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We discuss sectors on a surface of curvature bounded above by a constant and derive an isoperimetric inequality for a proper sector on such a surface. With this isoperimetric inequality we derive an inequality involving the total length of the cut locus of a point on a closed surface.

1. Introduction.

There have been extensive studies on isoperimetric inequalities on a Riemannian manifold of dimension 2 (shortly, a *surface*) [**Al**, **BdC**, **CF**, **Fi**]. C. Bandle derived an isoperimetric inequality for a sector in the Euclidean plane $\mathbb{E}^2$ [**Ba1**, **Ba2**]: Let D be a sector in $\mathbb{E}^2$, which is a simply connected region enclosed by two line segments γ_1, γ_2 starting at a point p and a piecewise smooth simple curve segment Γ joining the end points of γ_1, γ_2. Let θ_0 denote the interior angle of D at p. C. Bandle showed that for a sector D with $\theta_0 \leq \pi$,

$$L^2(\Gamma) \geq 2\,\theta_0\, A(D)$$

with equality if and only if D is a circular sector, where $L(\Gamma)$ denotes the length of Γ and $A(D)$ the area of D.

In this paper, we consider an isoperimetric inequality for a sector on a surface M with curvature K bounded above by a constant C. By a *sector* on a surface M we mean a region of M enclosed by two geodesic segments γ_1, γ_2 and a piecewise smooth curve segment Γ, which together constitute a simple closed curve $\Gamma^* = \gamma_1 \cup \Gamma \cup \gamma_2$. On a general surface M, a sector needs not be simply connected nor bounded (e.g., the cylinder $\mathbb{R} \times \mathbb{S}^1$), or could be the whole surface (e.g., the torus T^2). On the other hand, such a simple closed curve Γ^* may enclose two bounded sectors (e.g., the sphere $\mathbb{S}^2$). For our purpose, we will restrict our attention to sectors on a surface M that are closed, simply connected and bounded ones. We will call such a sector by a *proper sector*, denoted by $D(\Gamma)$ or just D. We take parametrizations of two geodesic segments γ_i and a piecewise smooth curve segment Γ as $\gamma_i : [0, r_i] \to M$ $(i = 1, 2)$ and $\Gamma : [a, b] \to M$ such that $\gamma_1(0) = \gamma_2(0)$ and $\Gamma(a) = \gamma_1(r_1)$, $\Gamma(b) = \gamma_2(r_2)$ so that $\Gamma^* = \gamma_1 \cup \Gamma \cup \gamma_2$ is a simple closed curve with a suitable orientation. The *vertex* of D is the point $\gamma_1(0) = \gamma_2(0)$ where γ_1 and γ_2 cross. Our main result is the following:

Isoperimetric Inequality for a Sector. *Let M be a surface with curvature K bounded above by a constant C. Let D be a proper sector on M with interior angle $\theta_0 \leq \pi$ at the vertex. Then*

$$L^2(\Gamma) \geq 2\,\theta_0\,A(D) - C\,A^2(D).$$

Equality holds only when D is isometric to a geodesic sector on a surface of constant curvature C.

Generally, the cut locus Cut (p) of p on a closed surface M is a local tree which may have infinitely many edges [**M1**, **M2**, **GS**]. So, the Hausdorff 1-measure is used to measure a subset of Cut (p). It is known that every compact subset of Cut (p) of p on a complete surface M has finite total length (Hausdorff 1-measure) [**He2**, **I**]. As an application of our isoperimetric inequality, we derive an inequality involving the total length of the cut locus of a point on a closed surface.

2. Sectors on a Surface.

Definition 2.1. Let A be a subset of M and $p, q \in A$. The distance between p and q in A is defined by

$$d_A(p,q) = \inf_{c \in \Omega^A_{p,q}} \int_c ds,$$

where $\Omega^A_{p,q}$ is the set of all piecewise smooth curve segments contained in A joining p and q. Let $B \subset M$ such that $A \cap B \neq \emptyset$. Then $d_A(p,B) = \inf_{q \in A\cap B} d_A(p,q)$ denotes the distance from p to B in A.

Note that $d_A(\cdot,\cdot) \geq d_M(\cdot,\cdot)$ for any set $A \subset M$.

Definition 2.2. Let A be a compact subset of M and G a subset of the boundary ∂A of A. For $t \geq 0$, the *parallel* G_t of G in the distance t in A is defined by

$$G_t = \{q \in A : d_A(q,G) = t\}.$$

For a proper sector $D(\Gamma)$ in M, the parallel Γ_t of Γ in $D(\Gamma)$ is a piecewise smooth simple curve for small $t > 0$. As t gets larger, Γ_t can have several components.

