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We prove a Sobolev type inequality for real-valued weakly differentiable functions on $\mathbb{R}^n$, decaying to 0 at infinity, involving the integral of a convex function of the full gradient.

1. Introduction and main results.

Let $n \geq 2$ and let $A : \mathbb{R}^n \to [0, \infty]$ be any convex function satisfying the following properties:

$$A(0) = 0 \quad \text{and} \quad A(\xi) = A(-\xi) \quad \text{for} \;\; \xi \in \mathbb{R}^n; \tag{1.1}$$

$$\text{for every} \;\; t > 0, \; \{\xi \in \mathbb{R}^n : A(\xi) \leq t\} \text{ is a compact set whose interior contains } 0. \tag{1.2}$$

Observe that A need not depend on the length $|\xi|$ of ξ nor be the sum of functions of its components ξ_i, $i = 1, \dots, n$. The purpose of this note is to exhibit an inequality of Sobolev type, for real-valued weakly differentiable functions u on $\mathbb{R}^n$ decaying to 0 at infinity, which involves the gradient ∇u through the integral $\int_{\mathbb{R}^n} A(\nabla u)\, dx$. In the relevant inequality, a role will be played by the function $B : [0, \infty) \to [0, \infty]$ associated with A as follows. Denote by $A_\star : [0, \infty] \to [0, \infty]$ the left-continuous increasing function satisfying

$$|\{\xi \in \mathbb{R}^n : A(\xi) \leq 1\}| = |\{\xi \in \mathbb{R}^n : A_\star(|\xi|) \leq t\}| \quad \text{for every} \;\; t \geq 0,$$

where $|\cdot|$ stands for Lebesgue measure. Assume that

$$\int_0 \left(\frac{t}{A_\star(t)}\right)^{n'-1} dt < \infty, \tag{1.3}$$

where $n' = n/(n-1)$, and define $H : [0, \infty) \to [0, \infty)$ as

$$H(r) = \left(\int_0^r \left(\frac{t}{A_\star(t)}\right)^{n'-1} dt\right)^{1/n'} \quad \text{for} \;\; r \geq 0. \tag{1.4}$$

Then B is given by

$$B = A_\star \circ H^{-1}, \tag{1.5}$$

where H^{-1} is the left-continuous inverse of H. Note that

$$A_\star(s) = \sup\{t : |\{\xi \in \mathbb{R}^n : A(\xi) \leq t\}| < C_n s^n\} \quad \text{for} \;\; s \geq 0,$$

where $C_n = \pi^{n/2}/\Gamma(1+n/2)$, the measure of the n-dimensional unit ball. Thus, $A_\star$ is a Young function, i.e., a left-continuous convex function vanishing at 0, since, owing to the Brunn-Minkowski inequality, $|\{\xi \in \mathbb{R}^n : A(\xi) \leq t\}|^{1/n}$ is a (finite-valued) concave function of t. Inasmuch as H is concave, increasing and vanishes only at 0, then also H^{-1} and B are Young functions.

Theorem 1. *Under assumptions (1.1)-(1.3), there exists a constant K, depending only on n, such that*

$$\int_{\mathbb{R}^n} B\left(\frac{|u(x)|}{K\left(\int_{\mathbb{R}^n} A(\nabla u)dy\right)^{1/n}}\right)dx \leq \int_{\mathbb{R}^n} A(\nabla u)dx \tag{1.6}$$

for every real-valued weakly differentiable function u on $\mathbb{R}^n$ decaying to 0 at infinity, i.e., satisfying $|\{x \in \mathbb{R}^n : |u(x)| > t\}| < \infty$ for every $t > 0$.

Moreover, the result is sharp, in the sense that if inequality (1.6) holds, with B replaced by any Young function B_0, for every A satisfying (1.1)-(1.2) and with prescribed $A_\star$, then (1.3) must be true and there exists $c > 0$ such that $B_0(s) \leq B(cs)$ for $s \geq 0$.

Let us mention that earlier results concerning general anisotropic Sobolev inequalities are contained in [**Kl**], [**Ko**] and [**Tr**].

