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COVERING A CONVEX BODY BY ITS NEGATIVE HOMOTHETIC COPIES

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We estimate the number and ratio of negative homothetic copies of a d-dimensional convex body C sufficient for the covering of C. If the number of those copies is not very large, then our estimates are better than recent estimates of Rogers and Zong. Particular attention is paid to the 2-dimensional case. It is proved that every planar convex body can be covered by two copies of ratio $-\frac{4}{3}$ (this ratio cannot be lessened if C is a triangle).

Every convex body C in Euclidean d-space E^d can be covered by a homothetic copy of C of ratio $-d$. This immediately follows from the papers of Neumann [**10**] for $d = 2$, and Süss [**12**] in the general case. The covering by more than one negative homothetic copy was considered in [**3**], [**9**] and [**11**]. The present paper establishes a few additional estimates about covering by negative copies. We also consider covering by negative and positive homothetic copies.

1. Covering a d-dimensional body.

Lemma. *Let P be a parallelotope of the smallest possible volume containing a convex body $C \subset E^d$. Denote by $\mathbf{v}_1, \dots, \mathbf{v}_d$ the vectors determined by some d edges of P with a common origin. Let $\lambda_1, \dots, \lambda_d$ be positive real numbers such that $\lambda_1 + \cdots + \lambda_d = 1$. The body C contains a parallelotope S whose d edges with a common origin determine vectors $\lambda_1\mathbf{v}_1, \dots, \lambda_d\mathbf{v}_d$.*

Proof. From the considerations of [**4**] it follows that for every $i \in \{1, \dots, d\}$ there are boundary points a_i and b_i of C such that $\overrightarrow{a_ib_i} = \mathbf{v}_i$. The required parallelotope S has the 2^d vertices of the form $\lambda_1c_1 + \cdots + \lambda_dc_d$, where $c_i \in \{a_i, b_i\}$ for $i = 1, \dots, d$. □

By a *box* in E^d we understand any set of the form

$$\{(x_1, \dots, x_d);\ r_j \le x_j \le s_j \text{ for } j = 1, \dots, d\},$$

where $r_j < s_j$ for $j = 1, \dots, d$. In particular, if $r_1 = \cdots = r_d = 0$ and $s_1 = \cdots = s_d = 1$, we obtain the unit cube I^d.

Theorem 1. *Assume that the d-dimensional unit cube I^d can be covered by boxes $B_1, \dots, B_k$ and denote by p_{jm} the length of an edge of B_j parallel to the m-th coordinate axis, where $j \in \{1 \dots, k\}$ and $m \in \{1, \dots, d\}$. Then every convex body $C \subset E^d$ can be covered by k homothetic copies of C whose homothety ratios are $r_1, \dots, r_k$, where $|r_j| = p_{j1} + \dots + p_{jd}$ for $j = 1, \dots, k$.*

Proof. Let P denote a parallelotope of the smallest possible volume containing C. We do not make our considerations narrower by assuming that $P = I^d$ (if $P \neq I^d$, then we take an affine transformation τ such that $\tau(P) = I^d$ and we consider the body $C' = \tau(C)$ instead of C). We apply the Lemma. For each $j \in \{1, \dots, k\}$, take the numbers $\frac{p_{j1}}{|r_j|}, \dots, \frac{p_{jd}}{|r_j|}$ in place of $\lambda_1, \dots, \lambda_d$, respectively. We see that for every $j \in \{1, \dots, k\}$, the body C contains a parallelotope S_j, whose d independent edges determine vectors $\frac{p_{j1}\mathbf{v}_1}{|r_j|}, \dots, \frac{p_{jd}\mathbf{v}_d}{|r_j|}$. Hence for every $j \in \{1, \dots, k\}$, the set $r_j S_j$ is a translate of B_j. Thus B_j is a homothetic copy of S_j, where the ratio of the corresponding homothety h_j is equal to r_j. Since $S_j \subset C \subset P$ for $j = 1 \dots, k$, we conclude that C can be covered by homothetic copies $h_1(C), \dots, h_k(C)$ of C. □

The earlier mentioned covering by one copy of ratio $-d$ follows immediately from Theorem 1 by taking $k = 1$ and $B_1 = I^d$.

