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FUSS-CATALAN ALGEBRAS AND CHAINS OF
INTERMEDIATE SUBFACTORS

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FUSS–CATALAN ALGEBRAS AND CHAINS OF INTERMEDIATE SUBFACTORS

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We give a complete description of the generalized Fuss–Catalan algebras: colored generalizations of the Temperley–Lieb algebras, introduced by D. Bisch and V. Jones. For these chains of finite dimensional algebras, we describe a basis in terms of generators, and give a complete description, including the dimensions, of the irreducible representations.

We then consider an arbitrary subfactor containing a chain of intermediate subfactors. The higher relative commutants of a subfactor are an important tool for classifying the subfactor. We show the Fuss–Catalan algebras to be generically contained inside the higher relative commutants of the subfactor. Thus the Fuss–Catalan algebras provide an underlying structure for the higher relative commutants of any subfactor that contains a chain of intermediate subfactors.

1. Introduction.

Given an inclusion of type II_1 factors $N \subset M$ of finite index, the basic construction yields a tower of II_1 factors [Jo1]:

$$N \subset M \subset M_1 \subset M_2 \subset \dots$$

along with a sequence of projections e_1, e_2, e_3, \dots with $M_i = \{M_{i-1}, e_i\}''$ (and $M_1 = \{M, e_1\}''$). Intersecting this tower with the algebras N' and M' gives a double sequence of finite-dimensional algebras:

$$\begin{array}{ccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \dots \\ & & \cup & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & \dots \end{array}$$

known as the (opposite) *standard invariant* (see for example [Po1]). Much of the structure of $N \subset M$ (in some cases all of it [Po2]) is captured by the standard invariant, and therefore a full description of it is very desirable. It was pointed out in [Jo1] that $\{e_1, e_2, \dots, e_i\}'' \subset N' \cap M_i$ and $\{e_2, e_3, \dots, e_i\}'' \subset M' \cap M_i$ for any subfactor $N \subset M$ and thus a description of the algebras $\{1, e_1, e_2, \dots, e_i\}''$ provides an underlying structure for all standard invariants. These algebras, the Temperley–Lieb algebras, were fully described in this same paper [Jo1], giving the structure of the reduced

words, the Bratteli diagram for the inclusions, as well as the value of the trace.

D. Bisch and V. Jones analyzed the corresponding situation when a subfactor $N \subset M$ is known to have an intermediate subfactor $N \subset P \subset M$ [BiJo1]. Here, the presence of P leads to added structure (i.e., additional projections) in the standard invariant. This added structure was analyzed and shown to be strongly related (in the generic case isomorphic) to a “colored” string algebra termed the Fuss-Catalan algebra $FC_{k,n}$. A full analysis of these algebras was given, with the exception of describing a basis set. In addition, some of the work in [BiJo1] addressed the more general situation of a subfactor $N \subset M$ with a chain of k intermediate subfactors:

$$N \subset P^1 \subset P^2 \subset \dots \subset P^k \subset M$$

and it was stated that the main results can be generalized to this situation.

The following is this generalization. We analyze the structure imposed on the standard invariant by a chain of k intermediate subfactors. Three algebras are discussed:

- 1) An abstract algebra $AA_{k,n}$ that is defined by generators and relations,
- 2) The colored string algebras $FC_{k,n}$ introduced in [BiJo1],
- 3) The algebras $IA_{k,n}$ that are the concrete algebras generated by projections corresponding to the intermediate subfactors P^1, P^2, \dots, P^k .

Sections 2-5 define the algebras $AA_{k,n}$ and analyze their structure. Section 3 gives a spanning set of words. Sections 4 and 5 count these words by giving a correspondence with certain types of labeled planar trees and then counting these trees. Section 6 defines the algebra $FC_{k,n}$, giving a generating set. Section 7 puts this work together to establish the isomorphism between $AA_{k,n}$ and $FC_{k,n}$, and in the process establishes that the spanning set given for $AA_{k,n}$ is actually a basis. Section 8 mimics the elegant structural analysis for $FC_{1,n}$ done in [BiJo1], giving an inductive description of the Bratteli diagram for

$$FC_{k,0} \subset FC_{k,1} \subset FC_{k,2} \dots$$

Section 9 develops explicit formulas for the dimensions of the irreducible representations of $FC_{k,n}$. Section 10 again mimics work in [BiJo1], giving a description of the trace on $FC_{k,n}$. Section 11 establishes the connection with subfactors: defining $IA_{k,n}$ and establishing the isomorphism with $AA_{k,n}$ (and thus with $FC_{k,n}$) generically. In addition Section 11 addresses the non-generic case, showing $IA_{k,n}$ to be a quotient of $AA_{k,n}$.

It is worth pointing out the similarities and differences between what is contained in [BiJo1] and what is contained here. The work in [BiJo1], though mainly concerned with the $k = 1$ case, provided a very nice road map for what appears here. The broad structure of the argument in both papers is the same: defining $AA_{k,n}$ and $FC_{k,n}$, proving them isomorphic,

establishing the isomorphism (homomorphism in the non-generic case) to $IA_{k,n}$, and finally analyzing the structure of $FC_{k,n}$ (and hence the structure of $IA_{k,n}$). Sections 7, 8, 10, and 11 consist of arguments from [BiJo1] adapted to the general case. In addition, many details particular to the analysis of $FC_{k,n}$ (Section 8), and the analysis of the trace (Section 10), though presented in [BiJo1] only for $k = 1$, carry through merely by replacing statements with 2 variables to statements with k variables. These restatements are referenced to [BiJo1] and given without proofs.

What is completely new in this work falls into two main categories: the description of a basis for the algebras $AA_{k,n}$ (Sections 2 - 6) and the computation of the dimensions of the irreducible representations of $FC_{k,n}$ (Section 9) as well as the description of the Bratteli diagram for the algebras $FC_{k,n}$ (Theorem 10). The use of the basis for $AA_{k,n}$ establishes $AA_{k,n} \cong FC_{k,n}$ which had been conjectured in [BiJo1], and also significantly simplifies and clarifies the proof of the case $AA_{1,n} \cong FC_{1,n}$ [BiJo1]. Explicit formulas for the dimensions of the irreducible representations of $FC_{k,n}$ are given using combinatorial arguments. (This computation for $k = 1$ was done by other means in [BiJo2].)

Finally, because most of the material in Sections 1-10 does not involve subfactors, it is accessible to non-specialists. Moreover the combinatorial aspects of Sections 3 - 5 and 9 are of independent interest.

2. The Algebra $AA_{k,n}$.

We begin by defining an abstract algebra $AA_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ (this same algebra is considered independently in [DFr]).

Definition 1. Let $AA_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{C}$ be the abstract algebra generated by:

$$\{1, {}_aE_i\} \quad 1 \leq i \leq n - 1, \quad 0 \leq a \leq k,$$

subject to the relations:

0) $1 {}_aE_i = {}_aE_i 1 = {}_aE_i$

1) ${}_aE_i {}_bE_i = \max(a,b) E_i$

2) ${}_aE_i {}_bE_j = {}_bE_j {}_aE_i$ for $|i - j| \geq 2$ or
 $|i - j| = 1$ and
 $a + b < k$.

3) ${}_aE_i {}_bE_{i\pm 1} {}_aE_i = c(a, b) {}_aE_i {}_{k-a-1}E_{i\pm 1}$ for $a + b \geq k$ where
we set ${}_{-1}E_{i\pm 1} = 1$.

Here $c(a, b) = \frac{k-a-1c_{i+1} \ k-b-1c_i}{a^{c_i} \ b^{c_{i+1}}}$, with

$${}_a c_i = \begin{cases} \alpha_0 \alpha_1 \cdots \alpha_a & \text{for } i \text{ even,} \\ \alpha_k \cdots \alpha_{k-a} & \text{for } i \text{ odd,} \end{cases} \quad \text{and } {}_{-1} c_i = 1.$$

Remarks.

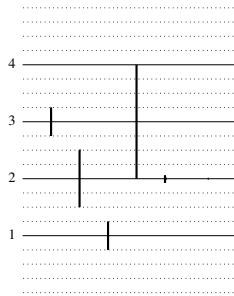
- 1) Relations 1) and 2) imply that ${}_a E_{kb} E_{k\pm 1a} E_k = {}_a E_{kb} E_{k\pm 1}$ for $a+b < k$.
- 2) Relations 1) and 3) imply ${}_a E_{kb} E_{k\pm 1d} E_k$ can always be reduced to a word with two or fewer terms.

Abacus Construction.

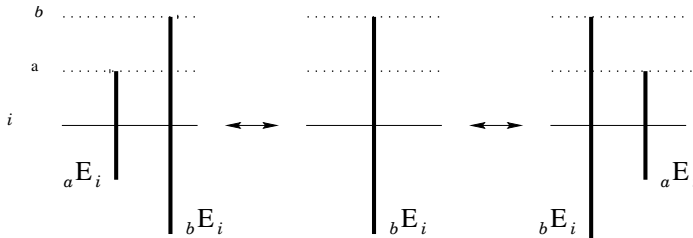
Abstractly these relations are slightly daunting, but there is a nice visualization of the words of this algebra. Consider an abacus with $n-1$ horizontal bars labeled from bottom to top by the integers 1 through $n-1$. In addition, have the bars of the abacus spaced k units apart. Now represent ${}_a E_j$ by a bead of radius a units, hanging on the j th horizontal bar. Thus the word

$${}_1 E_{32} E_{21} E_{14} E_{30} E_2 \in AA_{4,5}(\alpha_0, \dots, \alpha_4)$$

would be depicted by the following picture:

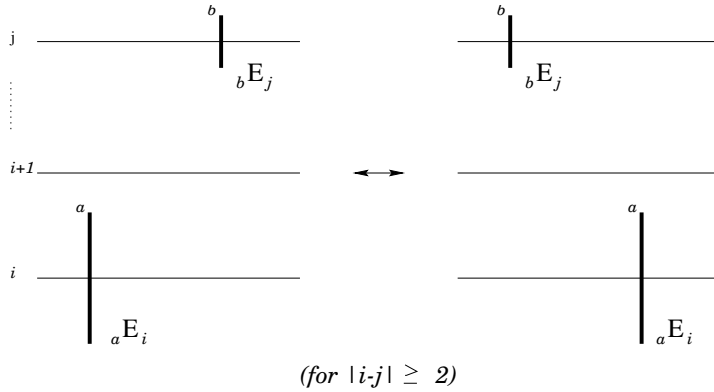


With this description, relation 1) has the following visual interpretation:

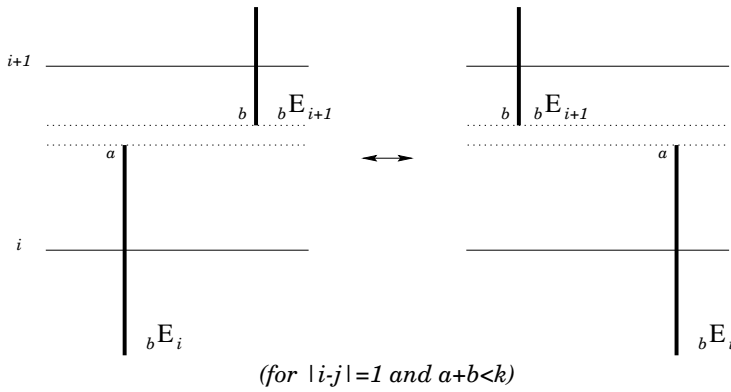


(assuming $a < b$).

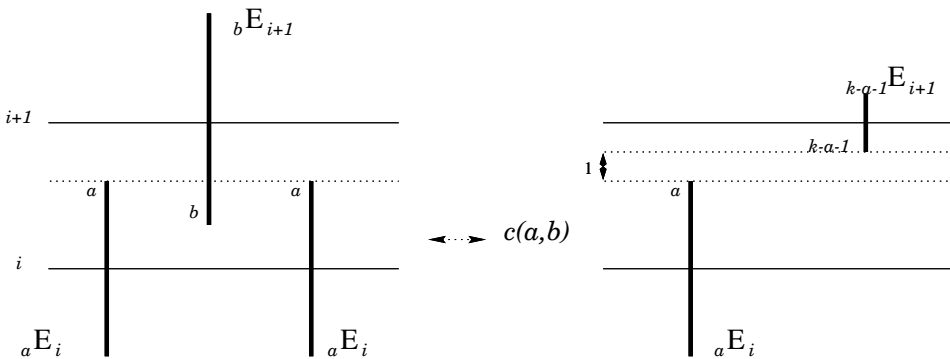
Relation 2) is given by:



and



Note. The abacus picture was motivated by interpreting this second relation. We allow two generators to commute (slide by each other) if there is space between their respective beads. We note that if there is no room (i.e., $a + b = k$), we do not allow them to slide by each other (commute). (If you like, imagine that the abacus is not that well made so that the beads are just a little larger than they are supposed to be.) Finally, relation 3) is visually interpreted as:



The abacus picture is very helpful for what follows, and the reader is encouraged to think of manipulating words in $AA_{k,n}(\alpha_0, \dots, \alpha_k)$ via these pictures.

We now analyze the structure of $AA_{k,n}$. In the next section, we will show that a certain subset of $AA_{k,n}$, defined to be the set of words of *reduced form*, linearly spans $AA_{k,n}$. In Sections 4 and 5 we will count the number of these reduced words. Later, in Section 7 we will show that the number of these words is equal to the dimension of $AA_{k,n}$ and that therefore the words of reduced form are a basis of $AA_{k,n}$.

3. A Spanning Set for $AA_{k,n}$.

Define a function s on the formal words in the $\{aE_k\}$ as follows:

$$\text{for } w = \prod_{i=1}^l a(i)E_{b(i)}, \quad \text{define } s(w) = \sum_{i=1}^l (a(i) + 1).$$

Note that we are defining s on the formal words, (i.e., the words in the free product of the generators), not on the elements of $AA_{k,n}$. We will call $s(w)$ the *size* of w .

Definition 2. A word w in the free product of the generators will be called *minimal* if $s(v) \geq s(w)$ for any $v \sim w$. Here we define $v \sim w$ to mean that as elements of $AA_{k,n}(\alpha_0, \dots, \alpha_k)$, $v = cw$ with c a nonzero scalar.

