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We will show that if $u_0 \in L^p_{\text{loc}}(R^2)$ for some constant $p > 1$, $0 \le u_0 \le (2/\beta)|x|^{-2}$, and $u_0(x) - (2/\beta)(|x|^2 + k')^{-1} \in L^1(R^2)$ for some constants $\beta > 0$, $k' > 0$, then the rescaled function $w(x,t) = e^{2\beta t}u(e^{\beta t}x, t)$ of the solution u of the Ricci flow equation $u_t = \Delta \log u$, $u > 0$, in $R^2 \times (0,\infty)$, $u(x,0) = u_0(x)$ in R^2, will converge to $\phi_{\beta,k_0}(x) = (2/\beta)(|x|^2 + k_0)^{-1}$ in $L^1(R^2)$ as $t \to \infty$ where $k_0 > 0$ is a constant chosen such that $\int_{R^2}(u_0 - \phi_{\beta,k_0})dx = 0$. Moreover if u_0 satisfies in addition the condition $\phi_{\beta,k_1} \le u_0 \le \phi_{\beta,k_2}$ for some constants $k_1 > 0$, $k_2 > 0$, then w will converge uniformly to ϕ_{β,k_0} on every compact subset of R^2 as $t \to \infty$.

In this paper we will study the asymptotic behaviour of solutions of the following degenerate parabolic equation

$$\begin{cases} u_t = \Delta \log u, u > 0 & \text{in } R^2 \times (0,\infty) \\ u(x,0) = u_0(x) & \forall x \in R^2 \end{cases} \tag{0.1}$$

as $t \to \infty$ where u_0 satisfies the condition

$$\begin{cases} 0 \le u_0 \le (2/\beta)|x|^{-2}, u_0 \in L^p_{\text{loc}}(R^2) \\ u_0(x) - (2/\beta)(|x|^2 + k')^{-1} \in L^1(R^2) \end{cases} \tag{0.2}$$

for some constants $p > 1$, $\beta > 0$, $k' > 0$. Recently there is a lot of interest on the above equation. [**ERV1**], [**V**], [**H1**], [**COR**] It arises as the singular limit of the famous porous medium equation [**H2**]

$$u_t = \Delta \left(\frac{u^m}{m} \right)$$

as $m \to 0$. We refer the reader to the survey papers of Aronson [**A**] and Peletier [**P**] for various results on the above porous medium equation. As shown by L.F. Wu [**W1**], [**W2**] Equation (0.1) also arises as the conformal factor of the metric on a complete manifold on R^2 evolving by the Ricci flow. Recently P.L. Lions and G. Toscani [**LT**] and T. Kurtz [**K**] have shown that (0.1) also appears as the singular limit for finite velocity Boltzmann kinetic models.

Existence of multiple solutions of (0.1) which extinct in finite time as well as the existence of a global infinite mass solution has been shown first by P. Daskalopoulos and M.A. Del Pino, [**DP**]. Existence of multiple solutions of (0.1) which extinct in finite time for $u_0 \in L^1(R^2) \cap L^p(R^2)$ for some constant $p > 1$ was also proved by K.M. Hui [**H1**] and for radially symmetric $u_0 \in L^1(R^2)$ by J.R. Esteban, A. Rodriguez, and J.L. Vazquez, [**ERV2**]. Regularity and some other properties of solution of (0.1) have been obtained by S.H. Davis, E. Dibenedetto, and D.J. Diller in the papers [**DD**] and [**DDD**]. Existence and uniqueness of global solution of (0.1) satisfying

$$\liminf_{r\to\infty} \frac{\log u(x,t)}{\log r} \ge -2 \text{ uniformly in } [t_1,t_2] \quad \forall t_2 > t_1 > 0 \tag{0.3}$$

and

$$u_t \le \frac{u}{t} \tag{0.4}$$

in $R^2 \times (0,\infty)$ under the very general condition

$$u_0 \notin L^1(R^2), u_0 \in L^p_{\text{loc}}(R^2), u_0 \ge 0, \quad \text{for some } p > 1 \tag{0.5}$$

was recently proved by S.Y. Hsu, [**Hs1**]. Existence of solution of (0.1) was also proved by L.F. Wu [**W1**], [**W2**] for $u_0 \notin L^1(R^2)$ and satisfying some geometric conditions. L.F. Wu also showed that under proper rescaling some rescaled function of the solution of (0.1) will have a subsequence that converges to a soliton solution of (0.1) of the form

$$\psi_{\beta,k}(x,t) = \frac{2}{\beta(|x|^2 + ke^{2\beta t})} \tag{0.6}$$

for some constants $\beta > 0$, $k > 0$, as $t \to \infty$.

In this paper we will use a modification of the dynamical system approach of J.T. Chayes, S.J. Osher, and J.V. Ralston, [**COR**] to prove that the rescaled function

$$w(x,t) = e^{2\beta t} u(e^{\beta t} x, t) \tag{0.7}$$

of the solution u of (0.1) satisfying (0.3), (0.4) with initial value u_0 satisfying (0.2) will converge in $L^1(R^2)$ to some function ϕ_{β,k_0} given by

$$\phi_{\beta,k}(x) = \frac{2}{\beta(|x|^2 + k)} \tag{0.8}$$

as $t \to \infty$ where the constant $k_0 > 0$ is uniquely determined by the equation

$$\int_{R^2} (u_0 - \phi_{\beta,k_0})\,dx = 0. \tag{0.9}$$

In other words when the initial value u_0 satisfies the condition (0.2), the solution u of (0.1) will tends to the soliton solution ψ_{β,k_0} of (0.1) for some constant $k_0 > 0$ satisfying (0.9) as $t \to \infty$. We will also show that if u_0 satisfies in addition the condition

$$\phi_{\beta,k_1} \le u_0 \le \phi_{\beta,k_2} \quad \text{in } R^2 \tag{0.10}$$

for some constants $k_1 > k_2 > 0$, then w will converge uniformly to ϕ_{β,k_0} on every compact subset of R^2 as $t \to \infty$.

The plan of the paper is as follows. In Section 1 we will give a new proof for the existence of solution of (0.1) with initial value u_0 satisfying (0.2) by a method different from that of [**Hs1**]. We do this by approximating solution of (0.1) by Dirichlet solutions of (0.1) in bounded cylindrical domains. We then use a modification of the dynamical system approach of [**COR**] to construct an appropriate Lyapunov functional for Equation (0.1) from the approximating solutions obtained in Section 1. The convergence result will then follow immediately from the form of the Lyapunov functional.

