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Suppose M is a compact manifold with boundary ∂M. Let $\tilde{M}$ be a normal covering of M. Suppose (A, T) is an elliptic differential boundary value problem on M with lift $(\tilde{A}, \tilde{T})$ to $\tilde{M}$. Then the von Neumann dimension of kernel and cokernel of this lift are defined. The main result of this paper is: These numbers are finite, and their difference, by definition the von Neumann index of $(\tilde{A}, \tilde{T})$, equals the index of (A, T). In this way, we extend the classical L^2-index theorem of Atiyah to elliptic differential boundary value problems.

1. Introduction.

In this paper, we study elliptic differential boundary value problems on coverings of compact manifolds. Let M be a compact Riemannian manifold with boundary ∂M. Suppose $E, F\downarrow M$ and $Y\downarrow\partial M$ are Riemannian vector bundles. Let $A : C^\infty(E) \to C^\infty(F)$ be a differential operator and $T : C^\infty(E) \to C^\infty(Y)$ a differential boundary operator so that the pair $\mathcal{P} := (A, T)$ is elliptic. The following definition will literally also be applied to non-compact spaces.

$$\ker\mathcal{P} := \{f \in L^2(E);\ f \in C^\infty, Af = 0 = Tf\} \qquad \text{and}$$
$$\operatorname{coker}\mathcal{P} := \{(F, f) \in L^2(F) \oplus L^2(Y);$$
$$(F, A\varphi)_{L^2(F)} + (f, T\varphi)_{L^2(Y)} = 0\ \ \forall\varphi \in C_0^\infty(E)\}.$$

The classical theory of elliptic boundary problems states that the dimensions of kernel and cokernel are finite and studies $\operatorname{ind}(\mathcal{P}) := \dim\ker\mathcal{P} - \dim\operatorname{coker}\mathcal{P}$. The index theorem (recalled below) provides deep connections between topological, geometrical and analytical properties of the manifold.

Suppose $\tilde{M}\downarrow M$ is a normal covering of M with deck transformation group Γ. Pull the bundles back to $\tilde{M}$ and lift the operators and metrics. We use the convention that corresponding objects on $\tilde{M}$ have the same notation decorated with an additional tilde. Note that Γ operates on the bundles, their sections and that $\tilde{\mathcal{P}} = (\tilde{A}, \tilde{T})$ is Γ-equivariant. Define the kernel and cokernel of $\tilde{\mathcal{P}}$ literally in the same way as for $\mathcal{P}$. They are in general infinite dimensional. But $\ker(\tilde{\mathcal{P}})$ and $\operatorname{coker}(\tilde{\mathcal{P}})$ have an additional structure: They are Hilbert modules over the group von Neumann algebra $\mathcal{N}(\Gamma)$. For these Hilbert modules, a normalized dimension $\dim_\Gamma$ with values in $[0,\infty]$ is defined. It vanishes exactly if the module is trivial, it is additive under direct sums, and

$$|\Gamma| < \infty \quad \implies \dim_\Gamma = \frac{1}{|\Gamma|} \dim_{\mathbb{C}} . \tag{1.1}$$

The following is the main result of this paper:

Theorem 1.2. *In the situation described above we have* $\dim_\Gamma \ker(\tilde{\mathcal{P}}) < \infty$, $\dim_\Gamma \operatorname{coker}(\tilde{\mathcal{P}}) < \infty$ *and*

$$\operatorname{ind}_\Gamma(\tilde{\mathcal{P}}) := \dim_\Gamma \ker(\tilde{\mathcal{P}}) - \dim_\Gamma \operatorname{coker}(\tilde{\mathcal{P}}) = \operatorname{ind}(\mathcal{P}).$$

Remarkably, $\operatorname{ind}_\Gamma(\tilde{\mathcal{P}})$*, the difference of two reals, is an integer.*

The theorem is particularly interesting because for $\operatorname{ind}(\mathcal{P})$ on M a *purely topological* expression exists, compare Atiyah/Bott [**2**, Theorem 2]: Every elliptic boundary problem (A,p) defines a K-theoretic symbol class $[\sigma(A,p)]$. One assigns to this symbol the topological index, which equals the analytical index. Cohomologically,

$$\operatorname{ind}_t(A,p) = \int_{S(M)} ch(\sigma(A))\pi^*\mathcal{T}(M) + \int_{B(M)|_{\partial M}} ch(\sigma(A,p))\pi^*\mathcal{T}(M),$$

where $\pi : TM \to M$ is the projection, $\mathcal{T}(M)$ is a Todd class of M, ch the Chern character, and $B(M)$ and $S(M)$ are the disc and sphere bundle of TM.

