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In Memory of Herbert Busemann

In this paper we prove a result connecting symmetric spaces on one hand and symmetry of Busemann functions and the co-ray relation on the other. We then apply the result to hyperbolic and Minkowski geometries thus completing a line of inquiry initiated jointly with Busemann but left unfinished during his lifetime.

1. Introduction.

H. Busemann cretated a beautiful theory of co-rays, asymptotes and parallels in a very general setting. In this theory certain functions constructed by Busemann (and now called Busemann functions) play a central role. In joint previous work [4] with Busemann we found that simple conditions on Busemann functions single out Minkowski's geometries when the parallel axiom is assumed to hold. We then wondered if we can single out the hyperbolic geometry also when the parallel postulate is not satisfied. In this paper we find an affirmative answer by first relating the symmetry of Busemann functions to symmetric spaces. More precisely we prove:

Theorem A. *Let R be a straight G-space in which the co-ray relation is symmetric and the Busemann functions are symmetric. Then R is a symmetric space.*

Theorem C. *A straight G-plane R in which the Busemann functions and the co-ray relation are symmetric is hyperbolic if the parallel axiom does not hold and is Minkowskian if the parallel axiom does hold.*

2. Busemann functions and co-rays.

We assume that the reader has access to Busemann's book [1] which contains precise formulations of the concepts and the basic framework in which we work. We will freely use the results of [1] but recapitulate here very briefly the main definitions and the notation.

Our general setting is that of a metric space R. We denote distance between points x and y by xy. In R a curve is called a segment (respectively a ray, a straight line) if it is isometric to an interval on the real line (respectively a half line, the entire real line). Let $\mathcal{T}$ denote the set of all segments, $\mathcal{R}$ the set of all rays and $\mathcal{L}$ the set of all straight lines in R.

We say that a metric space R is finitely compact if all closed balls in R are compact. A finitely compact metric space is called a straight G-space if for any two points a, b there exists a unique segment $T(a, b)$ of length ab joining a to b and a unique straight line $L(a, b)$ containing a and b.

To indicate the wide variety of examples of straight G-spaces, we remark that all simply connected complete Riemann spaces of non-positive curvature are straight G-spaces. Among notable classes of non-Riemannian straight G-spaces are all Hilbert geometries (see [**1**, Section 18]) and all quasihyperbolic planes (see [**2**]). Another class of examples is provided by Teichmüller spaces of compact Riemann surfaces of genus $g > 1$, when metrized by the Teichmüller metric.

In a straight space given any two points a, b there exists a unique ray which contains a and b and which starts at a. For a ray B with origin z the Busemann function of B is defined as follows:

$$\alpha(B, x) = \lim(xb - zb)$$

where the limit is taken as b recedes to infinity along B, see [**1**, p. 131].

Busemann constructed these functions in order to study, upto an order of magnitude, the difference between the length of a ray and the length of a co-ray. Much later the Busemann functions proved very useful in other areas, in particular for the structure theory of Riemannian manifolds of non-negative curvature.

One of the results in Busemann's theory of parallels is [**1**, (22.16) and (22.20)]:

A ray A is a co-ray to the ray B if and only if for any two points p and q on A with q following p we have

$$\alpha(B, p) - \alpha(B, q) = pq.$$

In general it is not true that if a ray A is a co-ray to B then B is a co-ray to A (see examples in [**1**, pp. 140, 141]). We therefore introduce the symmetry condition for the co-ray relation:

We say that the co-ray relation is symmetric if whenever A is a co-ray to B, B is a co-ray to A.

The co-ray relation is symmetric in hyperbolic and Minkowskian geometries, it is also symmetric in any straight Desarguesian G-space and in many other spaces.

3. Symmetric Busemann functions.

We say that the Busemann functions in a G-space R are symmetric if whenever a line L is divided into two opposite rays B^+ and B^- with common origin z and whenever this point z is the midpoint of any points x and x' the following symmetry condition holds:

$$\alpha(B^+, x) = \alpha(B^-, x').$$

As observed by us in [4, p. 106] Busemann functions are symmetric in any symmetric space. We wondered about the converse implication but did not then find a simple answer. We now show that if the co-ray relation is symmetric and the Busemann functions are symmetric then the space R is a symmetric space.

To prove this we have to show that for any point z if z is midpoint of p, p' and of q, q' then $pq = p'q'$.

Let R^+ be the ray which starts at p and contains q. Let B^+ be the co-ray from z to R^+, L the line containing it and B^- the opposite ray on L starting at z, see Figure 1.