Consider two special cases of Theorem 1. Just put $k = 2^q$, where $q \in \{1, \dots, d\}$, and $p_{1\,m} = \dots = p_{k\,m} = \frac{1}{2}$ for $m \leq q$ and $p_{1\,m} = \dots = p_{k\,m} = 1$ for $m > q$. For the second special case take $k = t^d$ and $p_{j\,m} = \frac{1}{t}$ for all indexes, where $t \in \{1, 2, \dots\}$. We obtain the following corollary.

Corollary 1. *Every convex body in E^d can be covered by 2^q homothetic copies of ratio $-d + \frac{1}{2}q$ for every $q \in \{0, 1, \dots, d\}$. It can be also covered by t^d homothetic copies of ratio $-\frac{d}{t}$ for every $t \in \{1, 2, \dots\}$.*

A particular case of both statements of Corollary 1 is when we cover a convex body by 2^d homothetic copies of ratio $-\frac{1}{2}d$. Another particular case of the first statement is about covering by two homothetic copies of ratio $-d + \frac{1}{2}$.

Similarly, we can evaluate the homothety ratio for the covering by any particular number of negative copies (see Corollary 4 for such a general formula in E^2). For instance, every convex body in E^d can be covered by 3 homothetic copies of ratio $-d + \frac{3}{4}$. This follows by taking $p_{1\,1} = p_{2\,1} = \frac{1}{2}$, $p_{1\,2} = p_{2\,2} = \frac{3}{4}$, $p_{3\,2} = \frac{1}{4}$, and $p_{i\,j} = 1$ in remaining cases.

We conjecture that every convex body in E^d can be covered by two negative homothetic copies of ratio $-d+1$ for d odd, and of ratio $-d+1-\frac{1}{d+1}$ for d even. Those values are attained for a d-dimensional simplex, as a simple but time consuming calculation shows. Let us present only a hint of how the two negative copies S_1 and S_2 are situated. If d is odd, then S_2 is a translate of S_1 by vector $\frac{1}{d-1}\overrightarrow{ab}$, where a and b are the centroids of two opposite $\frac{d-1}{2}$-dimensional faces of S_1. If d is even, then S_2 is a translate of S_1 by vector $\frac{1}{d}\overrightarrow{ab}$, where a is the centroid of a $\frac{d}{2}$-dimensional face of S_1 and b is the centroid of the opposite $\frac{d-2}{2}$-dimensional face.

The estimates of Corollary 1 can be also expressed in the following form, where $\lceil x \rceil$ means the smallest integer which is greater than or equal to x.

Corollary 2. *Let $C \subset E^d$ be a convex body. If $-d \le \lambda \le -\frac{1}{2}d$, then some*

$$2^{\lceil 2d+2\lambda \rceil} \tag{1}$$

homothetic copies of C with ratio λ cover C. If $-\frac{1}{2}d \le \lambda < 0$, then

$$\left\lceil -\frac{d}{\lambda} \right\rceil^d \tag{2}$$

homothetic copies of ratio λ cover C.

If the number of equal negative homothetic copies is not very large, then the estimates (1) and (2) are better than the estimate

$$\left(1 - \frac{1}{\lambda}\right)^d (d \log d + d \log\log d + 5d), \quad \text{where} \quad d \ge 3, \tag{3}$$

a special case of the formula (6) from the paper of Rogers and Zong [**11**]. It is easy to check that the estimates (1) and (2) remain better than (3), asymptotically as $d \to \infty$, for a polynomial number of negative covering copies. In other words, for $-\lambda$ of order $\log d$. A calculation shows that if λ is sufficiently small and if $d \le 8$, then (2) should be applied for obtaining better estimates than (3), and if $d \ge 9$, then (1) should be applied for this purpose.