Relative Cut Locus. Let $D(\Gamma)$ be the proper sector with two geodesics γ_1, γ_2 on M and a piecewise smooth simple curve segment $\Gamma : [a,b] \to M$ with corners $\Gamma(s_i) = x_i$ at $a = s_1 < s_2 < \cdots < s_{n+1} = b$. Let $N(s)$ be the inward unit normal vector field along Γ with the right/left limits $N(s_i^{\pm})$ at the corners x_i of Γ, $i = 1, \ldots, n+1$. With notational conventions, $N(s_1^-) = -\gamma_1'(r_1)$ and $N(s_{n+1}^+) = -\gamma_2'(r_2)$, let $\mathcal{N}_i$ be the set of all inward unit tangent vectors in $T_{x_i}M$ between $N(s_i^-)$ and $N(s_i^+)$. Let

$$\mathcal{N} = \bigcup_{i=0}^{n+1} \mathcal{N}_i,$$

where $\mathcal{N}_0 = \{N(s) : s \in [a,b]\}$. For each $v \in \mathcal{N} \cap T_qM$, $q \in \Gamma$, let $\gamma_v : [0,r] \to M$ be a geodesic such that $\gamma_v(0) = q$ and $\gamma_v'(0) = v$. The point $z = \gamma_v(t)$ where γ_v stops minimizing the distance $d_{D(\Gamma)}(\gamma_v(t), \Gamma)$ is called the *relative cut point* of $v \in \mathcal{N}$ in $D(\Gamma)$. The set of all such relative cut points of $v \in \mathcal{N}$ in $D(\Gamma)$ is called the *relative cut locus* of Γ in $D(\Gamma)$, denoted by $\mathcal{C}_{\mathrm{rel}}(\Gamma)$. If the exterior angle of Γ at a corner x_i is positive, then for all $v \in \mathcal{N}_i$, the relative cut point of v in $D(\Gamma)$ is x_i itself. Note also that $\mathcal{C}_{\mathrm{rel}}(\Gamma)$ need not be a connected set.

Geodesic Sectors. For a point $p \in M$, let $\mathcal{U}$ be a (simply connected) normal neighborhood of p, and take *polar coordinates* (r,θ) on $\mathcal{U} \setminus \{p\}$ such that the metric can be written as

$$ds^2 = dr^2 + f^2\, d\theta^2,$$

where $f = f(r,\theta)$ is the positive-valued function satisfying the initial conditions

$$\lim_{r\to 0} f(r,\theta) = 0, \qquad \lim_{r\to 0} \frac{\partial f}{\partial r}(r,\theta) = 1.$$

Let γ_1, $\gamma_2 : [0,r] \to M$, $i = 1,2$, be two geodesics starting at p with the angle $0 < \theta \le \pi$ and $\beta : [0,\theta] \to \mathcal{U} \subset M$ a geodesic circular arc given by $\beta(s) = (r,s)$ in $\mathcal{U}$. The proper sector enclosed by γ_1, γ_2 and β is called a *geodesic sector* denoted by $S_{r,\theta}$. We will call β the *circular boundary* of $S_{r,\theta}$. The area $A_{r,\theta}$ and the arc length $L_{r,\theta}$ of the circular boundary of $S_{r,\theta}$ are respectively given by

$$A_{r,\theta} = \int_0^\theta \int_0^r f(t,s)\, dt ds, \quad L_{r,\theta} = \int_0^\theta f(r,s)\, ds. \tag{2.1}$$

Remark 2.3. A geodesic sector with vertex p may cross the *usual cut locus* $\mathrm{Cut}(p)$ of p. One can easily construct a geodesic sector $S_{r,\theta}$ crossing the cut locus $\mathrm{Cut}(p)$ of p on the cylinder $\mathbb{R} \times \mathbb{S}^1$.

Let $S^C_{r,\theta}$ denote a geodesic sector of radius r and angle θ on a surface M_C of constant curvature $K \equiv C$. Let $A^C_{r,\theta}$ and $L^C_{r,\theta}$ denote its area and the arc length of the circular boundary, respectively. The explicit expressions are:

$$\begin{array}{cccc} C: & a^2>0 & 0 & -a^2<0 \\[2ex] A^C_{r,\theta}: & 2\theta\dfrac{\sin^2\frac{ar}{2}}{a^2} & \dfrac{1}{2}\theta r^2 & 2\theta\dfrac{\sinh^2\frac{ar}{2}}{a^2} \\[3ex] L^C_{r,\theta}: & \theta\dfrac{\sin ar}{a} & \theta r & \theta\dfrac{\sinh ar}{a}. \end{array} \tag{2.2}$$