Remark 1. The integral inequality (1.6) is equivalent to the inequality

$$\|u\|_{L^B(\mathbb{R}^n)} \leq K\|\nabla u\|_{L^A(\mathbb{R}^n)}, \tag{1.7}$$

where

$$\|u\|_{L^B(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} B\left(\frac{1}{\lambda}|u|\right)dx \leq 1\right\}$$

and

$$\|\nabla u\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} A\left(\frac{1}{\lambda}\nabla u\right)dx \leq 1\right\}$$

are the Luxemburg norms. Indeed, (1.6) implies (1.7) by the very definition of the norms. Conversely, (1.6) follows on replacing $A(\xi)$ by $A_1(\xi) = A(\xi)/M$ in (1.7), with $M = \int_{\mathbb{R}^n} A(\nabla u)\,dx$, and observing that, if B_1 is the function defined as in (1.4)-(1.5) but with A replaced by A_1, then $B_1(s) = M^{-1}B(sM^{-1/n})$.

Remark 2. If assumption (1.3) is dropped, an inequality of type (1.7) still holds for functions supported in a set having finite measure, with K depending also on such a measure and on A: One has just to replace A, in the definitions of H and B, by any convex function, still satisfying (1.1)-(1.2) and equivalent with A near infinity, for which the integral in (1.3) converges. Actually, Luxemburg norms over sets of finite measure turn into equivalent norms if the defining convex functions are replaced by functions equivalent near infinity. Recall that two functions $f, g : \mathbb{R}^n \to [0,\infty]$

are called equivalent if there exist positive constants c_1 and c_2 such that $f(c_1\xi) \le g(\xi) \le f(c_2\xi)$ for every $\xi \in \mathbb{R}^n$; f and g are called equivalent near infinity if these inequalities are satisfied for large $|\xi|$. Analogous definitions hold for functions defined on $[0, \infty)$.

Remark 3. Notice that, if

$$\int^{\infty} \left(\frac{t}{A_\star(t)}\right)^{n'-1} dt < \infty,$$

then $B(s) = \infty$ for large t. Hence, in particular, inequality (1.6) tells us that u is bounded provided that $\int_{\mathbb{R}^n} A(\nabla u)dx < \infty$.

A particular case of Theorem 1 is when $A(\xi)$ depends only on $|\xi|$. This case was considered in [**C**] and includes the standard (first order) Sobolev inequality ($A(\xi) = |\xi|^p, \;\; p \ne n$), Trudinger' s inequality ($A(\xi) = |\xi|^n$) and results form [**FLS**] and [**EGO**] ($A(\xi) = |\xi|^p \log^q(e + |\xi|)$). Let us emphasize that, even in this special situation, the proof given here is simpler than that of [**C**], which relies on interpolation techniques.

Another noticeable special case is when A has the form

$$A(\xi) = \sum_{i=1}^{n} A_i(|\xi_i|), \tag{1.8}$$

where A_i are Young functions. If (1.8) holds, elementary geometric considerations enable us to verify that $A_\star$ is equivalent to the left-continuous function $\bar{A}$ whose inverse is defined by

$$\bar{A}^{-1}(r) = \left(\prod_{i=1}^{n} A_i^{-1}(r)\right)^{1/n} \qquad \text{for} \;\; r \ge 0 \tag{1.9}$$

(all inverses are taken right-continuous in (1.9)). Thus, if we denote by $\bar{B}$ the function defined as B, save that $A_\star$ is replaced by $\bar{A}$ in (1.4)-(1.5), then we easily get from Theorem 1 the following:

Corollary. *Let A_i, $i = 1, \dots, n$, be Young functions such that*

$$\int_0 \left(\frac{t}{\bar{A}(t)}\right)^{n'-1} dt < \infty. \tag{1.10}$$

Then a positive constant K, depending only on n, exists such that

$$\int_{\mathbb{R}^n} \bar{B}\left(\frac{|u(x)|}{K\left(\sum_{i=1}^n \int_{\mathbb{R}^n} A_i(|u_{x_i}|)\, dy\right)^{1/n}}\right) dx \le \sum_{i=1}^{n} \int_{\mathbb{R}^n} A_i(|u_{x_i}|)\, dx \tag{1.11}$$

for every real-valued weakly differentiable function u on $\mathbb{R}^n$ decaying to 0 at infinity. Here, u_{x_i} denotes the partial derivative of u with respect to x_i.

Moreover, condition (1.10) is necessary and the function $\bar{B}$ is optimal for inequality (1.11) to hold for every n-tuple of functions A_i with prescribed $\bar{A}$.

Let us point out that a direct proof of the Corollary can also be supplied via Theorem 2 of the Appendix and the same argument as in the proof of Theorem 1 (Section 2).