We first start with some definition. We say that u is a solution of

$$u_t = \Delta \log u \tag{0.11}$$

in $R^2 \times (0,T)$ if $u > 0$ in $R^2 \times (0,T)$ and is a classical solution of (0.11) in $R^2 \times (0,T)$. For any $u_0 \in L^1_{\text{loc}}(R^2)$, $u_0 \ge 0$, we say that u is a solution of (0.1) if $u \in C([0,\infty); L^1_{\text{loc}}(R^2))$ and u is a solution of (0.11) in $R^2 \times (0,\infty)$ with $\|u(\cdot,t) - u_0(\cdot)\|_{L^1(K)} \to 0$ as $t \to 0$ for any compact set $K \subset R^2$. For any $R > 0$, $T > 0$, $n \in \mathcal{Z}^+$, we let $B_R = B_R(0) = \{x \in R^n : |x| < R\}$, $Q_R^T = B_R \times (0,T)$, and $Q_R = B_R \times (0,\infty)$. For any $u_0 \in L^1(B_R)$, $u_0 \ge 0$, $u_0 \not\equiv 0$, and $g \in C^\infty(\partial B_R \times [0,\infty))$, $g > 0$ on $\partial B_R \times [0,\infty)$, we say that u is a solution of the Dirichlet problem

$$\begin{cases} u_t = \Delta(\log u) & \text{in } B_R \times (0,\infty) \\ u > 0 & \text{on } \overline{B}_R \times (0,\infty) \\ u(x,t) = g(x,t) & \text{on } \partial B_R \times (0,\infty) \\ u(x,0) = u_0(x) & \forall x \in B_R \end{cases} \tag{0.12}$$

if $u \in C^\infty(\overline{B}_R \times (0,\infty))$ satisfies (0.11) in Q_R in the classical sense with $u(x,t) = g(x,t)$ on $\partial B_R \times (0,\infty)$ and $u(\cdot,t) \to u_0$ in distribution sense as $t \to 0$. For any set A, we let χ_A be the characteristic function of the set A.

Section 1.

In this section we will prove the existence of solution of (0.1) by approximating the solution of (0.1) by solutions of the Dirichlet problem (0.12) in bounded cylindrical domains. We will assume that $\phi_{\beta,k}$ is given by (0.8) for the rest of the paper. We first recall a result of [**Hs1**]:

Theorem 1.1 (Theorem 1.5 of [**Hs1**])**.** *If u_1, u_2, are solutions of (0.1) satisfying (0.3), (0.4) with initial values $u_{0,1} \le u_{0,2}$, then $u_1 \le u_2$. In particular if $u_{0,1} = u_{0,2}$, then $u_1 \equiv u_2$.*

By Theorem 1.4 of [**Hs1**] and Theorem 2.1 of [**ERV3**] we have:

Theorem 1.2. *Let u_1, u_2, be solutions of (0.1) satisfying (0.3), (0.4) with initial values $u_{0,1}$ and $u_{0,2}$ respectively. If $u_{0,1} - u_{0,2} \in L^1(R^2)$, then*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^1(R^2)} \le \|u_{0,1} - u_{0,2}\|_{L^1(R^2)} \quad \forall t > 0.$$

Theorem 1.3. *Suppose $u_0 \in L^\infty(R^2)$ satisfies the condition*

$$u_0(x) \ge \phi_{\beta,k}(x) \quad \forall x \in R^2 \tag{1.1}$$

for some constants $\beta > 0$ and $k > 0$. Then for each $R > 0$, there exists a solution u_R of (0.12) in Q_R with $g(x,t) = \psi_{\beta,k}(x,t)$ on $\partial B_R \times (0,\infty)$ satisfying (0.4) and

$$u_{R'} \ge u_R \ge \psi_{\beta,k} \quad \text{in } \overline{Q_R} \quad \forall R' \ge R > 0. \tag{1.2}$$

Moreover $\{u_R\}_{R>R_0}$ will increase monotonically to the unique solution of (0.1) satisfying (0.3), (0.4) as $R \to \infty$.

Proof. Since the proof is similar to the proof of Lemma 1.3 of [**Hs1**], we will only sketch the proof here. We first observe that since $\psi_{\beta,k}$ satisfies

$$(\psi_{\beta,k})_t \le \frac{\psi_{\beta,k}}{t} \quad \text{in } R^2 \times (0,\infty),$$

by the proof of Lemma 1.3 of [**Hs1**] for each $0 < \varepsilon < 1$, $R > 0$, there exists a solution $u_{R,\varepsilon}$ of (0.12) in Q_R with initial value $u_{R,\varepsilon}(x,0) = u_0(x) + \varepsilon$ and boundary value $g(x,t) = \psi_{\beta,k}(x,t)$ on $\partial B_R \times (0,\infty)$ satisfying (0.4) and

$$\min(\varepsilon, \psi_{\beta,k}(R,T)) \le u_{R,\varepsilon} \le \max(\|u_0\|_{L^\infty(R^2)} + 1, \phi_{\beta,k}(R)) \quad \text{in } Q_R^T \ \forall T > 0. \tag{1.3}$$

Moreover $u_{R,\varepsilon}$ will decrease monotonically to a solution u_R of (0.12) in Q_R satisfying (0.4) with initial value $u_R(x,0) = u_0(x)$ and boundary value $g(x,t) = \psi_{\beta,k}(x,t)$ on $\partial B_R \times (0,\infty)$ as $\varepsilon \to 0$. Since $\psi_{\beta,k}$ also satisfies (0.11) in Q_R, by (1.1), (1.3) and the proof of Lemma 2.3 of [**DK**] we have

$$\begin{aligned} & u_{R,\varepsilon} \ge \psi_{\beta,k} \quad \text{in } \overline{Q_R} \\ \Rightarrow \quad & u_R \ge \psi_{\beta,k} \quad \text{in } \overline{Q_R} \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{1.4}$$

By (1.4) for any $R' \ge R > 0$, we have

$$u_{R',\varepsilon}(x,t) \ge \psi_{\beta,k}(x,t) = u_{R,\varepsilon}(x,t) \quad \forall (x,t) \in \partial B_R \times (0,\infty).$$

Hence by Lemma 2.3 of [**DK**] again we have

$$u_{R,\varepsilon} \le u_{R',\varepsilon} \le \max(\|u_0\|_{L^\infty(R^2)} + 1, \phi_{\beta,k}(R)) \quad \text{in } \overline{Q_R} \quad \forall 0 < \varepsilon < 1, R' \ge R > 0$$

$$\Rightarrow u_R \le u_{R'} \le \max(\|u_0\|_{L^\infty(R^2)} + 1, \phi_{\beta,k}(R)) \quad \text{in } \overline{Q_R} \quad \forall R' \ge R > 0 \quad \text{as } \varepsilon \to 0.$$

Thus u_R will increase monotonically to a solution u of (0.11) in $R^2 \times (0,\infty)$ satisfying (0.4) and

$$u(x,t) \ge \psi_{\beta,k}(x,t) \quad \forall (x,t) \in R^2 \times (0,\infty) \tag{1.5}$$

as $R \to \infty$. By (1.5) u satisfies (0.3). By the same argument as proof of Lemma 3.6 of [**H1**], $u(\cdot,t) \to u_0$ in distribution sense as $t \to 0$. By Lemma 3.2 of [**H1**] $u(\cdot,t) \to u_0$ in $L^1_{loc}(R^2)$ as $t \to 0$. Hence by Theorem 1.1 u is the unique solution of (0.1) satisfying (0.3), (0.4) and the theorem follows. □