One can easily verify the following formula:

$$(L^C_{r,\theta})^2 = 2\theta A^C_{r,\theta} - C(A^C_{r,\theta})^2. \tag{2.3}$$

For the geodesic sectors on a surface with curvature $K \le C$, we may have the following lemmas as immediate consequences of the formulas (2.1) and Lemma 7 in [**Os**]:

Lemma 2.4. *Let M be a surface with curvature $K \le C$. Let $S_{r,\theta}$ be a geodesic sector on M of radius r and angle θ. Then*

$$A_{r,\theta} \ge A^C_{r,\theta}$$

with equality if and only if $S_{r,\theta}$ is isometric to $S^C_{r,\theta}$ on a surface M_C of constant curvature C.

Lemma 2.5. *Under the same assumptions as in Lemma 2.4, we have*

$$L_{r,\theta} \ge L^C_{r,\theta}.$$

If $L_{\xi,\theta} = L^C_{\xi,\theta}$ for all $\xi \in (0, r]$, then $S_{r,\theta}$ is isometric to $S^C_{r,\theta}$ on a surface M_C of constant curvature C.

3. Isoperimetric Inequalities for Sectors on a Surface.

Let $\beta : [a, b] \to M$ be a unit speed simple curve, and let $\mathbf{n}$ be a unit normal vector field along β. Then one can find a variation $\mathcal{X} : [a, b] \times (-\delta, \delta) \to M$ of β given by

$$\mathcal{X}(s, \xi) = \exp_{\beta(s)} \xi \mathbf{n}(s) \tag{3.1}$$

for some $\delta > 0$.

Let $L_{\mathcal{X}}(\xi)$ denote the arc length of the curve $\beta_\xi(s) = \mathcal{X}(s, \xi)$, which will be called a *geodesic parallel* of β. Then, from the first variation formula

$$L'_{\mathcal{X}}(0) = \int_a^b g(\beta', \nabla_{\beta'}\mathbf{n})\, ds = -\int_a^b \kappa\, ds,$$

where ∇ is the Levi-Civita connection of M, we have

$$L_{\mathcal{X}}(\xi) = L_{\mathcal{X}}(0) - \xi \int_\beta \kappa\, ds + o(\xi), \tag{3.2}$$

where κ is the geodesic curvature of β.

For the computation of the length of the parallel Γ_t of Γ, we write $\Gamma = \sum_{i=1}^n \beta^i$ where β^i $(i = 1, \dots, n)$ are smooth curves with inward unit normal N. For $t > 0$ small, the parallel Γ_t of Γ in $D(\Gamma)$ consists of parts of the geodesic parallels β^i_t of β^i in $D(\Gamma)$ together with the geodesic circular arcs of radius t. Let

$$t_* = \sup_{p\in\Gamma,\ q\in\mathcal{C}_{\mathrm{rel}}(\Gamma)} d_{D(\Gamma)}(p, q),$$

and for $0 \le t \le t_*$, let

$$D(\Gamma_t) = \{q \in D(\Gamma) : d_{D(\Gamma)}(q, \Gamma) \ge t\}.$$

Notice that $D(\Gamma_t)$ may not be a proper sector since it is not a connected set in general. We will denote the arc length of Γ_t by $L(t)$ and the area of $D(\Gamma_t)$ by $A(t)$ as functions of t.

Lemma 3.1. *Let M be a surface with curvature $K \le C$. Let $D(\Gamma)$ be a proper sector on M with interior angle $\theta_0 \le \pi$ at the vertex. Then*

$$L'(0) \le CA(0) - \theta_0.$$

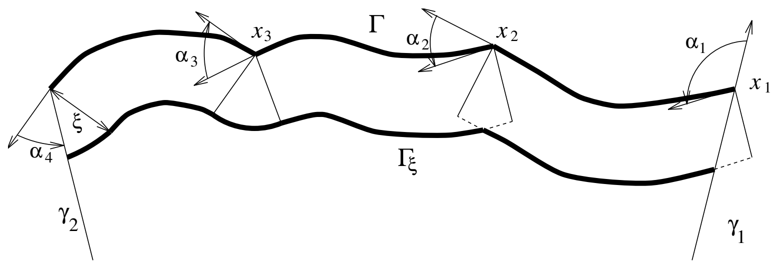

Figure 1.

