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When an algebra is graded by a group, any additive character of the group induces a diagonalizable derivation of the ring. This construction is studied in detail for the case of a path algebra modulo relations and its fundamental group. We describe an injection of the character group into the first cohomology group following Assem-de la Peña. Rather general conditions are determined, in this context, which guarantee that a diagonalizable derivation is induced from the fundamental group.

This paper is the second installment in a series devoted to diagonalizable derivations. Suppose that R is a finite-dimensional algebra over the field k. We will denote by $\mathrm{Der}(R)$ the space of k-algebra derivations from R to itself. Diagonalizable derivations arise naturally whenever R is graded by a group H. Indeed, every additive character $\chi \in \mathrm{Hom}(H, k^+)$ can be assigned a derivation $D_\chi \in \mathrm{Der}(R)$ according to the rule

$$D_\chi(r) = \chi(g)r$$

for every $r \in R$ in the homogeneous component of "degree" $g \in H$. Obviously, D_χ is diagonalizable. Conversely, if D is a diagonalizable derivation of the k-algebra S and H is the additive subgroup of k generated by the eigenvalues of D then, for the inclusion map $\iota : H \to k^+$, we have $D = D_\iota$.

In our first paper [**FGGM**], we proved that the span of all diagonalizable derivations of R comprise a Lie ideal of $\mathrm{Der}(R)$ whenever k has characteristic zero or is algebraically closed of positive characteristic. This result turned out to be a powerful tool in describing what we called *spanned-by-split* derivations (i.e., those which are sums of diagonalizable derivations) in several classes of algebras. In what follows, we shall use the notation

$$\mathrm{SPDer}(R)$$

for the subspace of $\mathrm{Der}(R)$ consisting of spanned-by-split derivations.

This paper describes some of the examples which motivated our original paper. Consider R presented as $k\Gamma/I$, a path algebra modulo relations, graded by the fundamental group $\pi_1(\Gamma, I)$. In section one, we review and clarify notions of fundamental group. The second section is devoted to tightening two results of Assem and de la Peña:

- The map which assigns to each character in $\mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ a derivation in $\mathrm{SPDer}(k\Gamma/I)$ is injective.
- The induced map of $\mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ to $H^1(k\Gamma/I)$ is injective.

Next we give a partial characterization of those diagonalizable derivations which arise in the form D_χ. Essentially, we require that the underlying algebra have its radical stabilized by the derivation and that the algebra of constants for the derivation be indecomposable.

We close with a short section placing our constructions in the context of Hopf algebras.

1. Fundamental Groups.

Let Γ be a finite connected directed graph. Temporarily forget about the orientation of arrows, obtaining the undirected graph Γ^{un}. If α is a walk from vertex x to vertex y and β is a walk from vertex y to vertes z then the concatenation $\alpha\beta$ is a walk from x to z and the reverse α^{-1} is a walk from y to x. Consider the "homotopy relation", the smallest equivalence relation on walks which is compatible with right and left concatenation (whenever they make sense) and for which $\sigma\sigma^{-1}$ is equivalent to the trivial walk at w for any walk σ beginning at w. For a fixed vertex x, the classes of closed walks from x to itself comprise the fundamental group, $\pi_1(\Gamma)$. (Different choices of x yield isomorphic groups.)

We will need a more traditional description of the fundamental group. Fix a vertex v, the "base point". For each vertex $w \in \Gamma$, choose a walk $\gamma_{v,w}$ from v to w in the underlying undirected graph Γ^{un}; we require that $\gamma_{v,v}$ be the empty walk from v to itself. The set

$$\gamma = \{\gamma_{v,w} \mid w \text{ is a vertex}\}$$

will be referred to as a choice of *parade data*. If f is any walk in Γ^{un} from x to y then we define

$$c_\gamma(f) = \gamma_{v,x} f \gamma_{v,y}^{-1},$$

the walk which begins at v, takes the parade route to x, follows f from x to y, and then reverses the parade route to return to v from y. Observe that if g is already a closed walk from v to v then $c_\gamma(g) = g$. Also, every $c_\gamma(f)$ lies in the subgroup generated by

$$\{c_\gamma(a) \mid a \text{ is an arrow in } \Gamma\}.$$

Suppose the parade walk $\gamma_{v,w}$ is a sequence

$$a_1^{\varepsilon(1)}, a_2^{\varepsilon(2)}, \ldots, a_t^{\varepsilon(t)}$$

of edges where each a_j is an arrow in Γ and $\varepsilon(j) = \pm 1$ according to whether the original orientation is preserved or reversed in the walk. Then

$$c_\gamma(a_1)^{\varepsilon(1)} c_\gamma(a_2)^{\varepsilon(2)} \cdots c_\gamma(a_t)^{\varepsilon(t)} = 1.$$

We call the word on the left-hand side of the last equation a *parade walk relator*.