Theorem 1.4. *Suppose u_0 satisfies* (0.2) *and $u_{0,j} = u_0\chi_{R^2\setminus B_j} + \phi_{\beta,j}\chi_{B_j}$, $u_{0,j,m} = u_0\chi_{A_{j,m}} + \phi_{\beta,j}\chi_{B_j} + \phi_{\beta,1/m}\chi_{E_m}$ where $A_{j,m} = \{x \in R^2 : \phi_{\beta,j}(x) \le u_0(x) \le \phi_{\beta,1/m}(x)\}$, $B_j = \{x \in R^2 : u_0(x) < \phi_{\beta,j}(x)\}$, $E_m = \{x \in R^2 : u_0(x) > \phi_{\beta,1/m}(x)\}$, for all $j,m = 1,2,\dots$. If $u_{j,m}$ is the unique solution of* (0.1) *satisfying* (0.3), (0.4) *with initial value $u_{0,j,m}$ given by Theorem* 1.3, *then for each $j = 1,2,\dots$ $u_{j,m}$ will increase monotonically to the unique solution u_j of* (0.1) *satisfying* (0.3), (0.4) *with initial value $u_{0,j}$ as $m \to \infty$ and u_j will decrease monotonically to the unique solution u of* (0.1) *satisfying* (0.3), (0.4) *with initial value u_0 as $j \to \infty$.*

Proof. By Theorem 1.3 for each $j,m = 1,2,\dots$, there exists an unique solution $u_{j,m}$ of (0.1) satisfying (0.3)(0.4) with initial value $u_{0,j,m}$. Since $u_{0,j,m} \le u_{0,j,m+1}$ for all $j,m = 1,2,\dots$, by Theorem 1.1 $u_{j,m} \le u_{j,m+1}$ for all $j,m = 1,2,\dots$. Since

$$u_{0,j} = u_0\chi_{R^2\setminus B_j} + \phi_{\beta,j}\chi_{B_j} \quad \Rightarrow \quad u_{0,j}^p = u_0^p\chi_{R^2\setminus B_j} + \phi_{\beta,j}^p\chi_{B_j} \le u_0^p + \phi_{\beta,1}^p \quad \forall j = 1,2,\dots,$$

by the same argument as the proof of Theorem 2.1.1 of [**Hs2**] for any $R \ge 1$, there exists a constant $C_1 > 0$ depending only on R and p such that

$$\begin{aligned}\left(\int_{B_{2R}} u_{j,m}(x,t)^p dx\right)^{1/p} &\le \left(\int_{B_{3R}} u_{0,j,m}^p dx\right)^{1/p} + C_1 t \\ &\le \left(\int_{B_{3R}} u_{0,j}^p dx\right)^{1/p} + C_1 t \\ &\le \left(\int_{B_{3R}} u_0^p dx\right)^{1/p} + \left(\int_{B_{3R}} \phi_{\beta,1}^p dx\right)^{1/p} + C_1 t\end{aligned}$$

holds for all $j,m = 1,2,\dots, t > 0$. By the $L^p - L^\infty$ regularity result of [**H1**], for any $T > t_1 > 0$, $R \ge 1$, there exists a constant $C_2 > 0$ depending only on R, t_1, and $\int_{B_{3R}} u_0^p dx$ such that

$$0 < u_{j,m}(x,t) \le C_2 \quad \forall |x| \le R, t_1 \le t < T, j,m = 1,2\dots.$$

Hence for each $j = 1, 2, \dots,$ $u_{j,m}$ will increase monotonically to a solution u_j of (0.11) in $R^2 \times (0,\infty)$ satisfying (0.4) as $m \to \infty$. Since $u_j \ge u_{j,1}$ and $u_{j,1}$ satisfies (0.3), u_j will also satisfy (0.3). By the same argument as the proof of Theorem 1.2 of [**Hs1**] $u_j(\cdot,t) \to u_0$ in $L^1_{\text{loc}}(R^2)$ as $t \to 0$. Thus for each $j = 1, 2, \dots,$ u_j is the unique solution of (0.1) with initial value $u_{0,j}$ and satisfying (0.3), (0.4).

Since u_0 satisfies (0.2), u_0 will satisfy (0.5), by Theorem 1.2 of [**Hs1**] and Theorem 1.1, there exists a unique solution u of (0.3), (0.4) with initial value value u_0. Since $u_{0,j} \ge u_{0,j+1} \ge u_0$ for all $j = 1, 2, \dots,$ by Theorem 1.1 we have

$$u_j \ge u_{j+1} \ge u \quad \forall j = i, 2, \dots . \tag{1.6}$$

Hence u_j will decrease monotonically to a solution $\overline{u}$ of (0.11) satisfying (0.4) in $R^2 \times (0,\infty)$. Letting $j \to \infty$ in (1.6), we get $\overline{u} \ge u$. Since u satisfies (0.3), $\overline{u}$ will also satisfy (0.3). By the same argument as the proof of Theorem 1.2 of [**Hs1**], $\overline{u}(\cdot,t) \to u_0$ in $L^1_{\text{loc}}(R^2)$ as $t \to 0$. Hence $\overline{u}$ is the unique solution of (0.1) with initial value u_0 and satisfying (0.3), (0.4). Thus $\overline{u} = u$ and the theorem follows. □

Section 2.

In this section we will use a modification of the dynamical system approach of [**COR**] to prove the large time behaviour of solution of (0.1) with initial value satisfying (0.2). We will do this by constructing an appropriate Lyapunov functional for (0.1). We first assume that u_0 satisfies (0.10) and let u be the solution of (0.1) given by Theorem 1.3 which satisfies (0.3), (0.4). We will also let w be given by (0.7) for the rest of the paper. Since $\psi_{\beta,k}$ is the solution of (0.1) satisfying (0.3), (0.4) with initial value $\phi_{\beta,k}$, by Theorem 1.1 and (0.10) we have

$$\psi_{\beta,k_1} \le u \le \psi_{\beta,k_2} \quad \Rightarrow \quad \phi_{\beta,k_1} \le w \le \phi_{\beta,k_2} \quad \text{in } R^2 \times (0,\infty). \tag{2.1}$$

Since u satisfies (0.1), a direct computation then shows that w satisfies

$$w_t(x,t) = \Delta \log w(x,t) + \beta \text{div } (w(x,t)x) \quad \text{in } R^2 \times (0,\infty). \tag{2.2}$$

Let $v = w - \phi_{\beta,k_1}$. Then v satisfies

$$v_t = \Delta(a(x, v(x,t))) + \beta \text{div } (v(x,t)x) \quad \text{in } R^2 \times (0,\infty) \tag{2.3}$$

and

$$0 \le v \le \phi_{\beta,k_2} - \phi_{\beta,k_1} = \frac{2(k_1 - k_2)}{\beta(|x|^2 + k_1)(|x|^2 + k_2)} \in L^1(R^2) \quad \text{in } R^2 \times (0,\infty) \tag{2.4}$$

where

$$a(x,\mu)=\log(\mu+\phi_{\beta,k_1}(x))-\log(\phi_{\beta,k_1}(x)).$$

We would then like to find functions $b(x,\mu)>0$, $F(x,\mu)$, which satisfies the following equation

$$\begin{aligned}(2.5)\quad &\nabla_x\{a(x,v(x,t))\}+\beta v(x,t)x\\ &\qquad=b(x,v(x,t))\nabla_x\{F(x,v(x,t))\}\quad\forall(x,t)\in R^2\times(0,\infty).\end{aligned}$$