Proof. Let x_i $(i = 1, 2, \ldots, n+1)$ be the corners of Γ including end points and let α_i denote the exterior angle of Γ^* at the corner x_i (See Figure 1). We may assume that $\alpha_i \ne \pi$. Let $\mathcal{S} = \{2, \ldots, n\}$, and let $\mathcal{A} = \{i \in \mathcal{S} : \alpha_i \le 0\}$ and $\mathcal{B} = \{j \in \mathcal{S} : \alpha_j > 0\}$. For sufficiently small $\xi > 0$, using the linear approximations for dotted parts of β^i_ξ (say, at x_2 in Figure 1), we have, with the help of (3.2),

$$\begin{aligned} L(\xi) \;\; &= \;\; L(0) - \xi \int_\Gamma \kappa ds - \sum_{i \in \mathcal{A}} \xi \alpha_i - \sum_{j \in \mathcal{B}} 2\xi \tan(\alpha_j/2) \\ &\quad\; -\xi \widetilde{\alpha}_1 - \xi \widetilde{\alpha}_{n+1} + o(\xi), \end{aligned}$$

where κ is the geodesic curvature of Γ and

$$\widetilde{\alpha}_k = \begin{cases} \alpha_k - \pi/2 & \text{if } \alpha_k \le \pi/2, \\ \tan(\alpha_k - \pi/2) & \text{if } \alpha_k > \pi/2. \end{cases}$$

Using that $\tan \alpha \ge \alpha$ for $\alpha \ge 0$ and $\widetilde{\alpha}_k \ge \alpha_k - \pi/2$, we have

$$\begin{aligned} L'(0) \;\; &= \;\; -\int_\Gamma \kappa ds - \sum_{i \in \mathcal{A}} \alpha_i - \sum_{j \in \mathcal{B}} 2 \tan(\alpha_j/2) - \widetilde{\alpha}_1 - \widetilde{\alpha}_{n+1} \\ &\le \;\; -\int_\Gamma \kappa ds - \sum_{i=1}^{n+1} \alpha_i + \pi. \end{aligned}$$

Since γ_1, γ_2 are geodesics, by the Gauss-Bonnet formula,

$$L'(0) \leq \int_{D(\Gamma)} K\, dA - \theta_0.$$

Finally, we have

$$L'(0) \leq CA(0) - \theta_0$$

from the curvature condition that $K \leq C$. □

Note that, for sufficiently small $t > 0$, Γ_t has one component of a piecewise smooth non-closed simple curve segment. Computation as in Lemma 3.1 thus gives that

$$L'(t) \leq CA(t) - \theta_0 \tag{3.3}$$

for sufficiently small $t > 0$.

Note also that Γ_t could be the union of at most finite number of piecewise smooth non-closed curves, piecewise smooth closed curves and points in general. For a piecewise smooth closed curve, by the same way as in Lemma 3.1 we have the following:

Lemma 3.2. *Let M be a surface with curvature $K \leq C$ and D a nondegenerate compact subset of M with the boundary $\partial D = G$ which is a piecewise smooth simple curve. Let G_ξ denote the parallel of $G = \partial D$ in D and $\ell(t) = L(G_t)$. Then*

$$\ell'(0) \leq CA(D) - 2\pi.$$

The following facts come from the results of [**Fi**, pp. 303-332] by a slight modification (also see [**CF**, p. 86]): $L(t)$ is continuous for all but at most a finite number of t in $[0, t_*]$ at which $L(t)$ has a jump discontinuity, however $A(t)$ is continuous; $A'(t) = -L(t)$ for almost all $t \in [0, t_*]$ (cf. [**Ha**, p. 706]).

Theorem 3.3. *Let M be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on M with interior angle $\theta_0 \leq \pi$ at the vertex. Then*

$$L'(t) \leq CA(t) - \theta_0 \tag{3.4}$$

for almost all $t \in [0, t_*]$.

Proof. Let n_t and m_t be the numbers of components of piecewise smooth non-closed curves Γ_t^i and piecewise smooth closed curves Ω_t^j of Γ_t, respectively. For almost all $t \in [0, t_*]$, we may write

$$\Gamma_t = \sum_{i=1}^{n_t} \Gamma_t^i + \sum_{j=1}^{m_t} \Omega_t^j.$$

Note that each end point of Γ_t^i is either on γ_1 or on γ_2 so that Γ_t^i and γ_1 and/or γ_2 bound a simply connected compact set, denoted by $D(\Gamma_t^i)$. Each

Ω_t^j itself also bounds simply connected compact set, denoted by $D(\Omega_t^j)$. Thus we may write

$$D(\Gamma_t) = \left(\bigcup_{i=1}^{n_t} D(\Gamma_t^i)\right) \bigcup \left(\bigcup_{j=1}^{m_t} D(\Omega_t^j)\right). \tag{3.5}$$

