi)} \sum_{v \in W(\varphi)} \sum_{w \in W(\varphi)\backslash W(\Phi)} \epsilon^G_\varphi(vw\lambda)\ (\mathrm{Lift}^G_\varphi \Theta^\varphi_{vw\lambda})(g).$$

Now $\epsilon^G_\varphi(vw\lambda) = \epsilon^G_\varphi(w\lambda), v \in W(\varphi)$, so that

$$\sum_{w \in W(\Phi)} \Theta_{w\lambda}(g)$$

$$= c(g) \sum_{\varphi \in \mathcal{T}(\Phi)} \sum_{w \in W(\varphi)\backslash W(\Phi)} \epsilon^G_\varphi(w\lambda) \left(\mathrm{Lift}^G_\varphi \left[\sum_{v \in W(\varphi)} \Theta^\varphi_{vw\lambda} \right] \right) (g).$$

Thus we have expressed the stable character $\sum_{w \in W(\Phi)} \Theta_{w\lambda}$ in terms of lifts of stable characters $\sum_{v \in W(\varphi)} \Theta^\varphi_{vw\lambda}$. However, the orbit mapping and lifting theory for real groups are much more complicated than those for complex groups, and Theorem 1.1 gives a much simpler formula than this stabilized formula. Thus it is worthwhile knowing that in the linear case, the stable theory can be obtained directly using the simpler orbit mapping for complex groups. Moreover, in the linear case, the Shelstad's theory of endoscopy [**S1**, **S2**, **S3**] can be used to recover formulas for individual discrete series characters given formulas for stable discrete series characters.

The organization of this paper is as follows. In §2 we define the stable discrete series class functions $T_\lambda, \lambda \in L'_B$, on $G'_{\mathbf{C}}$. In §3 we recall the formulas for stable discrete series characters on real forms of $G_{\mathbf{C}}$ and prove that they are obtained via restriction from the functions T_λ. In §4 we define the two-structure groups $G_{\varphi,\mathbf{C}}$, the orbit mappings $F_{\varphi,\mathbf{C}}$, and the lifting of class functions from $G'_{\varphi,\mathbf{C}}$ to $G'_{\mathbf{C}}$. Then we restate Theorem 1.1 in more detail as Theorems 4.7, 4.8, and 4.11. In §5 we give the proofs for Theorems 4.7, 4.8, and 4.11.

2. Definition of T_λ.

Let $G_{\mathbf{C}}$ be a complex connected reductive Lie group. Given any subgroup $H_{\mathbf{C}}$ of $G_{\mathbf{C}}$ we will use the corresponding lower case German letter $\underline{h}_{\mathbf{C}}$ for the Lie algebra of $H_{\mathbf{C}}$. Let $B_{\mathbf{C}}$ be a Cartan subgroup of $G_{\mathbf{C}}$, and let Φ denote the roots of $\underline{g}_{\mathbf{C}}$ with respect to $\underline{b}_{\mathbf{C}}$. For any root subsystem $\Psi \subset \Phi$ we write $W(\Psi)$ for the Weyl group of Ψ.

Fix a real subalgebra $\underline{b} \subset \underline{b}_{\mathbf{C}}$ such that

$$(2.1) \qquad \underline{b}_{\mathbf{C}} = \underline{b} \oplus i\underline{b} \ \text{ and } \ \alpha(H) \in i\mathbf{R} \ \forall\ H \in \underline{b},\ \alpha \in \Phi.$$

Since $G_{\mathbf{C}}$ is reductive, $\underline{g}_{\mathbf{C}} = \underline{z}_{\mathbf{C}} + [\underline{g}_{\mathbf{C}}, \underline{g}_{\mathbf{C}}]$ where $\underline{z}_{\mathbf{C}}$ is the center of $\underline{g}_{\mathbf{C}}$. For each $\alpha \in \Phi$ we let H_α be the element of $i\underline{b} \cap [\underline{g}_{\mathbf{C}}, \underline{g}_{\mathbf{C}}]$ dual to α. If $G_{\mathbf{C}}$ is semisimple, $\underline{b} = \sum_{\alpha\in\Phi} \mathbf{R} i H_\alpha$ is uniquely determined by (2.1). In general, a choice of $\underline{b}$ corresponds to the choice of a real form of $\underline{z}_{\mathbf{C}}$.

Recall that a subset S of Φ is called strongly orthogonal if for any $\alpha, \beta \in S, \alpha \pm \beta \notin \Phi$. Let $\mathcal{SO}_{\mathbf{C}}(\Phi)$ denote the set of all strongly orthogonal subsets of Φ. For $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ we define

$$(2.2a) \qquad \underline{t}_S = \{H \in \underline{b} : \alpha(H) = 0\ \forall\ \alpha \in S\}; \quad \underline{b}_S = \sum_{\alpha\in S} i\mathbf{R}H_\alpha.$$

Then $\underline{b} = \underline{t}_S \oplus \underline{b}_S$. Define

$$(2.2b) \qquad B = \exp(\underline{b}) \subset B_{\mathbf{C}}; \quad T^1_S = \{t \in B : e^\alpha(t) = 1\ \forall\ \alpha \in S\}.$$

The identity component of T^1_S is $T^0_S = \exp(\underline{t}_S)$. Finally, we set

$$(2.2c) \qquad B(S) = \{b \in B_{\mathbf{C}} : b = t\exp(iH), t \in T^1_S, H \in \underline{b}_S\}.$$

For $S, S' \in \mathcal{SO}_{\mathbf{C}}(\Phi)$, we write $S \equiv S'$ if $\underline{t}_S = \underline{t}_{S'}$. This is equivalent to the condition that S and S' span the same linear subspace of $i\underline{b}^*$. Let $G'_{\mathbf{C}}$ denote the set of regular semisimple elements of $G_{\mathbf{C}}$, and write $B'(S) = B(S) \cap G'_{\mathbf{C}}, S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$.

Lemma 2.1. *Let $S, S' \in \mathcal{SO}_{\mathbf{C}}(\Phi), b \in B(S) \cap B(S')$. Then there are unique $H \in \underline{b}_S \cap \underline{b}_{S'}$ and $t \in T^1_S \cap T^1_{S'}$ such that $b = t\exp(iH)$. Further, if $B'(S) \cap B'(S') \neq \emptyset$, then $S \equiv S'$.*

Proof. Let $b \in B(S) \cap B(S')$. Then there are $t \in T^1_S, t' \in T^1_{S'}, H \in \underline{b}_S, H' \in \underline{b}_{S'}$ such that $b = t\exp(iH) = t'\exp(iH')$. Let $\alpha \in \Phi$. Then $|e^\alpha(t)| = |e^\alpha(t')| = 1$. Further, $\alpha(iH)$ and $\alpha(iH')$ are real. Thus

$$e^\alpha(t)\exp(\alpha(iH)) = e^\alpha(t')\exp(\alpha(iH'))$$

implies that $\alpha(H) = \alpha(H')$. Since H and H' are in $\underline{b} \cap [\underline{g}_{\mathbf{C}}, \underline{g}_{\mathbf{C}}]$, this implies that $H = H' \in \underline{b}_S \cap \underline{b}_{S'}$. Now we must also have $t = t' \in T^1_S \cap T^1_{S'}$.

Now suppose that $b \in B'(S) \cap B'(S')$, and write $b = t \exp(iH)$ where $H \in \underline{b}_S \cap \underline{b}_{S'}$ and $t \in T^1_S \cap T^1_{S'}$. Let

$$\Psi = \{\alpha \in \Phi : e^\alpha(t) = 1\}.$$

Then $S, S' \subset \Psi$. Let

$$w_S = \prod_{\alpha \in S} s_\alpha, \quad w_{S'} = \prod_{\alpha \in S'} s_\alpha$$

where for any $\alpha \in \Phi$, s_α denotes the reflection in α. Then $w_S, w_{S'} \in W(\Psi), w_S^2 = w_{S'}^2 = 1$, and

$$\underline{b}_S = \{H \in \underline{b} : w_S H = -H\}, \quad \underline{b}_{S'} = \{H \in \underline{b} : w_{S'} H = -H\};$$

$$\underline{t}_S = \{H \in \underline{b} : w_S H = H\}, \quad \underline{t}_{S'} = \{H \in \underline{b} : w_{S'} H = H\}.$$

Since $H \in \underline{b}_S \cap \underline{b}_{S'}$ we must have $w_S w_{S'} H = -w_S H = H$. Let $\alpha \in \Psi$. Since $b \in G'_{\mathbf{C}}$,

$$e^\alpha(b) = \exp(\alpha(iH)) \neq 1,$$

so that $\alpha(H) \neq 0$. Thus H is regular with respect to Ψ so that $w_S w_{S'} H = H$ implies that $w_S w_{S'} = 1$. Thus $w_S = w_{S'}$ so that $\underline{b}_S = \underline{b}_{S'}, \underline{t}_S = \underline{t}_{S'}$. □

Note that the case $S = S' \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ of Lemma 2.1 shows that for $b \in B(S)$, there are unique $t \in T^1_S$ and $H \in \underline{b}_S$ such that $b = t \exp(iH)$. When we write $b = t \exp(iH) \in B(S)$, we will always mean that $t \in T^1_S, H \in \underline{b}_S$.

Let $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ and define

$$\Phi_R(S) = \{\alpha \in \Phi : \alpha(H) = 0 \ \forall \ H \in \underline{t}_S\}.$$

Then $S \subset \Phi_R(S)$ and rank $\Phi_R(S) = [S]$. For $b = t \exp(iH) \in B(S)$, we define

$$\Phi_{b,S} = \{\alpha \in \Phi_R(S) : e^\alpha(t) = 1\}; \quad \Phi^+_{b,S} = \{\alpha \in \Phi_{b,S} : \alpha(H) > 0\}. \tag{2.3}$$

Then $S \subset \Phi_{b,S} \subset \Phi_R(S)$, and since rank $\Phi_R(S) = [S]$, we also have rank $\Phi_{b,S} = [S]$. Thus S is a set of strongly orthogonal roots spanning $\Phi_{b,S}$. Write

$$B(\Phi) = \cup_{S \in \mathcal{SO}_{\mathbf{C}}(\Phi)} B(S), \quad B'(\Phi) = B(\Phi) \cap G'_{\mathbf{C}}. \tag{2.4}$$

Let $b \in B'(\Phi)$, and let $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ such that $b \in B'(S)$. Then we write

$$\Phi_b = \Phi_{b,S}, \quad \Phi^+_b = \Phi^+_{b,S}. \tag{2.5}$$

Note that if $b \in B'(S) \cap B'(S')$, by Lemma 2.1 we have $S \equiv S'$ so that $\Phi_R(S) = \Phi_R(S')$ and $\Phi_{b,S} = \Phi_{b,S'}$. Thus the definition of Φ_b does not depend on the choice of S with $b \in B'(S)$. Further, as in the proof of Lemma 2.1, for $\alpha \in \Phi_b$, $\alpha(H) \neq 0$. Thus Φ^+_b is a choice of positive roots for Φ_b.