For each $x\in R^2$ and $0<k\le k_1$ we let

$$z(x,k)=\phi_{\beta,k}(x)-\phi_{\beta,k_1}(x). \tag{2.6}$$

Then for each $x\in R^2$, $z(x,k)$ is a strictly monotone decreasing smooth function of k. Hence for each $x\in R^2$, $\mu\in[0,(2/\beta)|x|^{-2}-\phi_{\beta,k_1}(x))$, there exists a unique $k(\mu)\in(0,k_1]$ such that $\mu=z(x,k(\mu))$ and $k(\mu)$ is a smooth function of $\mu\in[0,(2/\beta)|x|^{-2}-\phi_{\beta,k_1}(x))$. We can then let F be defined by

$$F(x,\mu)=k_1-k(\mu)\ge 0\quad\forall 0\le\mu<(2/\beta)|x|^{-2}-\phi_{\beta,k_1}(x). \tag{2.7}$$

Then

$$F(x,z(x,k))=k_1-k\quad\forall 0<k\le k_1 \tag{2.8}$$

and $F(x,\mu)$ is a smooth function of $(x,\mu)\in\{(x,\mu):0\le\mu(2/\beta)|x|^{-2}-\phi_{\beta,k_1}(x)\}$. A direct computation shows that

$$\nabla_x\{a(x,z(x,k))\}+\beta z(x,k)x\equiv 0\quad\forall x\in R^2,0<k\le k_1. \tag{2.9}$$

Differentiating (2.8) with respect to x_i for $i=1$, 2, we get

$$F_{x_i}(x,z(x,k))+F_\mu(x,z(x,k))z_{x_i}=0\quad\forall i=1,2. \tag{2.10}$$

By (2.9) we have

$$a_{x_i}(x,z(x,k))+a_\mu(x,z(x,k))z_{x_i}+\beta z(x,k)x_i=0\quad\forall i=1,2. \tag{2.11}$$

Eliminating z_{x_i} from (2.10) and (2.11),

$$\begin{aligned}(2.12)\quad &a_\mu(x,z(x,k))F_{x_i}(x,z(x,k))\\ &\qquad=(a_{x_i}(x,z(x,k))+\beta z(x,k)x_i)F_\mu(x,z(x,k))\quad\forall i=1,2.\end{aligned}$$

Since $v(x,t)$ satisfies (2.4), for each $x\in R^2$, $t>0$, there exists $k_2\le k\le k_1$ such that $v(x,t)=z(x,k)$. Hence by (2.12)

$$\begin{aligned}(2.13)\quad &a_\mu(x,v(x,t))F_{x_i}(x,v(x,t))\\ &\qquad=(a_{x_i}(x,v(x,t))+\beta v(x,t)x_i)F_\mu(x,v(x,t))\quad\forall i=1,2.\end{aligned}$$

We next observe that (2.5) is equivalent to

$$\begin{aligned}&a_\mu(x,v(x,t))v_{x_i}+a_{x_i}(x,v(x,t))+\beta v(x,t)x_i\\ &\qquad=b(x,v(x,t))(F_{x_i}(x,v(x,t))+F_\mu(x,v(x,t))v_{x_i})\end{aligned}$$

for all $i = 1, 2$. Hence (2.5) will hold if both

$$a_\mu(x, v(x,t)) = b(x, v(x,t))F_\mu(x, v(x,t)) \tag{2.14}$$

and

$$a_{x_i}(x, v(x,t)) + \beta v(x,t)x_i = b(x, v(x,t))F_{x_i}(x, v(x,t)) \quad \forall i = 1, 2 \tag{2.15}$$

holds. Differentiating (2.8) with respect to k, we get

$$F_\mu(x, z(x,k))\frac{\partial z}{\partial k}(x,k) = -1$$
$$\Rightarrow \quad F_\mu(x, z(x,k)) = -\left(\frac{\partial z}{\partial k}\right)^{-1} \neq 0 \quad \text{by (2.6)}.$$

So we can let

$$\begin{aligned} b(x,\mu) &= a_\mu(x,\mu)(F_\mu(x,\mu))^{-1} = -a_\mu(x, z(x, k(\mu)))\frac{\partial z}{\partial k}(x, k(\mu)) \\ &= -\frac{\partial}{\partial k}(a(x, z(x,k)))\bigg|_{k=k(\mu)} \\ &= -\frac{\partial}{\partial k}(\log\phi_{\beta,k} - \log\phi_{\beta,k_1})\bigg|_{k=k(\mu)} \\ &= \frac{1}{|x|^2 + k(\mu)} > 0. \end{aligned} \tag{2.16}$$

With such choice of b, (2.14) holds. By (2.12)(2.14) we see that (2.15) also holds. Hence (2.5) holds. We next state a technical lemma.

Lemma 2.1. *If u_0 satisfies (0.2), there exists a unique constant $k_0 > 0$ satisfying (0.9).*

Proof. Since $\phi_{\beta,k'} - \phi_{\beta,k} \in L^1(R^2)$ for any $k > 0$, $k' > 0$, by (0.2) we have $u_0 - \phi_{\beta,k} \in L^1(R^2)$ for all $k > 0$. Hence for any $k > 0$ we can define

$$f(k) = \int_{R^2}(u_0 - \phi_{\beta,k})dx.$$

Then for any $k > 0$, $k' > 0$, we have

$$f(k) = f(k') + \frac{2(k-k')}{\beta}\int_{R^2}\frac{1}{(|x|^2 + k)(|x|^2 + k')}dx.$$

Hence $f(k)$ is a continuous strictly monotone increasing function of $k > 0$. Since the last term on the right hand side above will tends to $-\infty$ or ∞ as k tends to zero or infinity, $f(k)$ will tends to negative infinity or positive infinity as k tends to zero or infinity respectively. By the intermediate value theorem there exists a unique $k_0 > 0$ such that $f(k_0) = 0$ and the lemma follows. $\square$

Theorem 2.2. *If u_0 satisfies (0.2) and (0.10) for some constants $\beta > 0$, $k' > 0$, $k_1 > k_2 > 0$, then the rescaled function w of the solution u of (0.1) satisfying (0.3), (0.4) given by (0.7) will converge uniformly on every compact subset of R^2 and also in $L^1(R^2)$ to ϕ_{β,k_0} as $t \to \infty$ where ϕ_{β,k_0} is given by (0.8) and $k_0 > 0$ is a constant uniquely detemined by (0.9).*

Proof. We first observe that since u_0 satisfies (0.10), the solution u of (0.1) and w will satisfy (2.1). Hence Equation (2.2) satisfied by w is uniformly parabolic on $\overline{B}_{2R} \times (1/2, \infty)$ for any $R > 0$. By the Schauder's estimates [**LSU**] w is equi-Hölder continuous on $\overline{B}_R \times [1, \infty)$ for any $R > 0$. Hence by the Ascoli's Theorem and a diagonalization argument any sequence $\{w(\cdot, t_i)\}$, $t_i \to \infty$ as $i \to \infty$, of $\{w(\cdot, t)\}$ will have a convergent subsequence $\{w(\cdot, t_i')\}$ converging uniformly on every compact subset of R^2 as $i \to \infty$. Without loss of generality we may assume that $\{w(\cdot, t_i)\}$ converges uniformly on every compact subset of R^2 as $i \to \infty$ and $t_i \ge 1$ for all $i = 1, 2, \dots$.