Corollary 1.3 (of Theorem 1.2)**.** *The index of elliptic differential boundary problems is multiplicative under finite coverings.*

Proof. This follows from the multiplicativity (1.1) of $\dim_\Gamma$. □

In Theorem 1.2 we can replace $\operatorname{coker}(\tilde{\mathcal{P}})$ with the kernel of an adjoint boundary problem by Theorem 6.1. Sometimes it is easier to deal with kernels. As an application we compute the Euler characteristic of M in terms of L^2-harmonic forms on $\tilde{M}$ in Theorem 6.4. Dodziuk [**5**] and Donnelly/Xavier [**6**] have computed the sign of the Euler characteristic of closed negatively curved manifolds in this way. An extension to manifolds with boundary is given in [**11**, Section 6].

Our index theorem is the generalization of Atiyah's L^2-index theorem [**1**] to manifolds with boundary. The proof is along the lines of Atiyah's proof. In order to deal with boundary problems, we replace the calculus of pseudo-differential operators by the Boutet de Monvel calculus. As another foundation, in Section 2 we study traces for endomorphisms of Hilbert $\mathcal{N}(\Gamma)$-modules. We use the theory of Sobolev spaces to simplify the work with regularizing operators and especially with their traces. An important result, which should be valuable also in other contexts, is:

Theorem 1.4 (compare Theorem 3.4)**.** *If $r > \dim M/2$, then the inclusion of Sobolev spaces $H^{s+r}(\tilde{M}) \hookrightarrow H^s(\tilde{M})$ is a Γ-trace class operator.*

The idea for the proof of the index theorem is: To $\mathcal{P}$ construct an inverse $\mathcal{Q}$ (modulo smoothing operators) in the BdM calculus which can be lifted to $\tilde{M}$, i.e., $\mathcal{PQ} = \mathbf{1} - \mathcal{S}_1$, $\mathcal{QP} = \mathbf{1} - \mathcal{S}_0$ and $\tilde{\mathcal{P}}\tilde{\mathcal{Q}} = \mathbf{1} - \tilde{\mathcal{S}}_1$, $\tilde{\mathcal{Q}}\tilde{\mathcal{P}} = \mathbf{1} - \tilde{\mathcal{S}}_0$. Then the following two results prove the theorem:

- $\mathrm{ind}_\Gamma(\tilde{\mathcal{P}}) = \mathrm{Sp}_\Gamma \tilde{\mathcal{S}}_0 - \mathrm{Sp}_\Gamma \tilde{\mathcal{S}}_1$ (and the corresponding formula on the base with $\Gamma = \{1\}$).
- For lifts of smoothing operators, we have $\mathrm{Sp}_\Gamma \tilde{\mathcal{S}} = \mathrm{Sp}\mathcal{S}$.

Note that our index theorem does not generalize the Atiyah-Patodi-Singer index theorem [**3**]. They deal with a specific non-local boundary condition. There is also an L^2-version of this type of index theorem, proved by Ramachandran [**9**]. He deals with Dirac type operators and the APS-boundary conditions. Contrariwise, our result is valid for arbitrary elliptic *differential* boundary problems, but we only deal with local boundary conditions. In particular, we can not handle the signature.

This work is part of the Dissertation [**11**] of the author. I thank my advisor Prof. Wolfgang Lück for his constant support.

Throughout the paper, we use the following notation:

Definition 1.5. For $c > 0$ we define

$$a \overset{c}{\leq} b \quad\Longleftrightarrow\quad a \leq c \cdot b$$

and similarly $a \overset{c}{<} b$, ... In a longer chain of inequalities, the same symbol (e.g., c) may be used for different constants.