Note that if $b \in B(S)$ and $w \in W(\Phi)$, we have $wS \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ and $wb \in B(wS)$. Thus $B(\Phi)$ is a $W(\Phi)$-invariant subset of $B_{\mathbf{C}}$. For $g \in G_{\mathbf{C}}$, let $\mathcal{O}_{\mathbf{C}}(g) = \{xgx^{-1} : x \in G_{\mathbf{C}}\}$ denote the orbit of g in $G_{\mathbf{C}}$. Define

$$G'_{\mathbf{C}}(\Phi) = \{g \in G_{\mathbf{C}} : \mathcal{O}_{\mathbf{C}}(g) \cap B'(\Phi) \neq \emptyset\}. \tag{2.6}$$

We will see in Lemma 3.4 below that $G'_{\mathbf{C}}(\Phi)$ is the set of all $g \in G'_{\mathbf{C}}$ such that there is a real form G of $G_{\mathbf{C}}$ with $G \cap B_{\mathbf{C}} = B$ and $\mathcal{O}_{\mathbf{C}}(g) \cap G \neq \emptyset$.

Fix a set of positive roots Φ^+ for Φ and let

$$\rho = \rho(\Phi^+) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Let L'_B denote the set of all $\lambda \in i\underline{b}^*$ such that

$$e^{\lambda-\rho} \text{ is well-defined on } B \text{ and } \langle \alpha, \lambda \rangle \neq 0 \ \forall \ \alpha \in \Phi. \tag{2.7}$$

If $\underline{g}$ is semisimple, then our assumption that $\alpha(H) \in i\mathbf{R}$ for all $\alpha \in \Phi$ guarantees that the kernel of $\exp : \underline{b}_{\mathbf{C}} \to B_{\mathbf{C}}$ is contained in $\underline{b}$. However, when the center $\underline{z}_{\mathbf{C}}$ of $\underline{g}_{\mathbf{C}}$ is non-trivial, this need not be true. Thus the fact that $e^{\lambda-\rho}$ is well-defined on B does not necessarily imply that it has a well-defined extension to $B_{\mathbf{C}}$. However we can extend to $B(\Phi)$. Let $b \in B(\Phi)$. Then $b = t\exp(iH) \in B(S)$ for some $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$. For $\lambda \in L'_B$, define

$$e^{\lambda-\rho}(b) = e^{\lambda-\rho}(t)\ e^{(\lambda-\rho)(iH)}.$$

Since $t \in T^1_S \subset B$ and $H \in \underline{b}_S$ are unique by Lemma 2.1, and do not depend on the choice of S, this gives a well-defined extension of $e^{\lambda-\rho}$ to $B(\Phi)$.

As in [**K**, XIII, §4], stable discrete series constants $\overline{c}(\lambda : \Psi^+)$ can be defined for any $\lambda \in E'(\Phi) = \{\tau \in i\underline{b}^* :< \tau, \alpha >\neq 0 \ \forall \ \alpha \in \Phi\}$, root subsystem $\Psi \subset \Phi$ which is spanned by strongly orthogonal roots, and choice Ψ^+ of positive roots. They are uniquely determined by the following properties:

$$\overline{c}(\lambda : \emptyset) = 1 \ \forall \ \lambda \in E'(\Phi); \tag{2.8a}$$

$$\overline{c}(\lambda : \Psi^+) = 0 \text{ if } \langle \lambda, \alpha \rangle > 0 \ \forall \ \alpha \in \Psi^+; \tag{2.8b}$$

$$\overline{c}(\lambda : \Psi^+) + \overline{c}(s_\alpha\lambda : \Psi^+) = 2\overline{c}(\lambda : \Psi^+_\alpha) \tag{2.8c}$$

where α is a simple root for Ψ^+ and $\Psi^+_\alpha = \{\beta \in \Psi^+ : \langle \beta, \alpha \rangle = 0\}$. As a consequence of this uniqueness, it is easy to see that

$$\overline{c}(w\lambda : w\Psi^+) = \overline{c}(\lambda : \Psi^+) \ \forall \ w \in W(\Phi). \tag{2.8d}$$

Now $L'_B \subset E'(\Phi)$ and for any $b \in B'(\Phi)$, $\Phi_b \subset \Phi$ is a root system spanned by strongly orthogonal roots. Thus the stable discrete series constants $\overline{c}(\lambda : \Phi^+_b), \lambda \in L'_B, b \in B'(\Phi)$, are defined.

Write

$$\Delta'(\Phi^+ : b) = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}(b)), \quad b \in B_{\mathbf{C}}; \tag{2.9}$$

$$\epsilon(\Phi^+ : \lambda) = \text{sign} \prod_{\alpha\in\Phi^+} \langle \alpha, \lambda\rangle, \lambda \in L'_B. \tag{2.10}$$

For $\lambda \in L'_B$ we can now define a class function on $G'_{\mathbf{C}}$ as follows. For $b \in B'(\Phi)$, set

$$T_\lambda(b) = \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1} e^{\lambda-\rho(\Phi^+)}(wb)\ \overline{c}(\lambda : \Phi^+_{wb}). \tag{2.11}$$

From the formula it is clear that T_λ is a $W(\Phi)$-invariant function on $B'(\Phi)$, and so can be extended uniquely to a class function on $G'_{\mathbf{C}}$ which is zero for $g \notin G'_{\mathbf{C}}(\Phi)$.

Lemma 2.2. *Let $\lambda \in L'_B$. Then the definition of T_λ in* (2.11) *does not depend on the choice Φ^+ of positive roots for Φ. Further, we have $T_{w\lambda} = T_\lambda$ for all $w \in W(\Phi)$.*

Proof. Let $u \in W(\Phi)$ so that $u\Phi^+$ is another choice of positive roots for Φ. Then for any $b \in B'(\Phi), \lambda \in L'_B$, it is easy to check from the definitions that

$$\epsilon(u\Phi^+ : \lambda) = \det u\epsilon(\Phi^+ : \lambda),$$
$$\Delta'(u\Phi^+ : b)\ e^{\rho(u\Phi^+)-\rho(\Phi^+)}(b) = \det u\Delta'(\Phi^+ : b).$$

Thus the definition is independent of the choice of Φ^+.

Now since $T_{u\lambda}$ is independent of the choice of Φ^+, we can use $u\Phi^+$ in (2.11) to write

$$\begin{aligned} &T_{u\lambda}(b) \\ &= \epsilon(u\Phi^+ : u\lambda) \sum_{w\in W(\Phi)} \Delta'(u\Phi^+ : wb)^{-1} e^{u\lambda-\rho(u\Phi^+)}(wb)\ \overline{c}(u\lambda : \Phi^+_{wb}) \\ &= \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \Delta'(\Phi^+ : u^{-1}wb)^{-1} e^{\lambda-\rho(\Phi^+)}(u^{-1}wb)\ \overline{c}(u\lambda : \Phi^+_{wb}). \end{aligned}$$

Now the result follows from a change of variables $w \mapsto uw$ since using (2.8d),

$$\overline{c}(u\lambda : \Phi^+_{uwb}) = \overline{c}(u\lambda : u\Phi^+_{wb}) = \overline{c}(\lambda : \Phi^+_{wb}).$$

□

We will show in the next section how T_λ is related to stable discrete series characters on real forms of $G_{\mathbf{C}}$.

3. Stable Discrete Series Characters.

Define $\underline{b}$ as in (2.1), and let G be a real form of $G_{\mathbf{C}}$ such that $G \cap B_{\mathbf{C}} = B = \exp(\underline{b})$. Given any subgroup H of G we will use the corresponding lower case German letter $\underline{h}$ for the real Lie algebra of H, and $\underline{h}_{\mathbf{C}}$ for its complexification. We will write $H_{\mathbf{C}}$ for the connected subgroup of $G_{\mathbf{C}}$ corresponding to $\underline{h}_{\mathbf{C}}$.

By our choice of $\underline{b}$, $\alpha(H) \in i\mathbf{R}$ for all $\alpha \in \Phi, H \in \underline{b}$. Thus B is compact modulo the center Z_G of G, and we can pick a Cartan involution θ of G as in [**W**] so that B is contained in the fixed point set K of θ. In the case that G has compact center, K is a maximal compact subgroup of G and $B \subset K$ is a compact Cartan subgroup of G. In general, K and B contain Z_G and are compact modulo Z_G. Let $\Phi_K = \Phi(\underline{k}_{\mathbf{C}}, \underline{b}_{\mathbf{C}})$ denote the roots of $\underline{b}_{\mathbf{C}}$ in $\underline{k}_{\mathbf{C}}$. Roots in Φ_K are called compact roots of G. Since $B \subset K$ is a Cartan subgroup for both K and G, we have rank G = rank K so that G has discrete series representations. In the case that Z_G is not compact, these are sometimes called relative discrete series representations.

The discrete series representations of G are parameterized by the set L'_B defined in (2.7). For $\lambda \in L'_B$, let Θ_λ denote the discrete series character of G corresponding to λ. We know that $\Theta_{w\lambda} = \Theta_\lambda$ for any $w \in W(\Phi_K)$. We define a stable discrete series character corresponding to λ by

$$\overline{\Theta}_\lambda = \sum_{w \in W(\Phi_K)\backslash W(\Phi)} \Theta_{w\lambda} = [W(\Phi_K)]^{-1} \sum_{w \in W(\Phi)} \Theta_{w\lambda}. \tag{3.1}$$

In this section we will prove the following theorem.

Theorem 3.1. *For all $g \in G'$,*

$$\overline{\Theta}_\lambda(g) = (-1)^{q_G} T_\lambda(g)$$

where $q_G = 1/2 \dim(G/K)$.

Let

$$\mathcal{SO}(\Phi) = \{S \in \mathcal{SO}_{\mathbf{C}}(\Phi) : S \subset \Phi \backslash \Phi_K\}.$$

Each $S \in \mathcal{SO}(\Phi)$ corresponds to a Cartan subgroup H_S of G as follows.

For each noncompact $\alpha \in \Phi$ fix a Cayley transform c_α as in [**K**, p. 418]. Fix $S \in \mathcal{SO}(\Phi)$ and let $c_S = \prod_{\alpha \in S} c_\alpha$. Define $\underline{t}_S$ and $\underline{b}_S$ as in (2.2a). Then H_S is the Cartan subgroup of G with Lie algebra

$$\underline{h}_S = \underline{t}_S \oplus \underline{a}_S \text{ where } \underline{a}_S = c_S(i\underline{b}_S). \tag{3.2}$$

It satisfies $(\underline{h}_S)_{\mathbf{C}} = c_S(\underline{b}_{\mathbf{C}})$. Define $T_S = H_S \cap K$ and $A_S = \exp(\underline{a}_S)$. Then $H_S = T_S A_S$. Note that T_S need not be connected. The identity component T_S^0 of T_S is contained in B, but in general not every connected component of T_S will lie in B. By [**H4**, Lemma 2.1],

$$T_S \cap B = T_S^1 = \{b \in B : e^\alpha(b) = 1 \ \forall \ \alpha \in S\}.$$

Write $H_S^1 = T_S^1 A_S$. When we write $h = ta \in H_S^1$ we always mean that $t \in T_S^1$ and $a \in A_S$. Recall from [**H4**, Lemma 2.4] that every regular semisimple element of G can be conjugated into H_S^1 for some $S \in \mathcal{SO}(\Phi)$.

