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We extend the work done by Bolker and Roth in calculating the dimensions of the stress spaces of complete bipartite frameworks. We will present results which are analogous to those known for complete bipartite frameworks, yet hold for a much wider class of bipartite frameworks. The main results give the dimensions of the stress spaces for certain classes of frameworks, which are easily calculated using only the number of bars, the number of joints, and knowledge of the geometry of the specific realization of the framework.

1. Introduction.

A bar and joint framework in d-space is a pair (J, E), where J is an indexed set of points $\{a_1, a_2, a_3, \dots, a_v\}$ from $\mathbb{P}^d$ (projective d-space), called the joints, and E is a set of unordered pairs $\{\{a_i, a_j\}, \dots\}$ called the bars. This also defines an obvious graph associated with the bar and joint framework called the underlying graph.

Definition 1.1. A *bipartite framework* is a framework whose underlying graph is bipartite.

For the remainder of this paper we use K_{mn} to denote both the complete bipartite framework and the underlying complete bipartite graph K_{mn}.

One reason the rigidity of bipartite frameworks is of such interest is because they are examples of frameworks that contain no triangles. In fact, around the turn of the century it was known that the framework K_{33} in the plane was stress-free if the joints were in generic position; yet if the 6 joints of the framework fell on a conic, there would be a stress and the framework would become nonrigid or flexible [1]. Moreover, in 1978 Whiteley made a similar conjecture for K_{46} in 3-space. In 1980, Bolker and Roth wrote the paper "When is a bipartite graph a rigid framework?" [1] which gives a formula for the dimension of the stress space of any realization of a complete bipartite framework in d-space. The result of Bolker and Roth will be stated after some notation is presented. The purpose of this paper is to generalize the results of Bolker and Roth to some classes of bipartite frameworks which are not complete.

We study the dimension of the stress space because the existence of a stress implies that some bars are redundant. Identifying the number of redundant bars is crucial to determining the infinitesimal rigidity of a framework. For more on combinatorial rigidity one should see [5, 4, 2, 6, 7].

For a bipartite framework G with independent vertex sets A and B, a stress is a real valued function λ_{ab} on the bars such that

1. $\sum_{a\in A} \lambda_{ab}ab = 0$ for all $b \in B$
2. $\sum_{b\in B} \lambda_{ab}ab = 0$ for all $a \in A$

where $\lambda_{ab} = 0$ when $\{a, b\} \notin G$ and ab denotes the Plücker coordinates of the wedge product of the points a and b from $\mathbb{P}^d$.

We denote the stress space of a bipartite framework G by Ω_G. Hence Ω_G is the space of real $|A| \times |B|$ matrices satisfying Equations 1 and 2, given above, and having zeros in the prescribed positions.

2. Notations.

We have had to generalize the notation of Bolker and Roth to make sense in this more general setting. For simplicity most of the needed notation is stated here.

Let G be a bipartite framework and $\lambda \in \Omega_G$. Denote $\rho_a = \sum_{b\in B} \lambda_{ab}$ for every $a \in A$ and $\gamma_b = \sum_{a\in A} \lambda_{ab}$ for every $b \in B$, respectively called the row sums and column sums of a stress.

For each $b \in B$ define $A_b(G) = \{a \in A|\{a, b\} \in G\}$ and for each $a \in A$ define $B_a(G) = \{b \in B|\{a, b\} \in G\}$.

Given a bipartite framework on the vertex sets $A = \{a_1, a_2, a_3, \ldots , a_m\}$ and $B = \{b_1, b_2, b_3, \ldots , b_n\}$ define the linear map $\tau_G : \Omega_G \to \mathbb{R}^{m+n}$ by

$$\tau_G(\lambda) = (\rho_{a_1}, \rho_{a_2}, \rho_{a_3}, \ldots , \rho_{a_m}, \gamma_{b_1}, \gamma_{b_2}, \gamma_{b_3}, \ldots , \gamma_{b_n}).$$

Much of our work will be done in projective $d-$space. To simplify matters, unless otherwise stated, we will use the standard homogeneous coordinates for points from $\mathbb{P}^d$. The standard homogeneous coordinates for a nonzero point $x \in \mathbb{P}^d$ with $x = (x_1, x_2, \ldots , x_{d+1})$ is the coordinates of $\frac{1}{x_j}x$ where x_j is the value of latest nonzero entry of x.