Here is also a comparison of (2) with (3) for $d = 3$. By (2), every 3-dimensional convex body can be covered by $14^3 = 2744$ homothetic copies of ratio $-\frac{3}{14} = -0.2142\ldots$, while by (3) we need 2815 such copies. For $d = 3$ and $\lambda \le -\frac{3}{14}$ formula (2) always gives fewer copies than (3), while (3) gives fewer copies for $\lambda > -\frac{3}{14}$.

Corollary 3. *Every convex body in E^d can be covered by $d^d + 1$ homothetic copies of ratio $-1 + d^{-d}(d+1)^{-1}$. Any desired number of those copies can be exchanged for copies of ratio $1 - d^{-d}(d+1)^{-1}$.*

Proof. Let $x_k = \frac{1}{d} - \frac{1}{d^k(d+1)}$ for $k = 1, \ldots, d$, and $y_k = \frac{1}{d} + \frac{d-1}{d^k(d+1)}$ for $k = 2, \ldots, d$. It is easy to check that $(d-1)x_k + y_k = 1$ and that $(k-1)\frac{1}{d} + x_k + y_{k+1} + \cdots + y_d = 1 - d^{-d}(d+1)^{-1}$.

In order to apply Theorem 1, we will dissect the cube I^d into $d^d + 1$ convenient boxes. Here is how we provide the tiling. We represent I^d as the union of d horizontal strips of heights $x_d, \ldots, x_d, y_d$. We dissect each of the strips of height x_d into d^{d-1} boxes of successive widths $\frac{1}{d}, \ldots, \frac{1}{d}, x_d$.

At the second stage, the strip of height y_d is dissected into d strips by hyperplanes parallel to the $(d-1)$-st coordinate axis. The $(d-1)$-st widths of successive strips are $x_{d-1}, \ldots, x_{d-1}, y_{d-1}$. Each of the strips of the $(d-1)$-th width equal to x_{d-1} is dissected into boxes of successive widths $\frac{1}{d}, \ldots, \frac{1}{d}, x_{d-1}, y_d$.

Similarly, we make tilings in successive stages. At the $(d-k+1)$-st stage we get $(d-1)d^k$ boxes of successive widths $\frac{1}{d}, \ldots, \frac{1}{d}, x_k, y_{k+1}, \ldots, y_d$.

At the last, d-th stage, we obtain $d-1$ boxes of successive widths $x_1, y_2, \ldots, y_d$. The total number of boxes so obtained is $(d-1)d^{d-1} + (d-1)d^{d-2} + \cdots + (d-1)d + d + 1 = d^d + 1$. From the equalities at the beginning of the proof we see that the sum of the lengths of the independent edges of each box is $1 - d^{-d}(d+1)^{-1}$. Thus from Theorem 1 we obtain the claim of Corollary 3. □

In particular, when all the copies in Corollary 3 are positive, we obtain an estimate for the well known problem of Hadwiger [**5**] which asks if every convex body $C \subset E^d$ can be covered by 2^d smaller positive homothetic copies. For $d = 3$ we know only some estimates of the number of those copies C, see [**7**], [**8**] and [**11**]. For $d \geq 3$ the estimate of the number of copies of positive ratio smaller than 1 presented in Corollary 3 is better than the estimate from [**7**], but for $d = 3$ it is weaker than that from [**8**], and for $d \geq 6$ it is weaker than the estimate $\binom{2d}{d}(d \log d + d \log\log d + 5d)$ presented in [**2**], [**11**] and [**13**]. Thus here we get the best estimates 257 in E^4 and 3126 in E^5. The advantage of the estimate of Corollary 3 is that we have a universal ratio of homothety. Remember that such estimates with a universal homothety ratio were known only for $d \leq 3$: Every 2-dimensional convex body can be covered by 4 copies of ratio $\sqrt{2}/2$ (see [**6**]), and every 3-dimensional convex body can be covered by 24 copies of a universal positive ratio smaller than 1 (see [**8**]).