Lemma 3.2. *Let $S \in \mathcal{SO}(\Phi)$. Then there is $y_S \in G_{\mathbf{C}}$ such that $Ad(y_S) = c_S$, $H_S^1 = y_S B(S) y_S^{-1}$, and for $h = ta \in H_S^1$, $y_S^{-1} h y_S = t \exp(c_S^{-1} \log a)$.*

Proof. For each $\alpha \in S$, by [**K**, p. 418] there are $X_\alpha \in (\underline{g}_{\mathbf{C}})_\alpha, Y_\alpha \in (\underline{g}_{\mathbf{C}})_{-\alpha}$ such that $c_\alpha = Ad \exp(\pi/4)(Y_\alpha - X_\alpha)$. Let $y_\alpha = \exp(\pi/4)(Y_\alpha - X_\alpha) \in G_{\mathbf{C}}$ and $y_S = \prod_{\alpha\in S} y_\alpha$. Then $Ad(y_S) = c_S$, and since $e^\alpha(t) = 1$ for all $\alpha \in S, t \in T^1_S$, y_S centralizes T_S. Thus $y_S B(S) y_S^{-1} = y_S T^1_S \exp(i\underline{b}_S) y_S^{-1} = T^1_S \exp(c_S i \underline{b}_S) = H^1_S$. □

Proof of Theorem 3.1. Suppose first that $G_{\mathbf{C}}$ is simply connected. In this case G is acceptable, that is $e^{\rho(\Phi^+)}$ is well-defined on B, and we have the formula for $\overline{\Theta}_\lambda$ given in [**K**, (13.39)]. Note that Knapp's $\Theta^*_\lambda = (-1)^{q_G}$ $\epsilon(\Phi^+ : \lambda)\overline{\Theta}_\lambda$. Let $\lambda \in L'_B, S \in \mathcal{SO}(\Phi), h = ta \in H^1_S \cap G'$. Write

$$b = c_S^{-1} h = t \exp(c_S^{-1} \log a) \in B'(S).$$

Then in our notation [**K**, (13.39)] can be written as

$$\Delta(\Phi^+ : b)\overline{\Theta}_\lambda(h) = (-1)^{q_G}\ \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \det w\ e^{w\lambda}(b)\ \overline{c}(w\lambda : \Phi^+_b)$$

where $\Delta(\Phi^+ : b) = e^{\rho(\Phi^+)}(b)\Delta'(\Phi^+ : b)$. Since $\Delta(\Phi^+ : w^{-1}b) = \det w$ $\Delta(\Phi^+ : b), w \in W(\Phi)$, and $\overline{c}(w\lambda : \Phi^+_b) = \overline{c}(\lambda : \Phi^+_{w^{-1}b})$ using (2.8d), we can rewrite this as

$$\begin{aligned} &\overline{\Theta}_\lambda(h) \\ &= (-1)^{q_G}\ \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \Delta(\Phi^+ : w^{-1}b)^{-1} e^\lambda(w^{-1}b)\ \overline{c}(\lambda : \Phi^+_{w^{-1}b}) \\ &= (-1)^{q_G}\ \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1} e^{\lambda-\rho(\Phi^+)}(wb)\ \overline{c}(\lambda : \Phi^+_{wb}). \end{aligned}$$

Now suppose that $G_{\mathbf{C}}$ is arbitrary. Then for $\lambda \in L'_B$, λ is also a discrete series parameter for the covering of G contained in the simply connected covering of $G_{\mathbf{C}}$, and the formula on the cover is given as above. But the formula has been written so that all terms are well-defined on G, and so is valid on G as well.

For $g \in G'$, let $\mathcal{O}_G(g) = \{xgx^{-1} : x \in G\}$. Fix $g \in G'$. Then there are $S \in \mathcal{SO}(\Phi)$ and $h \in H^1_S$ such that $g \in \mathcal{O}_G(h) \subset \mathcal{O}_{\mathbf{C}}(h)$. But using Lemma 3.2, $\mathcal{O}_{\mathbf{C}}(h) = \mathcal{O}_{\mathbf{C}}(b)$ where $b = c_S^{-1} h \in B'(S) \subset B'(\Phi)$. Thus

$$\begin{aligned} &\overline{\Theta}_\lambda(g) = \overline{\Theta}_\lambda(h) \\ &\qquad = (-1)^{q_G}\ \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda-\rho}(wb)\ \overline{c}(\lambda : \Phi^+_{wb}) \end{aligned}$$

and from formula (2.11)

$$T_\lambda(g) = T_\lambda(b) = \epsilon(\Phi^+ : \lambda) \sum_{w\in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1} e^{\lambda-\rho}(wb)\ \overline{c}(\lambda : \Phi^+_{wb}).$$

□

Lemma 3.3. *Let $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$. Then there is a real form G of $G_{\mathbf{C}}$ such that $G \cap B_{\mathbf{C}} = B$ and S consists of noncompact roots for G.*

Proof. Let G_0 denote a maximally split real form of $G_{\mathbf{C}}$ containing B, and let S_0 be a set of strongly orthogonal noncompact roots of Φ such that H_{S_0} is a maximally split Cartan subgroup of G_0. Then S_0 is of maximal rank in $\mathcal{SO}_{\mathbf{C}}(\Phi)$, that is $[S_0] \geq [S']$ for all $S' \in \mathcal{SO}_{\mathbf{C}}(\Phi)$. Let $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ be of maximal rank. Then $S = wS_0$ for some $w \in W(\Phi)$. Let $x_w \in N_{G_{\mathbf{C}}}(B_{\mathbf{C}})$ represent w. Now $G = x_w G_0 x_w^{-1}$ is a real form of $G_{\mathbf{C}}$ containing B, and the roots in $S = wS_0$ are noncompact for G. Thus the lemma is true when S is of maximal rank.

Now suppose S is an arbitrary element of $\mathcal{SO}_{\mathbf{C}}(\Phi)$. By [**H4**, Lemma 5.9] there is $\varphi \in \mathcal{T}(\Phi)$ such that $S \subset \varphi$. Let S_0 be a basis for φ consisting of one root from every irreducible factor of type A_1 and two orthogonal long roots from every irreducible factor of type B_2. Then by [**H4**, Lemma 5.1], $S_0 \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ and is of maximal rank. Let G_0 be a real form of $G_{\mathbf{C}}$ containing B so that the roots in S_0 are noncompact. Let $\alpha \in S$. If α is in an irreducible factor of type A_1 of φ, or is a long root in an irreducible factor of type B_2 of φ, then $\pm\alpha \in S_0$, so that α is noncompact. Suppose that α is a short root in an irreducible factor φ_0 of type B_2. Since φ_0 is spanned by strongly orthogonal noncompact roots, one short root of φ_0 is compact and one is noncompact. Now $S \cap \varphi_0 = \{\alpha\}$, and there is $v \in W(\varphi_0)$ such that $v\alpha$ is noncompact. Thus there is $w \in W(\varphi) \subset W(\Phi)$ such that wS consists of noncompact roots for G_0. Now as above, $G = x_w G_0 x_w^{-1}$ is a real form of $G_{\mathbf{C}}$ containing B, and the roots in S are noncompact for G. □

Lemma 3.4. *Let $g \in G'_{\mathbf{C}}$. Then $g \in G'_{\mathbf{C}}(\Phi)$ if and only if there is a real form G of $G_{\mathbf{C}}$ such that $G \cap B_{\mathbf{C}} = B$ and $\mathcal{O}_{\mathbf{C}}(g) \cap G \neq \emptyset$.*

Proof. Let G be a real form of $G_{\mathbf{C}}$ such that $G \cap B_{\mathbf{C}} = B$, and let $x \in \mathcal{O}_{\mathbf{C}}(g) \cap G$. Since $x \in G'$, as in the proof of Theorem 3.1, there are $S \in \mathcal{SO}(\Phi)$ and $h \in H_S^1$ such that $x \in \mathcal{O}_{\mathbf{C}}(h) = \mathcal{O}_{\mathbf{C}}(c_S^{-1}h)$ where $c_S^{-1}h \in B'(S) \subset B'(\Phi)$. Thus $c_S^{-1}h \in \mathcal{O}_{\mathbf{C}}(g) \cap B'(\Phi)$, so that $g \in G'_{\mathbf{C}}(\Phi)$.

Conversely, suppose that $g \in G'_{\mathbf{C}}(\Phi)$. Then there are $S \in \mathcal{SO}_{\mathbf{C}}(\Phi), b \in B'(S)$, such that $b \in \mathcal{O}_{\mathbf{C}}(g)$. By Lemma 3.3 there is a real form G of $G_{\mathbf{C}}$ such that $G \cap B_{\mathbf{C}} = B$ and S consists of noncompact roots for G. Now, using the notation of Lemma 3.2, $h = y_S b y_S^{-1} \in H_S^1 \subset G$. Thus $h \in G \cap \mathcal{O}_{\mathbf{C}}(g)$. □

4. Two-structure Groups.

Let Φ be any root system. Then a root subsystem $\varphi \subset \Phi$ is called a two-structure for Φ if it satisfies the following two conditions.

(i) Every irreducible factor of φ is of type A_1 or $B_2 \simeq C_2$.

(ii) Let φ^+ be any choice of positive roots for φ. Then if $w \in W(\Phi)$ with $w\varphi^+ = \varphi^+$ we have $\det w = 1$.

Let $\mathcal{T}(\Phi)$ denote the set of all two-structures for Φ.

The sets $\mathcal{T}(\Phi)$ for irreducible Φ can be described as follows. If Φ has one root length or is of type G_2, then $\mathcal{T}(\Phi)$ consists of all root subsystems of Φ of type A_1^k where k is the size of a maximal set of orthogonal roots in Φ. If Φ is of type $B_{2k}, C_{2k}, k \geq 1$, or $F_{2k}, k = 2$, then $\mathcal{T}(\Phi)$ consists of all root subsytems of Φ of type B_2^k. Finally, if Φ is of type $B_{2k+1}, C_{2k+1}, k \geq 1$, then $\mathcal{T}(\Phi)$ consists of all root subsytems of Φ of type $B_2^k \times A_1$.

Note that $\varphi \in \mathcal{T}(\Phi)$ is a root subsystem of Φ, that is a subset of Φ which is closed under its own reflections. A root subsystem $\varphi \subset \Phi$ is called a subroot system of Φ if for $\alpha, \beta \in \varphi$, $\alpha \pm \beta \in \varphi$ if and only if $\alpha \pm \beta \in \Phi$.