Given fixed parade data γ, the earlier remark about closed walks through v implies that $\pi_1(\Gamma)$ is generated by $\{c_\gamma(a) \mid a \text{ is an arrow}\}$. It is less obvious that $\pi_1(\Gamma)$ is the free group on the formal symbols $c_\gamma(a)$ modulo the parade walk relators. (Let F_γ be the free group on arrows modulo the parade walk relations for γ. The obvious map from walks in Γ^{un} to F_γ respects the homotopy equivalence relation. If we regard $\pi_1(\Gamma)$ as the group of equivalence classes of closed walks through v then the restriction of the factored map is a group homomorphism $\phi : \pi_1(\Gamma) \to F_\gamma$. For any arrow a from x to y,

$$\begin{aligned}\phi(c_\gamma(a)) &= \phi(\gamma_{v,x} a \gamma_{v,y}^{-1}) \\ &= \phi(\gamma_{v,x}) \phi(a) \phi(\gamma_{v,y})^{-1} \\ &= \phi(a) \\ &= \overline{a}\end{aligned}$$

where $\overline{a}$ is the image of the symbol a in F_γ. Thus ϕ is surjective. But $\pi_1(\Gamma)$ is generated by the collection of all such $c_\gamma(a)$ and they are subject to the parade walk relations. It follows that ϕ is an isomorphism.) When the context is clear, we drop the subscript γ.

Suppose that I is an ideal of the path algebra $k\Gamma$ and that I is generated as an ideal by a set of relations ρ. The fundamental group $\pi_1(\Gamma, \rho)$ will turn out to be a certain image of $\pi_1(\Gamma)$. While it is possible to describe $\pi_1(\Gamma, \rho)$ abstractly ([**S**]), we will assume that $\pi_1(\Gamma)$ is already described using parade data γ. Let $N(\rho)$ be the normal subgroup of $\pi_1(\Gamma)$ generated by $c_\gamma(p)c_\gamma(q^{-1})$ as p and q range over all paths in the support of the same member of ρ. Then

$$\pi_1(\Gamma, \rho) = \pi_1(\Gamma)/N(\rho).$$

We denote the canonical homomorphism from $\pi_1(\Gamma)$ to $\pi_1(\Gamma, \rho)$ (which depends on γ) by ξ.

A choice of parade data γ induces a $\pi_1(\Gamma, \rho)$-grading on $k\Gamma/I$. Explaining this gives us the opportunity to introduce the useful notion of weight ([**G**]). Suppose that Γ is a finite directed graph and H is a group. A *weight function* for Γ with values in H is an assignment W from the arrows of Γ to H. If we extend W multiplicatively so that vertices have weight $1 \in H$ then the domain of the extension (also called W) consists of all directed paths in Γ. The weight function now induces an H-grading on $k\Gamma$. We say that an ideal I of $k\Gamma$ is *homogeneous for W* provided it is homogeneous with respect to this grading. For such an ideal, the weight induces a grading on $k\Gamma/I$. In the case of fundamental groups, we can consider the weight with values in $\pi_1(\Gamma,\rho)$ which sends an arrow a to $\xi(c_\gamma(a))$. The ideal I is homogeneous by the construction of $N(\rho)$.

Unfortunately, $\pi_1(\Gamma,\rho)$ is dependent on the choice of relations for I. This can be remedied as follows. We say that a nonzero element $r \in I$ is *support minimal* if it cannot be written as a sum of two elements of I, each of whose supports are proper subsets of the support of r. (If $\sum \alpha_p p \in I$ is support minimal where the sum runs over paths p with scalars $\alpha_p \neq 0$ then we cannot erase any summands and maintain the subsum in I.) It is an immediate consequence of the next proposition that the fundamental group is the same for any two choices of ρ which consist of support minimal relations; this common group is denoted $\pi_1(\Gamma, I)$.