2. Covering a two-dimensional body.

Observe that every positive integer n can be represented either in the form $n = m^2 + k$, where m and k are positive integers such that $0 \leq k \leq m-1$, or in the form $n = m(m+1) + k$, where m and k are positive integers such that $0 \leq k \leq m$.

Corollary 4. *Every convex body in E^2 can be covered by n homothetic copies of ratio $\frac{-2m^2-2m+k}{m^2(m+1)}$ provided $n = m^2 + k$, where $0 \leq k \leq m-1$, and of ratio $\frac{-2m^2-3m+k-1}{m(m+1)^2}$ provided $n = m(m+1) + k$, where $0 \leq k \leq m$. Any desired number of those copies can be exchanged for copies with ratio of the opposite sign.*

Figure 1 shows the idea of Corollary 4 and how it results from Theorem 1. We consider here only n fulfilling $2^2 \le n \le 3^2$ in order to fix our attention. We see how homothety ratios $-1, -\frac{11}{12}, -\frac{5}{6}, -\frac{7}{9}, -\frac{13}{18}, -\frac{2}{3}$ are obtained for $n = 4, \dots, 9$, respectively.

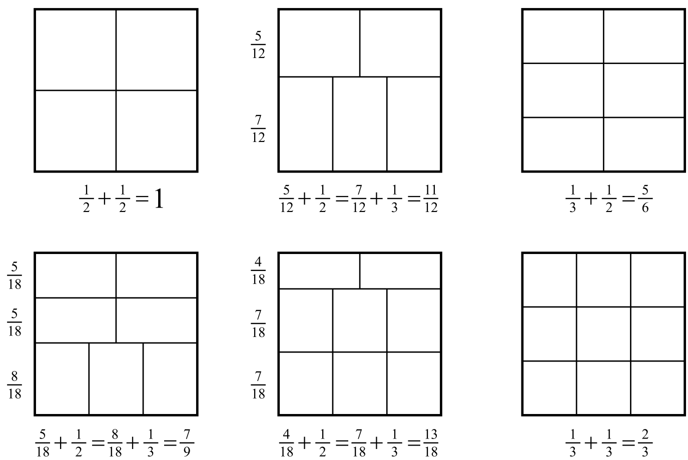

Figure 1.

We conjecture that a covering by 5 copies of ratio $-\frac{2}{3}$ is always possible (this value cannot be improved for a triangle). Better estimates than corresponding estimates for $n = 2, 3$ and 4 are obtained in [**3**] and [**9**]. They are $-\sqrt{2}$ for 2 copies, -1 for 3 copies (this ratio cannot be improved, as the example of a triangle shows), and less than -1 for 4 copies. Recall the conjecture from [**9**] that every planar convex body can be covered by 4 copies of ratio $-\frac{4}{5}$. Below we present improvements of the estimates for covering by 2 and by 7 negative copies. The example of a triangle shows that the following estimate $-\frac{4}{3}$ for covering by two copies is the best possible. Let us add that the estimate was conjectured in [**9**].

Theorem 2. *Every convex body $C \subset E^2$ can be covered by two homothetic copies of ratio $-\frac{4}{3}$.*

Proof. Let $C \subset E^2$ be a convex body. Let cde be a triangle contained in C with the greatest possible area. In order to simplify further computations, we will make some convenient assumptions. Since the affine image of this triangle is a triangle of maximum area in the corresponding transformed body, we loose no generality in assuming that $c = c(-1, 0), d = d(1, 0), e = e(0, 1)$. As usual, the numbers in brackets denote the coordinates of a given point. The triangle with vertices $t_1(0, -1), t_2(2, 1), t_3(-2, 1)$, contains C (see