Lemma 4.1. *Let $\varphi \in \mathcal{T}(\Phi)$. Then every irreducible factor of φ is a subroot system of Φ. Further, if Φ contains no irreducible factors of type $B_n, n \geq 3$, or F_4, then φ is a subroot system of Φ.*

Proof. Let φ_0 be an irreducible factor of φ, and let Φ_0 denote the intersection of Φ with the linear subspace of $i\underline{b}^*$ spanned by φ_0. Then Φ_0 is a subroot system of Φ with the same rank as φ_0. Since there are no root systems of the same rank properly containing a root system of type A_1 or B_2, we must have $\varphi_0 = \Phi_0$.

For the second part, we may as well assume that Φ is irreducible, not of type $B_n, n \geq 3$, or F_4. Suppose that $\alpha, \beta \in \varphi$ with $\alpha \pm \beta \in \Phi$. By the first part, if α, β are in the same irreducible factor of φ, we have $\alpha \pm \beta \in \varphi$. Suppose they are in different irreducible factors of φ. Then α and β are orthogonal roots in Φ with $\alpha \pm \beta \in \Phi$. This can't occur when Φ has one root length, is of type G_2, or when at least one of α, β is long and Φ of is of type C_n. Suppose that Φ is of type C_n. Then each short root is contained in a unique subroot system of type C_2, and is strongly orthogonal to any short root outside that subroot system. Thus in this case we also can't have $\alpha \pm \beta \in \Phi$ when α, β are in different irreducible factors of φ. □

Let $G_{\mathbf{C}}$ be any complex connected reductive Lie group and fix a Cartan subgroup $B_{\mathbf{C}}$ of $G_{\mathbf{C}}$. Let Φ denote the roots of $\underline{g}_{\mathbf{C}}$ with respect to $\underline{b}_{\mathbf{C}}$. We want to associate to every $\varphi \in \mathcal{T}(\Phi)$ a complex group $G_{\varphi,\mathbf{C}}$ with a Cartan subgroup $B_{\varphi,\mathbf{C}}$ isomorphic to $B_{\mathbf{C}}$ and root system φ. Fix $\varphi \in \mathcal{T}(\Phi)$, and write $\varphi = \varphi_1 \cup ... \cup \varphi_k$ for its decomposition into irreducible factors. Each $\varphi_i, 1 \leq i \leq k$, is a subroot system of Φ by Lemma 4.1, and so corresponds to a Lie subalgebra $\underline{g}_{i,\mathbf{C}}$ of $\underline{g}_{\mathbf{C}}$ as follows.

For each $\alpha \in \Phi$ we have the root space $(\underline{g}_{\mathbf{C}})_\alpha$ of $\underline{g}_{\mathbf{C}}$ and the root vector $H_\alpha \in \underline{b}_{\mathbf{C}} \cap [\underline{g}_{\mathbf{C}}, \underline{g}_{\mathbf{C}}]$ dual to α. Now define

$$\underline{b}_{i,\mathbf{C}} = \sum_{\alpha \in \varphi_i} \mathbf{C} H_\alpha; \quad \underline{g}_{i,\mathbf{C}} = \underline{b}_{i,\mathbf{C}} + \sum_{\alpha \in \varphi_i} (\underline{g}_{\mathbf{C}})_\alpha.$$

Since φ_i is of type A_1 or $B_2 \simeq C_2$, $\underline{g}_{i,\mathbf{C}}$ is isomorphic to either $\underline{s}l(2,\mathbf{C})$ or $\underline{s}o(5,\mathbf{C}) \simeq \underline{s}p(4,\mathbf{C})$. We also define

$$\underline{g}_{0,\mathbf{C}} = \underline{b}_{0,\mathbf{C}} = \{H \in \underline{b}_{\mathbf{C}} : \alpha(H) = 0 \ \forall \ \alpha \in \varphi\}.$$

Let $G_{i,\mathbf{C}}$ be the connected subgroup of $G_{\mathbf{C}}$ corresponding to $\underline{g}_{i,\mathbf{C}}$, $B_{i,\mathbf{C}} = \exp(\underline{b}_{i,\mathbf{C}}) = G_{i,\mathbf{C}} \cap B_{\mathbf{C}}$. Let $G_{0,\mathbf{C}} \times G_{1,\mathbf{C}} \times \cdots \times G_{k,\mathbf{C}}$, respectively $B_{0,\mathbf{C}} \times \cdots \times B_{k,\mathbf{C}}$, denote the abstract direct product of the groups $G_{i,\mathbf{C}}$, respectively $B_{i,\mathbf{C}}, 0 \leq i \leq k$. Define $f : B_{0,\mathbf{C}} \times \cdots \times B_{k,\mathbf{C}} \to B_{\mathbf{C}}$ by

$$f(b_0, \ldots, b_k) = b_0 \cdots b_k, \quad b_i \in B_{i,\mathbf{C}}, 0 \leq i \leq k.$$

Here $b_0 \cdots b_k$ denotes the product in $B_{\mathbf{C}}$ of the elements $b_i \in B_{i,\mathbf{C}} \subset B_{\mathbf{C}}$. Since $B_{\mathbf{C}}$ is abelian, f is a group homomorphism. Let Z denote the kernel of this homomorphism, and let Z_i denote the center of $G_{i,\mathbf{C}}, 0 \leq i \leq k$.

Lemma 4.2. *$f : B_{0,\mathbf{C}} \times \cdots \times B_{k,\mathbf{C}} \to B_{\mathbf{C}}$ is surjective and $Z \subset Z_0 \times \cdots \times Z_k$ is a central subgroup of $G_{0,\mathbf{C}} \times \cdots \times G_{k,\mathbf{C}}$.*

Proof. The proof is the same as that for [**H4**, Lemma 4.1]. □

Define

$$G_{\varphi,\mathbf{C}} = (G_{0,\mathbf{C}} \times \cdots \times G_{k,\mathbf{C}})/Z, \quad B_{\varphi,\mathbf{C}} = (B_{0,\mathbf{C}} \times \cdots \times B_{k,\mathbf{C}})/Z. \tag{4.1}$$

Then $G_{\varphi,\mathbf{C}}$ is a complex connected reductive Lie group and $B_{\varphi,\mathbf{C}}$ is a Cartan subgroup of $G_{\varphi,\mathbf{C}}$. The Lie algebra $\underline{g}_{\varphi,\mathbf{C}} = \sum_{i=0}^{k} \underline{g}_{i,\mathbf{C}}$ of $G_{\varphi,\mathbf{C}}$ can be identified with a subset, but not necessarily a subalgebra, of $\underline{g}_{\mathbf{C}}$. By Lemma 4.1, in the case that Φ contains no irreducible factors of type $B_n, n \geq 3$, or F_4, φ is a subroot system of Φ, so that $\underline{g}_{\varphi,\mathbf{C}}$ is a subalgebra of $\underline{g}_{\mathbf{C}}$ and $G_{\varphi,\mathbf{C}}$ can be identified with a subgroup of $G_{\mathbf{C}}$.

Let exp denote the exponential mapping from $\underline{g}_{\mathbf{C}}$ into $G_{\mathbf{C}}$, and $\exp_\varphi$ denote the exponential mapping of $\underline{g}_{\varphi,\mathbf{C}}$ into $G_{\varphi,\mathbf{C}}$. The Cartan subalgebra $\underline{b}_{\mathbf{C}} = \sum_{i=0}^{k} \underline{b}_{i,\mathbf{C}}$ can be identified with the Lie algebra of $B_{\varphi,\mathbf{C}}$ and is a Cartan subalgebra of $\underline{g}_{\varphi,\mathbf{C}}$. Further, $\Phi(\underline{g}_{\varphi,\mathbf{C}}, \underline{b}_{\mathbf{C}}) = \varphi$. The Weyl group $W(\varphi)$ can be identified with a subgroup of $W(\Phi)$.

Lemma 4.3.

(i) *The mapping $f_B : B_{\varphi,\mathbf{C}} \to B_{\mathbf{C}}$ induced by f is an isomorphism.*
(ii) *$f_B(\exp_\varphi(H)) = \exp(H)$ for all $H \in \underline{b}_{\mathbf{C}}$.*
(iii) *$f_B(vb) = v f_B(b)$ for all $b \in B_{\varphi,\mathbf{C}}, v \in W(\varphi)$.*

Proof. (i) Since Z is the kernel of f, f factors through $B_{\varphi,\mathbf{C}}$ to give an isomorphism.

(ii) For any $H = \sum_{0 \leq i \leq k} H_i \in \underline{b}_{\mathbf{C}}$,

$$f_B(\exp_\varphi(H)) = f(\exp(H_0), ..., \exp(H_k)) = \exp(H_0) \cdots \exp(H_k) = \exp(H).$$

(iii) For $H \in \underline{b}_{\mathbf{C}}, b = \exp_\varphi(H), v \in W(\varphi)$, using (ii),

$$f_B(vb) = f_B(\exp_\varphi(vH)) = \exp(vH) = vf_B(b).$$

□

Because of Lemma 4.3 we will identify $B_{\mathbf{C}}$ and $B_{\varphi,\mathbf{C}}$ using the isomorphism f_B. Thus even though $G_{\varphi,\mathbf{C}}$ is not necessarily a subgroup of $G_{\mathbf{C}}$, we will think of $B_{\mathbf{C}}$ as being a Cartan subgroup of both $G_{\varphi,\mathbf{C}}$ and $G_{\mathbf{C}}$. This identification respects the exponential map from $\underline{b}_{\mathbf{C}}$ to $B_{\mathbf{C}}$ and the action of $W(\varphi) \subset W(\Phi)$.

Let $G'_{\varphi,\mathbf{C}}$ denote the set of regular semisimple elements of $G_{\varphi,\mathbf{C}}$. For any $x \in G_{\varphi,\mathbf{C}}$, let $\mathcal{O}_{\varphi,\mathbf{C}}(x)$ denote the orbit of x in $G_{\varphi,\mathbf{C}}$. Let $x \in G'_{\varphi,\mathbf{C}}$. Then there exists $b \in B_{\mathbf{C}} \cap \mathcal{O}_{\varphi,\mathbf{C}}(x)$. We define

$$F_{\varphi,\mathbf{C}}(\mathcal{O}_{\varphi,\mathbf{C}}(x)) = \mathcal{O}_{\mathbf{C}}(b). \tag{4.2}$$

Suppose that $b, b' \in \mathcal{O}_{\varphi,\mathbf{C}}(x) \cap B_{\mathbf{C}}$. Then there is $v \in W(\varphi)$ such that $b' = vb$. Now $W(\varphi) \subset W(\Phi)$ and by Lemma 4.3 the actions are consistent with our identification of $B_{\varphi,\mathbf{C}}$ and $B_{\mathbf{C}}$. Thus $\mathcal{O}_{\mathbf{C}}(b') = \mathcal{O}_{\mathbf{C}}(b)$ and so the orbit mapping is independent of the choice of b.