Let $\overline{w}(x) = \lim_{i\to\infty} w(x, t_i)$ and $\overline{v} = \overline{w} - \phi_{\beta,k_1}$. We claim that $\overline{w} = \phi_{\beta,k_0}$ for some constant $k_0 > 0$ satisfying (0.9). To prove the claim, we observe that by Theorem 1.3, u is the uniform limit of the solution u_R of (0.12) in Q_R with $g(x,t) = \psi_{\beta,k_1}(x,t)$ on $\partial B_R \times (0, \infty)$ as $R \to \infty$. Let $v = w - \phi_{\beta,k_1}$ and $v_R = w_R - \phi_{\beta,k_1}$ where $w_R(x,t) = e^{2\beta t} u_R(e^{\beta t} x, t)$. Then v_R satisfies (2.3) in $D_R = \{(x,t) : |e^{\beta t} x| \le R, t > 0\}$. Moreover by (0.10)(1.2) and Lemma 2.3 of [**DK**], we have

$$
\psi_{\beta,k_1} \le u_R \le \psi_{\beta,k_2} \quad \Rightarrow \quad \begin{cases} \phi_{\beta,k_1} \le w_R \le \phi_{\beta,k_2} \\ 0 \le v_R \le \phi_{\beta,k_2} - \phi_{\beta,k_1} \end{cases} \quad \text{in } Q_R. \tag{2.17}
$$

Multiplying (2.3) for $v = v_R$ by $F(x, v_R(x,t))$ and integrating over the set $D_R^T = \{(x,t) : |e^{\beta t} x| \le R, 1 \le t \le T\}$, $T > 1$, we get after integrating by parts,

(2.18)
$$
\begin{aligned}
&\int_{|x| \le Re^{-\beta T}} G(x, v_R(x,T))\,dx \\
&\qquad + \iint_{D_R^T} b(x, v_R(x,t)) |\nabla_x \{F(x, v_R(x,t))\}|^2 dxdt \\
&= \int_{|x| \le Re^{-\beta}} G(x, v_R(x,1))\,dx \\
&\qquad + \int_1^T \int_{|x| = Re^{-\beta t}} b(x, v_R(x,t)) F(x, v_R(x,t)) \frac{\partial}{\partial n} \{F(x, v_R(x,t))\} d\sigma dt
\end{aligned}
$$

where

$$G(x,\mu)=\int_0^\mu F(x,s)ds.$$

Now for $|x|=Re^{-\beta t}$, $t>0$,

$$\begin{aligned} v_R(x,t)&=w_R(x,t)-\phi_{\beta,k_1}(x)=e^{2\beta t}u_R(e^{\beta t}x,t)-\phi_{\beta,k_1}(x)\\ &=e^{2\beta t}\phi_{\beta,k_1}(e^{\beta t}x)-\phi_{\beta,k_1}(x)=\phi_{\beta,k_1}(x)-\phi_{\beta,k_1}(x)=0\\ \Rightarrow\quad F(x,v_R(x,t))&=F(x,0)=0\qquad\forall |x|=Re^{-\beta t},t>0.\end{aligned}$$

Hence the last term on the right hand side of (2.18) is equal to 0. Since both $F(x,\mu)$ and $G(x,\mu)$ are monotone increasing functions of μ, by (2.17) we have

$$\begin{aligned} G(x,v_R(x,t))&\le G(x,\phi_{\beta,k_2}(x)-\phi_{\beta,k_1}(x))\\ &\le(\phi_{\beta,k_2}(x)-\phi_{\beta,k_1}(x))F(x,z(x,k_2)).\end{aligned}\tag{2.19}$$

Thus by (2.18), (2.19), we have

$$\begin{aligned}&\int_{|x|\le Re^{-\beta t}}G(x,v_R(x,T))dx\\ &\qquad+\iint_{D_R^T}b(x,v_R(x,t))|\nabla_x\{F(x,v_R(x,t))\}|^2dxdt\\ &\le\int_{R^2}(\phi_{\beta,k_2}(x)-\phi_{\beta,k_1}(x))F(x,z(x,k_2))dx\\ &\le\frac{2(k_1-k_2)^2}{\beta}\int_{R^2}\frac{1}{(r^2+k_1)(r^2+k_2)}dx<\infty.\end{aligned}$$

Letting $R\to\infty$, we get by Fatou's lemma,

$$\begin{aligned}&\int_{R^2}G(x,v(x,T))dx+\int_1^T\int_{R^2}b(x,v(x,t))|\nabla_x\{F(x,v(x,t))\}|^2dxdt\\ &\quad\le\frac{2(k_1-k_2)^2}{\beta}\int_{R^2}\frac{1}{(r^2+k_1)(r^2+k_2)}dx<\infty\\ \Rightarrow\quad&\int_1^\infty\int_{R^2}b(x,v(x,t))|\nabla_x\{F(x,v(x,t))\}|^2dxdt\\ &\quad\le\frac{2(k_1-k_2)^2}{\beta}\int_{R^2}\frac{1}{(r^2+k_1)(r^2+k_2)}dx<\infty\qquad\text{as }T\to\infty.\end{aligned}\tag{2.20}$$

We next claim that for any $L>0$, $\{t_i\}$ has a subsequence $\{t_i'\}$ such that

$$\int_{|x|\le L}b(x,v(x,t_i'))|\nabla_x\{F(x,v(x,t_i'))\}|^2dx\to0\qquad\text{as }i\to\infty.\tag{2.21}$$

Suppose the claim is not true. Then there exist $L > 0$, $\delta > 0$, such that

$$\int_{|x|\le L} b(x, v(x, t_i))|\nabla_x\{F(x, v(x, t_i))\}|^2 dx \ge \delta \quad \forall i = 1, 2 \dots. \tag{2.22}$$

Since by (2.1), for any $R > 0$ (2.2) is uniformly parabolic on $\overline{B}_{2R} \times [1/2, \infty)$, by standard parabolic theory [**LSU**], w, ∇w, $\partial_{x_i}\partial_t w$, are uniformly bounded on $\overline{B}_R \times [1, \infty)$. Thus v, ∇v, and $\partial_{x_i}\partial_t v$ are uniformly bounded on $K \times [1, \infty)$ for any compact subset $K \subset R^2$. Hence

$$\sup_{t\ge 1}\left|\partial_t \int_{|x|\le L} b(x, v(x, t))|\nabla_x\{F(x, v(x, t))\}|^2 dx\right| \le C < \infty.$$