For the sake of brevity, we use the notations: $L_i(t) = L(\Gamma_t^i)$, $A_i(t) = A(D(\Gamma_t^i))$, $\ell_j(t) = L(\Omega_t^j)$ and $B_j(t) = A(D(\Omega_t^j))$. Then by the same computation as (3.3) we have

$$L_i'(t) \leq CA_i(t) - \theta_i, \tag{3.6}$$

where $\theta_0 \leq \theta_i \leq \pi$. By Lemma 3.2,

$$\ell_j'(t) \leq CB_j(t) - 2\pi. \tag{3.7}$$

Thus,

$$\begin{aligned} L'(t) &= \sum_i^{n_t} L_i'(t) + \sum_j^{m_t} \ell_j'(t) \\ &\leq \sum_i^{n_t} (CA_i(t) - \theta_i) + \sum_j^{m_t} (CB_j(t) - 2\pi) \\ &\leq CA(t) - \theta_0 \end{aligned}$$

for almost all $t \in [0, t_*]$. □

Theorem 3.4. *Let M be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on M with interior angle $\theta_0 \leq \pi$ at the vertex. Then*

$$L^2(s) - L^2(t) \geq 2\theta_0(A(s) - A(t)) - C(A^2(s) - A^2(t)) \tag{3.8}$$

for $s < t \in [0, t_*]$.

Proof. By multiplying $A'(t) = -L(t) \leq 0$ to the inequality (3.4), we get

$$-L(t)L'(t) \geq CA(t)A'(t) - \theta_0 A'(t) \tag{3.9}$$

for almost all $t \in [0, t_*]$. Note that $L(t)$ is continuous on $[0, t_*]$ except for a finite number of points $0 < t_1 < t_2 < \cdots < t_m < t_*$. Let $I_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, m+1$, where $t_0 = 0$, $t_{m+1} = t_*$. For $s < t$ in $[0, t_*]$, we may assume that $s \in I_i$ and $t \in I_j$ for some $i \leq j$. By direct computation, we have

$$-\int_s^t L(t)L'(t)\,dt = \frac{1}{2}(L^2(s) - L^2(t)) + \frac{1}{2}\sum_{k=i}^{j-1}(L^2(t_k^+) - L^2(t_k^-)),$$

where $h(r^{\pm})$, as usual, stands for the right/left limits of a function h at r. Notice that $L(t_k^+) < L(t_k^-)$ and $A(t_k^+) = A(t_k^-)$. Thus we have

$$-\int_s^t L(t)L'(t)dt \leq \frac{1}{2}(L^2(s) - L^2(t)). \tag{3.10}$$

Similarly,

$$\int_s^t A(t)A'(t)\,dt = -\frac{1}{2}(A^2(s) - A^2(t)), \tag{3.11}$$

$$-\int_s^t A'(t)\,dt = A(s) - A(t). \tag{3.12}$$

Therefore, from (3.9)–(3.12), we have

$$\begin{aligned} L^2(s) - L^2(t) &\geq -2\int_s^t L(t)L'(t)\,dt \\ &\geq 2C\int_s^t A(t)A'(t)\,dt - 2\theta_0\int_s^t A'(t)\,dt \\ &= 2\theta_0(A(s) - A(t)) - C(A^2(s) - A^2(t)). \end{aligned}$$

□

We are now ready to state and prove our main result.

Theorem 3.5. *Let M be a surface with curvature $K \leq C$. Let $D(\Gamma)$ be a proper sector on M with interior angle $\theta_0 \leq \pi$ at the vertex. Then*

$$L^2(\Gamma) \geq 2\theta_0 A(D(\Gamma)) - CA^2(D(\Gamma)), \tag{3.13}$$

where equality holds only when $D(\Gamma)$ is isometric to a geodesic sector on a surface M_C of constant curvature $K \equiv C$.

Proof. By setting $t \to t_*$ in (3.8) of Theorem 3.4, we get

$$L^2(s) \geq 2\theta_0 A(s) - CA^2(s). \tag{3.14}$$

Now at $s = 0$, we get the inequality (3.13).

For a geodesic sector on a surface M_C of constant curvature $K \equiv C$, it is quite clear that the equality holds in (3.13) by (2.3).