We introduce the notation $r_{M(i),m(i)}$ to mean a word of the form

$$r_{M(i),m(i)} = \prod_{j=M(i)}^{m(i)} a(i,j)E_j \quad \text{with } m(i) \leq M(i), \quad 0 \leq a(i,j) \leq k,$$

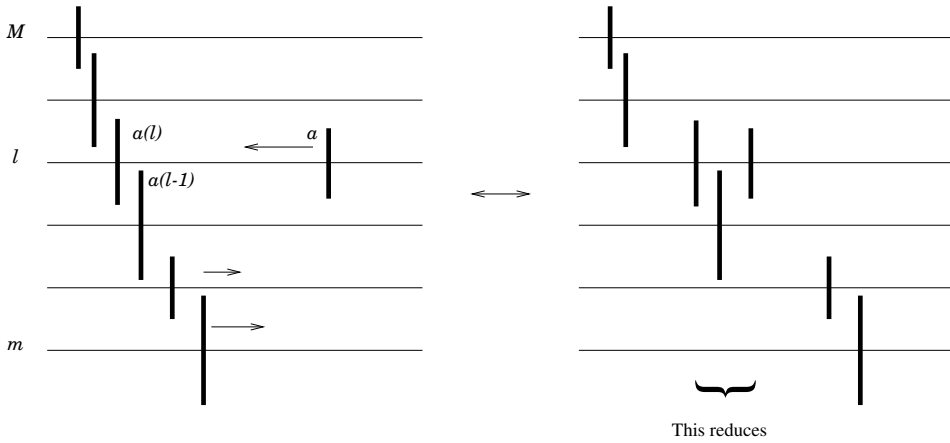
where we take the product in decreasing order from $j = M(i)$ to $j = m(i)$. Pictorially $r_{M(i),m(i)}$ represents a word with its first bead on the $M(i)$ th level, and each subsequent bead on the next lower level, with the last bead on the $m(i)$ th level.

Lemma 1. For $m(i) \leq l \leq M(i)$, the words $r_{M(i),m(i)} aE_l$ and $aE_l r_{M(i),m(i)}$ are not minimal.

Proof. For $l \neq m(i)$,

$$r_{M(i),m(i)} aE_l = \left(\prod_{j=M(i)}^{l+1} a(i,j)E_j \right) a(i,l)E_l \cdot a(i,l-1)E_{l-1} \cdot aE_l \left(\prod_{j=l-2}^{m(i)} a(i,j)E_j \right)$$

which is not minimal since $a(i,l)E_l a(i,l-1)E_{l-1} aE_l$ is not minimal by the remark after Definition 1. The abacus picture that corresponds to the above calculation is as follows:



For $k = m(i)$ we have

$$r_{M(i),m(i)} \cdot {}_a E_{m(i)} = \left(\prod_{j=M(i)}^{m(i)+1} a_{(i,j)} E_j \right) \cdot a_{(i,m(i))} E_{m(i)} a E_{m(i)}$$

which reduces by relation 1. By turning the pictures upside down we have a proof for ${}_a E_k \cdot r_{M(i),m(i)}$. \square

Lemma 2. *Given v minimal, then $v \sim w$ with $s(v) = s(w)$ and w of the form*

$$w = \prod_{i=1}^l r_{M(i),m(i)}, \quad \text{with}$$

$$\begin{aligned} M(1) &< M(2) < M(3) \dots, \\ m(1) &< m(2) < m(3) \dots \end{aligned}$$

Proof. Given $v = \prod_{i=1}^l a_{(i)} E_{b(i)}$, consider the related word \hat{v} in the Temperley-Lieb algebra: $\hat{v} = \prod_{i=1}^l e_{b(i)}$ [Jo1]. Now reduce \hat{v} using the Temperley-Lieb relations:

- 1') $e_i^2 = e_i$,
- 2') $e_i e_j = e_j e_i, \quad |i - j| \geq 2$,
- 3') $e_i e_{i \pm 1} e_i = \lambda e_i$.

Any use of 2') can be mirrored by the use of relation 2) on v . If the reduction uses 1') or 3') on \hat{v} , the corresponding use of relations 1), 3) or the remarks immediately following the definition of $AA_{k,n}$ will reduce the size of v , violating the minimality of v . Thus, using only relation 2'), \hat{v} can be put in the reduced form described in [Jo1, p. 14]. Mirroring these moves on v gives us a word of the form described in the statement of the lemma. \square

We give the following two definitions. For now we are interested in the definition of a *reduced block*. The definition of an *n-block* will facilitate a later discussion.

In what follows, l is an integer, m and M are maps from $\{1, 2, \dots, l\}$ to the integers and a is a map from $A = \{(i, j) : 1 \leq i \leq l, m(i) \leq j \leq M(i)\}$ to $\{0, 1, \dots, k\}$.

Definition 3. A word w in the generators of $AA_{k,n}$ is called a *block*, and denoted by (m, M, l, a) , if it is of the form:

$$w = \prod_{i=1}^l r_{M(i),m(i)} \quad \text{with}$$

- b1. $M(1) < M(2) < M(3) \cdots < M(l)$,
 $m(1) < m(2) < m(3) \cdots < m(l)$,
 $m(i + 1) \leq M(i)$.
- b2. $a(i, j) = a(i - 1, j - 2)$ whenever $a(i, j), a(i, j - 1), a(i - 1, j - 1), a(i - 1, j - 2)$ are all defined (i.e., $(i, j), (i, j - 1), (i - 1, j - 1), (i - 1, j - 2) \in A$).

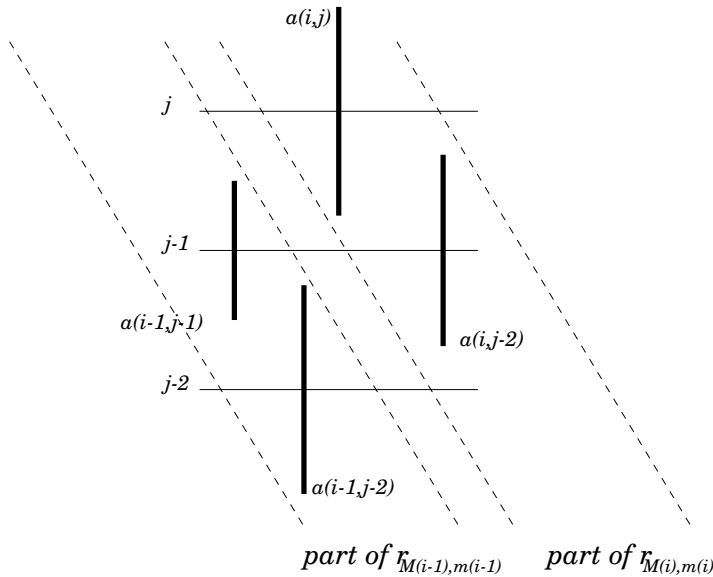
A block with $m(1) = 1$ and $M(l) = n$ is called an *n-block*.

Definition 4. A word w in the generators of $AA_{k,n}$ is called a *reduced block*, and denoted by (m, M, l, a) if it is a block and satisfies the additional condition:

- b3. $a(i, j) + a(i, j - 1) \geq k$,
 $a(i, j) + a(i + 1, j + 1) \geq k$.

Remark.

We present a brief visual description of the reduced blocks in the abacus picture. A reduced block is a sequence of descending strings of beads (each descending string being an $r_{M(i),m(i)}$ with beads starting at the $M(i)$ th level and ending at the $m(i)$ th level). Condition b1 tells us that the next descending string starts (resp. ends) at a higher level than the previous one starts (resp. ends). In addition, the ending of the next descending string cannot be higher than the start of the previous one. Condition b2 requires that if we ever see a diamond shape of beads as follows:



the top $(a(i, j))$ and bottom $(a(i - 1, j - 2))$ bead must have the same length. Condition b3 tells us that any bead at level j will be large enough to touch the closest beads to it that hang at the $j \pm 1$ level.

We now describe how to write a word in $AA_{k,n}$ as a product of reduced blocks. Suppose we have $w = \prod_{i=1}^l r_{M(i),m(i)}$ minimal, of the form of Lemma 2, and with the added requirement that l is maximal. Let $L = \{i : m(i+1) > M(i)\}$. Label the elements of L in increasing order l_1, l_2, \dots, l_d , so that $1 \leq l_1 < l_2 \dots l_d < l$. Define

$$\begin{aligned}
 w_1 &= \prod_{i=1}^{l_1} r_{M(i),m(i)}, \\
 w_i &= \prod_{i=l_{i-1}+1}^{l_i} r_{M(i),m(i)}, & \text{for } 2 \leq i \leq d, \\
 w_{d+1} &= \prod_{i=l_d+1}^l r_{M(i),m(i)}.
 \end{aligned}$$

Thus we have

$$w = \prod_{m=1}^{d+1} w_m.$$

Lemma 3. *The w_m as defined above are reduced blocks.*

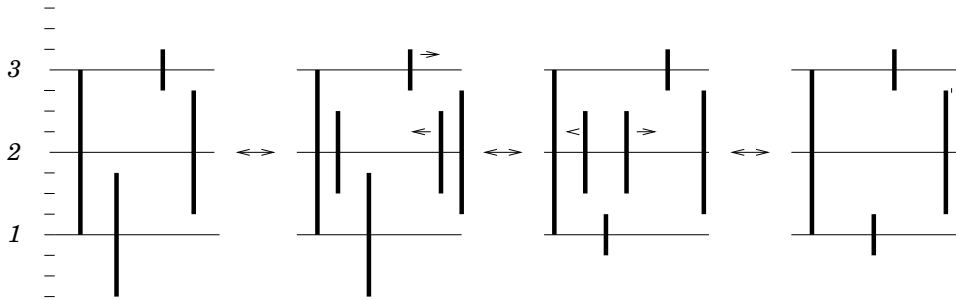
Proof. The w_m clearly satisfy conditions b1: the first two because w does, and the last due to the choice of the l_m . To show b2 we assume the existence

of $a(i, j)$, $a(i, j - 1)$, $a(i - 1, j - 1)$, $a(i - 1, j - 2)$. We can use the commuting relations to write:

$$w_i = \cdots a(i-1, j-1) E_{j-1} a(i-1, j-2) E_{j-2} a(i, j) E_j a(i, j-1) E_{j-1} \cdots$$

But this word is not of minimal size unless $a(i - 1, j - 2) = a(i, j)$. We demonstrate this with an example in $AA_{4,n}(\alpha_0, \alpha_1 \dots \alpha_4)$ along with its abacus interpretation.

$$\begin{aligned} & {}_4E_{23}E_{11}E_{33}E_2 \rightarrow ({}_4E_{22}E_2)_3E_{11}E_3({}_2E_{23}E_2) \\ & \rightarrow {}_4E_2({}_2E_{23}E_{12}E_2)_1E_{33}E_2 \rightarrow c_3(2, 3)_4E_2({}_2E_{21}E_1)_2E_{33}E_2 \\ & \rightarrow c_3(2, 3)_4E_{21}E_{11}E_{33}E_2 \end{aligned}$$



The same series of manipulations works on a general word of this form, though the notation becomes somewhat cumbersome.

Finally, for b3, suppose $a(i_0, j_0) + a(i_0, j_0 - 1) < k$. Thus $a(i_0, j_0)E_{j_0}$ commutes with $a(i_0, j_0 - 1)E_{j_0 - 1}$. Commuting $a(i_0, j_0 - 1)E_{j_0 - 1}$ to the left will contradict minimality of w (Lemma 1) unless $M(i_0 - 1) < j_0 - 1$. Commuting $a(i_0, j_0)E_{j_0}$ to the right will yield a contradiction by the same lemma unless $m(i_0 + 1) > j_0$. In this case, let

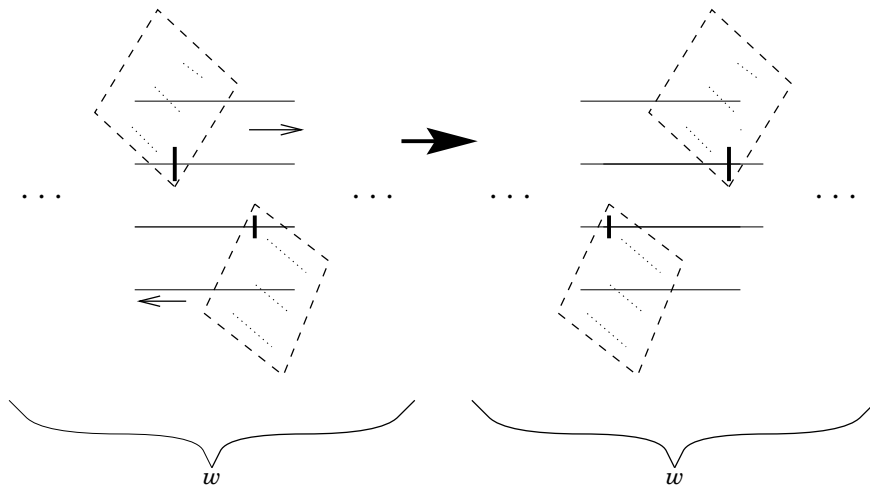
$$r_1 = \prod_{j=M(i_0)}^{j_0} a(i_0, j) E_j, \quad r_2 = \prod_{j=j_0-1}^{m(i_0)} a(i_0, j) E_j,$$

so that $r_{M(i_0), m(i_0)} = r_1 r_2$. Note r_1 and r_2 commute because $a(i_0, j_0) + a(i_0, j_0 - 1) < k$. Then we can rewrite

$$\begin{aligned} w = & w_1 w_2 \dots w_{m-1} \left(\prod_{j=l_{m-1}+1}^{i_0-1} r_{M(j), m(j)} \right) \\ & \cdot r_2 r_1 \left(\prod_{j=i_0+1}^{l_m} r_{M(j), m(j)} \right) w_{m+1} w_{m+2} \dots w_d \end{aligned}$$

which is of the desired form with one more term, thus contradicting the maximality of l .

To clarify, we include the corresponding picture:



The two beads drawn are ${}_{a(i_0, j_0+1)}E_{j_0+1}$ and ${}_{a(i_0, j_0)}E_{j_0}$. The two dotted boxes are meant to surround those beads that represent r_1 and r_2 . Because $M(i_0 - 1) < j_0 - 1$ and $m(i_0 + 1) > j_0$, the two boxes can slide by each other to give a configuration that contradicts the maximality of l , the number of r 's in w .

A slightly quicker argument holds for the second part of condition b3. Suppose $a(i, j) + a(i + 1, j + 1) < k$, then ${}_{a(i, j)}E_j$ and ${}_{a(i + 1, j + 1)}E_{j + 1}$ commute. Commuting ${}_{a(i, j)}E_j$ to the right and using Lemma 1 we get a contradiction on the minimality of w unless $m(i + 1) = j + 1$. Commuting ${}_{a(i + 1, j + 1)}E_{j + 1}$ to the left will again contradict minimality of w unless $M(i) = j$. But this is impossible since $M(i) = j < m(i + 1)$ means that ${}_{a(i, j)}E_j$ and ${}_{a(i + 1, j + 1)}E_{j + 1}$ would be in different w_i . \square

We are now ready to describe what will turn out to be a basis for $AA_{k, n}$.