Thus

$$\int_{|x|\le L} b(x, v(x, t))|\nabla_x\{F(x, v(x, t))\}|^2 dx$$

is a uniformly continuous function of $t \in [1, \infty)$. In particular there exists $0 < \varepsilon < 1/2$ such that

$$\begin{aligned} &\left|\int_{|x|\le L} b(x, v(x, t))|\nabla_x\{F(x, v(x, t))\}|^2 dx \right.\\ &\qquad \left. - \int_{|x|\le L} b(x, v(x, t'))|\nabla_x\{F(x, v(x, t'))\}|^2 dx\right| \\ &\le \delta/2 \quad \forall t, t' \ge 1, |t - t'| \le \varepsilon. \end{aligned} \tag{2.23}$$

By (2.22), (2.23), we have

$$\int_{|x|\le L} b(x, v(x, t))|\nabla_x\{F(x, v(x, t))\}|^2 dx \ge \delta/2 \quad \forall |t - t_i| \le \varepsilon, i = 1, 2 \dots. \tag{2.24}$$

By passing to a subsequence if necessary we may assume without loss of generality that

$$t_i \ge 2, |t_i - t_j| \ge 2\varepsilon \quad \forall i, j = 1, 2, \dots.$$

Integrating (2.24) over $(t_i - \varepsilon, t_i + \varepsilon)$ and summing over $i = 1, 2, \dots$, we get

$$\begin{aligned} &\int_1^\infty \int_{|x|\le L} b(x, v(x, t))|\nabla_x\{F(x, v(x, t))\}|^2 dxdt \\ &\ge \sum_{i=1}^\infty \int_{t_i-\varepsilon}^{t_i+\varepsilon} \int_{|x|\le L} b(x, v(x, t))|\nabla_x\{F(x, v(x, t))\}|^2 dxdt \\ &\ge \sum_{i=1}^\infty \delta/2 \cdot 2\varepsilon = \infty. \end{aligned}$$

This contradicts (2.20). Hence $\{t_i\}$ has a subsequence $\{t_i'\}$ such that (2.21) holds. We next observe that since $v(\cdot,t_i)=w(\cdot,t_i)-\phi_{\beta,k_1}$ converges uniformly on every compact subset of R^2 to $\overline{v}$ as $i\to\infty$, by (2.4) we have

$$
\begin{aligned}
(2.25)\qquad & v(\cdot,t_i)\to\overline{v} \quad \text{in } L^1(R^2) \quad \text{as } i\to\infty\\
\Rightarrow\quad & \int_{R^2}\overline{v}dx=\lim_{i\to\infty}\int_{R^2}v(x,t_i)dx.
\end{aligned}
$$

Now let $\eta\in C_0^\infty(R^2)$, $0\le\eta\le 1$, be such that $\eta(x)=1$ for $|x|\le 1$ and $\eta(x)=0$ for $|x|\ge 2$ and for any $R>1$, let $\eta_R(x)=\eta(x/R)$. Then $|\nabla\eta_R|\le C/R$, $|\Delta\eta_R|\le C/R^2$, for some constants $C>0$. Multiplying (2.3) by η_R and integrating by parts, by (2.4) we get

$$
\begin{aligned}
&\int_{R^2}v(x,t)\eta_R(x)dx-\int_{R^2}v(x,0)\eta_R(x)dx\\
&=\int_0^t\int_{B_{2R}}a(x,v(x,s))\Delta\eta_R(x)dxds-\beta\int_0^t\int_{B_{2R}}v(x,s)\nabla\eta_R\cdot xdxds\\
\Rightarrow\quad &\left|\int_{R^2}v(x,t)\eta_R(x)dx-\int_{R^2}v(x,0)\eta_R(x)dx\right|\\
&\le\frac{C}{R^2}\int_0^t\int_{R\le|x|\le 2R}(\log w-\log\phi_{\beta,k_1})dxds\\
&\qquad+C\int_0^t\int_{R\le|x|\le 2R}\frac{1}{(|x|^2+k_1)(|x|^2+k_2)}dxds\\
&\le\frac{C}{R^2}\int_0^t\int_{R\le|x|\le 2R}(\log\phi_{\beta,k_2}-\log\phi_{\beta,k_1})dxds+\frac{Ct}{R^2}\\
&\le\frac{C}{R^2}\int_0^t\int_{R\le|x|\le 2R}\log\left(1+\frac{k_1-k_2}{|x|^2+k_2}\right)dxds+\frac{Ct}{R^2}\\
&\le\frac{C'}{R^2}\int_0^t\int_{R\le|x|\le 2R}\frac{k_1-k_2}{|x|^2+k_2}dxds+\frac{Ct}{R^2}\\
&\le\frac{C''t}{R^2}\qquad\qquad\forall R>\sqrt{k_1}.
\end{aligned}
$$

Letting $R\to\infty$ we get

$$
\begin{aligned}
(2.26)\qquad &\int_{R^2}v(x,t)dx=\int_{R^2}v(x,0)dx\quad\forall t>0\\
\Rightarrow\quad &\int_{R^2}\overline{v}dx=\int_{R^2}v(x,0)dx\quad \text{by (2.25)}.
\end{aligned}
$$

By (2.4) for each $x\in R^2$, $i=1,2,\dots$, we can choose a constant $k_i'\in[k_2,k_1]$ such that

$$v(x,t_i')=z(x,k_i').$$

Then by (2.16) we have

$$b(x,v(x,t_i'))=b(x,z(x,k_i'))=\frac{1}{|x|^2+k_i'}\ge\frac{1}{L^2+k_1}\quad\forall |x|\le L, i=1,2,\dots.$$

Hence by (2.21), (2.4), and Fatou's lemma, we have

$$\begin{aligned}
&\int_{|x|\le L}|\nabla_x\{F(x,v(x,t_i'))\}|^2dx\to 0\quad\text{ as } i\to\infty\quad\forall L>0\\
\Rightarrow\quad&\int_{|x|\le L}|\nabla_x\{F(x,\overline{v}(x))\}|^2dx=0\quad\forall L>0\\
\Rightarrow\quad&F(x,\overline{v}(x))=k_1-k_0\ge 0\quad\forall x\in R^2
\end{aligned}$$

for some constant $k_0\in[k_2,k_1]$. Since for each $x\in R^2$, $F(x,\mu)$ is a strictly monotone increasing function of μ and $F(x,z(x,k_0))=k_1-k_0$, hence

$$\begin{aligned}
&\overline{v}(x)=z(x,k_0)=\phi_{\beta,k_0}(x)-\phi_{\beta,k_1}(x)\quad\forall x\in R^2\\
\Rightarrow\quad&\begin{cases}\int_{R^2}(\phi_{\beta,k_0}-\phi_{\beta,k_1})dx=\int_{R^2}\overline{v}dx=\int_{R^2}v(x,0)dx=\int_{R^2}(u_0-\phi_{\beta,k_1})dx\\ \overline{w}\equiv\phi_{\beta,k_0}\end{cases}\\
\Rightarrow\quad&\int_{R^2}(u_0-\phi_{\beta,k_0})dx=0.
\end{aligned}$$