Let $S = \{s_1, s_2, s_3, \ldots , s_t\} \subset \mathbb{P}^d$; fix the homogeneous coordinates of the points of S; and define

$$\mathbb{D}(S) = \left\{ (\alpha_1, \alpha_2, \alpha_3, \ldots , \alpha_t) \Big| \sum_{i=1}^{t} \alpha_i s_i = 0 \right\} \subset \mathbb{R}^t.$$

Note that the t-tuple of zeros is in $\mathbb{D}(S)$. Furthermore, $\mathbb{D}(S)$ is closed under addition and scalar multiplication and is therefore a vector space over $\mathbb{R}$. We call $\mathbb{D}(S)$ the vector space of dependencies of S.

We also will make use of the Kronecker product of an l-tuple with a k-tuple which is defined as follows: Given two points

$$a = (a_1, a_2, a_3, \dots, a_l) \in \mathbb{P}^{l-1}$$

and

$$b = (b_1, b_2, b_3, \dots, b_k) \in \mathbb{P}^{k-1},$$

using the standard homogeneous coordinates, define $a \otimes b$ to be the $l \times k$ matrix M with $M_{ij} = a_i \cdot b_j$. Moreover, given two sets $E \subset \mathbb{P}^{l-1}$ and $F \subset \mathbb{P}^{k-1}$ we use $E \otimes F$ to denote the vector space of finite linear combinations of elements of the form $e \otimes f$ with $e \in E$ and $f \in F$.

Finally, we will use the following notations:

$$\mathbb{D}(C_G^2) = \mathbb{D}(\{c \otimes c | c \in C_G\}),$$

where C_G is given by

$$C_G = \{a \in A | a \in \langle B_a(G) \rangle\} \cup \{b \in B | b \in \langle A_b(G) \rangle\}.$$

Now we may state the Bolker and Roth result for bipartite frameworks.

Let K_{mn} be a complete bipartite framework on the vertex sets A and B such that $|A| = m$ and $|B| = n$. Define $C = (A \cap \langle B \rangle) \cup (\langle A \rangle \cap B)$. Note that $C = C_{K_{mn}}$.

Theorem 2.1. $\dim(\Omega_{K_{mn}}) = \dim(\mathbb{D}(A)) \dim(\mathbb{D}(B)) + \dim(\mathbb{D}(C^2))$.

In order to apply these results on the stress spaces of general bipartite frameworks we must determine $\dim(\mathbb{D}(C_G^2))$. For points in $\mathbb{P}^2$ Crapo has a nice geometric interpretation of $\dim(\mathbb{D}(C_G^2))$ [3]. The table below summarizes these results for $C \subset \mathbb{P}^2$

$\dim \mathbb{D}(C^2)$	Realization and cardinality of the set C
$\lvert C\rvert - 6$	$\lvert C\rvert \geq 6$ in general position
$\lvert C\rvert - 5$	$\lvert C\rvert \geq 5$ points fall on a conic
$\lvert C\rvert - 4$	$\lvert C\rvert \geq 4$ points fall on four distinct points, no three collinear or all fall on a line and a point off the line
$\lvert C\rvert - 3$	$\lvert C\rvert \geq 3$ fall on three non-collinear points or on a line
$\lvert C\rvert - 2$	$\lvert C\rvert \geq 2$ fall on two points
$\lvert C\rvert - 1$	$\lvert C\rvert \geq 1$ fall on one point

Table 1.

In general, it is complicated to calculate $\dim(\mathbb{D}(C_G^2))$ when the points of C_G lie in $\mathbb{P}^d$ with $d > 2$.

3. Some Introductory Results.

For the remainder of this paper we will use the sets A and B to indicate the vertex sets of a given bipartite framework.

As in the Bolker and Roth paper we find the dimension of the stress space Ω_G by calculating both the $\dim(\ker(\tau)_G)$ and $\dim(\mathrm{Im}(\tau)_G)$.

With this end in mind, we state the following three lemmas. They are stated separately from the main results because they hold for all bipartite frameworks G.

Lemma 3.1. *If G is a bipartite framework, then* $\ker(\tau)_G \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$.