Proposition 1.1. *Let ρ be any generating set for I and suppose that $s \in I$ is support-minimal. Then $c(p)c(q)^{-1} \in N(\rho)$ for all p and q in the support of s.*

Proof. By definition, if $r \in \rho$ and σ is a path in the support of r then

$$c(\tau) \equiv c(\sigma) \pmod{N(\rho)}$$

for all τ in the support of r. We will abuse notation and write

$$\mathrm{supp}(r) \equiv c(\sigma) \pmod{N(\rho)}.$$

For any two paths α and β, we then have

$$\mathrm{supp}(\alpha r \beta) \equiv c(\alpha\sigma\beta) \pmod{N(\rho)}.$$

An arbitrary $s \in I$ is a linear combination of expressions $\alpha r \beta$ for paths α, β and for $r \in \rho$. Given $d \in \pi_1(\Gamma,\rho)$, set $s(d)$ to be the subcombination of all those $\alpha r \beta$ whose support lies in d, regarded as a coset. Then

$$s = \sum_d s(d)$$

with $s(d) \in I$ and the supports of the $s(d)$ pair-wise disjoint. Hence if s is support-minimal it must be equal to a single $s(d)$. □

We mention one last time that we will only be able to speak about a $\pi_1(\Gamma, I)$-grading of $k\Gamma/I$ in the presence of parade data γ. Thus if $\Psi \in \mathrm{Hom}(\pi_1(\Gamma, I), k^+)$, then the induced derivation defined in the introduction depends on some choice of γ and, so, will frequently be written $D_{\Psi,\gamma}$.

2. Injectivity Theorems.

In this section, make the standing assumption that I is an admissible ideal of $k\Gamma$. (That is, we assume that $k\Gamma/I$ is finite-dimensional and I lies inside the square of the ideal generated by all arrows.)

Proposition 2.1. *Choose parade data for the connected directed graph* Γ*. Then* D_Ψ *is diagonalizable for every* $\Psi \in \mathrm{Hom}(\pi_1(\Gamma, I), k^+)$*. Moreover the map*

$$\mathrm{Hom}(\pi_1(\Gamma, I), k^+) \to \mathrm{SPDer}(k\Gamma/I)$$

is injective.

Proof. It is obvious that D_Ψ is diagonalizable, so $\mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ maps into $\mathrm{SPDer}(k\Gamma)$. Thus the issue is injectivity. By the standing assumption, every arrow in Γ survives modulo I. Since $\pi_1(\Gamma, I)$ is generated by

$$\{\xi(c(a)) \mid a \text{ is an arrow}\},$$

we see that $\pi_1(\Gamma, I)$ is generated by degrees which genuinely occur. (In the literature, the grading is sometimes referred to as *full*.)

However if R is any H-graded algebra and

$$\{h \in H \mid \text{the } h\text{-component of } R \text{ is not } 0\}$$

generates H then the map $\mathrm{Hom}(H, k^+) \to \mathrm{Der}(R)$ is always injective. □

The next theorem provides a more significant injectivity result, which is based on a similar statement of Assem-de la Peña ([**AP**]).

Theorem 2.1. *Suppose* $\chi \in \mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ *and fix parade data for* $\pi_1(\Gamma)$*. If the associated derivation* D_χ *on* $k\Gamma/I$ *is inner then* $\chi = 0$*. Hence the induced map*

$$\mathrm{Hom}(\pi_1(\Gamma, I), k^+) \to H^1(k\Gamma/I)$$

is injective.

Proof. We shall write $\Lambda = k\Gamma/I$. Then $\Lambda = \Lambda_0 \oplus \mathrm{rad}\Lambda$ where we identify Λ_0 with $(k\Gamma)_0$: A commutative subalgebra with basis consisting of orthogonal idempotents $e(w)$, one for each vertex w of Γ.