Fig. 2). The reason is that the vertices of the triangle cde are in the sides of the triangle $t_1t_2t_3$ which has parallel sides (thus a point of C outside $t_1t_2t_3$ would permit the construction of a triangle of a greater area in C). Denote by o the centroid of the triangle cde. Let p_1, p_2, p_3 be the boundary points of C on the segments ot_1, ot_2, ot_3, respectively. Without loss of generality we can also assume that $|ot_3|/|op_3| \geq |ot_1|/|op_1|$ and $|ot_3|/|op_3| \geq |ot_2|/|op_2|$. If this assumption is not satisfied, we can apply an affine transformation which changes the order of the vertices c, d, e.

In order to shorten further explanations, we introduce the following notation. A homothetic copy of a set with the homothety ratio $-\frac{4}{3}$ will be called *a copy*. We say that a point is *on the left* (*on the right*) of a non-horizontal line L if its first coordinate is not greater (not smaller) than the first coordinate of the corresponding point of L on the same horizontal level. If a point is denoted by a symbol, then its first and second coordinates are denoted by x and y with just this symbol as the index.

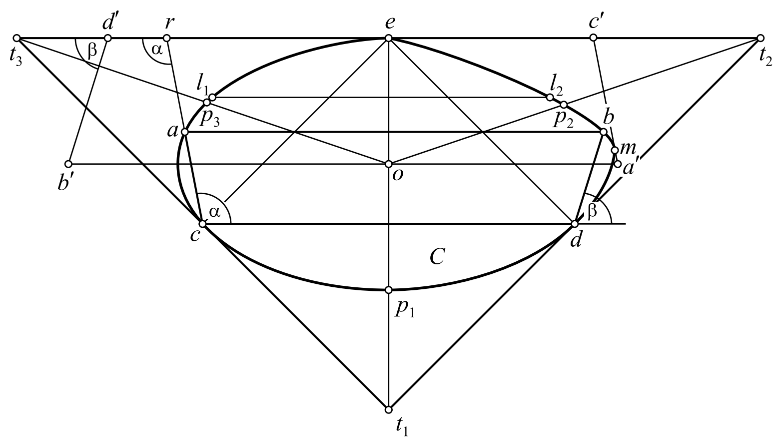

Figure 2.

Denote by C_1, C_2 and C_3 those parts of C whose points (x, y) fulfill the inequalities $y \leq \frac{1}{3}$, $\frac{1}{3} \leq y \leq \frac{2}{3}$ and $\frac{2}{3} \leq y$, respectively. Observe that the part of the triangle $t_1t_2t_3$ whose points (x, y) fulfill the inequality $y \leq \frac{1}{3}$ can be covered by one copy of the triangle cde. Thus C_1 can be covered by a copy of C. In order to prove that another copy of C is able to cover $C_2 \cup C_3$, it is sufficient to show that $C_2 \cup C_3$ can be covered by a copy of the trapezoid $cdba$, where $a(x_a, \frac{1}{2})$ and $b(x_b, \frac{1}{2})$ (with $x_a < x_b$) are points on the boundary of C (see Fig. 2). The required copy is the trapezoid $T = c'd'b'a'$, where the copy of the segment cd is the segment $d'c'$ contained in the line $y = 1$, such that all the points of C are on the left of the straight line containing $a'c'$, and such that a boundary point $m(x_m, y_m)$ of C belongs to the segment $a'c'$. Obviously, $-\frac{3}{2} \leq x_a \leq -\frac{1}{2}$ and $\frac{1}{2} \leq x_b \leq \frac{3}{2}$.

Let l_1, l_2 be the boundary points of C on the line $y = \frac{2}{3}$ (see Fig. 2). We have $|l_1 l_2| \leq 2$. Here is why. If $y_{p_1} \geq -\frac{1}{3}$, then $x_{p_2} \leq 1$ and $x_{p_3} \geq -1$ which implies $|l_1 l_2| \leq 2$. Also $y_{p_1} < -\frac{1}{3}$ gives $|l_1 l_2| \leq 2$ since the opposite leads to the conclusion that the area of the triangle $l_1 l_2 p_1$ is greater than the area of the triangle cde.