Definition 5. A word w will be called in *reduced form* if:

- 1) $w = 1$,
- or
- 2) $w = \prod_{i=1}^d w_i$ with
 - a) the w_i reduced blocks,
 - b) $\min(w_{i+1}) > \max(w_i)$.

where $\min(w) = \min\{j : {}_aE_j \text{ appears in } w\}$,
 $\max(w) = \max\{j : {}_aE_j \text{ appears in } w\}$.

Definition 6. Let $S_{k, n}$ to be the set of words of reduced form inside $AA_{k, n}$.

So far we have shown that any word can be put in the above reduced form (Lemma 3). Thus $S_{k, n}$ is a spanning set for $AA_{k, n}$ and $\dim(AA_{k, n}) \leq |S_{k, n}|$. Our next task is to count these words.

Our plan of attack for counting words of reduced form is as follows:

- 1) Show a 1-1 correspondence between reduced blocks and certain pictures we call Kauffman (k,n) -diagrams.
- 2) Then establish a 1-1 correspondence between the Kauffman (k,n) -diagrams and planar (k,n) -trees with labeled vertices.
- 3) Finally, count the number of planar (k,n) -trees using a recursion relation. Combine this with the relationship between reduced blocks and elements of $S_{k,n}'$ to count the words of reduced form.

4. Kauffman Diagrams and Planar Trees.

In this section we give the 1-1 correspondences listed as items 1 and 2 in the above plan. First, we introduce the notion of a Kauffman diagram, an object defined in [Ka]:

Definition 7. Let D_n be a rectangle with n marked points on the top of the boundary and n marked points on the bottom. We define a *Kauffman n -diagram* to be a picture sitting inside D_n , consisting of n non-intersecting curves that begin and end at distinct marked boundary points. We will consider two such diagrams equal if they are isotopically equivalent (keeping the boundary fixed).

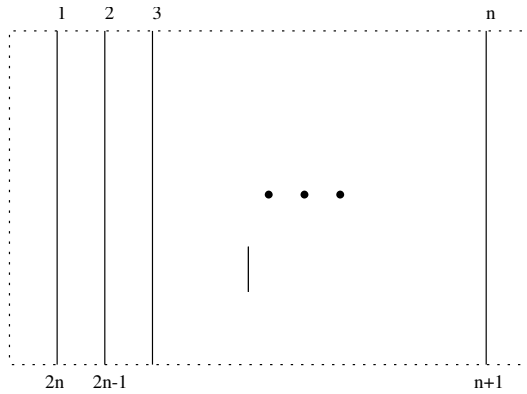
We note that a Kauffman diagram is completely characterized by which marked boundary points are joined together. For simplicity we label the marked boundary points 1 through $2n$ clockwise starting at the top left and ending at the bottom left. Then the Kauffman n -diagrams are in one-to-one correspondence with functions $J : [1, 2n] \rightarrow [1, 2n]$ with the following three properties:

- 1) $J(i) \neq i$,
- 2) $J^2 = \text{id}$,
- 3) If $i \leq j \leq J(i)$ then $i \leq J(j) \leq J(i)$.

We note that condition 3 characterizes the fact that the curves in a Kauffman diagram do not intersect (that is, the planarity of the Kauffman diagram). We will call a function J with the three properties listed above a *planar n -function*.

We note that the curves in a Kauffman n -diagram divide D_n into $n + 1$ pieces. We will call a *labeled Kauffman (k, n) -diagram* a Kauffman diagram equipped with a labeling of each of its $n + 1$ pieces by an integer in the interval $[0, k]$.

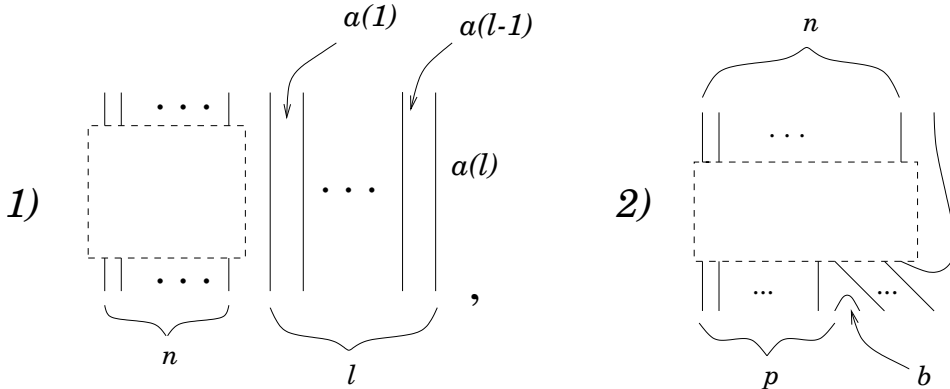
Lemma 4. *With the exception of the identity Kauffman n -diagram:*



all Kauffman diagrams have at least one curve connecting adjacent points on the bottom of D_n (i.e., there exists $i : n + 1 \leq i < 2n$ with $J(i) = i + 1$).

Proof. For a given Kauffman diagram consider the smallest $l \geq n + 1$ with $J(l) \neq 2n + 1 - l$. (If no such l exists then the diagram is the identity diagram.) The curve joining l to $J(l)$ divides D_n into two parts. Consider the part that includes the point $l + 1$. By choice of l , unless $J(l) = l + 1$ (in which case we are done), this part contains more points on the bottom row (i.e., with label $> l$) than points on the top row (i.e., with labels $< 2n + 1 - l$). Thus two points on the bottom must be joined together. Take a pair of points on the bottom that are joined together and that are minimal with respect to the number of points between them. By minimality, the points must be consecutive (since the planarity of Kauffman diagrams ensures that all the points between would be connected to each other). \square

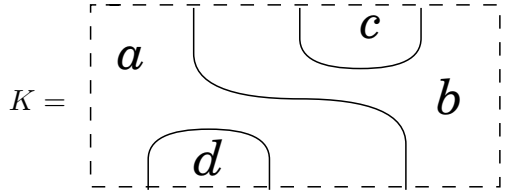
Consider the following two pictures:



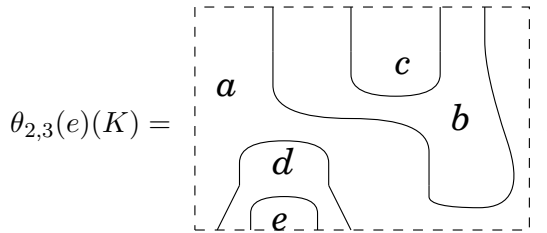
with $a(1), \dots, a(l), b$ integers in the interval $[0, k]$. Both can be thought of as operators on labeled Kauffman (k, n) -diagrams. For a given labeled Kauffman (k, n) -diagram K , place K inside the dotted rectangle aligning the marked boundary points with the strands leaving the rectangle. The resulting bigger rectangle is a labeled Kauffman $(k, n + l)$ -diagram in the

first case and a labeled Kauffman $(k, n + 1)$ -diagram in the second case. We will call the first operator $I_{l,n}(a(1), \dots, a(l))$ and the second $\theta_{p,n}(b)$. We provide an example for clarification:

Given



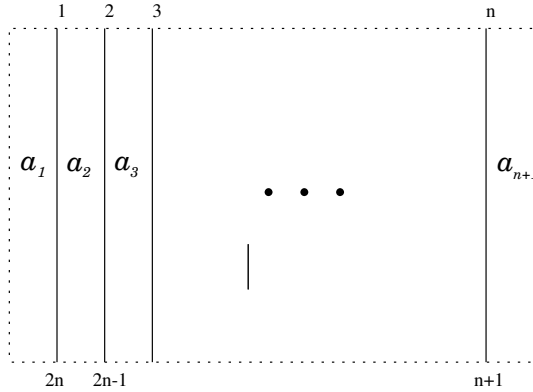
we have



We now establish a 1-1 correspondence between Kauffman (k, n) -diagrams and $n + 1$ - blocks (recall Definition 3). Let $\psi : \{ \text{Kauffman } n\text{-diagrams} \} \rightarrow \{ n + 1\text{- blocks} \}$ be inductively defined as follows:

Given a labeled Kauffman (k, n) -diagram K ,

- 1) If K is the Kauffman (k, n) -diagram:



then $\psi(K) = (m, M, l, a)$ with $l = 1, m(1) = 1, M(1) = n + 1, a(1, i) = a_i$.

- 2) Otherwise choose j maximal so that K is in the image of $I_{j,n-j}(\ast)$ (i.e., j is the number of vertical strands at the right of K). Write $K = I_{j,n-j}(a(1), \dots, a(j))(K')$ with labels $a(1), \dots, a(j)$ and Kauffman $(k, n - j)$ -diagram K' uniquely determined from K and j .
- 3) By Lemma 4, K' has two consecutive bottom boundary points that are connected and thus K' is in the image of $\theta_{p,n-j-1}(\ast)$ for at least

one p . Choose p maximal and write $K' = \theta_{p,n-j-1}(b)(L)$ with label b and Kauffman $(k,n-j-1)$ -diagram L uniquely determined from K' and p .

- 4) Set $\psi(L) = (m', M', l', a')$. Define $\psi(K) = (m, M, l, a)$ inductively as:
- $l = l' + 1$
 - $m(i) = m'(i), M(i) = M'(i)$ for $1 \leq i \leq l'$
 - $m(l) = p, M(l) = n$
 - $a(i', j') = a'(i', j')$ for $1 \leq i' \leq l'$
 - $a(l, j') = a(l - 1, j' - 2)$ for $p < j' \leq M'(l) + 1$
 - $a(l, p) = b$
 - $a(l, M'(l) + 1 + i) = a(i)$ for $1 \leq i \leq j$.

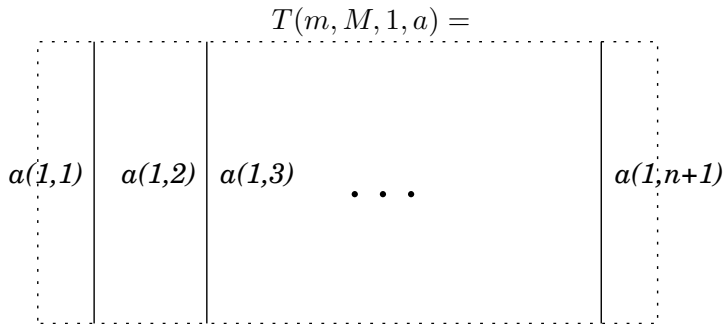
We define what will turn out to be the inverse of ψ . Let the map

$$T : \{ n + 1\text{-blocks} \} \rightarrow \{ \text{Kauffman } (k, n)\text{-diagrams} \}$$

be inductively defined as follows:

Given an $n + 1$ -block (m, M, l, a) ,

- 1) If $l = 1$ then set



- 2) Otherwise, let

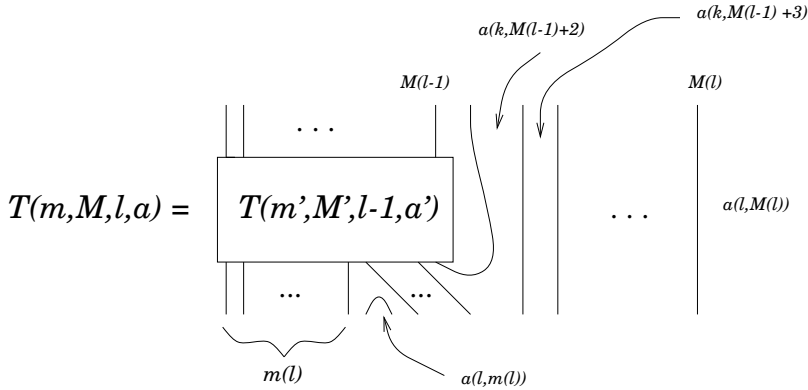
$$j = M(l) - M(l - 1) - 1,$$

$$\bar{a} = (a(l, M(l - 1) + 2), a(l, M(l - 1) + 3), \dots, a(l, M(l - 1) + j + 1))$$

and define

$$T(m, M, l, a) = I_{j, M(l-1)}(\bar{a})(\theta_{m(l), n-j-1}(T(m', M', l - 1, a')))$$

where the functions $m'(i)$ and $M'(i)$ are the functions $m(i)$ and $M(i)$ restricted to the domain $\{i : 1 \leq i \leq l - 1\}$ and a' is just a restricted to the part of the domain of a with the first coordinate between 1 and $l - 1$. We include the visual interpretation of the inductive definition of T for clarification:



These maps are clearly well defined and inverses and thus we have:

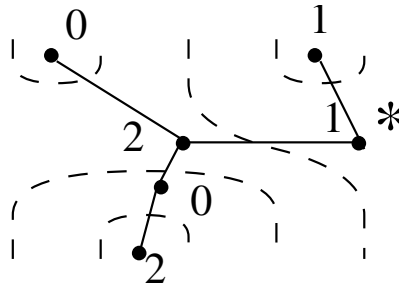
Theorem 1. *The maps ψ and T establish a one-to-one correspondence between $n+1$ -blocks and Kauffman (k, n) -diagrams. Moreover when this correspondence is restricted to reduced $(k, n+1)$ -blocks, it yields a one-to-one correspondence between reduced $(k, n+1)$ -blocks and Kauffman (k, n) -diagrams with the added condition that the sum of labels of any two adjacent pieces is $\geq k$.*

Proof. This follows from the inductive definitions of T and ψ . □

Next we need a correspondence well known to combinatorialists. We define a *planar (k, n) -tree* to be a rooted planar tree with n edges and vertices labeled by integers in the interval $[0, k]$.