Remark 4. Inequality (1.10) is equivalent to the product inequality

$$\|u\|_{L^{\bar{B}}(\mathbb{R}^n)} \leq K_0 \left(\prod_{i=1}^{n} \|u_{x_i}\|_{L^{A_i}(\mathbb{R}^n)} \right)^{1/n}, \tag{1.12}$$

where K_0 is a positive constant depending only on n. Notice that $\bar{B}$ need not be convex, so that $\|\cdot\|_{L^{\bar{B}}(\mathbb{R}^n)}$ is in general only a quasi-norm, which can however be turned into an equivalent norm on replacing $\bar{B}$ by the equivalent Young function defined with $\int_0^s \bar{A}(r)/r dr$ in the place of $\bar{A}(s)$. In order to deduce (1.12) from (1.11), one can apply the latter to the function $u(\lambda_1 x_1, \dots, \lambda_n x_n)$, with $\lambda_i > 0$ for $i = 1, \dots, n$, and get, after a change of variables,

$$\int_{\mathbb{R}^n} \bar{B}\left(\frac{\left(\prod_{i=1}^n \lambda_i\right)^{1/n} |u(x)|}{K \left(\sum_{i=1}^n \int_{\mathbb{R}^n} A_i(\lambda_i |u_{x_i}|) dy \right)^{1/n}} \right) dx \leq \sum_{i=1}^n \int_{\mathbb{R}^n} A_i(\lambda_i |u_{x_i}|)\, dx.$$

Choosing $\lambda_i = 1/\|u_{x_i}\|_{L^{A_i}(\mathbb{R}^n)}$ in the last inequality easily implies (1.12). Conversely, inequality (1.11) follows on replacing $A_i(s)$ by $A_i(s)/M$ in (1.12), with $M = \sum_{i=1}^n \int_{\mathbb{R}^n} A_i(|u_{x_i}|)\, dx$.

When the functions A_i are powers, i.e.,

$$A(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}$$

for some $p_i \geq 1$, $i = 1, \dots, n$, the Corollary reproduces results from [**N**] and [**Tro**] if $\sum_{i=1}^n \frac{1}{p_i} \neq 1$, and yields a Trudinger type inequality if $\sum_{i=1}^n \frac{1}{p_i} = 1$ (see also [**Kl**]).

An example (generalizing one from [**Tr**]) of a convex function A, satisfying (1.1)-(1.2), which is neither radial nor of type (1.8), is given by

$$A(\xi_1, \xi_2) = |\xi_1 - \xi_2|^p + |\xi_1|^q \log^\alpha(c + |\xi_1|), \qquad (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Here $p, q \geq 1$, $\alpha \in \mathbb{R}$ or $\alpha \in [0, \infty)$ according to whether $q > 1$ or $q = 1$, and c is a positive number so large that A is convex. For such an A, inequality (1.7) holds, for functions u supported in a set of fixed finite measure, with $B(s)$ equivalent near infinity to: $s^{\frac{2pq}{p+q-pq}} \log^{\frac{p\alpha}{p+q-pq}}(c + s)$, if $pq < p + q$; $\exp\left(s^{\frac{2(p+q)}{p+q-p\alpha}} \right) - 1$ if $pq = p+q$ and $p\alpha < p+q$; $\exp(\exp(s^2)) - e$ if $pq = p+q$ and $p\alpha = p+q$. If either $pq = p+q$ and $p\alpha > p+q$, or $pq > p+q$, then (1.7) holds with $\|\cdot\|_{L^B(\mathbb{R}^n)} = \|\cdot\|_{L^\infty(\mathbb{R}^n)}$. The preceding conclusions can be derived from Theorem 1 and the subsequent remarks, on taking into account that $A_\star(s)$ is equivalent to $s^{\frac{2pq}{p+q}} \log^{\frac{p\alpha}{p+q}}(c+s)$ near infinity. In the case where $\alpha = 0$, such an equivalence follows from the fact that the straight lines $\xi_1 = \pm t^{1/q}$ and $\xi_2 = \xi_1 \pm t^{1/p}$ are tangent to the (convex) level set $\{\xi \in \mathbb{R}^2 : A(\xi) \le t\}$ for every $t \ge 0$, so that

$$2\, t^{1/p+1/q} \le |\{\xi \in \mathbb{R}^2 : A(\xi) \le t\}| \le 4\, t^{1/p+1/q} \qquad \text{for } \ t \ge 0.$$

A similar argument works also in the case where $\alpha \ne 0$.