Case 1: *When* $y_m \leq \frac{2}{3}$.

First we show that $C_2 \subset T$.

Take the point $s(x_s, \frac{1}{3})$ on the straight line through a and e and the point $t(x_t, \frac{1}{3})$ on the straight line through b and e. Let $b^- a^- c^- d^-$ be the copy of the trapezoid $bacd$ with $a^- = t$ and $b^- = s$. The trapezoid $b^- a^- c^- d^-$ covers C_2. Obviously, $x_m \leq x_t$. Consequently, $C_2 \subset T$.

Now we show that $C_3 \subset T$.

Denote by α the angle $\angle acd$ and let $\beta = 180^o - \angle cdb$. We can assume that $\alpha \geq \beta$. The reason is that if $\alpha < \beta$ and if C_3 is not a subset of T, then the convexity of C implies that $a \notin T$, a contradiction to the inclusion $C_2 \subset T$.

Take the point $r(x_r, 1)$ on the straight line through a and c. Since $\alpha \geq \beta$, in order to show the inclusion $C_3 \subset T$, it is sufficient to show that $x_r \geq x_{d'}$ (see Fig. 2). The rest of Case 1 is devoted to this aim.

We omit an elementary calculation which gives $x_m \leq \frac{5}{3}$ and

$$x_r - x_{d'} = \frac{5}{3} + (2x_a + 2)y_m - x_m. \tag{4}$$

If $x_a > -1$, then $x_r - x_{d'} \geq \frac{5}{3} + (2x_a + 2)y_m - \frac{5}{3} > 0$. Thus we need consider only the case when $x_a \leq -1$.

First assume that $x_b < 1$. If $y_m \leq \frac{1}{2}$, then $x_m \leq x_t = \frac{4}{3}x_b$. Moreover, m is on the left of the straight line through e and b. Thus, $y_m \leq 1 - \frac{x_m}{2x_b}$ and by (4) we get $x_r - x_{d'} \geq \frac{5}{3} + (2x_a + 2)(1 - \frac{x_m}{2x_b}) - x_m \geq \frac{2}{3}x_a - \frac{4}{3}x_b + \frac{7}{3} > 0$. If $y_m > \frac{1}{2}$, then $x_m \leq x_b < 1$. Putting $x_m = 1, y_m = \frac{2}{3}$ and $x_a = -\frac{3}{2}$ in (4) we obtain $x_r - x_{d'} > 0$.

Next assume that $x_b \geq 1$. Consider three subcases.

Subcase 1: *When* $\frac{1}{2} < y_m \leq \frac{2}{3}$.

Assume that C_3 is not a subset of T. Then the point $u(x_u, \frac{2}{3})$ from the segment cd' belongs to C. The point m is on the right of the straight line parallel to the segment ac and passing through b, since otherwise we have the false conclusion that $b \notin T$. Thus $x_m \geq (y_m - \frac{1}{2})(2x_a + 2) + x_b$. Of course, the point $v(x_v, \frac{2}{3})$ from the segment em belongs to C. We obtain

$$|uv| = \frac{x_m}{3 - 3y_m} - \frac{4}{3}x_a + \frac{4}{3}y_m(x_a + 1) - \frac{2}{3}x_m + \frac{7}{9}.$$

Taking $(y_m - \frac{1}{2})(2x_a + 2) + x_b$ in place of x_m and $\frac{1}{2}$ in place of y_m we obtain $|uv| > -\frac{2}{3}x_a + \frac{13}{9} > 2$. Since u and v are points on the segment $[l_1, l_2]$, we obtain a contradiction to the inequality $|l_1 l_2| \le 2$ shown before starting Case 1.