Theorem 2. *There is a one-to-one correspondence between Kauffman (k, n) -diagrams and planar (k, n) -trees. Furthermore, in this correspondence, labels of adjacent pieces of the Kauffman diagram label neighboring vertices on the corresponding planar tree.*

Proof. We use the well-known bijection between Kauffman $(0, n)$ -diagrams and planar $(0, n)$ -trees, [St, Exercise 6.19 e & o, p. 256]. In this bijection, the vertices of a planar tree are in one-to-one correspondence with the pieces of the corresponding Kauffman diagram. Thus the bijection extends to a bijection between Kauffman (k, n) -diagrams and planar (k, n) -trees. The result follows from the fact that labels of adjacent vertices of a tree get sent to adjacent pieces of the Kauffman diagram and vice-versa. For clarity, we include the following picture that illustrates the correspondence between a Kauffman $(2, 5)$ -diagram (dotted lines) and a planar $(2, 5)$ -tree (solid lines, the tree is turned on its side with the root at the right):



□

Theorem 3. *There is a one-to-one correspondence between reduced $(k, n + 1)$ -blocks and (k, n) planar trees for which the sum of labels of any two adjacent vertices is $\geq k$.*

Proof. This follows from Theorems 1 and 2, and the fact that reduced (k, n) -blocks are (k, n) blocks with the added condition that

$$\begin{aligned} \text{b3. } & a(i, j) + a(i, j - 1) \geq k \\ & a(i, j) + a(i + 1, j + 1) \geq k. \end{aligned}$$

□

Corollary 3.1. *There is a one-to-one correspondence between reduced $(k, n + 1)$ -blocks and (k, n) planar trees for which the sum of labels of any two adjacent vertices is $\leq k$.*

Proof. Replacing every label of a vertex by k minus that label gives a one-to-one correspondence between

$$\begin{aligned} & \{ (k, n) \text{ planar trees with the sum of labels of any two adjacent vertices} \\ & \qquad \qquad \qquad \geq k \} \\ & \text{and} \\ & \{ (k, n) \text{ planar trees with the sum of labels of any two adjacent vertices} \\ & \qquad \qquad \qquad \leq k \}. \end{aligned}$$

The result then follows from Theorem 3. □

We are now ready to complete the plan of counting the number of reduced words described at the end of Section 3.

5. Counting.

We have established a one-to-one correspondence between the reduced (k, n) blocks and (k, n) -planar trees for which the sum of labels of any two adjacent vertices is $\leq k$. Using this correspondence we will now show that the generating function

$$T(x) = 1 + \sum_{i=1}^{\infty} |S_{k,i}| x^i$$

satisfies the relation:

$$(1) \quad xT^{k+2} = T - 1.$$

Let A_i^n be the number of planar $(k, n - 1)$ trees with

- 1) the root vertex labeled by i ,
- 2) the sum of labels of any two adjacent vertices $\leq k$.

Thus n will index the number of vertices on the tree. Let $A_i(x) = \sum_{n=1}^{\infty} A_i^n x^n$, and let $A(x) = \sum_{i=0}^k A_i(x)$. Writing $A(x) = \sum_{i=1}^{\infty} a_i x^i$ we have from Corollary 3.1 that a_i is the number of reduced (k, i) -blocks.

Recall (Definition 5) that a word $w \in S_{k,n}$ has the form:

- 1) $w = 1$,
- or
- 2) $w = \prod_{i=1}^d w_i$ with
 - a) the w_i reduced blocks,
 - b) $\min(w_{i+1}) > \max(w_i)$.

The elements $w = \prod_{i=1}^d w_i \in S_{k,n}$ fall into two disjoint categories:

- i) $\max(w_d) < n - 1$, in which case w can be any element of $S_{k,n-1}$,
- ii) $\max(w_d) = n - 1, \min(w_d) = j$, in which case w can be described as the product of an element of $S_{k,j}$ (i.e., $\prod_{i=1}^{d-1} w_i$) and a reduced block w_d with $\max(w_d) = n - 1$ and $\min(w_d) = j$.

Thus we have the relation for $n > 1$,

$$|S_{k,n}| = |S_{k,n-1}| + \sum_{j=1}^{n-1} |S_{k,j}| a_{n-j}.$$

The above and the fact that $|S_{1,k}| = 1$, translate to the generating function equation:

$$\sum_{i=1}^{\infty} |S_{k,i}| x^i = x \sum_{i=1}^{\infty} |S_{k,i}| x^i + \left(\sum_{i=1}^{\infty} |S_{k,i}| x^i \right) A(x) + x.$$

Thus

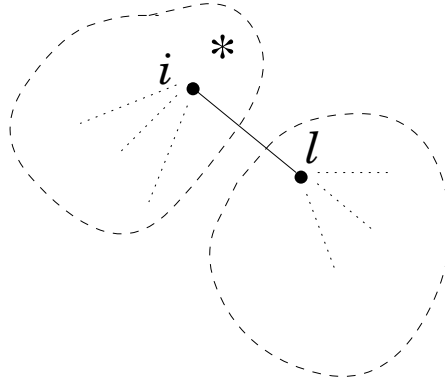
$$T(x) - 1 = x(T(x) - 1) + (T(x) - 1)A(x) + x$$

and solving for $T(x)$ and $A(x)$ respectively yields:

$$T(x) = \frac{1 - A(x)}{1 - A(x) - x},$$

$$(2) \quad A(x) = 1 - \frac{xT(x)}{T(x) - 1}.$$

Consider a tree counted by A_i^n . The root is labeled by i and the first vertex (as the tree is traced clockwise from the root), if one exists, is labeled by l :



The tree can be thought of as the joining of two trees (the dotted circles), one with root labeled i and the other with root labeled l for some l with $i + l \leq k$. We therefore have the recursion:

$$A_i^n = \sum_{j=1}^{n-1} A_i^j \left(\sum_{l=0}^{k-i} A_l^{n-j} \right) \quad \text{for } n > 0.$$

In terms of $A_i(x)$, this relation becomes

$$(3) \quad A_i(x) = x + \sum_{l=0}^{k-i} A_i(x)A_l(x)$$

with the initial x to allow for the tree with exactly one vertex. Setting $L_i(x) = 1 - \sum_{l=0}^i A_l(x)$, (3) becomes

$$(4) \quad (1 - L_0(x))L_k(x) = x$$

$$(5) \quad (L_{i-1}(x) - L_i(x))L_{k-i}(x) = x \quad 1 \leq i \leq k.$$

For convenience of notation, we will omit the dependence on x whenever possible (i.e., we will denote $L(x)$ by L). Rewriting (5) we have

$$(6) \quad L_i L_{k-i} = L_{i-1} L_{k-i} - x.$$

Evaluating (6) at the two values $i = j$ and $i = k - (j - 1)$ gives

$$L_j L_{k-j} = L_{j-1} L_{k-j} - x = (L_{k-(j-1)} L_{j-1} + x) - x = L_{k-(j-1)} L_{j-1},$$

and thus $L_j L_{k-j} = L_0 L_k$ for $1 \leq j \leq k$. Using (4) to express $L_0 L_k$ in terms of L_k we have

$$(7) \quad L_i L_{k-i} = L_k - x.$$

It follows from (6) that

$$(8) \quad L_{i-1} L_{k-i} = L_k - x + x = L_k.$$

Combining (7) and (8) yields

$$L_i = \left(\frac{L_k - x}{L_k} \right) L_{i-1}$$

and thus

$$(9) \quad L_i = \left(\frac{L_k - x}{L_k} \right)^i L_0 = \left(\frac{L_k - x}{L_k} \right)^{i+1}.$$

For $i = k$ the above becomes

$$(10) \quad L_k = \left(\frac{L_k - x}{L_k} \right)^{k+1}.$$

Combining (10),(2) and $L_k = 1 - A$ implies that T , the generating function for $|S_{k,n}|$, satisfies the relation:

$$xT^{k+2} = T - 1.$$

6. $FC_{k,n}$.

We now define the Fuss-Catalan algebras of [BiJo1]. Recall that in a Kauffman n -diagram we had labeled the boundary points $1, 2, \dots, 2n$ as we traveled clockwise around the picture starting at the top left, ending at the bottom left.

Definition 8 ([BiJo1]). Label the boundary points of the Kauffman $(k + 1)n$ -diagrams clockwise by the labels:

$$a_0, a_1, a_2 \dots a_k, a_k, a_{k-1} \dots a_1, a_0, a_0 \dots$$

(Thus boundary point $2(k+1)n$ is labeled by a_0 .) Define $\mathcal{B}(k, n, a_0, a_1, \dots, a_k)$ to be the set of all Kauffman $(k + 1)n$ -diagrams such that for each i , the boundary points i and $J(i)$ have the same “a” label. (This is equivalent to $i + J(i) \equiv 1 \pmod{2(k + 1)}$.)

Definition 9 ([BiJo1]). Let $FC_{k,n}(\alpha_0, \alpha_1 \dots \alpha_k)$, $\alpha_i \in \mathbb{C} - 0$ be the complex linear span of $\mathcal{B}(k, n, a_0, a_1 \dots a_k)$, with the following multiplicative structure:

Given $D_1, D_2 \in \mathcal{B}(k, n)$. Define $D_3 \in \mathcal{B}(k, n)$ as follows: stack D_2 on top of D_1 so that the boundary points at the bottom of D_2 are aligned with those at the top of D_1 . Fuse the curves of D_2 and D_1 together at these boundary points. Remove all closed loops and let D_3 be the resulting diagram.

We define

$$D_1 D_2 = (\alpha_0)^{n_0} (\alpha_1)^{n_1} \dots (\alpha_k)^{n_k} D_3$$

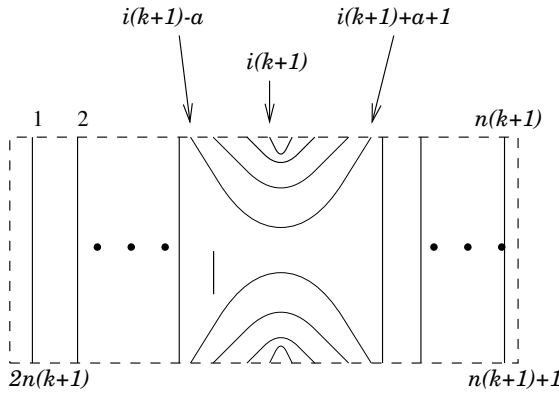
where n_i is the number of removed closed loops of curves of type “ a_i ”.

Note. The notation differs slightly from [BiJo1] in that the labels of the curves are a_i whereas the scalars associated to loops of type “ a_i ” are the α_i . In [BiJo1], a_i represented both these quantities. We do this to clarify that the types “ a_i ” are all distinct whereas the scalars α_i need not be. From now on we will refer to $\mathcal{B}(k, n, a_0, a_1 \dots a_k)$ as $\mathcal{B}(k, n)$, suppressing the labeling set of the vertices.

Definition 10 ([BiJo1]). We define a $*$ -operation on $FC_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbf{R}$ as follows: if $D \in FC_{k,n}$ is a diagram, denote by D^* the diagram obtained by the reflection of D over the top boundary. Thus the top (resp. bottom) boundary of D^* is the bottom (resp. top) boundary of D . Extend the $*$ -operation to all of $FC_{k,n}$ by conjugate linearity.

We note that $D^{**} = D$ and $(D_1 D_2)^* = D_2^* D_1^*$ (since the parameters are real) and so $*$ is a conjugate linear involution of $FC_{k,n}$.

Definition 11. Let ${}_a e_i$, $1 \leq i \leq n - 1$, $0 \leq a \leq k$ denote the idempotent in $FC_{k,n}$ that is $\frac{1}{{}_a c_i}$ times the following picture:

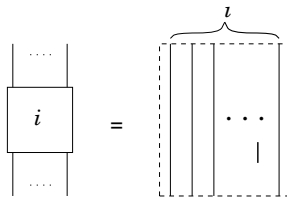


where

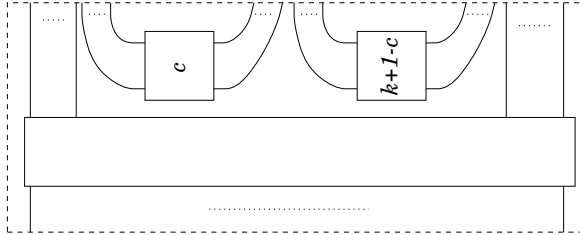
$${}_a c_i = \begin{cases} \alpha_0 \alpha_1 \dots \alpha_a & \text{for } i \text{ even,} \\ \alpha_{k-a} \alpha_{k-a+1} \dots \alpha_k & \text{for } i \text{ odd.} \end{cases}$$

We want to show that $FC_{k,n}$ is generated by the set $\{1, {}_a e_i\}$. To do so, we use the following notation and lemma:

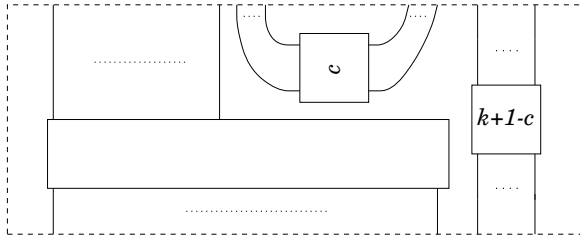
Denote by a box labeled i a Kauffman diagram consisting of i vertical strings:



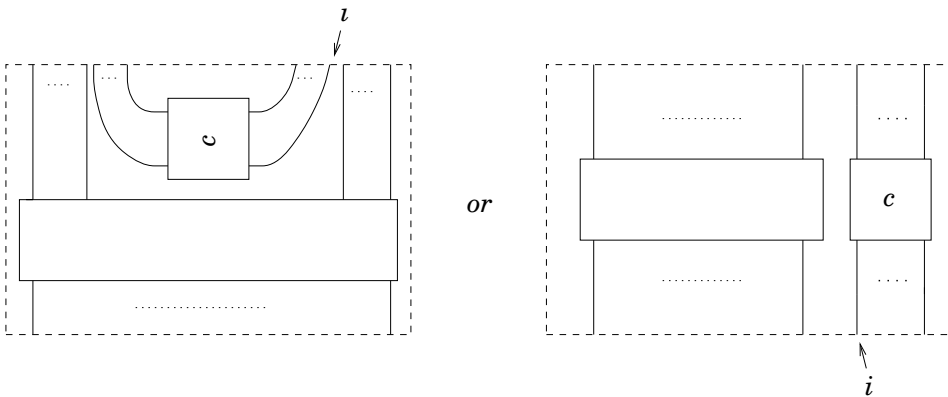
Lemma 5. *Given $K \in \mathcal{B}(k, n)$ with corresponding planar function J . Suppose that $|J(i) - i| < 2(k + 1)$ for all $i \in S = \{i_0, i_0 + 1, \dots, i_0 + k\}$. Then up to horizontal and/or vertical symmetry (i.e., reflection in the x and/or y axes) there exists an integer c , $0 \leq c \leq k + 1$ such that K can be drawn in one of the following two ways:*



or



Proof. Choose i so that $i \in S$ and $|i - J(i)|$ is maximal (i is not necessarily unique). Let us assume $J(i) < i$, an identical argument holds otherwise. Because there are only two boundary points of each type between $J(i)$ and i , we must have K of the form:



with $c = (i - J(i) + 1)/2$. If $|i - J(i)| = 2(k + 1) - 1$ we are done (since $c = k + 1$). Otherwise either $J(i) - 1$ or $i + 1$ is in S . By symmetry let us assume $i + 1 \in S$. We then have $J(i + 1) > (i + 1)$ (otherwise $J(i + 1) < J(i)$ contradicting the choice of i), and $J(i + 1) - (i + 1) < 2(k + 1)$. Again, since