2. Proof of Theorem 1.

We shall prove the statement of Theorem 1 with condition (1.3) replaced by

$$\int_0 \frac{\tilde{A}_\star(t)}{t^{1+p'}}\, dt < \infty, \tag{2.1}$$

and with B replaced by the function D defined as

$$D(s) = \left(sJ^{-1}\left(s^{n'}\right)\right)^{n'} \qquad \text{for } \ s \ge 0, \tag{2.2}$$

where J^{-1} is the left-continuous inverse of the function given by

$$J(r) = n' \int_0^r \frac{\tilde{A}_\star(t)}{t^{1+n'}}\, dt \qquad \text{for } \ r \ge 0. \tag{2.3}$$

Here $\tilde{A}_\star(s) = \sup\{rs - A_\star(r) : \ r \ge 0\}$, the Young conjugate of $A_\star$. The theorem will then follow, owing to Lemma 2 below.

The proof of inequality (1.6) proceeds according to the following scheme: A weak type version of the inequality is first established; inequality (1.6) is then derived from this weak type inequality by means of a discretization and truncation argument. Let us mention that such an argument is related to the approach recently used in various papers (including [**BCLS**], [**FGW**], [**FPW**], [**GN**], [**HK**], [**Tar**]) to deal with Sobolev inequalities in non-standard situations.

A basic ingredient of our proof is an extension of the Pólya-Szegö principle, contained in [**Kl**], which tells us that if A is a convex function satisfying (1.1)-(1.2) and u is a weakly differentiable function on $\mathbb{R}^n$ decaying to 0 at infinity and such that $\int_{\mathbb{R}^n} A(\nabla u)\, dx < \infty$, then its decreasing rearrangement u^* is locally absolutely continuous on $(0,\infty)$ and there exists a positive constant c, depending only on n, such that

$$\|\nabla u\|_{L^A(\mathbb{R}^n)} \ge \left\| c r^{1/n'} \left(-\frac{du^*}{dr}\right)\right\|_{L^{A_\star}(0,\infty)}. \tag{2.4}$$

Recall that

$$u^*(s) = \sup\{t \ge 0 : \ |\{x \in \mathbb{R}^n : |u(x)| > t\}| > s\} \qquad \text{for } \ s \ge 0.$$

Incidentally, let us point out that when A has the form (1.8), a direct short proof can be given of an inequality of type (2.4) with $A_\star$ replaced by $\bar{A}$ — see the Appendix.

Let u be any function as in the statement. We may obviously assume that $\int_{\mathbb{R}^n} A(\nabla u)\,dx < \infty$. Then the absolute continuity of u^* ensures that

$$u^*(s) = \int_s^\infty -\frac{du^*}{dr}\,dr \qquad \text{for} \quad s \geq 0.$$

By the Hölder inequality for Luxemburg norms and Lemma 1 below, we have

$$\begin{aligned} u^*(s) &= \int_s^\infty -\frac{du^*}{dr}\,dr \leq 2\,\|r^{-1/n'}\|_{L^{\tilde{A}_\star}(s,\infty)} \left\| r^{1/n'}\left(-\frac{du^*}{dr}\right)\right\|_{L^{A_\star}(0,\infty)} \\ &= 2D^{-1}(1/s)\left\| r^{1/n'}\left(-\frac{du^*}{dr}\right)\right\|_{L^{A_\star}(0,\infty)} \end{aligned} \tag{2.5}$$

for $s > 0$. Hence, owing to (2.4),

$$u^*(s) \leq \frac{2}{c} D^{-1}(1/s)\|\nabla u\|_{L^A(\mathbb{R}^n)} \qquad \text{for} \quad s > 0. \tag{2.6}$$

Since $|u|$ and u^* are equimeasurable functions, we deduce from (2.6) that

$$|\{|u| \geq \lambda\}|D\left(\frac{c}{2}\lambda/\|\nabla u\|_{L^A(\mathbb{R}^n)}\right) \leq 1 \qquad \text{for} \quad \lambda > 0. \tag{2.7}$$

Let us make use of (2.7) with $A(\xi)$ replaced by $A_1(\xi) = A(\xi)/M$, where $M = \int_{\mathbb{R}^n} A(\nabla u)\,dx$. Since the function D_1, defined as in (2.2)-(2.3) but with A_1 in the place of A, satisfies the equation $D_1(s) = M^{-1}D(sM^{-1/n})$ for $s \geq 0$, then we get the weak type inequality

$$|\{|u| \geq \lambda\}|D\left(\frac{c}{2}\lambda\left(\int_{\mathbb{R}^n} A(\nabla u)\,dx\right)^{-1/n}\right) \leq \int_{\mathbb{R}^n} A(\nabla u)\,dx \qquad \text{for} \quad \lambda > 0. \tag{2.8}$$