Suppose now that the equality $L^2(0) = 2\theta_0 A(0) - CA^2(0)$ holds. Then from (3.8) with $s = 0$ together with (3.14) we get

$$L^2(t) = 2\theta_0 A(t) - CA^2(t), \tag{3.15}$$

for all $0 \leq t \leq t_*$. Since Γ_{t_*} is in the relative cut locus $\mathcal{C}_{\mathrm{rel}}(\Gamma)$ of Γ in $D(\Gamma)$, $D(\Gamma_{t_*})$ is contained in $\mathcal{C}_{\mathrm{rel}}(\Gamma)$. That is, $A(t_*) = 0$ and so by (3.15) $L(t_*) = 0$. By differentiation,

$$L'(t) = -\theta_0 + CA(t) \tag{3.16}$$

for all $0 \le t \le t_*$. Therefore, equalities hold for all $0 \le t \le t_*$ in inequalities in the proof of Theorem 3.3. This implies that, for $t < t_*$, the exterior angles at the end points of Γ_t are less than or equal to $\pi/2$ and there are no corners (which are not end points) on Γ_t at which the exterior angle of Γ_t^* is positive. In addition, for each $v \in T_qM \cap \mathcal{N}$, which is the set defined as in Section 2, the geodesic $\gamma_v : [0, t_*] \to M$ such that $\gamma_v(0) = q \in \Gamma$, $\gamma_v'(0) = v$ satisfies $d_{D(\Gamma)}(\gamma_v(t), q) = t$ and $\gamma_v(t) \in \Gamma_t$ for each $t \in [0, t_*]$. That is, $\mathcal{C}_{\mathrm{rel}}(\Gamma) \subseteq \Gamma_{t_*}$. Therefore, $\mathcal{C}_{\mathrm{rel}}(\Gamma) = \Gamma_{t_*}$ is the set of a single point, say $\{p\}$. Moreover, no geodesics starting at p intersect before the distance t_* in $D(\Gamma)$, so $D(\Gamma)$ is a geodesic sector of radius t_* and angle θ_0 centered at p. By (3.15) and (3.16), $L : [0, t_*] \to \mathbb{R}$ satisfies the following ODE

$$L''(t) = -CL(t), \quad L(t_*) = 0, \quad L'(t_*) = -\theta_0. \tag{3.17}$$

By comparing the solution of (3.17) with $L^C_{t,\theta}$ in (2.2) for a geodesic sector on M_C, one can see that $D(\Gamma)$ is isometric to $S^C_{t_*,\theta_0}$ by Lemma 2.5. □

If $M = \mathbb{E}^2$ and we set $C = 0$, then Theorem 3.5 implies the result of C. Bandle [**Ba1**, **Ba2**]. Similar isoperimetric inequalities on Lorentzian surfaces were obtained by the authors [**BH**, **B**].

Remark 3.6. The condition that $\theta_0 \le \pi$ in Theorem 3.5 is essential. One can construct a proper sector for which the isoperimetric inequality (3.13) does not hold in the following way: In $\mathbb{E}^2$, consider a proper sector with $\Gamma = \Gamma^1 \cup \Gamma^2$, where Γ^1 is a semi-circle of radius r centered at q and Γ^2 is a circular arc of angle $0 < \varphi < \pi$ and radius αr $(0 < \alpha < 1)$ centered at p (see Figure 2). Take $C = 0$, then

$$L^2(\Gamma) = (\pi + \varphi\alpha)^2 r^2 < (\pi + \varphi)(\pi + \varphi\alpha^2) r^2 = 2\theta_0 A(D(\Gamma)).$$

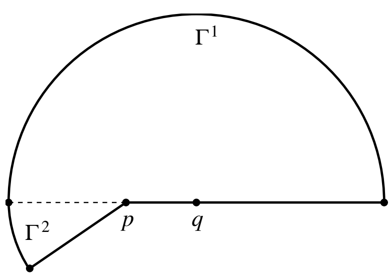

Figure 2.

Remark 3.7. The isoperimetric inequality (3.13) holds for a closed, simply connected, bounded set D having the boundary $\Gamma^* = \gamma_1 \cup \Gamma \cup \gamma_2$, where $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(r_1) = \gamma_2(r_2) = \Gamma(a) = \Gamma(b)$ by approximating Γ^* by Γ^*_ε, where Γ^*_ε is a closed curve obtained from Γ^* by changing the parts of Γ^* contained in a geodesic ball of radius ε at $\gamma_1(r_1) = \gamma_2(r_2)$ so that Γ^*_ε is piecewise smooth and simple.