If not stated otherwise, H is a Hilbert space, $\mathcal{B}(H_1, H_2)$ denotes the bounded operators from H_1 to H_2, $\mathcal{B}(H) = \mathcal{B}(H, H)$. M is a compact smooth manifold of dimension m with boundary ∂M and $E, F\downarrow M$, $Y\downarrow\partial M$ are vector bundles.

2. Traces for $\mathcal{N}(\Gamma)$-module morphisms.

The Hilbert space $l^2(\Gamma) = \{\sum_{g\in\Gamma} \lambda_g \cdot g |\ \sum_g |\lambda_g|^2 < \infty\}$ obviously admits commuting unitary left and right Γ-actions. $\mathcal{N}(\Gamma) = \mathcal{B}(l^2\Gamma)^\Gamma$ consists of all those operators which commute with the right action. A *Hilbert* $\mathcal{N}(\Gamma)$-*module* is a Hilbert space V with left Γ-action so that an isometric embedding $V \hookrightarrow l^2(\Gamma) \otimes H$ exists which is compatible with the Γ-actions. Here H is an arbitrary Hilbert space with trivial Γ-action (in this paper, tensor products are always completed to Hilbert spaces). If V, W are two Hilbert $\mathcal{N}(\Gamma)$-modules, a bounded linear map $f : V \to W$ which is compatible with the Γ-action is called an $\mathcal{N}(\Gamma)$-*module morphism*.

On the $\mathcal{N}(\Gamma)$-endomorphisms of $l^2(\Gamma)$, we have the canonical finite trace $\mathrm{tr}_\Gamma(a) = (a(e), e)_{l^2(\Gamma)}$. Moreover, on every Hilbert space H the trace $\mathrm{Sp}(A) = \sum_i (Ah_i, h_i)$ exists ((h_i) an orthonormal basis of H).

Definition 2.1. This yields a Γ-trace, called Sp_Γ, on the Γ-operators on $l^2(\Gamma) \otimes H$ which is defined by

$$\mathrm{Sp}_\Gamma(a \otimes A) = \mathrm{tr}_\Gamma(a) \cdot \mathrm{Sp}(A).$$

This makes sense only for positive operators and for operators in the Γ-trace class ideal (defined as usual, see [**4**, chapter I]). We also have the Γ-Hilbert Schmidt (HS) operators defined by

$$f \in \mathcal{B}(l^2\Gamma \otimes H)^\Gamma \text{ is } \Gamma\text{-HS} \quad \iff \quad \mathrm{Sp}_\Gamma(f^*f) < \infty.$$

Given an orthonormal base $\{u_i\}$ of H, we define isometric embeddings $U_i : l^2(\Gamma) \to l^2(\Gamma) \otimes H : x \mapsto x \otimes u_i$. An explicit formula for the Γ-trace of a positive or Γ-trace class Γ-operator f on $l^2(\Gamma) \otimes H$ is then given by

$$\mathrm{Sp}_\Gamma(f) = \sum_i \mathrm{tr}_\Gamma(U_i^* f U_i).$$

Definition 2.2. Let V_k be Hilbert $\mathcal{N}(\Gamma)$-modules with isometric Γ-embeddings $i_k : V_k \hookrightarrow l^2(\Gamma) \otimes H_k$. Set $p_k := i_k^*$ $(k = 1, 2)$. Let $f : V_1 \to V_2$ be a Hilbert $\mathcal{N}(\Gamma)$-module morphism.

We call $f : V_1 \to V_2$ Γ-*Hilbert Schmidt* (Γ-*HS*) if $\mathrm{Sp}_\Gamma(i_1 f^* f p_1) < \infty$, and we denote it Γ-*trace class* (Γ-*tr*) if Γ-HS morphisms $f_1 : V_1 \to V_3$ and $f_2 : V_3 \to V_2$ exist so that $f = f_2 f_1$.

If $V_1 = V_2$ and f is Γ-tr we set

$$\mathrm{Sp}_\Gamma(f) := \mathrm{Sp}_\Gamma(i_1 f p_1).$$

The following basic properties show in particular that this is well defined.