For $s = \sum_w \lambda_w e(w) \in \Lambda_0$ we compute $\mathrm{ad}\, s$. If $\overline{m} \in \Lambda$ is the image of a path m in Γ from vertex x to vertex y then

$$(\mathrm{ad}\, s)(\overline{m}) = (\lambda_x - \lambda_y)\overline{m}.$$

Thus $\mathrm{ad}\, s$ is always diagonalizable and all images of paths are among its eigenvectors.

Now suppose that $\chi \in \mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ and $D_\chi = \mathrm{ad}\, b$ for some $b \in \Lambda$. Set $b = s + n$ for $s \in \Lambda_0$ and $n \in \mathrm{rad}\Lambda$. The image $\overline{m}$ of every path is an eigenvector for D_χ corresponding to eigenvalue $(\chi \circ \xi)(c(m))$. Thus $\mathrm{ad}\, b$ is diagonalizable with a basis of eigenvectors which are images of paths. It follows that $\mathrm{ad}\, b$ and $\mathrm{ad}\, s$ must commute. But then $(\mathrm{ad}\, b) - (\mathrm{ad}\, s)$ is diagonalizable at the same time that it is equal to $\mathrm{ad}\, n$, which is nilpotent. We conclude that $D_\chi = \mathrm{ad}\, s$ for $s \in \Lambda_0$.

If $s = \sum_w \lambda_w e(w)$ then $(\chi \circ \xi)(c(m)) = \lambda_x - \lambda_y$ for every path m beginning at x and ending at y, whose image $\overline{m}$ is nonzero. In particular, if a is an arrow in Γ from x to y and $\varepsilon = \pm 1$ the $(\chi \circ \xi)(c(\overline{a})) = \varepsilon \cdot (\lambda_x - \lambda_y)$. It follows that if w is an arbitrary vertex in Γ and

$$a_1^{\varepsilon(1)}, \dots, a_t^{\varepsilon(t)}$$

is the parade walk from the base point v to w then

$$(\chi \circ \xi)\left(c\left(a_1^{\varepsilon(1)} \cdots c(a_t)^{\varepsilon(t)}\right)\right) = \lambda_v - \lambda_w.$$

On the other hand, $(\chi \circ \xi)(1) = 0$. We conclude that

$$s = \lambda_v \left(\sum_w e(w)\right) = \lambda_v \cdot 1.$$

Therefore $D_\chi = \operatorname{ad} s = 0$. □

Corollary 2.1 ([**BM**]). *Let $\Lambda = k\Gamma/I$ be a path-monomial algebra which is finite dimensional. If $H^1(\Lambda, \Lambda) = 0$ then Γ is a tree.*

Proof. We are assuming that the generating set ρ for I consists of monomials. As a consequence, $\pi_1(\Gamma, I)$ is a free group; it is trivial if and only if Γ is a tree. Thus if Γ is not a tree then $\operatorname{Hom}(\pi_1(\Gamma, I), k^+)$ is nonzero. By the theorem, H^1 is nonzero. □

3. Fundamental Derivations.

The argument presented in the previous theorem rests on the following property of the diagonalizable derivation D_χ. For any pair of vertices x and y there exists an undirected walk $a_1^{\varepsilon(1)}, \dots, a_t^{\varepsilon(t)}$ such that $\sum_j \varepsilon(j) D_\chi(\overline{a_j}) = 0$. This property turns out to be crucial in trying to characterize those diagonalizable derivations which arise from a fundamental group.

Definition 3.1. Let R be a finite-dimensional k-algebra. We say that a derivation $D \in \operatorname{Der}(R)$ is **fundamental** provided that there exists a finite directed graph Γ and an admissible ideal I of $k\Gamma$ such that $R \simeq k\Gamma/I$ and there is parade data γ together with some $\Psi \in \operatorname{Hom}(\pi_1(\Gamma, I), k^+)$ so that

$$D = D_{\Psi,\gamma}$$

under the identification given by the isomorphism.

There are some obvious things we can say about a fundamental derivation D of R. First, D must be diagonalizable. Indeed, the images of paths in R are all eigenvectors. As another consequence, $D(\operatorname{rad} R) \subseteq \operatorname{rad} R$. Notice that the algebra R is k-elementary, which means that $R/\operatorname{rad} R$ is a finite product of copies of k. More is true: The algebra complement in $k\Gamma$ to the ideal generated by all arrows, which coincides with the span of the vertex idempotents, survives as an algebra complement to rad$R$ in R. Thus rad$R$ has an algebra complement which lies inside the "subalgebra of constants"

$$R^D = \{r \in R \mid D(r) = 0\}.$$

We shall see that these properties come close to characterizing fundamental derivations.