We now consider a special case of a proper sector: Suppose that $\gamma_i : [0, r_i] \to M$ $(i = 1, 2)$ are two geodesic segments such that $\gamma_1(0) = \gamma_2(0) = p$, $\gamma_1(r_1) = \gamma_2(r_2) = q$ and $\gamma_1((0, r_1)) \cap \gamma_2((0, r_2)) = \emptyset$. Such a sector will be called an *oval sector*. Notice that there are no such oval sectors on a surface with curvature $K \leq 0$. Let θ_1, θ_2 be the interior angles of D at p, q, respectively. From Theorem 3.5, we have:

Corollary 3.8. *Let M be a surface with curvature $K \leq C$ for a positive constant C. Let D be an oval sector enclosed by γ_1, γ_2 with θ_1, $\theta_2 \leq \pi$ on M. Then*

$$A(D) \geq \frac{2\theta_*}{C}, \tag{3.18}$$

where $\theta_* = \max\{\theta_0, \theta_1\}$. Equality holds only when D is isometric to a geodesic sector of radius $\pi/\sqrt{C}$ and angle $\theta_1 = \theta_2$ on a surface of constant curvature $K \equiv C$.

The equality case of Corollary 3.8 is based on the following fact: For an oval sector D with $\theta_1, \theta_2 \leq \pi$ on a surface M with curvature $K \leq C$, by the Gauss-Bonnet formula, one can obtain the inequality

$$A(D) \geq \frac{1}{C}(\theta_0 + \theta_1), \tag{3.19}$$

where equality holds only when M is a surface of constant curvature C.

The following is an immediate consequence of Corollary 3.8:

Corollary 3.9. *Let M be a surface with curvature $K \leq C$ for a positive constant C. Suppose that γ_1, γ_2 are two geodesics starting at a point $p \in M$ with angle θ at p and D is a simply connected domain on M containing γ_1, γ_2 with area less than $\frac{2\theta}{C}$. Then γ_1 and γ_2 never meet again in D.*

4. Lengths of the cut locus.

Let M be a closed surface (i.e., a compact surface without boundary) with area $A(M)$. For $v \in T_pM$, $p \in M$, denote by γ_v the unique geodesic satisfying $\gamma_v'(0) = v$. Define $\rho(v) = \sup\{t \in \mathbb{R} : \gamma_v \text{ is minimal on } [0, t]\}$. Then $\rho(v)$ is continuous on the set $\mathbb{S}_p = \{v \in T_pM : \|v\| = 1\}$ and the values of ρ on $\mathbb{S}_p$ are bounded (by the diameter of M). Note that, if $w = \lambda v \in T_pM$ $(\lambda \geq 0)$, then $\rho(v) = \lambda\rho(w)$. Let

$$U_p = \{v \in T_pM : \rho(v) > 1\}.$$

Then U_p is a bounded set in T_pM and the (usual) cut locus of p is

$$\mathrm{Cut}(p) = \exp_p(\partial U_p).$$

It is well known that, for $p \in M$,

$$M = \mathcal{U}_p \,\dot\cup\, \mathrm{Cut}(p),$$

where $\mathcal{U}_p = \exp_p(U_p)$. Note that $\mathrm{Cut}(p)$ is a deformation retract of $M \setminus \{p\}$. Hence, on any orientable closed surface M of genus g, $\mathrm{Cut}(p)$ of any point $p \in M$ contains $2g$ closed curves, which form a set of generators for the fundamental group of M. It is known that any compact subset of $\mathrm{Cut}(p)$ of p on a closed surface M has finite Hausdorff 1-measure (cf. [**He2**, **I**]). Thus any path in $\mathrm{Cut}(p)$ is rectifiable ([**He1**, Proposition 5.1]) and the Hausdorff 1-measure of a path in $\mathrm{Cut}(p)$ is its arc length ([**Fa**, p. 29]). Using our isoperimetric inequality (4.1) in Theorem 4.2, we will derive an inequality involving the Hausdorff 1-measure of the cut locus of a point in a closed orientable surface.

Lemma 4.1. *Let M be a closed surface and $p \in M$. Then there is a geodesic segment through p such that its end points are in $\mathrm{Cut}(p)$ and it bisects $\mathcal{U}_p$ in area.*

Proof. For a unit vector $v \in T_pM$, denote $c_v : [-\rho(-v), \rho(v)] \to M$ the unique geodesic segment such that $c_v(s) = \gamma_{-v}(-s)$ for $s \in [-\rho(-v), 0]$ and $c_v(t) = \gamma_v(t)$ for $t \in [0, \rho(v)]$. Then $c_v(-\rho(-v)) = \gamma_{-v}(\rho(-v))$, $c_v(\rho(v)) = \gamma_v(\rho(-v))$ are in $\mathrm{Cut}(p)$ of p and $c_v(0) = p$. Clearly, for each $v \in \mathbb{S}_p$, c_v splits $\mathcal{U}_p$ into two pieces. We take one piece of these for each v continuously and name it D_v. Let $A(v)$ be the area of D_v. Then $A(v)$ is a continuous function on $\mathbb{S}_p$ since ρ is continuous on $\mathbb{S}_p$. By the mean value theorem, there is a $v_0 \in \mathbb{S}_p$ such that $A(v_0) = \frac{1}{2}A(M)$, and c_{v_0} is a desired one. □