Now, for $k \in \mathbb{Z}$, we denote by u_k the function defined in $\mathbb{R}^n$ by

$$u_k(x) = \begin{cases} 0 & \text{if } \ |u(x)| < 2^k \\ |u(x)| - 2^k & \text{if } \ 2^k \leq |u(x)| < 2^{k+1} \\ 2^{k+1} - 2^k & \text{otherwise.} \end{cases}$$

Applying inequality (2.8) to u_k, with $\lambda = 2^k$, yields

$$\begin{aligned} &|\{u_k \geq 2^k\}|D\left(\frac{c}{2}2^k\left(\int_{\{2^k\leq|u|<2^{k+1}\}} A(\nabla u)\,dx\right)^{-1/n}\right) \\ &\leq \int_{\{2^k\leq|u|<2^{k+1}\}} A(\nabla u)\,dx \qquad \text{for} \quad k \in \mathbb{Z}. \end{aligned} \tag{2.9}$$

Note that in (2.9) we have made use of assumption (1.1) and of the fact that $\nabla u_k(x)$ equals either $\nabla u(x)$ or $-\nabla u(x)$ if $2^k \leq |u(x)| < 2^{k+1}$, and vanishes otherwise. Owing to (2.9), one has

$$\begin{aligned}
&\int_{\mathbb{R}^n} D\left(\frac{c|u(x)|}{8\left(\int_{\mathbb{R}^n} A(\nabla u)\,dy\right)^{1/n}}\right)dx \\
&\leq \sum_{k=-\infty}^{\infty} |\{2^{k+1} \leq |u| < 2^{k+2}\}| D\left(\frac{c}{8}2^{k+2}\left(\int_{\mathbb{R}^n} A(\nabla u)\,dx\right)^{-1/n}\right) \\
&\leq \sum_{k=-\infty}^{\infty} |\{|u| \geq 2^{k+1}\}| D\left(\frac{c}{2}2^{k}\left(\int_{\{2^k \leq |u| < 2^{k+1}\}} A(\nabla u)\,dx\right)^{-1/n}\right) \\
&= \sum_{k=-\infty}^{\infty} |\{u_k \geq 2^{k}\}| D\left(\frac{c}{2}2^{k}\left(\int_{\{2^k \leq |u| < 2^{k+1}\}} A(\nabla u)\,dx\right)^{-1/n}\right) \\
&\leq \sum_{k=-\infty}^{\infty} \int_{\{2^k \leq |u| < 2^{k+1}\}} A(\nabla u)\,dx \\
&= \int_{\mathbb{R}^n} A(\nabla u)\,dx,
\end{aligned}$$

i.e., inequality (1.6) with $K = c/8$ and D in the place of B.

As far as the last part of the statement is concerned, assume that inequality (1.6) holds, with B replaced by some D_0, for every A with prescribed $A_\star$. Hence, by the definition of Luxemburg norms,

$$\|u\|_{L^{D_0}(\mathbb{R}^n)} \leq K\|\nabla u\|_{L^A(\mathbb{R}^n)} \tag{2.10}$$

for every such an A. Let us choose $A(\xi) = A_\star(|\xi|)$ in (2.10) and let us consider radially decreasing test functions u having the form

$$u(x) = \frac{1}{nC_n^{1/n}} \int_{C_n|x|^n}^{\infty} r^{-1/n'} f(r)\,dr$$

for some measurable function $f : [0, \infty) \to [0, \infty)$ such that $\|f\|_{L^{A_\star}(0,\infty)} < \infty$. Since $|\nabla u(x)| = f(C_n|x|^n)$, we get from (2.10)

$$\left\|\int_s^{\infty} r^{-1/n'} f(r)\,dr\right\|_{L^{D_0}(0,\infty)} \leq nC_n^{1/n} K\|f\|_{L^{A_\star}(0,\infty)}. \tag{2.11}$$