Lemma 3.1. *Let W be an H-valued weight on the arrows of Γ and let I be a W-homogeneous ideal of $k\Gamma$. Suppose that for a fixed vertex x and every other vertex y there exists a walk in Γ^{un},*

$$\gamma_{x,y}: \ b_1^{\varepsilon(1)}, \dots, b_t^{\varepsilon(t)}$$

from x to y such that

$$W(b_1)^{\varepsilon(1)} \cdots W(b_t)^{\varepsilon(t)} = 1$$

in H. Then there is a homomorphism $\theta : \pi_1(\Gamma, I) \to H$ such that

$$(\theta \circ \xi)(c_\gamma(a)) = W(a)$$

for all arrows a.

Proof. As we remarked earlier, $\pi_1(\Gamma)$ is isomorphic to the free group on $\{c_\gamma(a) \mid a \text{ is an arrow of } \Gamma\}$ modulo the parade walk relators

$$\{c_\gamma(\gamma_{x,y}) \mid x \neq y\}.$$

Hence W induces a group homomorphism $\theta : \pi_1(\Gamma) \to H$ such that

$$\theta(c_\gamma(a)) = W(a)$$

for all arrows a.

Suppose that ρ is a support-minimal set of relations for the homogeneous ideal I. We claim that the elements of ρ are homogeneous. If $r \in \rho$ write $r = \sum_h r_h$ where each r_h is a nontrivial linear combination of paths with weight h. By homogeneity, each r_h lies in I. But the support of r_h is clearly a subset of the support of r. Hence $r = r_h$ for some choice of h.

It follows that if p and q are paths in the support of some r in ρ then $W(p) = W(q)$. Therefore θ is the identity on $N(\rho)$, the normal subgroup generated by all possible $c_\gamma(p)c_\gamma(q)^{-1}$ of this sort. We conclude that θ factors though $\pi_1(\Gamma, I)$. □

Theorem 3.1. *Assume that I is an admissible ideal of $k\Gamma$. Suppose that*

(a) *E is a diagonalizable derivation of $k\Gamma/I$ which vanishes on images of vertices and for which the images of arrows are eigenvectors, i.e., for each arrow a in Γ there is a scalar $\omega(\overline{a})$ such that $E(\overline{a}) = \omega(\overline{a})\overline{a}$;*

(b) *there is a vertex x such that for every other vertex y there exists a walk*

$$\gamma_{x,y}: \ b_1^{\varepsilon(1)}, \dots, b_t^{\varepsilon(t)}$$

from x to y such that

$$\sum_j \varepsilon(j)\omega(\overline{b_j}) = 0.$$

Then there is some $\Psi \in \mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ with $E = D_{\Psi,\gamma}$ for $\gamma = \{\gamma_{x,y}\}$.

Proof. We apply the lemma with $H = k^+$. Since I is admissible, we see that different arrows cannot have the same images in $k\Gamma/I$. Thus it makes sense to define a function W on arrows via $W(a) = \omega(\overline{a})$, thereby lifting the k^+-grading to $k\Gamma$. We conclude from the lemma that there is an additive character Ψ on $\pi_1(\Gamma, I)$ such that

$$(\Psi \circ \xi)(c_\gamma(a)) = \omega(\overline{a}).$$

Thus the two derivations D_Ψ and E agree on the images of arrows and vertices. But these elements generate $k\Gamma/I$ as an algebra. □

The next result is a close relative to Theorem 3.4 in [**G**].