Theorem 4.2. *Let M be a closed surface with curvature $K \le C$ and ℓ the total length (the Hausdorff 1-measure) of the cut locus $\mathrm{Cut}(p)$ of $p \in M$. Then*

$$\ell^2 \ge \pi A(M) - \frac{C}{4} A^2(M). \tag{4.1}$$

Proof. Let $\gamma : [a, b] \to M$ be a geodesic segment bisecting U_p into D_1, D_2 with $A(D_1) = A(D_2)$ as in Lemma 4.1. Then there is a (continuous) path $\widetilde{\Gamma}$ in $\mathrm{Cut}(p)$ joining $\gamma(a)$ and $\gamma(b)$ so that γ and $\widetilde{\Gamma}$ constitute the common boundary of D_1 and D_2. As the path $\widetilde{\Gamma}$ may not be piecewise smooth and $\widetilde{\Gamma}$ is compact, for any $\varepsilon > 0$ we choose a piecewise simple curve segment Γ joining $\gamma(a)$ and $\gamma(b)$ such that Γ is contained in the ε-neighborhood of $\widetilde{\Gamma}$ and $L(\Gamma) \le L(\widetilde{\Gamma})$. Let D_1', D_2' be two domains with boundary γ and Γ corresponding to D_1, D_2, respectively. By Theorem 3.5 and Remark 3.7,

$$L^2(\Gamma) \ge 2\pi A(D_1') - CA^2(D_1'), \quad L^2(\Gamma) \ge 2\pi A(D_2') - CA^2(D_2').$$

Combining these with the fact that $A(D_i') = A(D_i) + O(\varepsilon^2)$ for $i = 1, 2$,

$$L^2(\widetilde{\Gamma}) \geq 2\pi A(D_1) - CA^2(D_1), \quad L^2(\widetilde{\Gamma}) \geq 2\pi A(D_2) - CA^2(D_2).$$

Since $\widetilde{\Gamma} \subset \mathrm{Cut}(p)$ and $L(\widetilde{\Gamma})$ is equal to the Hausdorff 1-measure of $\widetilde{\Gamma}$, which is less than or equal to the total length ℓ of $\mathrm{Cut}(p)$, we get

$$\ell^2 \geq \pi A(M) - \frac{C}{4} A^2(M).$$

□

Example 4.3. Let T^2 be the flat torus obtained by identifying the opposite sides of quadrilateral ABCD (see Figure 3). Then the cut locus $\mathrm{Cut}(p)$ of middle point $p \in T^2$ is the set formed by the line segments AB and BC, and so $\ell = a + b$, where a and b are the length of the line segments AB and BC, respectively. The area of T^2 is ab. Take $C = 0$ as usual, then (4.1) gives a well-known inequality

$$(a + b)^2 \geq \pi ab.$$

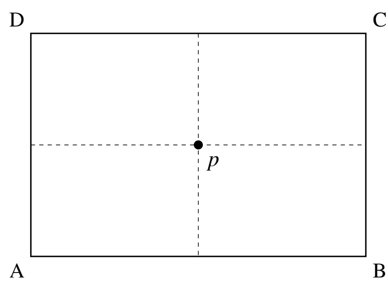

Figure 3.

Example 4.4. Let $P^2(\mathbb{R})$ be the projective plane of constant curvature $C = 1$, obtained by identifying the antipodal points of the unit sphere $\mathbb{S}^2$. The cut locus $\mathrm{Cut}(p)$ of a point p of $P^2(\mathbb{R})$ is the set obtained by identifying the antipodal points of the equator of p. Since $\ell = \pi$ and $A(P^2(\mathbb{R})) = 2\pi$, we get equality in (4.1). On $\mathbb{S}^2$, trivially we also get equality.

It is well known that any closed orientable surface of genus $g > 1$ carries a metric with constant negative curvature [**GHL**, p. 167], and that the area of a closed orientable surface of genus $g > 1$ and curvature -1 is $4\pi(g - 1)$ [**GHL**, p. 169]. Combining these and Theorem 4.2 we have:

Corollary 4.5. *Let M be a closed orientable surface of genus $g > 1$ with curvature -1. Then*

$$\ell^2 \geq 4\pi^2 g(g-1).$$

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