Theorem 2.3. *Let $f : V_1 \to V_2$, $g : V_2 \to V_3$, $e : V_0 \to V_1$ be Hilbert $\mathcal{N}(\Gamma)$-module morphisms. Then:*

1. f *Γ-tr* $\iff$ f^* *Γ-tr* $\iff$ $|f|$ *Γ-tr;* $\quad f$ *Γ-HS* $\iff$ f^* *Γ-HS.*
2. f *Γ-HS* $\implies$ gf, fe *Γ-HS.*
3. f *Γ-tr* $\implies$ gf, fe *Γ-tr.*
4. f *Γ-tr and* $V_1 = V_3 \implies g \mapsto \mathrm{Sp}_\Gamma(gf)$ *is ultra-weakly continuous.*
5. $V_1 = V_3$ *and either* f *Γ-tr or* f, g *Γ-HS* $\implies \mathrm{Sp}_\Gamma(gf) = \mathrm{Sp}_\Gamma(fg)$.
6. *If* $V_{1,2} = l^2(\Gamma) \otimes H$, a *is* Γ*-HS and* $B \in \mathcal{B}(H)$ *is HS, then* $f = a \otimes B$ *is* Γ*-HS. If* a *is* Γ*-tr and* B *is trace class, then* f *is* Γ*-tr with* $\mathrm{Sp}_\Gamma(f) = \mathrm{tr}_\Gamma(a)\mathrm{Sp}(B)$.

Proof. These are rather straightforward consequences of the proofs of the corresponding well known properties of tr_Γ and Sp. (For a detailed proof compare [**11**, 9.13].) Note in particular that the statements are standard if $V_1 = V_2 = V_3 = l^2(\Gamma) \otimes H$. In view of Definition 2.2 and the polar decomposition, the general case is based on the following fact:

If $u : l^2(\Gamma)\otimes H_2 \to l^2(\Gamma)\otimes H_1$ is a partial isometry and an $\mathcal{N}(\Gamma)$-morphism, and if $f : l^2(\Gamma) \otimes H_1 \to l^2(\Gamma) \otimes H_2$ is Γ-tr, then

$$\mathrm{Sp}_\Gamma(uf) = \mathrm{Sp}_\Gamma(fu). \tag{2.4}$$

First consider the case where u is injective (this implies $u^*u = \mathbf{1}$). Then $\mathrm{Sp}_\Gamma(uf) = \mathrm{Sp}_\Gamma(uu^*uf) = \mathrm{Sp}_\Gamma(u(fu)u^*)$ by the trace property on $l^2(\Gamma) \otimes H_1$. Since arbitrary trace class operators are linear combinations of positive operators, assume that $g = fu$ is positive. Then

$$\begin{aligned}\mathrm{Sp}_\Gamma(ugu^*) &= \mathrm{Sp}_\Gamma(u\sqrt{g}(u\sqrt{g})^*) \overset{(*)}{=} \mathrm{Sp}_\Gamma((u\sqrt{g})^*u\sqrt{g})\\ &= \mathrm{Sp}_\Gamma(\sqrt{g}\sqrt{g}) = \mathrm{Sp}_\Gamma(g).\end{aligned}$$

It remains to establish for f as above

$$\mathrm{Sp}_\Gamma(f^*f) = \mathrm{Sp}_\Gamma(ff^*). \tag{$*$}$$

For this, choose an orthonormal basis $\{h_i\}_{i\in I}$ of H_1. This gives rise to isometric embeddings $U_i : l^2(\Gamma) \to l^2(\Gamma)\otimes H_1 : x \mapsto x\otimes h_i$. Similarly, choose an orthonormal basis $\{v_j\}$ of H_2 and construct $V_j : l^2(\Gamma) \to l^2(\Gamma)\otimes H_2$. Then

$$\begin{aligned}\mathrm{Sp}_\Gamma(f^*f) &= \sum_{i\in I} \mathrm{tr}_\Gamma(U_i^* f^* f U_i) = \sum_{i\in I}\sum_{j\in J} \mathrm{tr}_\Gamma(U_i^* f^* V_j V_j^* f U_i)\\ &= \sum_{i,j} \mathrm{tr}_\Gamma(V_j^* f U_i U_i^* f^* V_j) = \sum_j \mathrm{tr}_\Gamma(V_j^* f f^* V_j) = \mathrm{Sp}_\Gamma(ff^*).\end{aligned}$$