If t is any fixed positive number and the support of f is contained in $[t, \infty)$, then

$$\|f\|_{L^{A_\star}(0,\infty)} = \|f\|_{L^{A_\star}(t,\infty)} \tag{2.12}$$

and

$$(2.13)\qquad \left\| \int_s^\infty r^{-1/n'} f(r)\,dr \right\|_{L^{D_0}(0,\infty)} \geq \int_t^\infty r^{-1/n'} f(r)\,dr\, \|1\|_{L^{D_0}(0,t)} = \int_t^\infty r^{-1/n'} f(r)\,dr\, \frac{1}{D_0^{-1}(1/t)},$$

where D_0^{-1} is the right-continuous inverse of D_0. Combining (2.11)-(2.13) and making use of the converse of the Hölder inequality yield

$$(2.14)\qquad KD_0^{-1}(1/t) \geq \sup_{f\in L^{A_\star}(t,\infty)} \frac{\int_t^\infty r^{-1/n'} f(r)\,dr}{\|f\|_{L^{A_\star}(t,\infty)}} \geq \|r^{-1/n'}\|_{L^{\tilde{A}_\star}(t,\infty)} \quad \text{for } \ t>0.$$

Hence, by Lemma 1 below, (2.1) holds and $D_0(s/K) \leq D(s)$ for $s \geq 0$.

Lemma 1. *Condition* (2.1) *holds if and only if* $\|r^{-1/n'}\|_{L^{\tilde{A}_\star}(s,\infty)}$ *for every* $s>0$. *Moreover,*

$$(2.15)\qquad \|r^{-1/n'}\|_{L^{\tilde{A}_\star}(s,\infty)} = D^{-1}(1/s) \qquad \textit{for } \ s>0,$$

where D^{-1} *is the the right-continuous inverse of the function defined by* (2.2).

Proof. We have

$$(2.16)\qquad \|r^{-1/n'}\|_{L^{\tilde{A}_\star}(s,\infty)} = \inf\left\{\lambda>0 : \int_s^\infty \tilde{A}_\star\left(\frac{r^{-1/n'}}{\lambda}\right) dr \leq 1\right\} = \inf\left\{\lambda>0 : n'\lambda^{-n'} \int_0^{r^{-1/n'}/\lambda} \frac{\tilde{A}_\star(t)}{t^{1+n'}}\,dt \leq 1\right\}.$$

Equations (2.16) tell us that $\|r^{-1/n'}\|_{L^{\tilde{A}_\star}(s,\infty)} < \infty$ if and only if (2.1) is fulfilled. Moreover, on setting

$$(2.17)\qquad I(r) = r^{n'} J(r) \qquad \text{for } \ r \geq 0$$

(where J is defined by (2.3)) and denoting by I^{-1} the right-continuous inverse of I, one infers from (2.16) that

$$(2.18)\qquad \|r^{-1/n'}\|_{L^{\tilde{A}_\star}(s,\infty)} = \frac{s^{-1/n'}}{I^{-1}(1/s)} \qquad \text{for } s>0.$$

By (2.2) and (2.17),

$$(2.19)\qquad D^{-1}(s) = \frac{s^{1/n'}}{I^{-1}(s)} \qquad \text{for } s>0.$$

Equation (2.15) follows from (2.18) and (2.19). □

Lemma 2. *Conditions* (1.3) *and* (2.1) *are equivalent. Moreover, there exist constants* c_1 *and* c_2*, depending only on* n*, such that*

$$B(c_1 s) \le D(s) \le B(c_2 s) \qquad \text{for } s \ge 0, \tag{2.20}$$

where B *and* D *are the functions defined by* (1.5) *and* (2.2)*, respectively.*

Proof. Denote by $A_\star^{-1}$ and $\tilde{A}_\star^{-1}$ the right-continuous inverses of $A_\star$ and $\tilde{A}_\star$, respectively. Set $E(s)=2s/A_\star^{-1}(s)$ and $F(s)=A_\star(s)/s$. Since $A_\star^{-1}(s)\tilde{A}_\star^{-1}(s) \le 2s$, then $\tilde{A}_\star^{-1}(s) \le E(s)$ for $s \ge 0$. Thus, if E^{-1} and F^{-1} are the left-continuous inverses of E and F, then $E^{-1}(s) \le \tilde{A}_\star(s)$ for $s \ge 0$. Moreover, on letting a be the increasing left-continuous function such that

$$A_\star(s) = \int_0^s a(r)\,dr \qquad \text{for } s \ge 0,$$

we have

$$\begin{aligned} J(s) &\ge n' \int_0^s \frac{E^{-1}(t)}{t^{1+n'}}\,dt \\ &= \left(\int_0^{E^{-1}(s)} \left(\frac{A_\star^{-1}(r)}{2r} \right)^{n'} dr - \frac{E^{-1}(s)}{s^{n'}} \right) \\ &\ge \left(\frac{1}{2^{n'}} \int_0^{F^{-1}(s/2)} \left(\frac{t}{A_\star(t)} \right)^{n'} a(t)\,dt - \frac{E^{-1}(s)}{s^{n'}} \right) \\ &\ge \left(\frac{1}{2^{n'}} \int_0^{F^{-1}(s/2)} \left(\frac{t}{A_\star(t)} \right)^{n'-1} dt - \frac{\tilde{A}_\star(s)}{s^{n'}} \right). \end{aligned} \tag{2.21}$$