Proof. Let $\lambda \in \Omega_G$ and assume $\lambda \in \ker(\tau)_G$. Hence λ satisfies the following:

1. For every $b \in B$ we have $\sum_a \lambda_{ab} a = 0$ and,
2. For every $a \in A$ we have $\sum_b \lambda_{ab} b = 0$.

The above shows that every row of λ is an element of $\mathbb{D}(B)$ and every column of λ is an element of $\mathbb{D}(A)$. We now prove $\lambda \in \mathbb{D}(A) \otimes \mathbb{D}(B)$.

Define an ordering of the entries M_{ij} of an $m \times n$ matrix M, according to the following ordering on the indices: M_{ij} is earlier than M_{kl} if

1. $i < k$ or
2. $i = k$ and $j < l$.

Let λ_{ij} be the earliest nonzero entry of λ and construct the following element of $\mathbb{D}(A) \otimes \mathbb{D}(B)$: Let $\mu_B = (\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in}) \in \mathbb{D}(B)$ and let $\mu_A = (\lambda_{1j}, \lambda_{2j}, \dots, \lambda_{mj}) \in \mathbb{D}(A)$. Hence, $w = \frac{1}{\lambda_{ij}} \mu_A \otimes \mu_B \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ and therefore $N = \lambda - w$ is a matrix whose rows and columns are still elements of $\mathbb{D}(A)$ and $\mathbb{D}(B)$ respectively. Now we show $N_{kl} = 0$ if N_{kl} is earlier in the ordering than N_{ij}. Assume first $N_{kl} \neq 0$ with $N_{kl} < N_{ij}$. This implies that

$$w_{kl} = \frac{\lambda_{il}\lambda_{kj}}{\lambda_{ij}} \neq 0.$$

Therefore both $\lambda_{il} \neq 0$ and $\lambda_{kj} \neq 0$. Now, if $k < i$, then $\lambda_{kj} \neq 0$, which contradicts the fact that λ_{ij} was the earliest nonzero entry. On the other hand, if $k = i$ and $l < j$ then $\lambda_{il} \neq 0$ brings us to the same contradiction. Therefore, the earliest nonzero position of N appears later than N_{ij}. Continuing in this manner we write λ as a linear combination of elements from $\mathbb{D}(A) \otimes \mathbb{D}(B)$. Consequently we see $ker(\tau)_G \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$, as required. $\square$

This gives us an upper bound on $\dim(\ker(\tau)_G)$.

The following lemma is proved in Chapter 11 of Crapo's book [2]. It is independent of the framework G.

Lemma 3.2. *Assume that λ is a matrix $\lambda : A \times B \to \mathbb{R}$, ρ and γ are maps $\rho : A \to \mathbb{R}$ and $\gamma : B \to \mathbb{R}$, and $B_1 \subset B$ such that B_1 is a basis for the span of B. If*

(1) $\sum_a \lambda_{ab} a = \gamma_b b$ *for all* $b \in B - B_1$, *and*

(2) $\sum_b \lambda_{ab} b = \rho_a a \qquad$ *for all* $a \in A$,

then the following are equivalent:

(i) $\sum_a \lambda_{ab} a = \gamma_b b \qquad$ *for all* $b \in B_1$,

(ii) $\sum_a \rho_a a \otimes a = \sum_b \gamma_b b \otimes b$.

This lemma is used later when showing that $\mathbb{D}(C_G^2) \subset \mathrm{Im}(\tau)_G$.

The following lemma will give an upper bound for $\dim(\mathrm{Im}(\tau)_G)$.

Lemma 3.3. *Let G be a bipartite framework. Then* $\mathrm{Im}(\tau)_G \subset \mathbb{D}(C_G^2)$.

Proof. Let $\lambda \in \Omega_G$. Hence, we have

1. $\sum_a \lambda_{ab} a = \gamma_b b$,
2. $\sum_b \lambda_{ab} b = \rho_a a$

where $\lambda_{ab} = 0$ if $\{a, b\} \notin G$. Hence $\tau(\lambda) = (\rho_a, \dots, \gamma_b, \dots)$. Clearly λ satisfies conditions 1, 2, and *i* of Lemma 3.2. Therefore we may conclude

$$\sum_a \rho_a a \otimes a = \sum_b \gamma_b b \otimes b$$

and

$$\mathrm{Im}(\tau)_G \subset \mathbb{D}((A \cup B)^2).$$

We conclude from 1, above, that if $b \notin C_G$ then $\gamma_b = 0$. Similarly, 2 yields, if $a \notin C_G$ then $\rho_a = 0$. Hence $\tau(\lambda) \in \mathbb{D}(C_G^2)$and therefore $\mathrm{Im}\,(\tau)_G \subset \mathbb{D}(C_G^2)$. We note that zeros can always be added in the appropriate positions so that $\mathbb{D}(C_G^2) \subset \mathbb{D}((A \cup B)^2)$. ☐