An element $g \in G_{\mathbf{C}}$ is called strongly regular if its centralizer in $G_{\mathbf{C}}$ is a Cartan subgroup. In particular, if $b \in B_{\mathbf{C}}$ is strongly regular, its centralizer in $G_{\mathbf{C}}$ is $B_{\mathbf{C}}$. Thus b is regular and no non-trivial element of $W(\Phi)$ fixes b. Write $G''_{\mathbf{C}}$ for the set of strongly regular elements in $G_{\mathbf{C}}, B''_{\mathbf{C}} = B_{\mathbf{C}} \cap G''_{\mathbf{C}}$.

Lemma 4.4. *For $b \in B'_{\mathbf{C}}$, we have*

$$F^{-1}_{\varphi,\mathbf{C}}(\mathcal{O}_{\mathbf{C}}(b)) = \{\mathcal{O}_{\varphi,\mathbf{C}}(wb) : w \in W(\Phi)\}.$$

If $b \in B''_{\mathbf{C}}$, then for $w, w' \in W(\Phi)$,

$$\mathcal{O}_{\varphi,\mathbf{C}}(wb) = \mathcal{O}_{\varphi,\mathbf{C}}(w'b)$$

if and only if $w' \in W(\varphi)w$.

Proof. Every orbit in $G'_{\varphi,\mathbf{C}}$ can be represented by an element $b' \in B_{\mathbf{C}}$. Now

$$F_{\varphi,\mathbf{C}}(\mathcal{O}_{\varphi,\mathbf{C}}(b')) = \mathcal{O}_{\mathbf{C}}(b') = \mathcal{O}_{\mathbf{C}}(b)$$

just in case there is $w \in W(\Phi)$ such that $b' = wb$. Now for $w, w' \in W(\Phi)$, $\mathcal{O}_{\varphi,\mathbf{C}}(wb) = \mathcal{O}_{\varphi,\mathbf{C}}(w'b)$ if and only if there is $v \in W(\varphi)$ such that $w'b = vwb$. But if b is strongly regular, $w'b = vwb$ implies that $w' = vw$. □

For $x \in G_{\mathbf{C}}$, write $\det(t - 1 + Ad(x)) = D(x)t^n +$ terms of higher degree, where t is an indeterminate. Then D is a class function on $G_{\mathbf{C}}$, and x is regular just in case $D(x) \neq 0$. We also write $D_\varphi(x), x \in G_{\varphi,\mathbf{C}}$, for the corresponding function on $G_{\varphi,\mathbf{C}}$.

Let $x \in G'_{\varphi,\mathbf{C}}, g \in G'_{\mathbf{C}}$ such that $F_{\varphi,\mathbf{C}}(\mathcal{O}_{\varphi,\mathbf{C}}(x)) = \mathcal{O}_{\mathbf{C}}(g)$. Then we define

$$D^\Phi_\varphi(x) = |D(g)|^{-\frac{1}{2}}|D_\varphi(x)|^{\frac{1}{2}}. \tag{4.3}$$

Since D is a class function on $G_{\mathbf{C}}$ and D_φ is a class function on $G_{\varphi,\mathbf{C}}$, this definition is independent of the choice of g and gives a class function on $G_{\varphi,\mathbf{C}}$. For $g \in G'_{\mathbf{C}}$, we let $X_{\varphi,\mathbf{C}}(g)$ denote a complete set of representatives for the $G_{\varphi,\mathbf{C}}$ orbits which map to $\mathcal{O}_{\mathbf{C}}(g)$ under the orbit correspondence $F_{\varphi,\mathbf{C}}$.

Let Θ be a class function defined on $G'_{\varphi,\mathbf{C}}$. Now for $g \in G'_{\mathbf{C}}$, we define

$$(4.4) \qquad (\mathrm{Lift}_\varphi^\Phi \Theta)(g) = \sum_{x \in X_{\varphi,\mathbf{C}}(g)} D_\varphi^\Phi(x)\Theta(x).$$

Since D_φ^Φ and Θ are class functions on $G_{\varphi,\mathbf{C}}$, the definition does not depend on the choice of $X_{\varphi,\mathbf{C}}(g)$. If $g, g' \in G'$ with $\mathcal{O}_{\mathbf{C}}(g) = \mathcal{O}_{\mathbf{C}}(g')$ we can take $X_{\varphi,\mathbf{C}}(g) = X_{\varphi,\mathbf{C}}(g')$. Thus $\mathrm{Lift}_\varphi^\Phi\Theta$ is a class function on $G'_{\mathbf{C}}$.

Fix a real subalgebra $\underline{b} \subset \underline{b}_{\mathbf{C}}$ satisfying the conditions of (2.1). In §2, we used $\underline{b}$ to define a subset $G'_{\mathbf{C}}(\Phi)$ of $G'_{\mathbf{C}}$ and class functions $T_\lambda, \lambda \in L'_B$, on $G'_{\mathbf{C}}$. Let $\varphi \in \mathcal{T}(\Phi)$. Since $G_{\varphi,\mathbf{C}}$ is a connected complex reductive Lie group with Cartan subgroup $B_{\mathbf{C}}$, we can carry out all the constructions of §2 for the group $G_{\varphi,\mathbf{C}}$. Note that $\mathcal{SO}_{\mathbf{C}}(\varphi)$ is not necessarily a subset of $\mathcal{SO}_{\mathbf{C}}(\Phi)$ since $S \subset \varphi$ can be strongly orthogonal in φ, but not in Φ. For $S \in \mathcal{SO}_{\mathbf{C}}(\varphi)$ we can define $\underline{t}_S, \underline{b}_S, T_S^1, B(S)$ as in (2.2). Write

$$(4.5a) \qquad B(\varphi) = \cup_{S \in \mathcal{SO}_{\mathbf{C}}(\varphi)} B(S), \quad B'(\varphi) = B(\varphi) \cap G'_{\varphi,\mathbf{C}}.$$

Define

$$(4.5b) \qquad G'_{\mathbf{C}}(\varphi) = \{g \in G_{\varphi,\mathbf{C}} : \mathcal{O}_{\varphi,\mathbf{C}}(g) \cap B'(\varphi) \neq \emptyset\}.$$

Lemma 4.5. *For any $\varphi \in \mathcal{T}(\Phi)$, $B(\varphi) \subset B(\Phi)$.*

Proof. We may as well assume that Φ is irreducible. Let $S \in \mathcal{SO}_{\mathbf{C}}(\varphi)$. If Φ is not of type $B_n, n \geq 3$, or F_4, by Lemma 4.1 φ a subroot system of Φ. Thus S is also strongly orthogonal in Φ, and so $B(S) \subset B(\Phi)$.

Suppose Φ is of type $B_n, n \geq 3$, or F_4. Then if β_1, β_2 are any orthogonal short roots of Φ, $\beta_1 \pm \beta_2$ are both long roots of Φ. Let $\beta_1, \dots, \beta_k$ denote the short roots in S and $\beta_{k+1}, \dots, \beta_n$ denote the long roots in S. For $1 \leq i \leq r = [k/2]$, set $\alpha_{2i-1} = \beta_{2i-1} + \beta_{2i}, \alpha_{2i} = \beta_{2i-1} - \beta_{2i}$. Then $S' = \{\alpha_1, \dots, \alpha_{2r}, \beta_{2r+1}, \dots, \beta_n\}$ is an orthogonal subset of Φ which contains at most one short root, and hence is a strongly orthogonal subset of Φ. Further $\underline{b}_S = \underline{b}_{S'}$, and $T_S^1 \subset T_{S'}^1$, so that $B(S) \subset B(S') \subset B(\Phi)$. □

Let Φ^+ denote a choice of positive roots for Φ and let $\varphi^+ = \Phi^+ \cap \varphi$. Then we define

$$(4.6) \qquad \rho = \rho(\Phi^+) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha,$$

$$\rho_\varphi = \rho(\varphi^+) = \frac{1}{2} \sum_{\alpha \in \varphi^+} \alpha,$$

$$\rho(\Phi^+,\varphi) = \rho(\Phi^+) - \rho(\varphi^+).$$

Recall from (2.7) that L'_B denotes the set of all $\lambda \in i\underline{b}^*$ such that $e^{\lambda-\rho}$ is well-defined on $B = \exp(\underline{b})$ and $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Phi$. Assume that Φ contains no irreducible factors of type $A_{2k}, k \geq 1$. Then by [**H4**, Theorem 5.7], $\rho(\Phi^+,\varphi)$ is in the root lattice of Φ, so that $e^{\rho(\Phi^+,\varphi)}$ is well-defined on $B_{\mathbf{C}}$. Thus for any $\lambda \in L'_B$, $e^{\lambda-\rho_\varphi} = e^{\lambda-\rho}\ e^{\rho(\Phi^+,\varphi)}$ is well-defined on B and $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \varphi$. Thus we can define a class function T^φ_λ on $G'_{\varphi,\mathbf{C}}$ as in (2.11). It is supported on $G'_{\mathbf{C}}(\varphi)$ and satisfies

(4.7)

$$T^\varphi_\lambda(b) = \epsilon(\varphi^+ : \lambda) \sum_{v \in W(\varphi)} \Delta'(\varphi^+ : vb)^{-1} e^{\lambda-\rho_\varphi}(vb)\ \bar{c}(\lambda : \varphi^+_{vb}), \quad b \in B'(\varphi).$$

Here, as in §2,

(4.8a)
$$\Delta'(\varphi^+ : b) = \prod_{\alpha \in \varphi^+} (1 - e^{-\alpha}(b)), \quad b \in B_{\mathbf{C}};$$

(4.8b)
$$\epsilon(\varphi^+ : \lambda) = \text{sign} \prod_{\alpha \in \varphi^+} \langle \alpha, \lambda \rangle, \lambda \in L'_B.$$

Further, for $S \in \mathcal{SO}_{\mathbf{C}}(\varphi), b = t\exp(iH) \in B'(S) = B(S) \cap G'_{\varphi,\mathbf{C}}$, we have

(4.9) $\quad \varphi_b = \{\alpha \in \varphi : e^\alpha(tt_0) = 1\ \forall\ t_0 \in T^0_S\}, \varphi^+_b = \{\alpha \in \varphi_b : \alpha(iH) > 0\}.$

As in §3, the restriction of T^φ_λ to any real form G_φ of $G_{\varphi,\mathbf{C}}$ with $G_\varphi \cap B_{\mathbf{C}} = B$ is, up to a sign, a stable discrete series character.

Associated to each $\varphi \in \mathcal{T}(\Phi)$ and choice of positive roots Φ^+ for Φ is a sign $\epsilon(\varphi : \Phi^+) = \pm 1$ defined as in [**H4**, (5.1)], [**K**, p. 501]. Define

(4.10)
$$\epsilon^\Phi_\varphi(\lambda) = \epsilon(\varphi : \Phi^+)\ \epsilon(\varphi^+ : \lambda)\ \epsilon(\Phi^+ : \lambda).$$

Lemma 4.6. *$\epsilon^\Phi_\varphi(\lambda)$ is independent of the choice Φ^+ of positive roots.*

Proof. This follows from [**H4**, Lemma 6.4]. □

The main results of this paper are the following theorems.