The fact that $\{h_i\}$ is an orthonormal basis implies $\sum_i U_iU_i^* = 1$ weakly. Moreover, we used the fact that tr_Γ is a trace and is normal. All summands are non-negative. Therefore, neither the order of summation nor convergence (allowing $+\infty$ as possible value) are an issue.

Back to the the proof of (2.4). Suppose now u is surjective. Then u^* is injective and

$$\mathrm{Sp}_\Gamma(uf) \overset{\text{trace on } l^2(\Gamma)\otimes H_1}{=} \mathrm{Sp}_\Gamma(f^*u^*) = \mathrm{Sp}_\Gamma(u^*f^*) = \mathrm{Sp}_\Gamma(fu).$$

If u is arbitrary, decompose u as follows:

$$u = p_2 \circ (\mathbf{1} \oplus u) : X \otimes H_2 \to (X \otimes H_2) \oplus (X \otimes H_2) \to X \otimes H_2.$$

$$\begin{aligned}\mathrm{Sp}_\Gamma(uf) = \mathrm{Sp}_\Gamma(p_2(\mathbf{1}\oplus u)f) &\overset{p_2 \text{ surjective}}{=} \mathrm{Sp}_\Gamma((\mathbf{1}\oplus u)fp_2)\\ &\overset{(\mathbf{1}\oplus u) \text{ injective}}{=} \mathrm{Sp}_\Gamma(fp_2(\mathbf{1}\oplus u)) = \mathrm{Sp}_\Gamma(fu).\end{aligned}$$

To complete the proof one has to do (quite a lot of) computations of similar spirit and apply (2.4) and the trace properties for operators on $l^2(\Gamma)\otimes H$. This does not seem to be very enlightening and is left as an exercise. □

As usual, armed with a Γ-trace we define the Γ-dimension:

Definition 2.5. Let V be a Hilbert $\mathcal{N}(\Gamma)$-module. Then

$$\dim_\Gamma(V) := \mathrm{Sp}_\Gamma(\mathrm{id}_V) \in [0,\infty].$$

We now come to an important result, which is essentially proved in Atiyah's paper [**1**, p. 67]. He does not state it explicitly and in full generality, but his proof works nearly literally. (This proof can also be found in [**11**, 9.16].)

Proposition 2.6. *Suppose V, W are Hilbert $\mathcal{N}(\Gamma)$-modules. Let $T_0 : V \to V$ and $T_1 : W \to W$ be bounded Γ-morphisms which are Γ-tr. Let $D : V \to W$ be a closed operator with domain $\mathcal{D}(D)$ which commutes with the action of Γ. Especially, we require that $\mathcal{D}(D)$ is Γ-invariant and dense. Suppose*

$$T_1 D \subset D T_0; \quad \ker D \subset \ker T_0; \quad \ker D^* \subset \ker T_1^*.$$

Then

$$\mathrm{Sp}_\Gamma(T_0) = \mathrm{Sp}_\Gamma(T_1).$$

3. L^2-Rellich lemma.

Let M be a compact m-dimensional manifold with boundary ∂M (possibly empty). Let $\tilde{M}$ be a normal covering of M with covering group Γ (acting by isometries). Let $E{\downarrow}M$ be a vector bundle with pullback $\tilde{E}{\downarrow}\tilde{M}$.