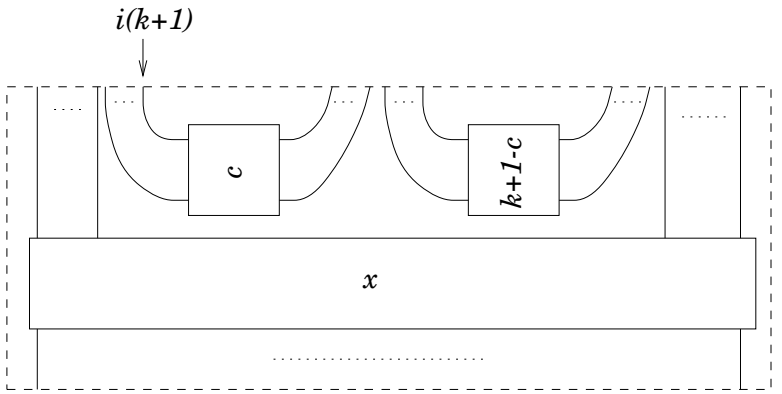
there are only two boundary points of each type between $i + 1$ and $J(i + 1)$ we are left with one of the two pictures we wanted. \square

Armed with the above lemma we have the following:

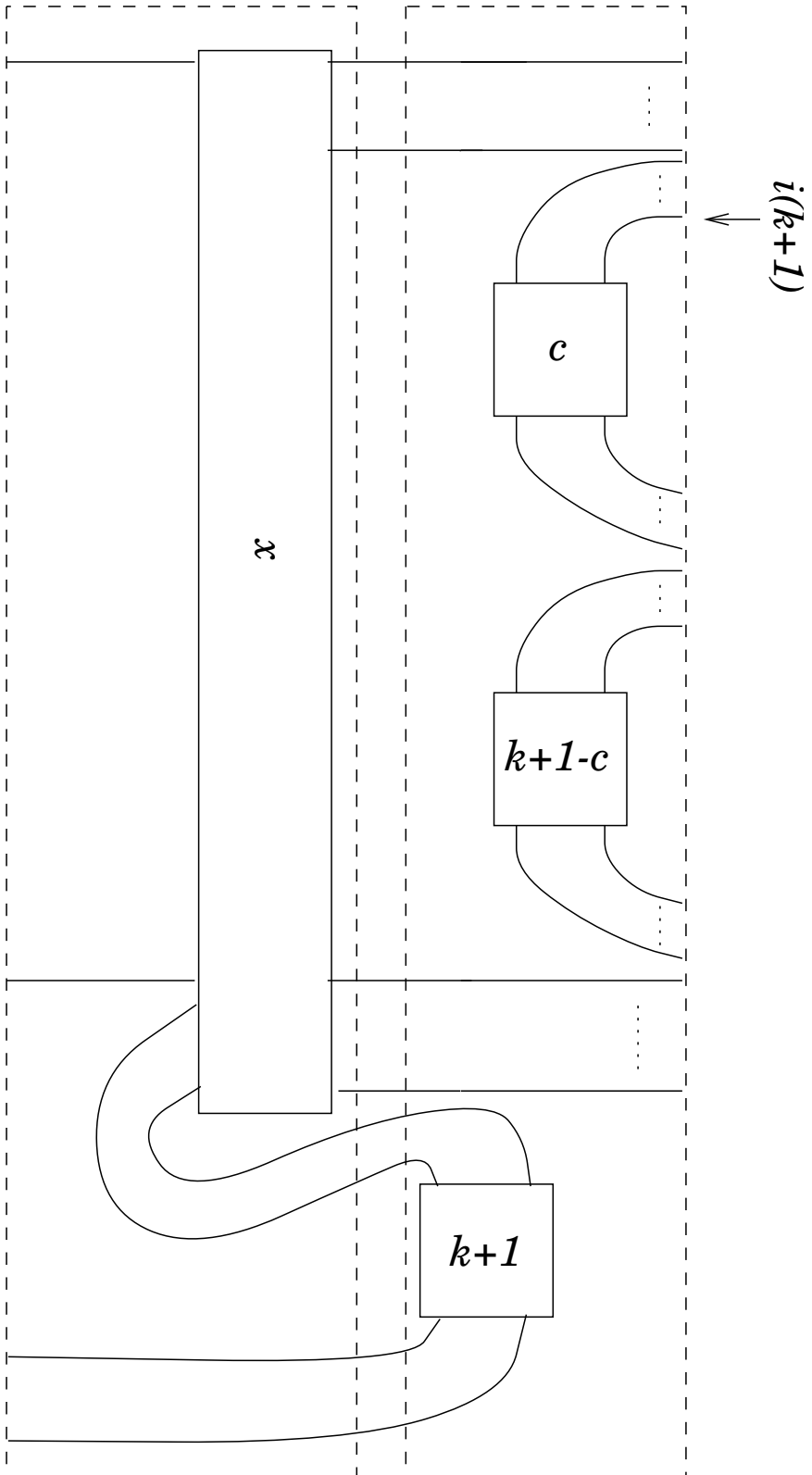
Theorem 4. *The set $\{1, a e_i\}$, with $1 \leq i \leq n - 1$ and $0 \leq a \leq k$ is a generating set for $FC_{k,n}$.*

Proof. We induct on n . The result is clear for $n = 1, 2$. We first show that all elements of $\mathcal{B}(k, n) \subset FC_{k,n}$ satisfy the conclusion of Lemma 5. Consider a pair of points $(i, J(i))$, $i < J(i)$ with $|i - J(i)|$ minimal among all pairs $(j, J(j))$ with $|j - J(j)| \geq 2(k + 1)$. If no such i exists then the hypothesis for Lemma 5 is satisfied for any string of $k + 1$ consecutive boundary points. If an i does exist then all the points $i \leq j \leq J(i)$, will have $i \leq J(j) \leq J(i)$ (by planarity of J). This contradicts the choice of i unless $|j - J(j)| < 2(k + 1)$ in which case we again have at least $k + 1$ points in a row with $|j - J(j)| \leq 2k$ and thus we can apply Lemma 5.

Up to reflection in the x and/or y axis, we then have one of two possibilities for the diagram of an arbitrary element of $\mathcal{B}(k, n)$. The first:



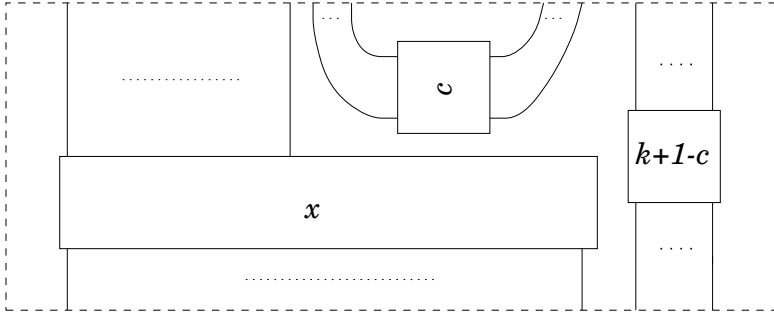
can be rewritten as



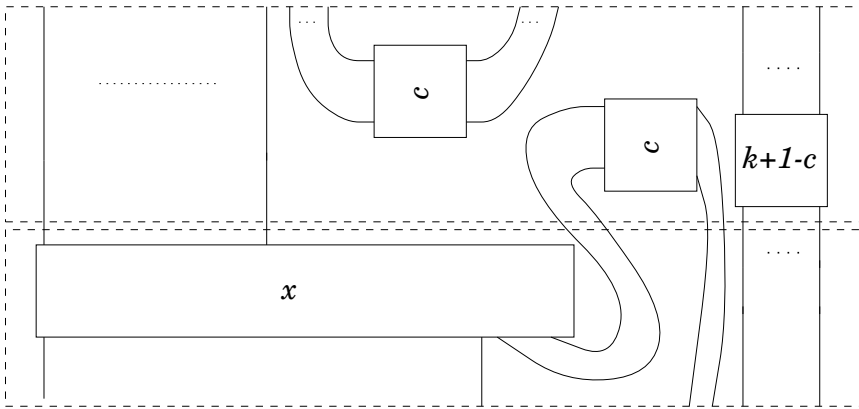
The upper dotted diagram is just a scalar multiple of

$$({}_k e_{n-1} \cdot {}_k e_{n-2} \cdots {}_k e_{i+1})_{c-1} e_i$$

while the lower dotted diagram is an element of $FC_{k,n-1}$ sitting inside $FC_{k,n}$ and thus by the induction hypothesis the whole diagram is a product of elements in $\{1, {}_i e_a\}$. Similarly the second possible picture for an element in $\mathcal{B}(k, n)$:



can be written as



The upper dotted diagram is just a scalar multiple of

$$c^{-1}e_{n-1}$$

while once again the lower dotted diagram is an element of $FC_{k,n-1}$ sitting inside $FC_{k,n}$. Once again the induction hypothesis allows us to conclude that the diagram is generated by a product of elements in $\{1, a e_i\}$. Similar arguments hold for x and/or y axis reflection of the diagrams. \square

Theorem 5 ([BiJo1]). $\dim FC_{k,n} = \frac{1}{(k+1)n+1} \binom{(k+2)n}{n} = c_n^{k+2}$ and $G(x) = \sum_{n=0}^{\infty} c_n^{k+2} x^n$ satisfies:

$$G(x) = x(G(x))^{k+2} + 1.$$

For a nice combinatorial proof of the above see [He].

7. $AA_{k,n}$ isomorphic to $FC_{k,n}$.

Theorem 6. *The algebras $AA_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ and $FC_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ are isomorphic. In addition, $S_{k,n}$ is a basis for $AA_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$.*

Proof. The map $\phi : AA_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k) \rightarrow FC_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ given by

$$\phi({}_a E_i) = {}_a e_i$$

is a homomorphism onto $FC_{k,n}$ due to the universality of the abstract algebra and the fact that the generators of $FC_{k,n}$ (Theorem 4) satisfy the same relations as the abstract ones. Since the generating functions for the dimension of $FC_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ (Theorem 5) and for $|S_{k,n}|$ (Section 5) are the same, we have that the dimension of $FC_{k,n}$ is $|S_{k,n}|$. We deduce that the dimension of $AA_{k,n}$ is equal to $|S_{k,n}|$ since it is both $\geq |S_{k,n}|$ (since ϕ is onto) and $\leq |S_{k,n}|$ (Section 3). It follows that ϕ is an isomorphism and that $S_{k,n}$ is a basis for $A_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$. \square

8. The Structure of $FC_{k,n}$.

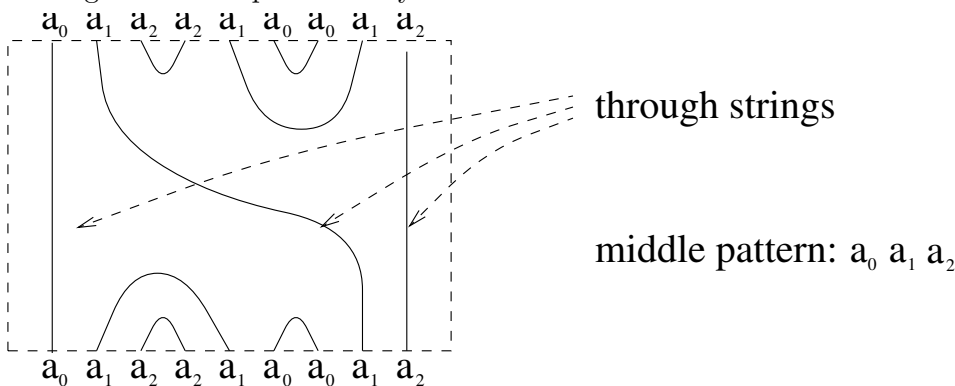
Lemma 6 ([BiJo1]). *The algebras $FC_{k,n}(\alpha_0, \alpha_1 \dots \alpha_k)$ are semisimple for an open dense (even Zariski open) subset of parameters $(\alpha_0, \alpha_1 \dots \alpha_k) \in \mathbb{C}^{k+1}$.*

We now describe the structure of $FC_{k,n}$ in the case when $FC_{k,n}$ is semi-simple (the generic case) giving the Bratteli diagram for the inclusion $FC_{k,0} \subset FC_{k,1} \subset FC_{k,2} \subset \dots$. In essence, this work was done in [BiJo1, Section 3], though only explicitly described for $k = 1$ (i.e., for 2 parameters α_0, α_1). We describe this structure, stating, but not proving, those results that are straightforward generalizations of arguments given in [BiJo1].

Definition 12 ([BiJo1]). A *through string* on a diagram $D \in \mathcal{B}(k, n)$ is a curve that connects a point on the top of D to one on the bottom. Let $t(D)$ be the number of through strings of D .

Definition 13 ([BiJo1]). The *middle pattern* on a diagram $D \in \mathcal{B}(k, n)$ is the well defined word, denoted by $m(D)$, obtained by reading from left to right the labels “ a_i ” of the through strings of D .

We give an example to clarify:



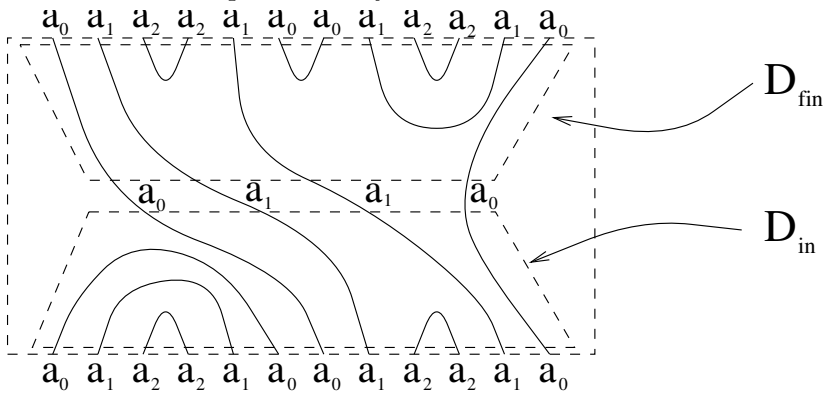
So far, we have dealt with planar diagrams that have the same number of boundary points on the top as on the bottom. We now define a more general picture:

Definition 14 ([BiJo1]). Let $D_{pl}(p, q)$ be the set of diagrams with p boundary points on the bottom and q on the top with non-intersecting curves (strings) connecting distinct boundary points.