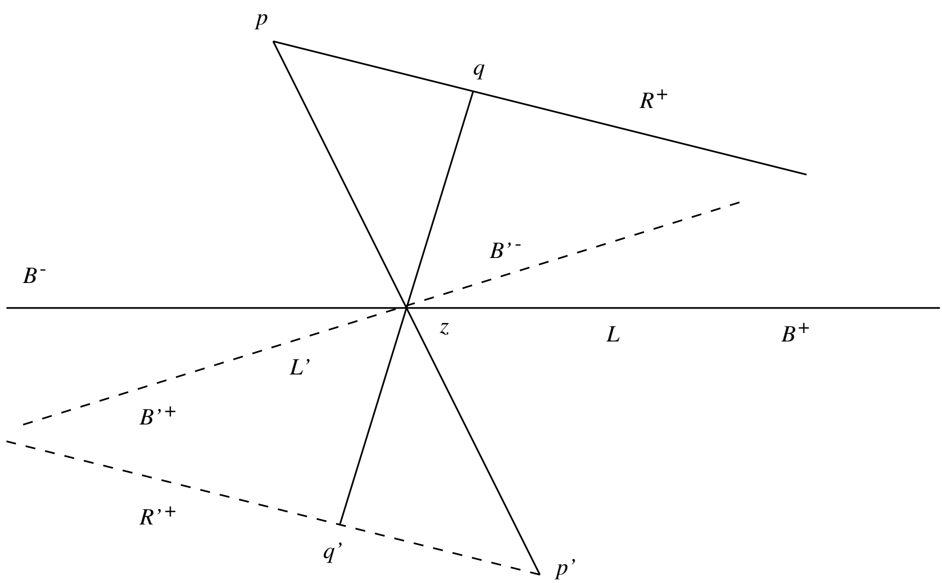

Figure 1.

Since we assume here that the co-ray relation is symmetric, it follows that R^+ is a co-ray to B^+ and q follows p. Hence by the results in [1, (22.16,17, 20)] we have

(i) $$pq = \alpha(B^+, p) - \alpha(B^+, q).$$

By the hypothesis of symmetry of Busemann functions we have

(ii) $$\alpha(B^+, p) - \alpha(B^+, q) = \alpha(B^-, p') - \alpha(B^-, q').$$

For any ray U and points x, y we always have (see [1, (22.5)]),

$$\alpha(U, x) - \alpha(U, y) \leq xy.$$

Hence

$$\text{(iii)} \qquad \alpha(B^-, p') - \alpha(B^-, q') \leq p'q'.$$

The relations (i), (ii), (iii) give us

$$pq \leq p'q'.$$

If we replace the roles of p, q with those of p', q' (see the dotted part of Figure 1), we have

$$p'q' \leq pq.$$

Thus

$$pq = p'q'.$$

This proves that R is symmetric. We therefore have:

Theorem A. *Let R be a straight G-space in which the co-ray relation is symmetric and the Busemann functions are symmetric. Then R is a symmetric space.*

In a symmetric straight plane all translations along all lines exist. (For the definition of a translation in this context see [1, p. 212].) In fact the composition of the symmetry in a point p with the symmetry in a point q is a translation along the line joining p to q by an amount equal to twice the distance between p and q. Now Busemann had proved in [1, (51.5)] the following:

Theorem B. *If in a straight plane R all translations along two lines G and H exist where H is not an asymptote to G in either orientation of G then R is Minkowskian or hyperbolic.*

The Minkowskian case occurs (see the proof in [1, (51.5)]) when the parallel axiom holds and the hyperbolic case occurs when the parallel axiom does not hold. Hence by combining Theorem A with Theorem B we have:

Theorem C. *A straight G-plane R in which the Busemann functions and the co-ray relation are symmetric is hyperbolic if the parallel axiom does not hold and is Minkowskian if the parallel axiom does hold.*

We note that when the parallel axiom holds the co-ray relation is, by part of the definition of the parallel axiom (see [1, p. 141]), automatically symmetric.

4. Conclusion.

In many spaces which arise naturally the shortest joins are not unique even locally but the sets $\mathcal{T}$ and $\mathcal{L}$ of all shortest joins and all straight lines contain respectively subsets $\mathcal{C}$ and $\mathcal{G}$ of distinguished shortest joins and distinguished straight lines which have the missing uniqueness properties. The theory of G-spaces can be extended to this wider class of spaces using a novel principle of selection first formulated by Busemann in [**3**] and developed further in detail by us in [**4**]. The spaces with distinguished shortest joins are called *Chord Spaces*. The idea of obtaining symmetries of the space from the symmetry properties of the Busemann functions remains valid in chord spaces also, but precise formulations would require the build up of an elaborate notation of chord space theory. For this reason the present work is restricted to the setting of straight G-spaces, keeping close to the origins of Busemann's geometry of geodesics.

References

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