Theorem 4.7. *Assume that Φ has no irreducible factors of type $A_{2k}, k \geq 1$, and let $\lambda \in L'_B$. Then for all $g \in G''_{\mathbf{C}}$,*

$$T_\lambda(g) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^\Phi_\varphi(\lambda)(\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda)(g).$$

Theorem 4.7 can be reformulated as follows. Fix $\varphi \in \mathcal{T}(\Phi)$. Since L'_B is stable under the action of $W(\Phi)$, for each $\lambda \in L'_B$ we can define

(4.11)
$$S^\varphi_\lambda = [W(\Phi,\varphi)]^{-1} \sum_{w \in W(\Phi)} \epsilon^\Phi_\varphi(w\lambda) T^\varphi_{w\lambda}$$

where $W(\Phi,\varphi) = \{w \in W(\Phi) : w\varphi = \varphi\}$.

Theorem 4.8. *Assume that* Φ *has no irreducible factors of type* $A_{2k}, k \geq 1$, *and let* $\lambda \in L'_B$. *Then for all* $g \in G''_{\mathbf{C}}$,

$$T_\lambda(g) = (\mathrm{Lift}^\Phi_\varphi S^\varphi_\lambda)(g).$$

Suppose that Φ contains an irreducible factor of type $A_{2k}, k \geq 1$. Then as in [**H4**, Theorem 5.7], $\rho(\Phi^+, \varphi)$ is not in the weight lattice of Φ so that $e^{\lambda - \rho_\varphi}$ is not well-defined on B for any $\lambda \in L'_B$. However, we can still define T^φ_λ and $\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda$ near the identity in $G'_{\varphi,\mathbf{C}}$ and $G'_{\mathbf{C}}$ respectively as follows. This construction is valid for general Φ, but is needed only in the case that Φ contains irreducible factors of type $A_{2k}, k \geq 1$.

Define

(4.12*a*)

$$\omega = \{X \in \underline{g}_{\mathbf{C}} : |\mathrm{Im}\lambda| < \pi \text{ for every eigenvalue } \lambda \text{ of ad } X\}, \quad \Omega = \exp(\omega).$$

Then as in [**HC1**, §3], ω is an invariant neighborhood of the identity in $\underline{g}_{\mathbf{C}}$, and Ω is an invariant neighborhood of the identity in $G_{\mathbf{C}}$. Define $\Omega' = \Omega \cap G'_{\mathbf{C}}$. For any $\varphi \in \mathcal{T}(\Phi)$, we define

$$(4.12b) \qquad \Omega'_\varphi = \{x \in G'_{\varphi,\mathbf{C}} : F_{\varphi,\mathbf{C}}(\mathcal{O}_{\varphi,\mathbf{C}}(x)) \subset \Omega'\}.$$

Clearly Ω'_φ is an invariant subset of $G'_{\varphi,\mathbf{C}}$ and $B_{\mathbf{C}} \cap \Omega'_\varphi = B_{\mathbf{C}} \cap \Omega'$. The following lemma is a direct consequence of the definition of Ω'_φ.

Lemma 4.9. *Let* $g \in \Omega'$. *Then* $X_{\varphi,\mathbf{C}}(g) \subset \Omega'_\varphi$.

Because of Lemma 4.9, if Θ is any class function on Ω'_φ, we can define a class function on Ω' by

$$(4.13) \qquad (\mathrm{Lift}^\Phi_\varphi \Theta)(g) = \sum_{x \in X_{\varphi,\mathbf{C}}(g)} D^\Phi_\varphi(x)\Theta(x), g \in \Omega'.$$

Lemma 4.10. *Let* $b \in \Omega' \cap B_{\mathbf{C}}$. *Then there is* $H \in \omega \cap \underline{b}_{\mathbf{C}}$ *such that* $b = \exp(H)$. *Suppose that* $H, H' \in \omega \cap \underline{b}_{\mathbf{C}}$ *such that* $b = \exp(H) = \exp(H')$. *Then* $\alpha(H) = \alpha(H')$ *for all* $\alpha \in \Phi$.

Proof. Let $b \in \Omega' \cap B_{\mathbf{C}}$. Then there is $H \in \omega$ such that $b = \exp(H)$. Now $H \in C_{\underline{g}_{\mathbf{C}}}(b) = \underline{b}_{\mathbf{C}}$ so that $H \in \omega \cap \underline{b}_{\mathbf{C}}$. Now suppose that $b = \exp(H) = \exp(H'), H, H' \in \omega \cap \underline{b}_{\mathbf{C}}$. Let $\alpha \in \Phi$. Then $\alpha(H - H') \in 2\pi i\mathbf{Z}$ since $\exp(H - H') = 1$. But since $H, H' \in \omega$,

$$|\mathrm{Im}\ \alpha(H - H')| \leq |\mathrm{Im}\ \alpha(H)| + |\mathrm{Im}\ \alpha(H')| < 2\pi,$$

so that $\alpha(H - H') = 0$. Thus $\alpha(H) = \alpha(H')$. □

Let Φ^+ be a choice of positive roots for Φ and define $\rho(\Phi^+, \varphi)$ as in (4.6). Because of Lemma 4.10 we can define $e^{\rho(\Phi^+,\varphi)}$ on $B_{\mathbf{C}} \cap \Omega'$ as follows. Let $b \in \Omega' \cap B_{\mathbf{C}}$. Then there is $H \in \omega \cap \underline{b}_{\mathbf{C}}$ such that $b = \exp(H)$. Define $e^{\rho(\Phi^+,\varphi)}(b) = \exp(\rho(\Phi^+, \varphi)(H))$. By Lemma 4.10 this is independent of the choice of H. Now for any $\lambda \in L'_B$, $e^{\lambda-\rho}$ is defined on $B(\Phi)$, so we can define $e^{\lambda-\rho_\varphi}$ on $B(\varphi) \cap \Omega'_\varphi \subset B(\Phi) \cap \Omega'$ by

$$e^{\lambda-\rho_\varphi}(b) = e^{\lambda-\rho}(b)\; e^{\rho(\Phi^+,\varphi)}(b). \tag{4.14}$$

Thus for each $\lambda \in L'_B$ we can define a class function T^φ_λ on Ω'_φ which is supported on $G'_{\mathbf{C}}(\varphi) \cap \Omega'_\varphi$ and satisfies

(4.15)

$$T^\varphi_\lambda(b) = \epsilon(\varphi^+ : \lambda) \sum_{v \in W(\varphi)} \Delta'(\varphi^+ : vb)^{-1} e^{\lambda-\rho_\varphi}(vb)\; \bar{c}(\lambda : \varphi^+_{vb}), \quad b \in B(\varphi) \cap \Omega'_\varphi.$$

T^φ_λ corresponds to stable discrete series characters of real forms of a two-fold cover of $G_{\varphi,\mathbf{C}}$. That is, there is a two-fold cover $\pi : \tilde{G}_{\varphi,\mathbf{C}} \to G_{\varphi,\mathbf{C}}$ such that $e^{\lambda-\rho_\varphi}$ is well defined on $\tilde{B} = \pi^{-1}(B)$. Now $\lambda \in L'_{\tilde{B}}$ and the usual construction gives a class function $\tilde{T}^\varphi_\lambda$ on $\tilde{G}_{\varphi,\mathbf{C}}$ which restricts to stable discrete series characters of real forms of $\tilde{G}_{\varphi,\mathbf{C}}$ with Cartan subgroup $\tilde{B}$. Let $\tilde{U}$ and U denote neighborhoods of the identity in $\tilde{G}_{\varphi,\mathbf{C}}$ and $G_{\varphi,\mathbf{C}}$ respectively so that the restriction π_U of π to $\tilde{U}$ gives an isomorphism onto U. Then

$$T^\varphi_\lambda(x) = \tilde{T}^\varphi_\lambda(\pi_U^{-1}(x)), \quad x \in U \cap \Omega'_\varphi.$$

We can also define

$$S^\varphi_\lambda = [W(\Phi,\varphi)]^{-1} \sum_{w \in W(\Phi)} \epsilon^\Phi_\varphi(w\lambda) T^\varphi_{w\lambda} \tag{4.16}$$

on Ω'_φ.

Theorem 4.11. *Let $\lambda \in L'_B$ and $g \in G''_{\mathbf{C}} \cap \Omega$. Then*

$$T_\lambda(g) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^\Phi_\varphi(\lambda)(\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda)(g)$$

and for any $\varphi \in \mathcal{T}(\Phi)$,

$$T_\lambda(g) = (\mathrm{Lift}^\Phi_\varphi S^\varphi_\lambda)(g).$$

5. Proof of Theorems 4.7, 4.8, and 4.11.

In this section we will prove Theorems 4.7, 4.8, and 4.11. In order to handle both cases at the same time, we will let Ω be defined as in (4.12a) when Φ contains an irreducible factor of type $A_{2k}, k \geq 1$. If Φ contains no irreducible factors of type A_{2k} we set $\Omega = G_{\mathbf{C}}$.

Let $\varphi \in \mathcal{T}(\Phi)$. Fix a set of positive roots Φ^+ and define $\rho(\Phi^+,\varphi)$ as in (4.6). If Φ contains no irreducible factors of type $A_{2k}, k \geq 1$, $\rho(\Phi^+,\varphi)$ is in the root lattice of Φ by [**H4**, Theorem 5.7]. Thus $e^{\rho(\Phi^+,\varphi)}$ gives a well-defined character of $B_{\mathbf{C}} = B_{\mathbf{C}} \cap \Omega$. Otherwise, we can define $e^{\rho(\Phi^+,\varphi)}$ on $B'_{\mathbf{C}} \cap \Omega$ as in (4.14) using Lemma 4.10.