4. Frameworks With Complete Bipartite Spanning Subframeworks.

We will be interested in realizations of bipartite frameworks on the vertex sets A and B which have the following property:

Definition 4.1. We say a bipartite framework G has a *complete bipartite spanning subframework* if there exists a complete bipartite subframework of G on the vertex sets A and $B_1 \subset B$, such that B_1 is a basis for $\langle B \rangle$.

Theorem 4.2. *Let G be a bipartite graph with a complete bipartite spanning subframework. Order the elements of the set B so that*

$$B_1 = \{b_{p+1}, b_{p+2}, b_{p+3}, \dots, b_n\},$$

then $\dim(\ker(\tau))_G = \sum_{j=1}^{p} \dim(\mathbb{D}(A_{b_j}(G)))$.

Proof. Take the following elements as a basis for $\mathbb{D}(B)$:

$$w_j = (0, 0, \dots, 1, \dots, 0, \beta_{p+1}, \beta_{p+2}, \dots, \beta_n)$$

where the 1 is in the j^{th} position. For each $j = 1, 2, 3, \dots, p$ choose $\{f_{1j}, f_{2j}, \dots, f_{k_j j}\}$, a basis for $\mathbb{D}(A_{b_j}(G))$ where $k_j = \dim(\mathbb{D}(A_{b_j}(G)))$. Here we mention that we can make each f_{ij} an m-tuple by adding the zeros in the appropriate positions. Thus, we will have $f_{ij} \in \mathbb{D}(A)$.

Let $\mathbf{D} = \{D_{ij}\}$ be the set of elements from $\mathbb{D}(A) \otimes \mathbb{D}(B)$ given by

$$D_{ij} = f_{ij} \otimes w_j.$$

We claim $\mathbf{D}$ is a basis for $\ker(\tau)_G$. It will be convenient to define the following: For any matrix M let $\mathrm{Col}_k(M)$ denote the k^{th} column of M. Note, by construction, $\mathrm{Col}_h(D_{ij}) = \mathbf{0}$ for $h \leq p$ except when $h = j$, in which case $\mathrm{Col}_j(D_{ij}) = f_{ij}$.

Clearly $|\mathbf{D}| = \sum_{j=1}^{p} \dim(\mathbb{D}(A_{b_j}(G)))$. We first show that $\mathbf{D}$ is independent. Assume

$$\sum_l \sum_i \nu_{il} D_{il} = \mathbf{0}.$$

Since $\mathrm{Col}_j(\sum_l \sum_i \nu_{il} D_{il}) = \sum_i \nu_{ij} f_{ij}$, we can conclude for each $j = 1, 2, 3, \ldots, p$ that we have $\sum_i \nu_{ij} f_{ij} = \mathbf{0}$. Therefore $\nu_{ij} = 0$ for every i and j because, for each j, $\{f_{1j}, f_{2j}, f_{3j}, \ldots, f_{k_j j}\}$ is independent. Hence, $\mathbf{D}$ itself is independent.

Now we show $\langle \mathbf{D} \rangle = \ker(\tau)_G$. By construction, every element of $\mathbf{D}$ has zeros in the positions i, j where $\{a_i, b_j\} \notin G$, hence for any coefficients ν_{ij} we have

$$\sum_{ij} \nu_{ij} D_{ij} \in \ker(\tau)_G.$$

Therefore, $\langle \mathbf{D} \rangle \subseteq \ker(\tau)_G$.