In (2.21) we have used the fact that $A_\star(s)/s \le a(s)$ for $s > 0$. Inequality (2.21) already shows that (2.1) implies (1.3). Moreover, since $\tilde{A}_\star(s)/s$ is an increasing function, we have

$$\tilde{A}_\star(s/2) = \frac{n' s^{n'}}{2^{n'}-1} \tilde{A}_\star(s/2) \int_{s/2}^s t^{-1-n'}\,dt \le \frac{n' s^{n'}}{2^{n'}-1} \int_0^s \frac{\tilde{A}_\star(t)}{t^{1+n'}}\,dt. \tag{2.22}$$

From (2.21)-(2.22) we deduce that a positive constant k exists such that $kJ(ks)^{1/n'} \ge H(F^{-1}(s))$, where H is defined by (1.4). Hence, if we set $G(s) = (sF(H^{-1}(s)))^{n'}$, we have

$$D(s) \le G(k_1 s) \qquad \text{for } s \ge 0 \tag{2.23}$$

for some constant $k_i > 0$. Since $G(s)/s$ increases, $G(s) \le \int_0^s G(r)/r dr \le G(2s)$. Now, one can perform a change of variable in the last integral and show that

$$\frac{1}{n'} B(s/2) \le G(s) \le B(2s) \qquad \text{for } s \ge 0. \tag{2.24}$$

Note that the proof of (2.24) requires the use of the inequality $cH^{-1}(s) \le H^{-1}(cs)$ for $c \ge 1$ and $s \ge 0$, which holds because H^{-1} is a Young function, and of the inequalities $A_\star(s) \le \int_0^s A_\star(r)/r\,dr \le A_\star(2s)$ for $s \ge 0$, which hold because $A_\star(s)/s$ increases. The second of inequalities (2.20) is now a consequence of (2.23)-(2.24).

On the other hand, since $\tilde{A}_\star(s)/s \le a^{-1}(s) \le F^{-1}(s)$, one has

$$\begin{aligned} J(s) \le n' \int_0^s \frac{a^{-1}(t)}{t^{n'}}\,dt &\le n \left(\int_0^{a^{-1}(s)} a(r)^{1-n'}\,dr \right) \\ &\le n \left(\int_0^{F^{-1}(s)} \left(\frac{1}{A_\star(t)} \right)^{n'-1} dt \right) \\ &= nH(F^{-1}(s))^{n'}. \end{aligned} \tag{2.25}$$

From (2.25) we deduce that (1.3) implies (2.1) and that $J^{-1}(r^{n'}) \ge F(H^{-1}(n^{-1/n'}r))$ for $r \ge 0$. Thus, owing to (2.24), $D(s) \ge nG(n^{-1/n'}s) \ge (n-1)B(n^{-1/n'}s/2)$, whence the first of inequalities (2.20) follows. □

Appendix.

Theorem 2. *Let A_i, $i = 1, \dots, n$, be Young functions and let $\bar{A}$ be the function defined by (1.9). If u is any real-valued weakly differentiable function on $\mathbb{R}^n$ decaying to 0 at infinity and such that $\sum_{i=1}^n \int_{\mathbb{R}^n} A_i(|u_{x_i}|)\,dx < \infty$, then u^* is locally absolutely continuous on $(0,\infty)$ and*

$$\sum_{i=1}^n \int_{\mathbb{R}^n} A_i(|u_{x_i}|)\,dx \ge \int_0^\infty \bar{A}\left(2s^{1/n'}\left(-\frac{du^*}{ds}\right)\right) ds. \tag{A.1}$$

Proof. The standard (multiplicative) Gagliardo-Niremberg inequality tells us that

$$\|v\|_{L^{n'}(\mathbb{R}^n)} \le \frac{1}{2} \left(\prod_{i=1}^n \|v_{x_i}\|_{L^1(\mathbb{R}^n)} \right)^{1/n} \tag{A.2}$$

for every $v \in W^{1,1}(\mathbb{R}^n)$. One can apply (A.2) to the function v defined, for fixed $s > 0$ and $h > 0$, by

$$v(x) = \begin{cases} 0 & \text{if } |u(x)| < u^*(s) \\ |u(x)| - u^*(s) & \text{if } u^*(s) \le |u(x)| < u^*(s-h) \\ u^*(s-h) - u^*(s) & \text{otherwise.} \end{cases}$$