Now for $b \in B'_{\mathbf{C}} \cap \Omega$, we can define

$$\Delta(\Phi^+, \varphi, b) = \Delta'(\Phi^+ : b)^{-1}\ \Delta'(\varphi^+ : b)\ e^{-\rho(\Phi^+,\varphi)}(b); \tag{5.1a}$$

$$\delta(\Phi^+, \varphi, b) = |\Delta(\Phi^+, \varphi, b)|\ \Delta(\Phi^+, \varphi, b)^{-1}. \tag{5.1b}$$

Then as in [**H4**, Lemma 6.6], we have

$$D^{\Phi}_{\varphi}(b) = |\Delta(\Phi^+, \varphi, b)| = \delta(\Phi^+, \varphi, b)\ \Delta(\Phi^+, \varphi, b), b \in B'_{\mathbf{C}} \cap \Omega. \tag{5.2}$$

Lemma 5.1. *Let $\varphi \in \mathcal{T}(\Phi), b \in B''_{\mathbf{C}} \cap \Omega, \lambda \in L'_B$. Then for any choice Φ^+ of positive roots for Φ,*

$$\begin{aligned}&\epsilon^{\Phi}_{\varphi}(\lambda)\ (\mathrm{Lift}^{\Phi}_{\varphi} T^{\varphi}_{\lambda})(b)\\ &= \epsilon(\Phi^+ : \lambda)\ \epsilon(\varphi : \Phi^+) \sum_{w \in W(\Phi,\varphi,b)} \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda - \rho(\Phi^+)}(wb)\\ &\qquad \cdot \delta(\Phi^+, \varphi, wb)\ \overline{c}(\lambda : \varphi^+_{wb}),\end{aligned}$$

where

$$W(\Phi, \varphi, b) = \{w \in W(\Phi) : wb \in B(\varphi)\}.$$

Proof. Since $b \in B''_{\mathbf{C}}$, by Lemma 4.4 we can take $X_{\varphi,\mathbf{C}}(b) = \{wb\}$ where w runs over a set of coset representatives for $W(\varphi)\backslash W(\Phi)$. Now using the definitions (4.4) and (4.10) we have

$$\begin{aligned}&\epsilon^{\Phi}_{\varphi}(\lambda)\ (\mathrm{Lift}^{\Phi}_{\varphi} T^{\varphi}_{\lambda})(b)\\ &= \epsilon(\Phi^+ : \lambda)\ \epsilon(\varphi^+ : \lambda)\ \epsilon(\varphi : \Phi^+) \sum_{w \in W(\varphi)\backslash W(\Phi)} D^{\Phi}_{\varphi}(wb)\ T^{\varphi}_{\lambda}(wb).\end{aligned}$$

But $T^{\varphi}_{\lambda}(wb) = 0$ unless $wb \in B(\varphi)$, that is $w \in W(\Phi, \varphi, b)$. Since $B(\varphi)$ is invariant under $W(\varphi)$, we will have $w \in W(\Phi, \varphi, b)$ if and only if $vw \in W(\Phi, \varphi, b)$ for all $v \in W(\varphi)$. Let $w \in W(\Phi, \varphi, b)$. Then by (4.7) or (4.15),

$$\begin{aligned}&D^{\Phi}_{\varphi}(wb)\ T^{\varphi}_{\lambda}(wb)\\ &= \epsilon(\varphi^+ : \lambda) \sum_{v \in W(\varphi)} D^{\Phi}_{\varphi}(vwb)\Delta'(\varphi^+ : vwb)^{-1} e^{\lambda - \rho_{\varphi}}(vwb)\ \overline{c}(\lambda : \varphi^+_{vwb}).\end{aligned}$$

Now, for all $v \in W(\varphi)$, using (5.1) and (5.2),

$$\begin{aligned}&D^{\Phi}_{\varphi}(vwb)\ \Delta'(\varphi^+ : vwb)^{-1}\ e^{\lambda - \rho_{\varphi}}(vwb)\\ &= \delta(\Phi^+, \varphi, vwb)\ \Delta'(\Phi^+, vwb)^{-1}\ e^{\lambda - \rho}(vwb).\end{aligned}$$

Thus

$$\begin{aligned}&\epsilon^{\Phi}_{\varphi}(\lambda)\ (\mathrm{Lift}^{\Phi}_{\varphi} T^{\varphi}_{\lambda})(b)\\ &= \epsilon(\Phi^+ : \lambda)\ \epsilon(\varphi : \Phi^+) \sum_{w \in W(\varphi)\backslash W(\Phi,\varphi,b)}\ \sum_{v \in W(\varphi)} \delta(\Phi^+, \varphi, vwb)\\ &\qquad \cdot \Delta'(\Phi^+ : vwb)^{-1}\ e^{\lambda - \rho}(vwb)\ \overline{c}(\lambda : \varphi^+_{vwb})\end{aligned}$$

$$= \epsilon(\Phi^+ : \lambda)\ \epsilon(\varphi : \Phi^+) \sum_{w \in W(\Phi,\varphi,b)} \delta(\Phi^+, \varphi, wb) \cdot \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda-\rho}(wb)\ \overline{c}(\lambda : \varphi^+_{wb}).$$

□

For $b \in B_{\mathbf{C}}$, define

$$\mathcal{T}(\Phi, b) = \{\varphi \in \mathcal{T}(\Phi) : b \in B(\varphi)\}. \tag{5.3}$$

Lemma 5.2. *Let $b \in B''_{\mathbf{C}} \cap \Omega, \lambda \in L'_B$. Then for any $\varphi_0 \in \mathcal{T}(\Phi)$,*

$$\begin{aligned} &\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^\Phi_\varphi(\lambda)\ (\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda)(b) \\ &= (\mathrm{Lift}^\Phi_{\varphi_0} S^{\varphi_0}_\lambda)(b) \\ &= \epsilon(\Phi^+ : \lambda) \sum_{w \in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda-\rho}(wb) \\ &\qquad \cdot \sum_{\varphi \in \mathcal{T}(\Phi,wb)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, wb)\ \overline{c}(\lambda : \varphi^+_{wb}). \end{aligned}$$

Proof. For any $\varphi \in \mathcal{T}(\Phi), w \in W(\Phi)$, $w \in W(\Phi, \varphi, b)$ if and only if $\varphi \in \mathcal{T}(\Phi, wb)$. Thus using Lemma 5.1 we have

$$\begin{aligned} &\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^\Phi_\varphi(\lambda)\ (\mathrm{Lift}^\Phi_\varphi T^\varphi_\lambda)(b) \\ &= \epsilon(\Phi^+ : \lambda) \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi : \Phi^+) \\ &\qquad \cdot \sum_{w \in W(\Phi,\varphi,b)} \delta(\Phi^+, \varphi, wb)\ \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda-\rho}(wb)\ \overline{c}(\lambda : \varphi^+_{wb}) \\ &= \epsilon(\Phi^+ : \lambda) \sum_{w \in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda-\rho}(wb) \\ &\qquad \cdot \sum_{\varphi \in \mathcal{T}(\Phi,wb)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, wb)\ \overline{c}(\lambda : \varphi^+_{wb}). \end{aligned}$$

Further, since lifting is clearly linear, we have using definition (4.11) or (4.16)

$$(\mathrm{Lift}^\Phi_{\varphi_0} S^{\varphi_0}_\lambda)(b) = [W(\Phi, \varphi_0)]^{-1} \sum_{s \in W(\Phi)} \epsilon^\Phi_{\varphi_0}(s\lambda)\ (\mathrm{Lift}^\Phi_{\varphi_0} T^{\varphi_0}_{s\lambda})(b).$$

Fix $s \in W(\Phi)$. Then, using Lemma 5.1 to evaluate $\epsilon^\Phi_{\varphi_0}(s\lambda)\ (\mathrm{Lift}^\Phi_{\varphi_0} T^{\varphi_0}_{s\lambda})(b)$ with the positive roots $s\Phi^+$, we have

$$\epsilon^\Phi_{\varphi_0}(s\lambda)\ (\mathrm{Lift}^\Phi_{\varphi_0} T^{\varphi_0}_{s\lambda})(b)$$

$$= \epsilon(s\Phi^+ : s\lambda)\ \epsilon(\varphi_0 : s\Phi^+) \cdot \sum_{w \in W(\Phi,\varphi_0,b)} \Delta'(s\Phi^+ : wb)^{-1}\ e^{s\lambda-\rho(s\Phi^+)}(wb) \cdot \delta(s\Phi^+, \varphi_0, wb)\ \overline{c}(s\lambda : (\varphi_0)^+_{wb}).$$

But it is easy to check that

$$\epsilon(s\Phi^+ : s\lambda) = \epsilon(\Phi^+ : \lambda), \quad \Delta'(s\Phi^+ : wb) = \Delta'(\Phi^+ : s^{-1}wb)$$
$$\delta(s\Phi^+, \varphi_0, wb) = \delta(\Phi^+, s^{-1}\varphi_0, s^{-1}wb).$$

Further, by [**H4**, Lemma 5.4] and (2.8d),

$$\epsilon(s\Phi^+ : \varphi_0) = \epsilon(\Phi^+ : s^{-1}\varphi_0), \quad \overline{c}(s\lambda : (\varphi_0)^+_{wb}) = \overline{c}(\lambda : s^{-1}(\varphi_0)^+_{wb}) = \overline{c}(\lambda : (s^{-1}\varphi_0)^+_{s^{-1}wb}).$$

Thus we have

$$\epsilon^{\Phi}_{\varphi_0}(s\lambda)\ (\mathrm{Lift}^{\Phi}_{\varphi_0} T^{\varphi_0}_{s\lambda})(b) = \epsilon(\Phi^+ : \lambda)\ \epsilon(s^{-1}\varphi_0 : \Phi^+) \sum_{w \in W(\Phi,\varphi_0,b)} \Delta'(\Phi^+ : s^{-1}wb)^{-1}\ e^{\lambda-\rho(\Phi^+)}(s^{-1}wb) \cdot \delta(\Phi^+, s^{-1}\varphi_0, s^{-1}wb)\ \overline{c}(\lambda : (s^{-1}\varphi_0)^+_{s^{-1}wb}).$$

But $w \in W(\Phi, \varphi_0, b)$ if and only if $wb \in B'(\varphi_0)$ if and only if $s^{-1}wb \in B'(s^{-1}\varphi_0)$ if and only if $s^{-1}w \in W(\Phi, s^{-1}\varphi_0, b)$. Thus

$$\epsilon^{\Phi}_{\varphi_0}(s\lambda)\ (\mathrm{Lift}^{\Phi}_{\varphi_0} T^{\varphi_0}_{s\lambda})(b) = \epsilon(\Phi^+ : \lambda)\ \epsilon(s^{-1}\varphi_0 : \Phi^+) \sum_{w \in W(\Phi,s^{-1}\varphi_0,b)} \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda-\rho(\Phi^+)}(wb) \cdot \delta(\Phi^+, s^{-1}\varphi_0, wb)\ \overline{c}(\lambda : (s^{-1}\varphi_0)^+_{wb})$$
$$= \epsilon^{\Phi}_{s^{-1}\varphi_0}(\lambda)\ (\mathrm{Lift}^{\Phi}_{s^{-1}\varphi_0} T^{s^{-1}\varphi_0}_{\lambda})(b)$$

by Lemma 5.1. Now every $\varphi \in \mathcal{T}(\Phi)$ is of the form $s^{-1}\varphi_0$ for some $s \in W(\Phi)$, and when we sum over $s \in W(\Phi)$, each $\varphi \in \mathcal{T}(\Phi)$ occurs $[W(\Phi, \varphi_0)]$ times. Thus

$$(\mathrm{Lift}^{\Phi}_{\varphi_0} S^{\varphi_0}_{\lambda})(b) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon^{\Phi}_{\varphi}(\lambda)\ (\mathrm{Lift}^{\Phi}_{\varphi} T^{\varphi}_{\lambda})(b).$$

□

Let Ψ be any root system. We define rank $\mathcal{T}(\Psi)$ to be the common rank of all $\psi \in \mathcal{T}(\Psi)$. Then rank $\mathcal{T}(\Psi) \le$ rank Ψ and the two are equal just in case Ψ is spanned by orthogonal roots. We let $\mathcal{T}_{\mathrm{aug}}(\Psi)$ denote the set of all root subsystems $\psi \subset \Psi$ such that

(i) every irreducible factor of ψ is of type A_1 or B_2;

(ii) rank $\psi =$ rank $\mathcal{T}(\Psi)$.