(The pl in D_{pl} refers to the “planarity” or non-intersecting property of the diagram.)

We note that an element of $D_1 \in D_{pl}(p, q)$ can be multiplied by an element in $D_2 \in D_{pl}(q, r)$ as long as the labeling of the top of D_1 coincides with the labeling of the bottom of D_2 . Thus an element $D \in \mathcal{B}(k, n)$ can be written uniquely as the composition of $D_{in} \in D_{pl}((k + 1)n, t(D))$, the *initial part*, and $D_{fin} \in D_{pl}(t(D), (k + 1)n)$, the *final part*, where the top of D_{in} and the bottom of D_{fin} are labeled by $m(D)$. (Of course the top of D_{fin} and the bottom of D_{in} are labeled by $a_0, a_1, a_2 \dots a_k, a_k, a_{k-1} \dots a_1, a_0, a_0 \dots$).

We include an example to clarify:



For two words w, w' we define $w \leq w'$ if w is obtained from w' by removing some (or no) letters of w' . Define v_n to be the following word with $(k + 1)n$ letters:

$$a_0 a_1 a_2 \dots a_k a_k a_{k-1} \dots a_1 a_0 a_0 \dots a_0 \text{ (or } a_k \text{)}.$$

The following is a straightforward observation:

Lemma 7. *The word w appears as a middle pattern for elements in $FC_{k,n}$ if and only if $w \leq v_n$ and $w \equiv v_n$ when reduced using the relations $a_i^2 = 1, 0 \leq i \leq k$.*

Note that the smallest such middle pattern is ϕ for n even and $a_0 a_1 \dots a_k$ for n odd.

Definition 15 ([BiJo1]). Let I_t be the two sided ideal of $FC_{k,n}$ spanned by diagrams with at most t through strings.

For obvious parity reasons, for n even we have $I_{2t} = I_{2t+1}$ and

$$\{0\} \subset I_0 \subset I_2 \subset I_4 \subset \dots \subset I_{n(k+1)} = FC_{k,n}(\alpha_0, \alpha_1 \dots \alpha_k).$$

For n odd we have

$$\{0\} = I_0 = I_1 = I_2 = \dots = I_k, \quad I_{k+2l+1} = I_{k+2l+2}$$

and

$$\{0\} \subset I_{k+1} \subset I_{k+3} \subset I_{k+5} \subset \dots \subset I_{n(k+1)} = FC_{k,n}(\alpha_0, \dots \alpha_k).$$

Thus, in the semisimple case, to describe the structure of $FC_{k,n}$ it will suffice to describe the appropriate algebras I_{t+2}/I_t , where t is the same parity as $(k + 1)n$.

Given $D \in \mathcal{B}(k, n)$ define the projection $p_M = \frac{1}{c_0}MM^*$ where $M = D_{in}$ and $c_0MM^* = MM^*MM^*$.

Lemma 8 ([BiJo1]). *The projection p_M has the following properties:*

- 1) $p_MFC_{k,n}p_M \equiv \mathbb{C}p_M \pmod{I_{t(D)-2}}$.
- 2) *Given M_1, M_2 initial parts of two diagrams with distinct middle patterns of length t , then $p_{M_1}Ep_{M_2} \equiv 0 \pmod{I_{t-2}}$ for E a diagram in $FC_{k,n}$.*
- 3) *Given M_1 and M_2 initial parts of two diagrams with the same middle pattern, we have $M_1 = cDM_2$ for some $D \in FC_{k,n}$, $c \in \mathbb{C}$.*

Definition 16 ([BiJo1]). Given a word w in the alphabet $\{a_0, a_1, \dots a_k\}$, define $\begin{bmatrix} n \\ w \end{bmatrix}$ to be the number of distinct initial parts of the diagrams in $\mathcal{B}(k, n)$ having middle pattern w . (Define $\begin{bmatrix} n \\ w \end{bmatrix} = 0$ if w is not a middle pattern.)

The above work leads us to the following:

Lemma 9 ([BiJo1]). *If w is a middle pattern of length t and M is an initial part of a diagram with middle pattern w , then the dimension of the left ideal $p_M(I_t/I_{t-2})$ is $\begin{bmatrix} n \\ w \end{bmatrix}$.*

Lemma 10 ([BiJo1]).

$$I_t/I_{t-2} = \bigoplus_w (I_t/I_{t-2})p_M(I_t/I_{t-2}),$$

where the sum is over all middle patterns of length t , and M is an initial part of a diagram with middle pattern w (one M for each w).

Theorem 7 ([BiJo1]). *If $(\alpha_0, \alpha_1 \dots \alpha_k)$ are such that $FC_{k,n}(\alpha_0, \alpha_1 \dots \alpha_k)$ is semisimple, we have*

$$FC_{k,n}(\alpha_0, \alpha_1 \dots \alpha_k) \cong \bigoplus_{w: \begin{bmatrix} n \\ w \end{bmatrix} > 0} M_{\begin{bmatrix} n \\ w \end{bmatrix}}(\mathbb{C}),$$

where the sum is over all middle patterns w of diagrams in $FC_{k,n}$, and the summands are matrix algebras over \mathbb{C} of dimension $\begin{bmatrix} n \\ w \end{bmatrix}$.

With this decomposition we can describe the irreducible representations. We need the following definition:

Definition 17 ([BiJo1]). Given w a word t letters long in the alphabet $\{a_0, a_1 \dots a_k\}$, with $\begin{bmatrix} n \\ w \end{bmatrix} > 0$. Let V_w be the complex vector space of dimension $\begin{bmatrix} n \\ w \end{bmatrix}$ having as basis distinct initial parts M of diagrams in $\mathcal{B}(k, n)$ with middle pattern w . For $D \in \mathcal{B}(k, n)$ define the representation π_w of $FC_{k,n}$ (extended by linearity) on V_w as follows:

$$\pi_w(D)M = \begin{cases} DM, & \text{if } DM \text{ has } t \text{ through strings,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $M \in V_w$ (where as usual we count a closed loop of type “ a_i ” as a scaling factor α_i).

With the above notation, we have:

Theorem 8 ([BiJo1]). *If $(\alpha_0, \alpha_1 \dots \alpha_k)$ are such that $FC_{k,n}(\alpha_0, \alpha_1 \dots \alpha_k)$ is semisimple, then all the representations π_w are irreducible and any irreducible representation of $FC_{k,n}$ is equivalent to a π_w .*

A description of the Bratteli diagram turns out to be quite simple for the following more general situation. Given an infinite word $w = \prod_{n=0}^\infty t_n$, $t_i \in \{a_0, a_1 \dots a_k\}$, let $w_i = \prod_{n=0}^i t_n$. Define $A_{w_i}(\alpha_0, \alpha_1 \dots \alpha_k)$ to be the complex algebra with basis the set of Kauffman n diagrams whose top and bottom row are labeled (both from left to right) by w_i , and whose curves connect boundary points of the same label (again, loops labeled by “ a_i ” are replaced by a scalar factor “ α_i ”). Define $\pi_v^{(i)}$ to be the irreducible representation of A_{w_i} whose vector space consists of all initial parts of diagrams with middle pattern v . Then we have:

Theorem 9 ([BiJo1]). *Suppose the algebras A_{w_i} defined above are semisimple. Let v be a word in the alphabet $\{a_0, a_1, \dots a_k\}$, let $t \in \{a_0, a_1, \dots a_k\}$ and suppose that vt is a middle pattern for $A_{w_{i+1}}$. Then*

$$\pi_{vt}^{(i+1)}|_{A_{w_i}} \equiv \begin{cases} \pi_v^{(i)} \oplus \pi_{vtt}^{(i)} & \text{if } t_{i+1} = t \\ \pi_{vtt_{i+1}}^{(i)} & \text{otherwise.} \end{cases}$$

This theorem determines the Bratteli diagram for $A_{w_0} \subset A_{w_1} \subset A_{w_2} \subset \dots$. The following sentence is the rule that describes the above theorem [BiJo1, p. 113]: “The words on the i th row are certain subwords of w_i . To connect the i -th row to the $(i+1)$ th, add t_{i+1} to all words and take it away, if possible”.

If we now return to the special case of $FC_{k,n}$, where our infinite word is

$$w = a_0 a_1 a_2 \dots a_k a_k a_{k-1} \dots a_1 a_0 a_0 \dots,$$

we have $FC_{k,n} = A_{w_{(k+1)n}}$, that is, we are looking for the inclusion every $k + 1$ steps. Recall that $FC_{k,n}$ is isomorphic to a direct sum of matrix

algebras indexed by the distinct middle patterns that occur for diagrams in $FC_{k,n}$. Thus the vertices of the Bratteli diagrams at the n -th level are indexed by these middle patterns as well. Let π_w^n denote the representation of π_w described in Definition 17 applied to the algebra $FC_{k,n}$. Let us denote by $N_{\pi_w^n, \pi_{w'}^{n+1}}$, the number of edges in the Bratteli diagram for $FC_{k,n} \subset FC_{k,n+1}$ between the vertex indexed by w in $FC_{k,n}$ and that of w' in $FC_{k,n+1}$. Alternatively, $N_{\pi_w^n, \pi_{w'}^{n+1}}$ is the number of times π_w^n occurs in $\pi_{w'}^{n+1}|_{FC_{k,n}}$. Using Theorem 9, we can describe the Bratteli diagram for $FC_{k,n} \subset FC_{k,n+1} \subset FC_{k,n+2}$:

Lemma 11 ([BiJo1]). *Assume that $FC_{k,n}$ are semisimple. Let w (resp. w') be a middle pattern for $FC_{k,n}$ (resp. $FC_{k,n+1}$). Then*

- i) $N_{\pi_w^n, \pi_{w'}^{n+1}} = N_{\pi_{w'}^{n+1}, \pi_w^{n+2}}$.
- ii) *If p is a minimal projection in $FC_{k,n}$ for π_w^n , then $p \cdot e_{n+1} = e_{n+1} \cdot p$ is a minimal projection in $FC_{k,n+2}$ for π_w^{n+2} .*

Thus the vertices at the n -th step (i.e., representing matrix algebras of $FC_{k,n}$) are naturally identified with some of the vertices at the $(n + 2)$ -th step (i.e., some of the matrix algebras of $FC_{k,n+2}$). We now characterize the rest of the vertices at the $(n + 2)$ -th step, that is, those vertices indexed by w so that $\dim(\pi_w^{n+2}) > 0$ but $\pi_w^n = 0$.

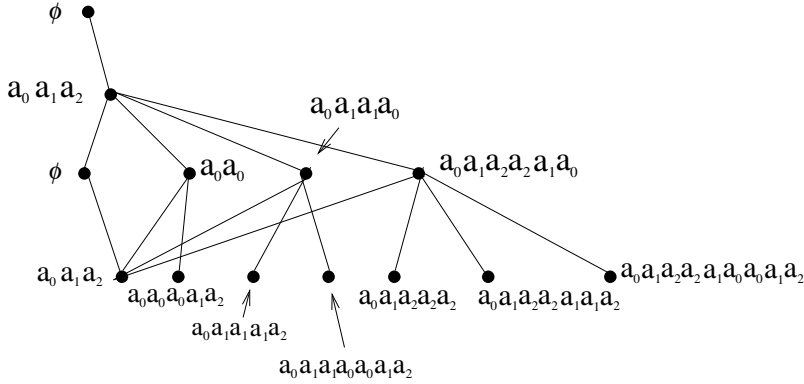
Lemma 12 ([BiJo1]). *The following are equivalent when w is a middle pattern for a diagram in $FC_{k,n}$. ($FC_{k,n}$ need not be semisimple.)*

- 1) $\pi_w^{n-2} = 0$
- 2) $\dim \pi_w^n = 1$.
- 3) $\pi_w^n(e_j) = 0$ for all j , $1 \leq j \leq n - 1$.
- 4) $\pi_w^n(e_j) = 0$ for some j , $1 \leq j \leq n - 1$.
- 5) *No basis diagram for π_w^n has $2(k + 1)$ consecutive non-through strings.*

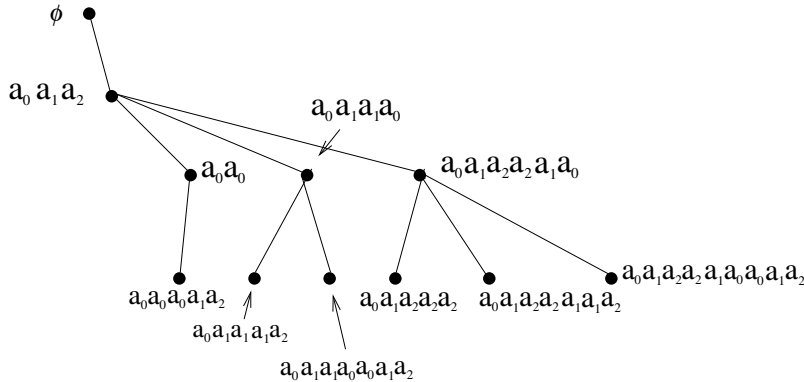
Piecing the above two lemmas together we arrive at the following method for generating the Bratteli diagram for $FC_{k,n+1} \subset FC_{k,n+2}$, given the diagram for $FC_{k,n} \subset FC_{k,n+1}$:

First reflect the Bratteli diagram for $FC_{k,n} \subset FC_{k+1,n}$ to get the “old stuff” in $FC_{k,n+2}$ (Lemma 11). Then add additional new vertices indexed by middle patterns that first occur in diagrams of $FC_{k,n+2}$. Each of these “new” vertices will be connected to only one vertex in $FC_{k,n+1}$ and that vertex will be one that was “new” in $FC_{k,n+1}$ (Lemma 12). Analyzing which middle patterns are “new” and which ones they are connected to is a straightforward application of Theorem 9.