So doing, one easily gets

$$2(s-h)^{1/n'}[u^*(s-h) - u^*(s)] \le \left(\prod_{i=1}^n \int_{\Omega_s^h} |u_{x_i}|\,dx \right)^{1/n}, \tag{A.3}$$

where we have set $\Omega_s^h = \{x \in \mathbb{R}^n : u^*(s) < |u(x)| < u^*(s-h)\}$. Observe that v actually belongs to $W^{1,1}(\mathbb{R}^n)$. Indeed, since the functions A_i grow at least linearly at infinity, $|\nabla u|$ is integrable over subsets of $\mathbb{R}^n$ having finite measure. The definition of $\bar{A}$ implies that

$$\bar{A}\left(\left(\prod_{i=1}^{n} s_i\right)^{1/n}\right) \leq \sum_{i=1}^{n} A_i(s_i) \qquad \text{for } \ s_i \geq 0, \ i = 1, \dots, n. \tag{A.4}$$

Inequality (A.4) and Jensen inequality yield

$$\begin{aligned} \bar{A}\left(\frac{1}{|\Omega_s^h|}\left(\prod_{i=1}^{n}\int_{\Omega_s^h}|u_{x_i}|\,dx\right)^{1/n}\right) &\leq \sum_{i=1}^{n} A_i\left(\frac{1}{|\Omega_s^h|}\int_{\Omega_s^h}|u_{x_i}|\,dx\right) \\ &\leq \frac{1}{|\Omega_s^h|}\sum_{i=1}^{n}\int_{\Omega_s^h} A_i(|u_{x_i}|)\,dx. \end{aligned} \tag{A.5}$$

Since $|\Omega_s^h| \leq h$ and since $\bar{A}(s)/s$ is an increasing function, we get from (A.5)

$$\bar{A}\left(\frac{1}{h}\left(\prod_{i=1}^{n}\int_{\Omega_s^h}|u_{x_i}|\,dx\right)^{1/n}\right) \leq \frac{1}{h}\sum_{i=1}^{n}\int_{\Omega_s^h} A_i(|u_{x_i}|)\,dx. \tag{A.6}$$

Dividing through by h in (A.3), making use of (A.6), and passing to the limit as $h \to 0^+$ yield, thanks to the lower semicontinuity of $\bar{A}$

$$\begin{aligned} &\bar{A}\left(2s^{1/n'}\left(-\frac{du^*}{ds}\right)\right) \\ &\leq \frac{d}{ds}\sum_{i=1}^{n}\int_{\{|u|>u^*(s)\}} A_i(|u_{x_i}|)\,dx \quad \text{for a.e.} \quad s > 0. \end{aligned} \tag{A.7}$$

Inequality (A.1) follows on integrating (A.7) between 0 and ∞.

As far as the local absolute continuity of u^* is concerned, given $a > 0$, consider any family of disjoint intervals (a_j, b_j), $j = 1, \dots, m$, with $a \leq a_j < b_j$, and set $\delta = \sum_{j=1}^{m}(b_j - a_j)$. On making use of (A.3) and of the inequality between geometric and arithmetic means, one obtains

$$\sum_{j=1}^{m}(u^*(a_j) - u^*(b_j)) \leq \frac{1}{2na^{1/n'}}\sum_{i=1}^{n}\int_{\cup_{j=1}^{m}\{u^*(b_j)<|u(x)|<u^*(a_j)\}}|u_{x_i}|\,dx. \tag{A.8}$$

Since $\max_{|E|=s}\int_E |u_{x_i}|\,dx = \int_0^s |u_{x_i}|^*(r)\,dr$ for $s > 0$, and since $|\cup_{j=1}^{m}\{x : u^*(b_j) < |u(x)| < u^*(a_j)\}| \leq \delta$, inequality (A.8) implies

$$\sum_{j=1}^{m}(u^*(a_j) - u^*(b_j)) \leq \frac{1}{2na^{1/n'}}\sum_{i=1}^{n}\int_0^{\delta}|u_{x_i}|^*(r)\,dr.$$

Hence, the absolute continuity of u^* on $[a, \infty)$ follows. □

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