Then $\mathcal{T}(\Psi) \subset \mathcal{T}_{\mathrm{aug}}(\Psi)$. Suppose that Ψ is irreducible. If two-structures of Ψ

are of type A_1^k, then $\mathcal{T}_{\rm aug}(\Psi) = \mathcal{T}(\Psi)$. However if two-structures of Ψ contain irreducible factors of type B_2, then $\mathcal{T}(\Psi)$ is a proper subset of $\mathcal{T}_{\rm aug}(\Psi)$, since $\mathcal{T}_{\rm aug}(\Psi)$ contains all root subsystems of Ψ of type $B_2^r \times A_1^s, 2r + s =$ rank $\mathcal{T}(\Psi)$.

Define $\mathcal{T}(\Phi, b), b \in B_{\mathbf{C}}$, as in (5.3).

Lemma 5.3. *Let $\varphi \in \mathcal{T}(\Phi), b \in B'(\Phi)$. Then $\varphi \in \mathcal{T}(\Phi, b)$ if and only if $\varphi \cap \Phi_b \in \mathcal{T}_{\rm aug}(\Phi_b)$.*

Proof. Let $b \in B'(\Phi), \varphi \in \mathcal{T}(\Phi)$, and let $\psi = \Phi_b \cap \varphi$. Then every irreducible factor of ψ is of type A_1 or B_2. Further, Φ_b is spanned by strongly orthogonal roots, so that rank $\mathcal{T}(\Phi_b) =$ rank Φ_b. Thus $\psi \in \mathcal{T}_{\rm aug}(\Phi_b)$ if and only if rank $\psi =$ rank Φ_b.

Suppose that $\varphi \in \mathcal{T}(\Phi, b)$. Then $b \in B(\varphi)$, so there is $S \in \mathcal{SO}_{\mathbf{C}}(\varphi)$ such that $b = t\exp(iH) \in B(S)$. As in the proof of Lemma 4.5, there is $S' \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ such that $[S] = [S'], \underline{t}_S = \underline{t}_{S'}$, and $b \in B(S')$. Thus rank $\Phi_b = [S'] = [S]$. Now for $\alpha \in S, t_0 \in T^0_{S'}, e^\alpha(tt_0) = 1$ since $t \in T^1_S$ and $t_0 \in T^0_{S'} = T^0_S$. Thus $S \subset \Phi_b$, and so $S \subset \psi = \varphi \cap \Phi_b$. Thus $[S] \leq$ rank $\psi \leq$ rank $\Phi_b = [S]$, and $\psi \in \mathcal{T}_{\rm aug}(\Phi_b)$.

Now suppose that $\psi \in \mathcal{T}_{\rm aug}(\Phi_b)$. Let S be a basis for ψ consisting of one root from every irreducible factor of ψ of type A_1 and two long orthogonal roots from every irreducible factor of ψ of type B_2. Since rank $\psi =$ rank Φ_b, we know that $[S] =$ rank Φ_b. Further, S is strongly orthogonal in ψ. Suppose that S is not strongly orthogonal in φ. Then there are $\alpha, \beta \in S$ with $\alpha \pm \beta \in \varphi$. But $\alpha, \beta \in \Phi_b, \alpha \pm \beta \in \Phi$ implies that $\alpha \pm \beta \in \Phi_b$. Thus $\alpha \pm \beta \in \psi$. This contradicts the fact that α, β are strongly orthogonal in ψ. Thus $S \in \mathcal{SO}_{\mathbf{C}}(\varphi)$. Since $b \in B(\Phi)$ there is $S' \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ so that $b = t\exp(iH) \in B(S')$. Now since $S \subset \Phi_b$, $e^\alpha(t) = 1$ for all $\alpha \in S$. Thus $t \in T^1_S$. Further, since S and S' are both orthogonal subsets of Φ_b with $[S'] = [S] =$ rank Φ_b, they must have the same linear span. Thus $\underline{b}_S = \underline{b}_{S'}$, and so $b \in B(S) \subset B(\varphi)$. □

For $b \in B'(\Phi) \cap \Omega$ and $\psi \in \mathcal{T}_{\rm aug}(\Phi_b)$, define

$$\mathcal{T}(\Phi, \psi) = \{\varphi \in \mathcal{T}(\Phi) : \varphi \cap \Phi_b = \psi\}.$$

By Lemma 5.3,

$$\mathcal{T}(\Phi, b) = \cup_{\psi \in \mathcal{T}_{\rm aug}(\Phi_b)} \mathcal{T}(\Phi, \psi). \tag{5.4}$$

Lemma 5.4. *Let $b \in B'(\Phi) \cap \Omega$ and $\psi \in \mathcal{T}_{\rm aug}(\Phi_b)$. Then for any choice Φ^+ of positive roots for Φ,*

$$\sum_{\varphi \in \mathcal{T}(\Phi, \psi)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, b) = \begin{cases} \epsilon(\psi : \Phi_b^+) & \text{if } \psi \in \mathcal{T}(\Phi_b); \\ 0 & \text{if } \psi \notin \mathcal{T}(\Phi_b). \end{cases}$$

Proof. Fix $b \in B'(\Phi) \cap \Omega$ and let $S \in \mathcal{SO}_{\mathbf{C}}(\Phi)$ such that $b \in B'(S)$. By Lemma 3.3 there is a real form G of $G_{\mathbf{C}}$ such that $G \cap B_{\mathbf{C}} = B$ and S consists of noncompact roots for G. Thus as in Lemma 3.2 we have $b = y_S^{-1} h y_S, h = ta \in H_S^1$. Now in the notation of [**H4**, §6], we have $\delta(\Phi^+, \varphi, b) = \delta(\Phi^+, \varphi, h), \Phi_b = \Phi_t$, and $\Phi_b^+ = \Phi_R^+(h)$. Thus the lemma follows directly from [**H4**, Lemma 7.2]. □

Finally, we will need the following theorem which was proven in [**H1**, Theorem 1]. Let E be a real vector space, let $\Psi \subset E$ be a root system spanned by strongly orthogonal roots, and let Ψ^+ be a choice of positive roots for Ψ. Let $E'(\Psi) = \{\lambda \in E :< \lambda, \alpha > \neq 0 \ \forall\ \alpha \in \Psi\}$.

Theorem 5.5. *For all* $\lambda \in E'(\Psi)$,

$$\bar{c}(\lambda : \Psi^+) = \sum_{\psi \in \mathcal{T}(\Psi)} \epsilon(\psi : \Psi^+)\ \bar{c}(\lambda : \psi \cap \Psi^+).$$

Proof of Theorems 4.7, 4.8 *and* 4.11. By Lemma 5.2, Theorems 4.7 and 4.8 are equivalent and the two parts of Theorem 4.11 are equivalent. Since both sides are class functions on $G'_{\mathbf{C}} \cap \Omega$, it suffices to prove the theorems for all $b \in B''_{\mathbf{C}} \cap \Omega$. Then using Lemma 5.2, for any $b \in B''_{\mathbf{C}} \cap \Omega$, we have

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon_\varphi^\Phi(\lambda)(\mathrm{Lift}_\varphi^\Phi T_\lambda^\varphi)(b) = \epsilon(\Phi^+ : \lambda) \sum_{w \in W(\Phi)} \Delta'(\Phi^+ : wb)^{-1}\ e^{\lambda - \rho(\Phi^+)}(wb) \times$$

$$\sum_{\varphi \in \mathcal{T}(\Phi, wb)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, wb)\ \bar{c}(\lambda : \varphi_{wb}^+).$$

Suppose that $b \notin B''(\Phi)$. Then $T_\lambda(b) = 0$. Let $w \in W(\Phi)$. Then $wb \notin B(\Phi)$ so that $wb \notin B(\varphi)$ for all $\varphi \in \mathcal{T}(\Phi)$ by Lemma 4.5. Thus $\mathcal{T}(\Phi, wb) = \emptyset$, so that

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon_\varphi^\Phi(\lambda)\ (\mathrm{Lift}_\varphi^\Phi T_\lambda^\varphi)(b) = 0.$$

Now suppose that $b \in B''(\Phi)$. Comparing the formula from Lemma 5.2 to that for $T_\lambda(b)$ in (2.11), we see that it is enough to prove that for all $w \in W(\Phi)$,

$$\sum_{\varphi \in \mathcal{T}(\Phi, wb)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, wb)\ \bar{c}(\lambda : \varphi_{wb}^+) = \bar{c}(\lambda : \Phi_{wb}^+).$$

Equivalently, we must show that for all $b \in B''(\Phi) \cap \Omega$,

$$\sum_{\varphi \in \mathcal{T}(\Phi, b)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, b)\ \bar{c}(\lambda : \varphi_b^+) = \bar{c}(\lambda : \Phi_b^+).$$

Fix $b \in B''(\Phi) \cap \Omega$. For every $\varphi \in \mathcal{T}(\Phi, b)$, we have $\varphi_b^+ = \varphi \cap \Phi_b^+ = \psi \cap \Phi_b^+$ where $\psi = \varphi \cap \Phi_b \in \mathcal{T}_{\mathrm{aug}}(\Phi_b)$ by Lemma 5.3.

Thus for any choice Φ^+ of positive roots and any $\lambda \in L'_B$, using (5.4) we can write

$$\sum_{\varphi \in \mathcal{T}(\Phi,b)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, b)\ \overline{c}(\lambda : \varphi_b^+)$$
$$= \sum_{\psi \in \mathcal{T}_{\text{aug}}(\Phi_b)} \overline{c}(\lambda : \psi \cap \Phi_b^+) \sum_{\varphi \in \mathcal{T}(\Phi,\psi)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, b).$$

Now by Lemma 5.4 and Theorem 5.5,

$$\sum_{\psi \in \mathcal{T}_{\text{aug}}(\Phi_b)} \overline{c}(\lambda : \psi \cap \Phi_b^+) \sum_{\varphi \in \mathcal{T}(\Phi,\psi)} \epsilon(\varphi : \Phi^+)\ \delta(\Phi^+, \varphi, b)$$
$$= \sum_{\psi \in \mathcal{T}(\Phi_b)} \epsilon(\psi : \Phi_b^+)\ \overline{c}(\lambda : \psi \cap \Phi_b^+)$$
$$= \overline{c}(\lambda : \Phi_b^+).$$

□

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