Thus the first few levels of the Bratteli diagram for $FC_{2,n}$ is given by:



The above description actually tells us that writing down the full Bratteli diagram is redundant since the “old” stuff is just the reflection of a previous part of the Bratteli diagram. We can fully describe the Bratteli diagram by just describing the “new” stuff that appears at every level. This smaller graph is called the *principal graph* [GHJ], a term familiar to those who study subfactors. To clarify, the principal graph for the first few levels of $FC_{2,n}$ is given by:



This brings us to a simple description of the Bratteli diagram for $FC_{k,n}$:

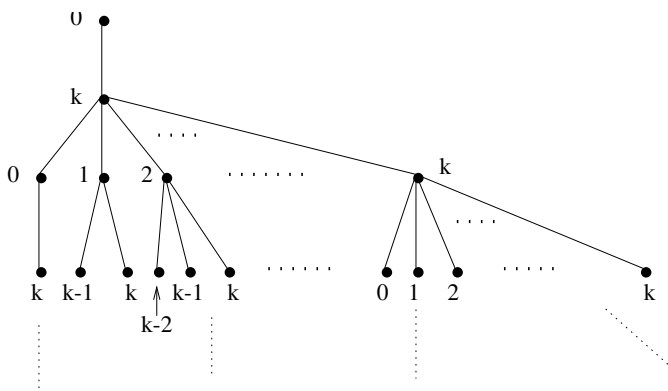
Theorem 10. *The principal graph Γ_k for the inclusion*

$$FC_{k,0} \subset FC_{k,1} \subset FC_{k,2} \subset \dots$$

is a tree inductively described as follows:

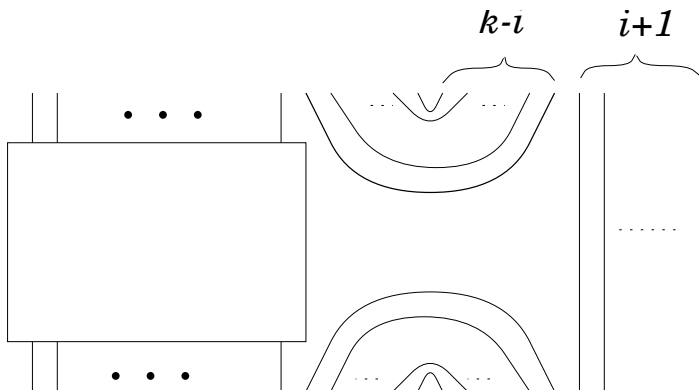
- 1) *The 0-th level has one vertex labeled by 0.*
- 2) *Given a vertex on the n-th level labeled by i. Create (i + 1) new vertices on the (n + 1)-th level with labels k - i, k - i + 1, . . . k.*
- 3) *Connect each of these new vertices to the given one on the n-th level.*

To clarify, we give the principal graph Γ_k for the first few levels:



Proof. We use induction, using the characterization of new words w as those for which no basis diagram for π_w^n has $2(k + 1)$ consecutive non-through strings (Lemma 12).

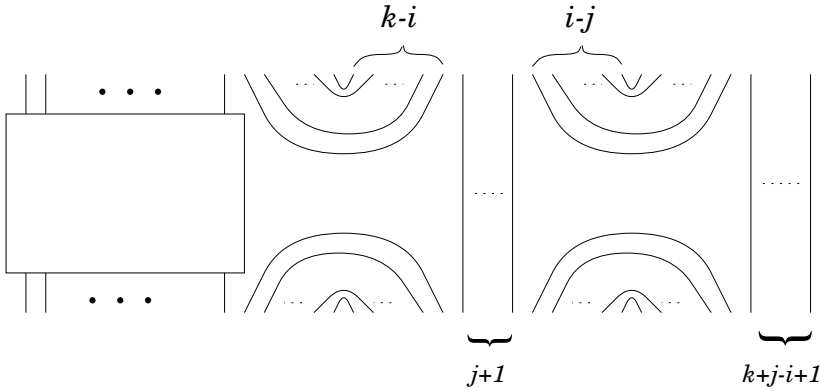
Let $P(n)$ be the statement: A single diagram in $FC_{k,n}$ having the “new” middle pattern w corresponding to a vertex on Γ_k labeled by i looks like:



Clearly since $\dim(FC_{k,1}) = 1$, and the vertex corresponding to the word $a_0a_1 \dots a_k$ is labeled by k , we have $P(1)$ true.

Given $P(n)$ true consider w corresponding to a vertex on the n th level of Γ_k labeled by i . By the induction hypothesis the single diagram in $FC_{k,n}$ with middle pattern w looks like the above figure. Without loss of generality, let us assume n is even and so the rightmost label on the top of the diagrams is a_k . (The identical argument holds for n odd if we replace a_k with a_0 .) Following the rule of Theorem 9 we can add or take away a_k from w . If we add a_k , we will have to add a_{k-1} and then $a_{k-2} \dots$ to get just the new word $wa_k a_{k-1} \dots a_0$. If we remove a_k we will have the option to add or remove a_{k-1} . We continue in this way until we consider adding or removing a_{k-i} . If we remove a_{k-i} then no matter what we do from then on we will have a middle pattern that has a diagram with $2(k + 1)$ consecutive non-through strings and thus by Lemma 12 it will not correspond to a new

middle pattern at the $(n + 1)$ -th level. Thus the only possibilities are the following $i + 1$ middle patterns, indexed by j , $0 \leq j \leq i$, none of which has $2(k + 1)$ consecutive non-through strings.



Labeling the vertex corresponding to the j -th middle pattern with $k - j + 1$ gives us $P(n + 1)$. □

9. Computation of $\begin{bmatrix} n \\ w \end{bmatrix}$.

To facilitate the ensuing discussion of paths on Γ_k we introduce the following notation:

- Given a vertex v of Γ_k , let $l(v)$ denote the label of v (an integer between 0 and k).
- A path P of length n on Γ_k will be written as $P = v_0 v_1 \dots v_n$ where the v_i are the successive vertices on Γ_k encountered when following the path P .
- Denote by $f(w)$, the smallest n such that w appears as a middle pattern in $FC_{k,n}$.
- Each middle pattern w corresponds to a vertex of the principal graph described above. There is a unique shortest path $P_w = v_0 v_1 \dots v_{f(w)}$ along the principal graph Γ_k (since the principal graph is a tree) from the root to the vertex indexed by w . Define $L(w)$ be the sequence $0, k, l(v_2), l(v_3) \dots l(v_{f(w)})$.

Corollary 10.1. *Write $w = a_{s(1)} a_{s(2)} \dots a_{s(d)}$ where the $s(i) \in [0, k]$. Let $S = \{i : s(i) = s(i + 1)\}$. Then $f(w) = |S| + 1$. Let $i_1 < i_2 < \dots < i_{|S|}$ be the elements of S . Set $P_w = v_0 v_1 \dots v_{f(w)}$ as above, the shortest path from the root to the vertex labeled by w corresponding to the summand in $FC_{k,f(w)}$*

indexed by w . Then we have:

$$l(v_j) = \begin{cases} 0 & \text{for } j = 0, \\ k & \text{for } j = 1, \\ s(i_{j-1}) & \text{for } j \text{ even,} \\ k - s(i_{j-1}) & \text{for } j \text{ odd.} \end{cases}$$

Proof. This follows from the argument given in Theorem 10. □

We have given an inductive description of the Bratteli diagram. In this section we get an explicit formula for $\begin{bmatrix} n \\ w \end{bmatrix}$.

Consider all paths of length $2n$ on Γ_k with the following two properties:

- 1) The paths begin and end at the same vertex v .
- 2) No vertex of the path is closer to the root than v .

We note that the structure of Γ_k guarantees that the same number of such paths will exist from two vertices v and w if $l(v) = l(w)$. This allows us to define c_i^n to be the number of such paths of length $2n$ starting and ending at a vertex v with $l(v) = i$. Let $C_i(x) = \sum_{n=0}^\infty c_i^n x^n$, with the convention $c_i^0 = 1$.

Lemma 13. *With the above notation we have*

$$C_i(x) = T^{i+1}(x)$$

where T is the generating function encountered previously (Section 5) satisfying:

$$T = xT^{k+2} + 1.$$

Proof. Let $P = v_0v_1 \dots v_{2n}$ be an arbitrary path counted by c_i^n . (This means that $v_0 = v_{2n}$ with $l(v_0) = i$ and no v_i closer to the root than v_0 .) Let $i_0 = \min \{i : i \geq 1 \text{ and } v_i = v_0\}$. Write $P = P_1P_2$ with $P_1 = v_0v_1 \dots v_{i_0}$ and $P_2 = v_{i_0+1}v_{i_0+2} \dots v_{2n}$. The nature of P and Γ_k ensures that $v_1 = v_{i_0-1}$ and thus the various possibilities for P_1 are counted by $c_{l(v_1)}^{\frac{i_0-2}{2}}$. On the other hand, the various possibilities for P_2 are counted by $c_i^{n-\frac{i_0}{2}}$. Since $l(v_1)$ can be any of $k - i, k - i + 1 \dots k$, we have the recursion:

$$c_i^n = \sum_{j=k-i}^k \sum_{m=0}^{n-1} c_j^m c_i^{n-m-1}$$

along with $c_i^0 = 1$.

This becomes:

$$(11) \quad C_i = 1 + \sum_{j=k-i}^k xC_jC_i.$$

Setting $D_{k-i} = xC_i$, and setting $i' = k - i$, (11) becomes

$$D_{i'} = x + \sum_{j'=0}^{k-i'} D_{j'} D_{i'},$$

the same recursion relation as $A_i(x)$ in Equation (3), and thus all the results of Section 5 apply. Since $xC_i = D_{k-i} = A_{k-i} = L_{k-i-1} - L_{k-i}$, it follows from (9) that

$$L_{k-i-1} - L_{k-i} = T^{-(k-i)} - T^{-(k-i+1)} = T^{i-k-1}(T - 1) = xT^{i+1},$$

where the last equality comes from substituting the known recursion relation for T given in Equation (1). □

Recall that given a middle pattern w , Corollary 10.1 allows us to calculate $f(w)$. In addition, it allows us to calculate $l(v_j)$ for $P = v_0v_1 \dots v_{f(w)}$ the unique shortest path from the root of Γ_k to the vertex corresponding to the summand labeled by middle pattern w . With this notation we have:

Theorem 11.

$$[n]_w = \text{the coefficient of } x^{n-f(w)} \text{ in } T^{\sum_{i=0}^{f(w)} (l(v_i)+1)} .$$

Setting $r = \sum_{i=0}^{f(w)} (l(v_i) + 1)$,

$$[n]_w = \binom{(k+2)(n-f(w)) + r}{n-f(w)} \frac{r}{(k+2)(n-f(w)) + r} .$$

Proof. An arbitrary path of length n from the root of Γ_k to the $v_{f(w)}$ can be described as composition of paths as follows:

$$\left(\text{a path counted by } c_{l(v_0)}^{n_0} \right) \left(\text{a path counted by } c_{l(v_1)}^{n_1} \right) \dots \left(\text{a path counted by } c_{l(v_{f(w)})}^{n_{f(w)}} \right)$$

with

$$n_0 + n_1 + \dots + n_{f(w)} = n - f(w).$$

Thus we have

$$[n]_w \text{ is the coefficient of } x^{n-f(w)} \text{ in } \prod_{i=0}^{f(w)} T^{l(v_i)+1} .$$

Finally [GKP] gives the closed form expression for T to an arbitrary power. □

10. The Trace.

The Fuss-Catalan algebras have a natural trace as described in the following definition:

Definition 18 ([BiJo1]). The *Markov trace* tr on $FC_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ is defined on a diagram $D \in \mathcal{B}(k, n)$ as follows: tie the top of D to the bottom (the “types” a_i will match up) and let n_i be the number of loops of “type” a_i . Then set

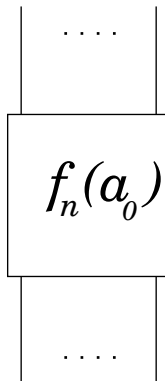
$$tr(D) = \prod_{i=1}^k \alpha_i^{n_i - n}.$$

Extend it by linearity to all of $FC_{k,n}$.

It is worth noting that this trace is consistent with the natural inclusion of $FC_{k,n} \subset FC_{k,n+1}$ given by adding $k + 1$ vertical lines to the end of diagrams in $FC_{k,n}$.

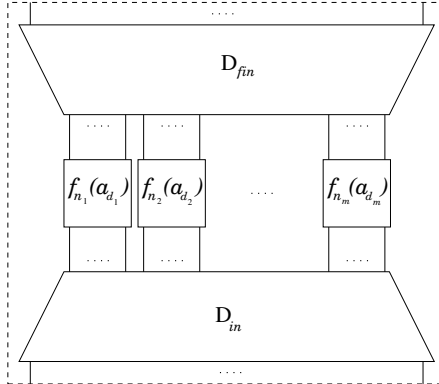
Up to normalization, there is just one trace on a full matrix algebra. Thus to describe the trace on a direct sum of matrix algebras one need only give the relative weights on each of the summands. The standard way to do this is to give the value of the trace on a minimal projection of each summand. Section 3.3 of [BiJo1] explicitly describes the minimal projections (and the corresponding trace values) for $FC_{1,n}$. The general situation follows virtually the identical argument. For this reason we will merely state the results for $FC_{k,n}$ without proving most of the details.

We begin by recalling the situation for the Temperley-Lieb algebra (i.e., $FC_{0,n}$) [GHJ]. Here an inductively defined series of polynomials is given by $P_0(\tau) = 0, P_1(\tau) = 1$ and $P_{l+1}(\tau) = P_l(\tau) - \tau P_{l-1}(\tau)$. Then if $1/(\alpha_0)^2$ is not a root of $P_l(\tau)$ for $l \leq n+1$ the minimal projection in $TL_n(\alpha_0)$ corresponding to the summand with middle pattern $(a_0)^n$ has trace of $P_n((\frac{1}{\alpha_0})^2)$. We denote this projection by $f_n(a_0)$ and draw it as



We note that $f_n(a_0)$ is a linear combination of elements of $\mathcal{B}(0, n)$.

Theorem 12. *Suppose the $FC_{k,n}(a_0, a_1, \dots, \alpha_k)$ are semisimple and none of the $(\alpha_i)^{-2}$ are zeros of $P_l(\tau)$, $1 \leq l \leq n$. Let w be a middle pattern appearing for the first time inside $FC_{k,n}$. Write $w = a_{d_1}^{n_1} a_{d_2}^{n_2} \dots a_{d_m}^{n_m}$ where $d_i \neq d_i + 1$. If we denote by D_{in} and D_{fin} the initial and final parts of the only element of $\mathcal{B}(k, n)$ with middle pattern w , then the following diagram*



multiplied by the scalar

$$\left(\frac{\alpha_{d_1}^{n_1} \dots \alpha_{d_m}^{n_m}}{(\alpha_0 \dots \alpha_k)^n} \right)^{\frac{1}{2}},$$

is the minimal idempotent corresponding to the simple summand indexed by w .