Conversely, let $\lambda \in \ker(\tau)_G$. We will show that there are coefficients ν_{ij} such that $\lambda = \sum_{ij} \nu_{ij} D_{ij}$. For any j with $1 \leq j \leq p$ we have $\mathrm{Col}_j(\lambda) \in \mathbb{D}(A_{b_j}(G))$. Therefore, there are coefficients ν_{ij} such that

$$\mathrm{Col}_j(\lambda) = \sum_{i=1}^{k_j} \nu_{ij} \mathrm{Col}_j(D_{ij}).$$

Hence, $\mathrm{Col}_j(\lambda - \sum_{i=1}^{k_j} \nu_{ij} D_{ij}) = \mathbf{0}$. Therefore, $\mathrm{Col}_j(\lambda - \sum_l \sum_{i=1}^{k_l} \nu_{il} D_{il}) = \mathbf{0}$ for $j = 1, 2, 3, \ldots, p$.

Let $E = \lambda - \sum_l \sum_{i=1}^{k_l} \nu_{il} D_{il}$. We see E is a linear combination of $m \times n$ matrices each having the property that their rows are elements of $\mathbb{D}(B)$. Therefore E itself has this property. But $\mathrm{Col}_j(E) = \mathbf{0}$ for $j = 1, 2, 3, \ldots, p$. Since each row of E is an element of $\mathbb{D}(B)$, which can be nonzero only on a basis of $\langle B \rangle$, we may conclude that $E = \mathbf{0}$. We have shown $\ker(\tau)_G = \langle \mathbf{D} \rangle$. Thus, $\dim(\ker(\tau)_G) = \sum_j \dim(\mathbb{D}(A_{b_j}(G)))$. $\square$

Theorem 4.2 gives the $\dim(\ker(\tau)_G)$ for any realization of a framework containing a complete bipartite spanning subframework. Next we find the $\dim(\mathrm{Im}(\tau)_G)$ for these frameworks.

Theorem 4.3. *A pair of vectors* $(\rho_a, a \in A)$ *and* $(\gamma_b, b \in B)$ *are row and column sums of a stress of a bipartite framework* G *with a complete spanning subframework if and only if*

(i) $\sum_a \rho_a a \otimes a = \sum_b \gamma_b b \otimes b$ *and*
(ii) $\rho_a = 0$ *if* $a \notin \langle B_a(G)\rangle$, $\gamma_b = 0$ *if* $b \notin \langle A_b(G)\rangle$.

Proof. Let $\lambda \in \Omega_G$. Then its row and column sums ρ_a and γ_b satisfy

(1) $\sum_a \lambda_{ab} a = \gamma_b b$ and
(2) $\sum_b \lambda_{ab} b = \rho_a a$.

Using Lemma 3.2 we see that property (i) holds for a stress of any bipartite framework.

Furthermore, since λ is a stress, it is clear from (2) that if $a \notin \langle B_a(G)\rangle$ then $\rho_a = 0$. Similarly, using (1) we find $\gamma_b = 0$ whenever $b \notin \langle A_b(G)\rangle$.

Conversely, assume (i) and (ii), and let $B_1 \subset B$ be the basis of $\langle B\rangle$ so that the subframework on the sets A and B_1 is a complete bipartite framework.

Note, for every $s \in B - B_1$, we know $\gamma_s s \in \langle A_s(G)\rangle$. Hence, for each $s \in B - B_1$, there exist scalars λ_{as} such that

$$\sum_a \lambda_{as} a = \gamma_s s$$

where $\lambda_{as} = 0$ if $\{a, s\} \notin G$.

Furthermore, since $\rho_a a \in \langle B_a\rangle \subset \langle B\rangle$ we have

$$\rho_a a - \sum_{s \in B - B_1} \lambda_{as} s \in \langle B\rangle \quad \text{for all} \quad a \in A.$$

Hence, for every $a \in A$, there exist scalars λ_{ax} with $x \in B_1$ such that

$$\sum_{x \in B_1} \lambda_{ax} x = \rho_a a - \sum_{s \in B - B_1} \lambda_{as} s.$$

Therefore,

$$\sum_{b \in B} \lambda_{ab} b = \rho_a a \quad \text{for all} \quad a \in A.$$

Again, using Lemma 3.2, we conclude that for all $x \in B_1$,

$$\sum_a \lambda_{ax} a = \gamma_x x.$$

Therefore λ is a stress with the required row and column sums. □

Corollary 4.4. *Let* G *be a bipartite framework with a complete bipartite spanning subframework. Order the elements of the set* B *so that*

$$B_1 = \{b_{p+1}, b_{p+2}, b_{p+3}, \ldots, b_n\}.$$

Then

$$\dim(\Omega_G) = \sum_{j=1}^{p} \dim \mathbb{D}(A_{b_j}(G)) + \dim \mathbb{D}(C_G^2).$$