There is a natural way to define Sobolev spaces on $\tilde{M}$:

Definition 3.1. Choose a finite covering of M by charts κ_i with subordinate partition of unity φ_i so that E is trivial over the domain of κ_i with trivialization t_i. Lift charts, partition of unity and trivializations to $\tilde{M}$. Then we define the Sobolev norm $|\cdot|_{H^s}$ by

$$|\sigma|^2_{H^s} := \sum_{\gamma\in\Gamma}\sum_i \left|\tilde{t}_i \circ (\tilde{\varphi}_i \cdot \gamma^*\sigma) \circ \tilde{\kappa}_i^{-1}\right|^2_{H^s(\mathbb{R}^m)} \qquad \sigma \in C_0^\infty(\tilde{E}).$$

The Sobolev space $H^s(\tilde{E})$ is defined as the completion of $C_0^\infty(\tilde{E})$ with respect to this norm. The inner product does depend on the choices, but not the topology.

We will show in this section that $H^s(\tilde{E})$ is a Hilbert $\mathcal{N}(\Gamma)$-module and that the inclusion $H^{s+r}(\tilde{E}) \hookrightarrow H^s(\tilde{E})$ is Γ-HS for $r > m/2$.

Let W be the double of M with reflection $\mathrm{fl} : W \to W$. Let $X{\downarrow}W$ be the double of E. The reflection fl extends as a bundle map to X. Construct similarly $\tilde{W}$ and $\tilde{X}$. Then $\tilde{W}$ is a normal covering of W with covering group Γ. Again we denote the reflection fl.

Lemma 3.2. *Fix $s \in \mathbb{R}$. There exists a bounded Γ-equivariant extension map $e : H^s(\tilde{M}) \to H^s(\tilde{W})$, i.e., $e(f)|_{\tilde{M}} = f \;\; \forall f \in H^s(\tilde{M})$. The restricition map is also Γ-equivariant and bounded.*

The corresponding statement holds for $\tilde{E}$.

Proof. A straightforward exercise. One uses a Γ-invariant covering of $\tilde{M}$ by charts and the corresponding extension map on Euclidian space (Taylor [**14**, I.5.1]). $\square$

Suppose $U \subset \tilde{M} \subset \tilde{W}$ is a fundamental domain for the covering $p : \tilde{M} \to M$. This means that U is open, $p|_U$ is injective and $M - p(U)$ is a set of measure zero. Choose U so that its closure is compact, and choose a compact submanifold with boundary $T \subset \tilde{W}$ of codimension zero, so that $U \cup \mathrm{fl}(U) \subset T$ and so that the interior of T is mapped surjectively onto W.

Lemma 3.3. *Suppose $s \in \mathbb{R}$. The map p defined by the composition*

$$\begin{array}{ccccc} H^s(\tilde{M}) & \xrightarrow{e} & H^s(\tilde{W}) & \xrightarrow{\bar{p}} & l^2(\Gamma) \otimes H^s(T) \\ \in & & \in & & \in \\ f & \longrightarrow & ef & \longrightarrow & \sum_{g\in\Gamma} g \otimes \underbrace{g^{-1}(ef)}_{=g^*(ef)}|_T \end{array}$$

is Γ-equivariant, and there exist $C_{1,2} > 0$ so that

$$|f|_{H^s(\tilde{M})} \overset{C_1}{\leq} |pf|_{l^2(\Gamma)\otimes H^s(T)} \overset{C_2}{\leq} |f|_{H^s(\tilde{M})}.$$

In particular, $H^s(\tilde{M})$ (with the pull back norm under p) is a Hilbert $\mathcal{N}(\Gamma)$-module. The corresponding statement holds for $\tilde{E}$.

Proof. By Lemma 3.2, e has the required properties. It remains to consider $\bar{p}$. Obviously, $\bar{p}$ is Γ-equivariant.

Because Γ is discrete and T is compact, it meets only finitely many, say N, of its translates $\{gT\}_{g\in\Gamma}$.