Subcase 2: *When* $\frac{1}{3} \le y_m \le \frac{1}{2}$ *and when* $x_a < \frac{1}{2}x_b^2 - \frac{2}{3}x_b - \frac{23}{24}$.

Assume that C_3 is not a subset of T. The point $w(\frac{2}{3}x_b, \frac{2}{3})$ from the segment eb belongs to C. Observe, that $|uw| = \frac{2}{3}x_b - \frac{4}{3}x_a + \frac{4}{3}y_m(x_a+1) - \frac{2}{3}x_m + \frac{7}{9}$. The point m is on the left of the straight line passing through e and b. Thus $y_m \le 1 - \frac{x_m}{2x_b}$. Let $q(x_q, y_q)$ be the common point of the straight lines containing the segments eb and dt_2. We get $x_m \le x_q = \frac{4x_b}{2x_b+1}$, which together with our assumption $x_a < \frac{1}{2}x_b^2 - \frac{2}{3}x_b - \frac{23}{24}$ leads to the false inequality

$$|uw| \ge \frac{12x_b^2 - 16x_b - 24x_a - 23}{18x_b + 9} + 2 > 2.$$

Subcase 3: *When* $\frac{1}{3} \le y_m \le \frac{1}{2}$ *and when* $x_a \ge \frac{1}{2}x_b^2 - \frac{2}{3}x_b - \frac{23}{24}$.

Taking $y_m = \frac{1}{2}, x_m = x_q$ and $x_a = \frac{1}{2}x_b^2 - \frac{2}{3}x_b - \frac{23}{24}$ in (4) we obtain that $x_r - x_{d'} > 0$.

Case 2: *When* $y_m > \frac{2}{3}$.

The proof of the inclusion $C_2 \subset T$ is similar to Case 1.

Finally, we show that $C_3 \subset T$.

Assume the contrary. If $\alpha < \beta$, then $a \notin T$, a contradiction. Let $\alpha \ge \beta$. Obviously the point $z(x_z, \frac{2}{3}) \in md$ belongs to C. Observe that $u \in C$. We have

$$(5) \qquad |uz| = \frac{2}{3}x_m\left(\frac{1}{y_m} - 1\right) - \frac{2}{3y_m} - \frac{4}{3}x_a + \frac{4}{3}y_m(x_a+1) + \frac{16}{9}.$$

Of course, m is on the right of the straight line by e and d. Thus, $x_m \ge 1 - y_m$. Taking $x_m = 1 - y_m$, $x_a = -\frac{1}{2}$ and $y_m = \frac{2}{3}$ in (5) we get $|uz| > 2$, a contradiction. □

The following estimate $-\frac{2}{3}$ is better than the estimate $-\frac{7}{9}$ resulting from Corollary 4. We conjecture that the best possible ratio here is $-\frac{10}{17} = 0.5882\ldots$. It is easy to show that a triangle can be covered by 7 copies of ratio $-\frac{10}{17}$ and it cannot be covered if the negative ratio is over $-\frac{10}{17}$.

Proposition. *Every convex body* $C \subset E^2$ *can be covered by* 7 *homothetic copies of ratio* $-\frac{2}{3}$.

Proof. We can inscribe an affine-regular hexagon $H = abcdef$ in C (see [**1**]). Three of the lines containing the sides of H bound a triangle T_1 containing H and the other three a triangle T_2 containing H. Since H is inscribed in C, we see that $C \subset T_1 \cup T_2$. We can assume that the center of symmetry of H is the origin o of E^2. Of course, $-\frac{2}{3}H \subset -\frac{2}{3}C$. Thus in order to show the promised estimate, it is sufficient to cover $T_1 \cup T_2$ by 7 translates of $-\frac{2}{3}H$. Observe that these are $-\frac{2}{3}H$ and its translates by vectors $2\,\overrightarrow{om_i}$ for $i = 1, \ldots, 6$, where $m_1, \ldots, m_6$ are midpoints of the sides of $-\frac{2}{3}H$. □

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