Lemma 3.2. *Let R be a finite-dimensional k-elementary algebra. Assume that D is a diagonalizable derivation of R such that $D(\mathrm{rad}R) \subseteq \mathrm{rad}R$ and R^D contains an algebra complement to* rad$R$. *Then there exists a finite directed graph Γ, an admissible ideal I of $k\Gamma$, and a derivation $\widetilde{D}$ of $k\Gamma$ such that*

(a) $\widetilde{D}(I) \subseteq I$;
(b) *$\widetilde{D}$ vanishes on the vertex idempotents of $k\Gamma$*;
(c) *each arrow is an eigenvector for $\widetilde{D}$*;
(d) *$R \simeq k\Gamma/I$ with $\widetilde{D}$ inducing D.*

Proof. Let $e(1), \ldots, e(n)$ be orthogonal idempotents whose sum is 1, which span a complement to rad$R$, and which satisfy $D(e(j)) = 0$ for $j = 1, \ldots, n$. Then

$$D(e(i)\mathrm{rad}R\, e(j)) \subseteq e(i)\mathrm{rad}R\, e(j)$$

for all i and j. An elementary eigenspace argument using the diagonalizability of D implies that the pair of D-stable spaces

$$e(i)\mathrm{rad}R\, e(j) \cap (\mathrm{rad}R)^2 \subseteq e(i)\mathrm{rad}R\, e(j)$$

splits with a D-stable vector space complement $A(i,j)$.

We argue that $\{A(i,j) \mid 1 \leq i, j \leq n\}$ generates rad$R$ as an algebra. Denote by A the algebra generated by these subspaces. Certainly $A \subseteq \mathrm{rad}R$. If the algebras do not coincide, the nilpotence of rad$R$ implies that there must be a largest m such that $(\mathrm{rad}R)^m$ does not lie in A. That is, there exist $r_1, \ldots, r_m \in \mathrm{rad}R$ such that

$$r_1 r_2 \cdots r_m \notin A.$$

We may assume that $r_j \in e_{f(j)}\text{rad}R\, e_{g(j)}$ for some choice of indices $f(j)$ and $g(j)$. Write $r_j = a_j + s_j$ with $a_j \in A(f(j), g(j))$ and $s_j \in (\text{rad}R)^2$. Then

$$r_1 r_2 \cdots r_m = a_1 a_2 \cdots a_m + z$$

where $z \in (\text{rad}R)^{m+1}$. In other words, $a_1 \cdots a_m + z \in A$. We have reached the contradiction that $r_1 \cdots r_m \in A$.

We can now describe Γ. Its vertices are the idempotents $e(1), \ldots , e(n)$. Choose a basis for each $A(i, j)$ which consists of eigenvectors for D. These basis vectors comprise the set of arrows which begin at $e(i)$ and end at $e(j)$. There is an obvious algebra map from $k\Gamma$ onto R. If a is one of the designated eigenvectors for D in $A(i, j)$ and $D(a) = \lambda a$ then we define the function $\widetilde{D}$ on the arrow a by $\widetilde{D}(a) = \lambda a$. It is easy to see that $\widetilde{D}$ extends uniquely to a derivation of $k\Gamma$ which vanishes on vertex idempotents.

The lemma follows with I the kernel of the map $k\Gamma \to R$. □

Theorem 3.2. *Let R be a finite-dimensional k-elementary algebra. Suppose that D is a diagonalizable derivation of R with $D(\text{rad}R) \subseteq \text{rad}R$. Suppose, further, that R^D contains an algebra complement to* rad$R$ *and that R^D is indecomposable as an algebra. Then D is fundamental.*

Proof. We carry over all of the notation in the previous lemma. Define a new graph G whose vertices are $e(1), \ldots e(n)$ (the vertices of Γ) and construct an arrow from $e(i)$ to $e(j)$ provided $e(i)R^D e(j) \neq 0$. The sum of the $e(h)$ over all those vertex idempotents in a connected component of G is a central idempotent of R^D. Since R^D is indecomposable, G is connected.

We claim that if $e(i)R^D e(j) \neq 0$ then there is a path m in Γ such that m begins at $e(i)$, ends at $e(j)$, and $D(\overline{m}) = 0$. To see this, suppose that $e(i)xe(j) \neq 0$ for $x \in R^D$. Write $x = \sum \alpha_p \overline{p}$ with α_p a nonzero scalar and $\overline{p}$ the image in R of a path p in $k\Gamma$ which begins at $e(i)$ and ends at $e(j)$. We may assume that the $\overline{p}$ which appear in the sum are linearly independent in R. Each such $\overline{p}$ is an eigenvector for D. Since eigenvectors for distinct eigenvalues are linearly independent, we conclude that $D(\overline{p}) = 0$ for every $\overline{p}$ which appears.