Thus k_0 satisfies (0.9). By Lemma 2.1 the constant $k_0>0$ is uniquely determined by (0.9) and is independent of the sequence $\{t_i\}$. Hence $w(x,t)\to\phi_{\beta,k_0}(x)$ uniformly on every compact subset of R^2 and in $L^1(R^2)$ as $t\to\infty$. □

Theorem 2.3. *If u_0 satisfies (0.2) for some constant $\beta>0$, $k'>0$, then the rescaled function w of the solution u of (0.1) satisfying (0.3), (0.4) given by (0.7) will converge to ϕ_{β,k_0} in $L^1(R^2)$ as $t\to\infty$ where ϕ_{β,k_0} is given by (0.8) and $k_0>0$ is chosen such that (0.9) holds.*

Proof. We will use a modification of the proof of [**Z**] to prove the theorem. Let $A_{j,m}$, B_j, E_m, $u_{0,j}$, $u_{0,j,m}$ be as in Theorem 1.4. Then by Theorem 1.4, for each $j,m=1,2,\dots$, there exists an unique solution $u_{j,m}$ of (0.1) satisfying (0.3), (0.4) with initial value $u_{0,j,m}$ such that for each $j=1,2,\dots$, $u_{j,m}$ will increase monotonically to the unique solution u_j of (0.1) satisfying (0.3), (0.4) with initial value $u_{0,j}$ as $m\to\infty$ and u_j will decrease monotonically to the unique solution u of (0.1) satisfying (0.3)(0.4) with initial value u_0 as $j\to\infty$.

Observe that for all $m>1/k'$, $j=1,2,\dots$,

$$\begin{aligned}
&|u_{0,j,m}-u_{0,j}|\le(u_0-\phi_{\beta,1/m})\chi_{E_m}\\
&\qquad\qquad\qquad\le|u_0-\phi_{\beta,k'}|\chi_{E_m}\le|u_0-\phi_{\beta,k'}|\in L^1(R^2)\\
\Rightarrow\quad&|u_{0,j,m}-u_{0,j}|\to 0\quad\text{ a.e. }\ x\in R^2\quad\text{ as } m\to\infty
\end{aligned}$$

and

$$\begin{aligned}|u_{0,j}-u_0|\le(\phi_{\beta,j}-u_0)\chi_{B_j}&\le|\phi_{\beta,k'}-u_0|\chi_{B_j}\\&\le|\phi_{\beta,k'}-u_0|\in L^1(R^2)\quad\forall j>k'\\\Rightarrow\quad|u_{0,j}-u_0|\to 0\quad&\text{a.e.}\quad x\in R^2\quad\text{as } j\to\infty.\end{aligned}$$

Hence by the Lebesgue dominated convergence theorem,

$$\left\{\begin{array}{l}\|u_{0,j,m}-u_{0,j}\|_{L^1(R^2)}\to 0\quad\text{as } m\to\infty\quad\forall j=1,2\dots\\\|u_{0,j}-u_0\|_{L^1(R^2)}\to 0\quad\text{as }\ j\to\infty.\end{array}\right.\tag{2.27}$$

Let $w_{j,m}$ and w_j be given by (0.7) with u replaced by $u_{j,m}$ and u_j respectively. By Theorem 2.2, for each $j,m=1,2\dots$ there exist a unique constant $k_{j,m}$ satisfying

$$\int_{R^2}(u_{0,j,m}-\phi_{\beta,k_{j,m}})dx=0\tag{2.28}$$

such that $w_{j,m}$ will converge uniformly to $\phi_{\beta,k_{j,m}}$ on every compact subset of R^2 and also in $L^1(R^2)$ as $t\to\infty$. We claim that for any $j>k'$, there exists constants $C_{2,j}>C_{1,j}>0$ such that

$$C_{1,j}\le k_{j,m}\le C_{2,j}\quad\forall m=1,2,\dots.\tag{2.29}$$

Suppose not. Then without loss of generality we may assume that there exists $j>k'$ such that either

$$k_{j,m}\to\infty\quad\text{as } m\to\infty\tag{2.30}$$

or

$$k_{j,m}\to 0\quad\text{as } m\to\infty.\tag{2.31}$$

Now by (2.27) and (2.28) we have

$$\begin{aligned}&\int_{R^2}(u_{0,j,m}-\phi_{\beta,k'})dx+\int_{R^2}(\phi_{\beta,k'}-\phi_{\beta,k_{j,m}})dx=0\quad\forall m=1,2,\dots\\\Rightarrow\quad&\int_{R^2}(u_{0,j}-\phi_{\beta,k'})dx+\lim_{m\to\infty}\int_{R^2}(\phi_{\beta,k'}-\phi_{\beta,k_{j,m}})dx=0\quad\text{as } m\to\infty.\end{aligned}\tag{2.32}$$

Now by (0.2),

$$\left|\int_{R^2}(u_{0,j}-\phi_{\beta,k'})dx\right|\le\int_{R^2}|u_0-\phi_{\beta,k'}|dx+\int_{R^2}|\phi_{\beta,j}-\phi_{\beta,k'}|dx<\infty.$$

By the proof of Lemma 2.1, the second term on the left hand side of (2.32) is equal to either positive infinity or negative infinity depending on either (2.30) or (2.31) holds. Hence contradiction arises. Thus (2.29) must hold. Hence for each $j>k'$, $\{k_{j,m}\}_{m=1}^{\infty}$ will have a subsequence converging to some constant k_j satisfying $C_{1,j}\le k_j\le C_{2,j}$ as $m\to\infty$. Without loss of generality we may assume that $k_{j,m} \to k_j$ as $m \to \infty$. Letting $m \to \infty$ in (2.28), by (2.27) we see that $k_j > 0$ satisfies

$$\int_{R^2} (u_{0,j} - \phi_{\beta,k_j})dx = 0. \tag{2.33}$$

By repeating the previous argument but with k_j replacing $k_{j,m}$ in the argument. There exist constants $C_2 > C_1 > 0$ such that

$$C_1 \le k_j \le C_2 \quad \forall j > k'$$

and $\{k_j\}$ will have a subsequence converging to some constant k_0 satisfying $C_1 \le k_0 \le C_2$ as $j \to \infty$. Without loss of generality we may assume that $k_j \to k_0$ as $j \to \infty$. Letting $j \to \infty$ in (2.33), by (2.27) we see that $k_0 > 0$ satisfies (0.9). By Theorem 1.2 we have

$$\begin{cases} \|u_{j,m}(\cdot,t) - u_j(\cdot,t)\|_{L^1(R^2)} \le \|u_{0,j,m} - u_{0,j}\|_{L^1(R^2)} & \forall j, m = 1, 2 \dots \\ \|u_j(\cdot,t) - u(\cdot,t)\|_{L^1(R^2)} \le \|u_{0,j} - u_0\|_{L^1(R^2)} & \forall j = 1, 2 \dots. \end{cases}$$