Notice that in the case where G is a complete bipartite framework $A_{b_j}(G) = A$ for every $b_j \in B_2$ and $p = \dim \mathbb{D}(B)$. Hence,

$$\sum_{j=1}^{p} \dim \mathbb{D}(A_{b_j}(G)) = \dim(\mathbb{D}(A) \otimes \mathbb{D}(B))$$

and $C_G = (A \cap \langle B \rangle) \cup (\langle A \rangle \cap B)$. Therefore if G is a complete bipartite framework this theorem gives the same result as the theorem of Bolker and Roth.

Obviously, not every bipartite framework has a complete bipartite spanning subframework. In fact, one should note that having a complete bipartite spanning subframework is dependent upon the particular realization of the framework. For example, let G be the framework obtained by removing the two bars, $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from K_{44} realized in the plane. Furthermore, assume that the set A has no three points collinear and the set B has no three points collinear. One can easily check that, for this realization, G has no complete bipartite spanning subframework. On the other hand, if G is realized so that B is collinear, then there is a complete bipartite spanning subframework. Simply choose $B_1 = \{b_3, b_4\}$.

Now, we answer the question: Does the formula from Corollary 4.4 yield the proper dimension of the stress space for frameworks not satisfying its conditions? The next two examples give both a case when a framework G has no complete bipartite spanning subframework and Corollary 4.4 does not yield the proper dimension and a case when a different realization of G, still having no complete bipartite spanning subframework, has Corollary 4.4 yielding the proper dimension.

As above, let G be a framework obtained by removing the two bars $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from K_{44} realized in the plane. Furthermore let the joints of G be in generic position. In this case $\dim(\mathbb{D}(C_G^2)) = 2$ from Table 1. From this we would hope $\dim(\mathrm{Im}(\tau)_G) = 2$. However, one can check that this realization has $\dim(\mathrm{Im}(\tau)_G) = 1$.

Now, assume that G is realized, as seen in Figure 1, where we have no three points of the set A being collinear, no three points of the set B being collinear, the points $\{a_1, a_3, a_4, b_2, b_3, b_4\}$ are on one conic, and the points $\{a_2, a_3, a_4, b_1, b_3, b_4\}$ are on a different conic. Here we have $C_G = A \cup B$, $\dim(\mathbb{D}(C_G^2)) = 2$ and we can show, in this case, we do have $\dim(\mathrm{Im}(\tau)_G) = 2$.

Alternatively, there are some frameworks for which Corollary 4.4 applies to all but a few very special realizations. For example, if we let $G = K_{44}$ in the plane with only the bar $\{a_1, b_1\}$ removed, then every realization of G with distinct points, such that b_1 is in the span of $B - \{b_1\}$, has a complete bipartite spanning subframework. Therefore, we can calculate the dimension of the stress space for G in any of these realizations. In fact we can use Corollary 4.4 to predict the realizations for which Ω_G would change. For instance,

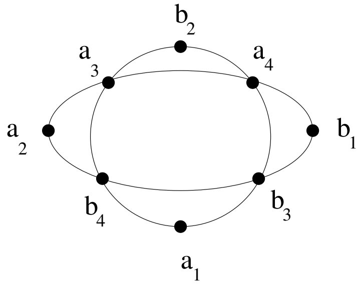

Figure 1. Special Realization of K_{44}.

if all the points of C_G fall on a circle, then we still have $\dim \mathbb{D}(A_{b_1}) = 0$ but now $\dim(\mathbb{D}(C_G^2)) = 3$.

5. Conclusion.

As one can see, the dimension of Ω_G may change dramatically for each realization of the framework G. Furthermore, the examples of the previous section show that the $\dim \mathbb{D}(C_G^2)$ is not capable of predicting these changes in frameworks with no complete bipartite spanning framework. Although there are techniques which allow us to calculate the dimension of the stress space of any bar and joint framework, there are still no results which yield the geometric insight that the Bolker and Roth's result and this result give for general bipartite frameworks.

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