We put the previous two paragraphs together. For each $i \neq j$ there is a walk from $e(i)$ to $e(j)$ in G with edge sequence

$$g_1^{\varepsilon(1)}, g_2^{\varepsilon(2)}, \ldots , g_v^{\varepsilon(v)} \; .$$

This walk gives rise to an “expanded” walk

$$m_1^{\varepsilon(1)}, m_2^{\varepsilon(2)}, \ldots , m_v^{\varepsilon(v)}$$

from $e(i)$ to $e(j)$ in Γ, where each m_d is a *path* with the same endpoints as g_d and $D(\overline{m_d}) = 0$. If we rewrite the second walk as

$$a_1^{\eta(1)}, a_2^{\eta(2)}, \ldots , a_t^{\eta(t)}$$

for arrows a_i in Γ then

$$\sum \eta(i) D(\overline{a_i}) = 0.$$

(The point is that if the sum of the eigenvalues along a path m_d is zero then the same is true for the sum of the negatives of those eigenvalues in reverse order along the path.) For each $2 \leq j \leq n$ pick such a walk $\gamma_{1,j}$ from the base point $e(1)$ to $e(j)$.

According to Theorem 3.1, there exists some $\Psi \in \mathrm{Hom}(\pi_1(\Gamma, I), k^+)$ such that $D = D_{\Psi,\gamma}$. □

Corollary 3.1. *Assume that k is an algebraically closed field of characteristic zero and R is a finite-dimensional local k-algebra. Then every diagonalizable derivation of R is fundamental.*

Proof. Since k is algebraically closed and R is local, we have

$$R/\mathrm{rad}R \simeq k.$$

It follows that R^D is local, and so, indecomposable. Finally, it is well known that $D(\mathrm{rad}R) \subseteq \mathrm{rad}R$ for any $D \in \mathrm{Der}(R)$, by virtue of $\mathrm{char}k = 0$. □

4. Hopf Algebras.

We end with a hint that there may be other classes of derivations beside diagonalizable ones for which an interesting theory exists. Every diagonalizable derivation of a k-algebra corresponds to a group grading by a subgroup of the additive group k^+. Equivalently, every diagonalizable derivation has the form D_χ where $\chi \in \mathrm{Hom}(G, k^+)$ for some group G which grades the algebra. A group grading for an algebra R is an example of an H-comodule algebra action on R, where H is a Hopf algebra. (In the special case, $H = kG$ with the standard Hopf structure.) In general, if H is any Hopf algebra then R is an H-comodule algebra provided that R is a left H-comodule, via

$$\lambda : R \to H \otimes R$$

(so $\lambda(a) = \sum a_0 \otimes a_1$ for $a \in R$) and

$$\lambda(ab) = \sum a_0 b_0 \otimes a_1 b_1 \quad \text{for} \quad a, b \in R;$$
$$\lambda(1) = 1 \otimes 1.$$

See [**Mo**], Section 4.1 for more details. In the particular case of the group algebra, R is a kG-comodule algebra if and only if it is G-graded as an algebra.

If ϵ is the augmentation for H then those functionals which are ϵ-derivations,

$$\mathrm{Der}_k^\epsilon(H, k) = \{f \in H^* \mid f(ab) = \epsilon(a)f(b) + f(a)\epsilon(b)\},$$

comprise a Lie algebra under the commutator $[f,g] = f * g - g * f$. (Here $*$ is convolution on H^*.) When $H = kG$ then

$$\mathrm{Der}_k^\epsilon(kG, k) = \mathrm{Hom}(G, k^+).$$

Back to the general set-up, for each $f \in \mathrm{Der}_k^\epsilon(H,k)$ define $D_f \in \mathrm{Hom}_k(R,R)$ by

$$D_f(a) = \sum f(a_0)a_1$$

for all $a \in R$. We leave it as an exercise that $D_f \in \mathrm{Der}(R)$ and the map $\mathrm{Der}_k^\epsilon(H,k) \to \mathrm{Der}(R)$ sending f to D_f is a Lie algebra homomorphism. This construction subsumes our earlier D_χ.

The subspace of derivations spanned by all D_f, as one runs over all comodule algebra actions of one or more Hopf algebras H, deserves future scrutiny.

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