Since

$$\begin{cases} \|w_{j,m}(\cdot,t) - w_j(\cdot,t)\|_{L^1(R^2)} = \|u_{j,m}(\cdot,t) - u_j(\cdot,t)\|_{L^1(R^2)} & \forall j, m = 1, 2 \dots \\ \|w_j(\cdot,t) - w(\cdot,t)\|_{L^1(R^2)} = \|u_j(\cdot,t) - u(\cdot,t)\|_{L^1(R^2)} & \forall j = 1, 2 \dots \end{cases}$$

hence

$$\begin{cases} \|w_{j,m}(\cdot,t) - w_j(\cdot,t)\|_{L^1(R^2)} \le \|u_{0,j,m} - u_{0,j}\|_{L^1(R^2)} & \forall j, m = 1, 2 \dots \\ \|w_j(\cdot,t) - w(\cdot,t)\|_{L^1(R^2)} \le \|u_{0,j} - u_0\|_{L^1(R^2)} & \forall j = 1, 2 \dots. \end{cases}$$

Thus

$$\begin{aligned}
& \|w(\cdot,t) - \phi_{\beta,k_0}\|_{L^1(R^2)} \\
& \le \|w(\cdot,t) - w_j(\cdot,t)\|_{L^1(R^2)} + \|w_j(\cdot,t) - w_{j,m}(\cdot,t)\|_{L^1(R^2)} \\
& \quad + \|w_{j,m}(\cdot,t) - \phi_{\beta,k_{j,m}}\|_{L^1(R^2)} \\
& \quad + \|\phi_{\beta,k_{j,m}} - \phi_{\beta,k_j}\|_{L^1(R^2)} + \|\phi_{\beta,k_j} - \phi_{\beta,k_0}\|_{L^1(R^2)} \\
& \le \|u_0 - u_{0,j}\|_{L^1(R^2)} + \|u_{0,j} - u_{0,j,m}\|_{L^1(R^2)} \\
& \quad + \|w_{j,m}(\cdot,t) - \phi_{\beta,k_{j,m}}\|_{L^1(R^2)} + \|\phi_{\beta,k_{j,m}} - \phi_{\beta,k_j}\|_{L^1(R^2)} \\
& \quad + \|\phi_{\beta,k_j} - \phi_{\beta,k_0}\|_{L^1(R^2)} \quad \forall j > k', m = 1, 2, \dots \\
\Rightarrow \quad & \limsup_{t\to\infty} \|w(\cdot,t) - \phi_{\beta,k_0}\|_{L^1(R^2)} \\
& \le \|u_{0,j} - u_0\|_{L^1(R^2)} + \|u_{0,j,m} - u_{0,j}\|_{L^1(R^2)} \\
& \qquad\qquad + \|\phi_{\beta,k_{j,m}} - \phi_{\beta,k_j}\|_{L^1(R^2)} + \|\phi_{\beta,k_j} - \phi_{\beta,k_0}\|_{L^1(R^2)} \\
\Rightarrow \quad & \lim_{t\to\infty} \|w(\cdot,t) - \phi_{\beta,k_0}\|_{L^1(R^2)} = 0 \qquad \text{as } m \to \infty, j \to \infty
\end{aligned}$$

and the theorem follows. □

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References

[A] D.G. Aronson, *The porous medium equation*, CIME Lectures, in 'Some problems in Nonlinear Diffusion', Lecture Notes in Mathematics, **1224**, Springer-Verlag, New York, 1986.

[COR] J.T. Chayes, S.J. Osher and J.V. Ralston, *On singular diffusion equations with applications to self-organised criticality*, Comm. on Pure and Applied Math., **XLVI** (1993), 1363-1377.

[DK] B.E.J. Dahlberg and C. Kenig, *Non-negative solutions of generalized porous medium equations*, Revista Matemática Iberoamericana, **2** (1986), 267-305.

[DP] P. Daskalopoulos and M.A. Del Pino, *On a singular diffusion equation*, Comm. in Analysis and Geometry, **3**(**3**) (1995), 523-542.

[DDD] S.H. Davis, E. Dibenedetto and D.J. Diller, *Some a priori estimates for a singular evolution equation arising in thin-film dynamics*, SIAM J. Math. Anal., **27**(**3**) (1996), 638-660.

[DD] E. Dibenedetto and D.J. Diller, *About a singular parabolic equation arising in thin film dynamics and in the Ricci flow for complete* R^2, Partial differential equations and applications, Lecture Notes in Pure and Applied mathematics, Vol. 177, 103-119, edited by P. Marcellini, Giorgio G.Talenti and E. Vesentini, Dekker, New York, 1996

[ERV1] J.R. Esteban, A. Rodriguez and J.L. Vazquez, *A nonlinear heat equation with singular diffusivity*, Comm. in P.D.E., **13**(**8**) (1988), 985-1039.

[ERV2] ———, *The fast diffusion equation with logarithmic nonlinearity and the evolution of conformal metrics in the plane*, Advances in Diff. Eq., **1**(**1**) (1996), 21-50.

[ERV3] ———, *The maximal solution of the logarithmic fast diffusion equation in two space dimensions*, Advances in Diff. Eq., **2**(**6**) (1997), 867-894.

[Hs1] S.Y. Hsu, *Global existence and uniqueness of solutions of the Ricci flow equation*, to appear in Differential and Integral Equations.

[Hs2] ———, *Some new results for the Ricci flow equation*, Ph.D. Thesis, The Chinese University of Hong Kong, December 1999.

[H1] K.M. Hui, *Existence of solutions of the equation* $u_t = \Delta \log u$, Nonlinear Analysis, TMA, **37** (1999), 875-914.

[H2] ———, *Singular limit of solutions of the equation* $u_t = \Delta(\frac{u^m}{m})$ *as* $m \to 0$, Pacific J. of Math., **187**(**2**) (1999), 297-316.

[K] T.G. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics*, Trans. Amer. Math. Soc., **186** (1973), 259-272.

[LSU] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Uraltceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Mono., **23**, Amer. Math. Soc., Providence, R.I., 1968.

[LT] P.L. Lions and G. Toscani, *Diffusive limit for finite velocity Boltzmann kinetic models*, Revista Matematica Iberoamericana, **13**(**3**) (1997), 473-513.

[P] L.A. Peletier, *The porous medium equation in Applications of Nonlinear Analysis in the Physical Sciences*, H.Amann, N.Bazley, K.Kirchgassner editors, Pitnam, Boston, 1981.

[V] J.L. Vazquez, *Nonexistence of solutions for nonlinear heat equations of fast-diffusion type*, J. Math. Pures Appl., **71** (1992), 503-526.

[W1] L.F. Wu, *The Ricci flow on complete* R^2, Comm. in Analysis and Geometry, **1** (1993), 439-472.

[W2] ———, *A new result for the porous medium equation*, Bull. Amer. Math. Soc., **28** (1993), 90-94.

[Z] H. Zhang, *On a nonlinear singular diffusion problem: Convergence to a travelling wave*, Nonlinear Analysis, TMA, **19**(**12**) (1992), 1111-1120.

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