By definition, $|\sum g \otimes f_g|^2_{l^2(\Gamma)\otimes H^s(T)} = \sum |f_g|^2_{H^s(T)}$. To show that $\bar{p}$ is bounded let $\{U_i\}_{i=1,\dots N}$ be open subsets of $\tilde{W}$ which cover T so that the covering projection maps each U_i injectively to W. Choose submanifold charts κ_i for $(U_i, U_i \cap T)$ and functions $0 \leq \varphi_i \leq 1$ with compact support in U_i so that $\sum_i \varphi_i = 1$ on T. Recognize that for every single i we can extend $(U_i, \varphi_i, \kappa_i)$ to a corresponding collection $(U^i_{\alpha,\gamma}, \varphi^i_{\alpha,\gamma}, \kappa^i_{\alpha,\gamma})_{\alpha,\gamma}$ which can be used to compute Sobolev norms on $\tilde{W}$. The norm will depend on the data (hence on i), but all such norms are equivalent. Therefore for $f \in H^s(\tilde{W})$

$$\begin{aligned} |\bar{p}f|^2_{l^2(\Gamma)\otimes H^s(T)} &= \sum_{i=0}^{N} \sum_{\gamma\in\Gamma} \left|\varphi_i \gamma^* f \circ \kappa_i^{-1}\right|^2_{H^s(\mathbb{R}^m_{\geq 0})} \\ &\leq \sum_i \sum_\gamma \sum_{\alpha=1}^{N_i} \left|(\varphi^i_{\alpha,\gamma} f) \circ (\kappa^i_{\alpha,\gamma})^{-1}\right|^2_{H^s(\mathbb{R}^m)} \end{aligned}$$

(since we have more and larger summands)

$$\overset{NC}{\leq} |f|^2_{H^s(\tilde{W})}.$$

On the other hand (fix i)

$$|f|^2_{H^s(\tilde{W})} = \sum_{\alpha=1}^{N_i} \sum_{\gamma} \left|(\varphi^i_{\alpha,\gamma} f) \circ (\kappa^i_{\alpha,\gamma})^{-1}\right|^2_{H^s(\mathbb{R}^m)}$$

(choose $U^i_{\alpha,\gamma}$ so small that each of them lies in the interior of some translate of T. Then we can for every fixed α add more positive summands to get (up to norm equivalence) $|\cdot|_{l^2(\Gamma)\otimes H^s(T)}$. Therefore:)

$$\overset{CN_i}{\leq} |f|_{l^2(\Gamma)\otimes H^s(T)}.$$

The computations for $\tilde{E}$ are similar, but notationally more complicated. □

Theorem 3.4. *Suppose $s, r \in \mathbb{R}$. The inclusion $\tilde{i} : H^{s+r}(\tilde{E}) \to H^s(\tilde{E})$ is Γ-HS if $r > m/2$, and is Γ-tr if $r > m$.*

Proof. Let $X{\downarrow}W$ be the double of E. The following diagram commutes by the geometric definition of p:

$$\begin{array}{ccc} H^{s+r}(\tilde{E}) & \xrightarrow{p_{s+r}} & l^2(\Gamma) \otimes H^{s+r}(\tilde{X}|_T) \\ \tilde{i}\big\downarrow & & \big\downarrow \mathbf{1}\otimes i \\ H^s(\tilde{E}) & \xrightarrow{p_s} & l^2(\Gamma) \otimes H^s(\tilde{X}|_T). \end{array}$$

Remember that we have equipped $H^s(\tilde{E})$ with the Hilbert space structure which makes p an isometric embedding, therefore $p^*p = \mathbf{1}$. This yields

$$\tilde{i} = p_s^* p_s \tilde{i} = p_s^*(\mathbf{1} \otimes i) p_{s+r}.$$

Now we apply Properties (2) and (6) of Theorem 2.3, together with the classical result that for bundles over compact manifolds the inclusion $H^{s+r} \hookrightarrow H^s$ is HS if $r > m/2$ and trace class if $r > m$. □

4. Boutet de Monvel calculus.

The *Boutet de Monvel* (*BdM*) calculus is a tool to deal with boundary value problems. It generalizes the calculus of pseudo-differential operators on manifolds without boundary. We will not go into the details but only give a reminder of those results which are essential for our applications. Detailed accounts can be found in [**10**] or [**13**] with proofs of the statements below. We will follow the notation of these sources, in particular [**13**].