It is thus straightforward to see that the trace of the projection given in the above theorem is:

$$\left(\frac{\alpha_{d_0}^{n_0} \dots \alpha_{d_m}^{n_m}}{(\alpha_0 \dots \alpha_k)^{2n}} \right)^{-\frac{1}{2}} \prod_{i=1}^m P_n((\alpha_i)^{-2}).$$

Furthermore, by Lemma 11, the trace of a minimal projection corresponding to a summand w inside $FC_{n',k}$ is the trace of the minimal projection corresponding to w when it first appeared, say in $FC_{k,n}$, multiplied by $(\alpha_0 \dots \alpha_k)^{2(n-n')}$.

11. Chains of Subfactors.

We now come to the connection to subfactors that motivates the interest in the algebras $FC_{k,n}$. We consider the following situation: a pair of subfactors $N \subset M$, $[M : N] < \infty$, with a chain of intermediate subfactors

$$N \subset P^1 \subset P^2 \subset \dots \subset P^k \subset M.$$

As mentioned in the introduction, much of the information about the subfactor (and in some cases all of it [Po2]) is contained in what is called the (opposite) *standard invariant* [Po1]:

In general, given a II_1 subfactor $N \subset M$, we can perform the basic construction to get a projection e_1 and a new subfactor $M_1 = \{M, e_1\}''$. Iterating this process we get a series of projections $e_1, e_2, e_3 \dots$ and a chain of subfactors (see [Jo1]):

$$N \subset M \subset M_1 \subset M_2 \dots$$

Intersecting this chain with N' and M' respectively gives a double sequence of finite-dimensional algebras:

$$\begin{matrix} N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset \dots \\ \cup & & \cup \\ M' \cap M \subset M' \cap M_1 \subset \dots \end{matrix}$$

known as the *standard invariant*. Now in any subfactor we have

$$\{e_1, e_2 \dots e_n\} \subset N' \cap M_n,$$

$$\{e_2, e_3 \dots e_n\} \subset M' \cap M_n.$$

Thus the description of the algebras $\{e_1, e_2 \dots e_n\}''$, the Temperley-Lieb algebras, gives an underlying basic structure for all standard invariants (see [GHJ]).

In the case of $N \subset M$ with a chain of k intermediate subfactors, it is natural to ask what additional basic structure of the standard invariant is ensured by the presence of such a chain. It is this question that was answered for one intermediate subfactor in [BiJo1]. We now answer the question in the general case using the work that has been developed here.

The presence of $P^1 \subset P^2 \subset \dots \subset P^k$ gives rise to the factors

$$M \subset P_1^k \subset P_1^{k-1} \dots P_1^1 \subset M_1,$$

with $P_1^a = J (P^a)' J$. Here, all the factors are acting on $L^2(M)$ and J is the completion of the canonical anti-isomorphism given by $J(x) = x^*$ on M , which is dense in $L^2(M)$. Alternatively we have $P_1^a = \{M, e_{P_1^a}^M\}''$ where $e_{P_1^a}^M$ is the projection from $L^2(M)$ to $L^2(P^a)$, i.e., P_1^a is the result of the basic construction for the inclusion $P^a \subset M$.

Thus when we iterate the basic construction we get the tower:

$$\begin{aligned} N \subset P^1 \subset P^2 \subset \dots \subset P^k \subset M \subset P_1^k \subset P_1^{k-1} \subset \dots \\ \subset P_1^1 \subset M_1 \subset P_2^1 \subset P_2^2 \subset \dots \end{aligned}$$

with

$$P_j^a = \{M_{j-1}, e_{P_{j-1}^a}^{M_{j-1}}\}'' .$$

To simplify notation we will let

$$e_{a,j} = \begin{cases} e_{M_{j-2}}^{M_{j-1}} & \text{for } a = k, \\ e_{P_{j-1}^{k-a}}^{M_{j-1}} & \text{for } j \text{ odd, } a \neq k, \\ e_{P_{j-1}^{a+1}}^{M_{j-1}} & \text{for } j \text{ even, } a \neq k. \end{cases}$$

(The notation is similar to the notation for the generators for $AA_{k,n}$ above. It is hoped that the similarity will not cause confusion but rather give away the punch line, namely that the map from ${}_a e_j$ to $e_{a,j}$ is generically an isomorphism.)

Thus for a subfactor with a chain of k intermediate subfactors we have:

$$\{1, e_{a,j} : 0 \leq a \leq k, 1 \leq j \leq n - 1\} \subset N' \cap M_n,$$

and

$$\{1, e_{a,j} : 0 \leq a \leq k, 2 \leq j \leq n - 1\} \subset M' \cap M_n.$$

It is the structure of the algebra generated by the set $\{1, e_{a,j} : 0 \leq a \leq k, 1 \leq j \leq n - 1\}$ that we seek.

Definition 19. Let

$$\begin{aligned} \alpha_0 &= [P^1 : N]^{\frac{1}{2}} \\ \alpha_a &= [P^{a+1} : P^a]^{\frac{1}{2}} \quad \text{for } 1 \leq a \leq k - 1 \\ \alpha_k &= [M : P^k]^{\frac{1}{2}} \end{aligned}$$

where $[M : N]$ is the index of the inclusion of $N \subset M$ (see [Jo1] for definition of index).

Let $IA_{k,n}(\alpha_0, \alpha_1, \dots, \alpha_k)$ be the algebra generated by the set $\{1, e_{a,j} : 0 \leq a \leq k, 1 \leq j \leq n - 1\}$.

Lemma 14. Given $IA_{k,n}$ as above we have:

- 1) $e_{a,i} e_{b,i} = e_{\max(a,b),i}$,
- 2) $e_{a,i} e_{b,j} = e_{b,j} e_{a,i}$ for $|i - j| \geq 2$ or, $|i - j| = 1$ and $a + b < k$,
- 3) $e_{a,i} e_{b,i \pm 1} e_{a,i} = c(a, b) e_{a,i} e_{k-a-1, i \pm 1}$ for $a + b \geq k$ ($e_{-1, i \pm 1} = 1$)

with $c(a, b)$ as in Definition 1:

$$c(a, b) = \frac{k-a-1c_{i+1} \ k-b-1c_i}{a c_i \ b c_{i+1}}$$

$$\text{where } a c_i = \begin{cases} \alpha_0 \cdots \alpha_a & \text{for } i \text{ even} \\ \alpha_k \cdots \alpha_{k-a} & \text{for } i \text{ odd.} \end{cases} \quad \text{and } -_1 c_i = 1.$$

Proof. We assume i even; identical arguments hold for i odd. Relation 1 follows from the fact that $P_{i-1}^{\max(a,b)+1} \subset P_{i-1}^{a+1}, P_{i-1}^{b+1}$. Relation 2 follows from $\{e_{a,i}\}' \cap M_{i-1} = P_{i-1}^{a+1}$. Relation 3 follows from [BiJo1, Prop. 5.1]. \square

Lemma 15. *Let tr be the trace on M_n restricted to $IA_{k,n}$. Then tr is a Markov trace, that is,*

$$\text{tr}(1) = 1,$$

$$\text{tr}(xe_{a,n}) = \frac{1}{(ac_n)^2} \text{tr}(x) \text{ for all } x \in IA_{k,n}.$$

Proof. The proof follows from the fact that the conditional expectations are trace preserving:

$$\text{tr}(xe_{a,n}) = \text{tr}(E_{M_{n-1}}^{M_n}(xe_{a,n})) = \text{tr}(xE_{M_{n-1}}^{M_n}(e_{a,n})) = \frac{1}{(ac_n)^2} \text{tr}(x).$$

\square

Theorem 13. *There is a surjective, trace-preserving, unital *-homomorphism*

$$\iota : FC_{k,n}(\alpha_0 \dots \alpha_k) \longrightarrow IA_{k,n}(\alpha_0 \dots \alpha_k)$$

given by

$$\iota(1) = 1,$$

$$\iota(e_{a,i}) = e_{a,i}, \quad 0 \leq a \leq k, \quad 1 \leq i \leq n - 1.$$

Proof. By definition of $IA_{k,n}$, ι is clearly onto. That ι is a *-homomorphism is established by Lemma 14. We now show ι is trace-preserving by showing that the properties:

$$\text{tr}(1) = 1,$$

$$\text{tr}(x_{ae_n}) = \frac{1}{(ac_n)^2} \text{tr}(x) \text{ for } x \in AA_{k,n},$$

completely determine the value of the trace on $\cup_{n=0}^\infty FC_{k,n}$. The form of words in the basis $S_{k,n}$ of $AA_{k,n} \cong FC_{k,n}$ indicates that a word $v \in FC_{k,n+1}$ can be written as $v = x_{ae_n} y$ with x and y in $FC_{k,n}$. Since the trace property gives us that

$$\text{tr}(v) = \text{tr}(x_{ae_n} y) = \text{tr}(yx_{ae_n}) = \frac{1}{(ac_n)^2} \text{tr}(yx),$$

we have by induction that the above properties determine the value of the trace everywhere. Since the trace on $FC_{k,n}$ clearly satisfies the above two properties, ι is trace preserving. \square

In many cases ι is an isomorphism:

Theorem 14. *Suppose $\alpha_i \geq 2$ for $0 \leq i \leq k$. Then the homomorphism $\iota : FC_{k,n} \longrightarrow IA_{k,n}$ in Theorem 13 is an isomorphism. Thus $FC_{k,n}$ is semisimple.*

Proof. Let $FC_0(\alpha) = \cup_{i=0}^\infty FC_{0,i}(\alpha)$. (The $FC_{0,i}(\alpha)$ are generically the Temperley-Lieb algebras.) Consider the trace TR on

$$FC_0(\alpha_0) \otimes FC_0(\alpha_1) \otimes \cdots \otimes FC_0(\alpha_k)$$

given by the tensor product of the traces on $FC_0(\alpha_i)$. It is known that the trace on the Temperley-Lieb algebra $TL(\alpha_i)$ is faithful for $\alpha_i \geq 2$ [GHJ]. Since the tensor product of faithful traces is faithful [Di], TR is faithful. We can think of $FC_{k,n} \subset FC_{0,n}(\alpha_0) \otimes FC_{0,n}(\alpha_1) \otimes \cdots \otimes FC_{0,n}(\alpha_k)$ in the following way: view a diagram in $FC_{k,n}$ as an overlay of $k+1$ diagrams, each diagram consisting of only strings labeled by a_i , $0 \leq i \leq k$. Each of these is an element of $FC_{0,n}(\alpha_i)$ and since multiplication in $FC_{k,n}$ does not combine strings with different labels, the multiplicative structure of $FC_{k,n}$ agrees with that of $FC_0(\alpha_0) \otimes FC_0(\alpha_1) \otimes \cdots \otimes FC_0(\alpha_k)$. It is clear that $TR|_{FC_{k,n}}$ is just the usual trace on $FC_{k,n}$, and thus tr on $FC_{k,n}$ is faithful. Since ι is trace preserving for a faithful trace, $\ker(\iota)=\{0\}$ and ι an isomorphism. \square

The important condition for the above isomorphism was the faithfulness of the trace. To determine the situation for the non-generic case we merely have to examine to what extent the trace on $FC_{k,n} \subset FC_{0,n}(\alpha_0) \otimes FC_{0,n}(\alpha_1) \otimes \cdots \otimes FC_{0,n}(\alpha_k)$ is not faithful.

Recall that Γ_k is the principal graph for the inclusion $FC_{k,0} \subset FC_{k,1} \subset FC_{k,2} \dots$. Each node of Γ_k is labeled by a “new” middle pattern w in the alphabet $\{a_0, a_1, \dots, a_k\}$. Let $v = (v_0, v_1, \dots, v_n) \in \{\mathbf{N} \cup \infty\}^k$. Define

$$W_v = \{w : w \text{ does not have a string of consecutive } a_i\text{'s longer than } v_i\}.$$

Define $\Gamma_{k,v}$ to be the subgraph of Γ_k consisting only of nodes labeled by $w \in W_v$.

Theorem 15. *Given*

$$(\alpha_0, \alpha_1, \dots, \alpha_k), \quad \alpha_i \in \left\{ 2 \cos \frac{\pi}{m} : m \geq 2 \right\} \cup [2, \infty),$$

set

$$v_i = \begin{cases} m & \text{if } \alpha_i = 2 \cos \frac{\pi}{m}, \\ \infty & \text{if } \alpha_i \in [2, \infty). \end{cases}$$

Then $\Gamma_{k,v}$ is the principal graph for the tower

$$\mathbb{C} \subset IA_{k,1} \subset IA_{k,2} \subset \dots$$

Proof. Define the ideal I_n^j to be the ideal generated by $\{x^*x : x \in F_{0,n}(\alpha_j) \text{ and } \text{tr}(x^*x) = 0\}$. Again considering $FC_{k,n} \subset FC_{0,n}(\alpha_0) \otimes FC_{0,n}(\alpha_1) \otimes \cdots \otimes FC_{0,n}(\alpha_k)$ as in the proof of Theorem 14, we have ι factors through the algebra:

$$B = FC_{0,n}(\alpha_0)/I_n^0 \otimes FC_{0,n}(\alpha_1)/I_n^1 \otimes \cdots \otimes FC_{0,n}(\alpha_k)I_n^k :$$

$$\iota : FC_{k,n} \xrightarrow{\iota_1} B \xrightarrow{\iota_2} IA_{k,n}.$$

Because the tensor product of faithful traces is faithful, we have that the trace on B is faithful. Therefore since ι_2 is trace preserving we have ι_2 an isomorphism. Thus for $\iota : FC_{k,n} \rightarrow IA_{k,n}$, $\ker(\iota)$ is the intersection of $FC_{k,n}$ and the ideal generated by

$$\{I_n^0 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \cup 1 \otimes I_n^1 \otimes 1 \otimes \cdots \otimes 1 \cup \cdots \cup 1 \otimes 1 \otimes \cdots \otimes I_n^k\}.$$

Let p be a diagram that is a projection in $FC_{k,n}$ with middle pattern w . It follows that $p \in \ker(\iota) \bmod FC_{k,n} \cong e_{n-1} FC_{k,n}$ if and only if $w \notin W$. The result now follows by applying this fact at every node of the Bratteli diagram Γ_k . \square

This completes the description of the underlying structure of the standard invariant given by the presence of a chain of intermediate subfactors. When considering a subfactor $N \subset M$, we can ask the following questions. Does $N \subset M$ contain chains of intermediate subfactors? If so, how can we use the added structural information obtained in this work to help determine the entire structure of the standard invariant? The answer to the first question is provided in [Bi1]. Work in the direction of the second question appears in [BiJo3], where dimensional criteria are given for the standard invariant to be exactly $FC_{1,n}$, and in [La], where for certain examples, the standard invariants of $N \subset P$ and $P \subset M$ are used to determine the standard invariant of $N \subset M$.

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