The main point of the Boutet de Monvel calculus is the introduction of an algebra of operators which includes the boundary problems we want to study and also their inverses. To do this, we have to consider matrices of operators:

Let M be a manifold with boundary ∂M. Let $E, F \downarrow M$ be vector bundles over M, $X, Y \downarrow \partial M$ bundles over the boundary. A BdM operator $\mathcal{P}$ has the shape

$$\mathcal{P} = \begin{pmatrix} A+G & K \\ T & p \end{pmatrix} : \begin{matrix} C_0^\infty(E) \\ \oplus \\ C_0^\infty(X) \end{matrix} \to \begin{matrix} C^\infty(F) \\ \oplus \\ C^\infty(Y) \end{matrix},$$

where A and p are pseudo-differential operators on M and ∂M, respectively. A boundary value problem (A, T) will give typical entries in the matrix above.

Every BdM operator has an order $\mu \in [-\infty, \infty)$ and a type $d \in \mathbb{N}_0$. The order is a generalization of the order of a (pseudo)differential operator, the type is determined by T and G and says "how much restriction to the boundary" is involved. It restricts the range of Sobolev spaces, to which $\mathcal{P}$ can be extended.

Up to smoothing operators, BdM operators are locally defined: $\mathcal{P}$ is BdM (of order $\leq \mu$ and type $\leq d$), if and only if for all cutoff functions φ and ψ ($\psi = 1$ on $\operatorname{supp}\varphi$) the operator $\varphi\mathcal{P}\psi$ is BdM (of order $\leq \mu$ and type $\leq d$), and if $\varphi\mathcal{P}(1-\psi)$ is a smoothing operator of type zero.

By definition, $\mathcal{P}$ is a smoothing operator (i.e., of order $-\infty$) of type d, if it has smooth integral kernels in the following sense: The pseudo-differential operators A and p have smooth integral kernels $a(x, y)$ and $p(x, y)$; and for $F \in C_0^\infty(E)$ and $f \in C_0^\infty(X)$ we have

$$\begin{aligned} GF(x) &= \sum_{i=1}^{d} \int_{\partial M} g_i(x, y')(\partial_\nu)^{i-1} F(y') dy' + \int_M g_0(x, y) F(y) dy \\ Kf(x) &= \int_{\partial M} k(x, y') f(y') dy' \\ TF(x') &= \sum_{i=1}^{d} \int_{\partial M} t_i(x', y')(\partial_\nu)^{i-1} F(y') dy' + \int_M t_0(x', y) F(y) dy, \end{aligned}$$

where ∂_ν denotes differentiation in inward unit normal direction.

Here $g_0 \in C^\infty(\operatorname{Hom}(p_2^* E, p_1^* F) \downarrow M \times M)$, and g_i, t_i and k are smooth sections of appropriate homomorphism bundles, too.

The following properties are basic extensions of corresponding properties of pseudo-differential operators. In compliance with our sources assume M is compact:

Let $\mathcal{P} : C^\infty(E) \oplus C^\infty(X) \to C^\infty(F) \oplus C^\infty(Y)$ and $\mathcal{Q} : C^\infty(F) \oplus C^\infty(Y) \to C^\infty(G) \oplus C^\infty(Z)$ be BdM operators of order μ and type d and μ', d' respectively. Then the composition $\mathcal{Q}\mathcal{P}$ is a BdM operator of order $\mu + \mu'$ and

type $\max\{d', d+\mu'\}$.
If $s > d - 1/2$, then $\mathcal{P}$ extends to a continuous operator

$$\mathcal{P} : H^s(E) \oplus H^s(X) \to H^{s-\mu}(F) \oplus H^{s-\mu}(Y).$$

We are interested in index problems. To do this, we have to define ellipticity: A BdM operator $\mathcal{P}$ of order $\mu \geq 0$ and type $d \leq \mu$ is elliptic if and only if there exists a BdM operator $\mathcal{Q} : C^\infty(F) \oplus C^\infty(Y) \to C^\infty(E) \oplus C^\infty(X)$ of order $-\mu$ and type zero so that

$$\mathcal{S}_0 := \mathcal{Q}\mathcal{P} - \mathbf{1} \qquad \text{and} \qquad \mathcal{S}_1 := \mathcal{P}\mathcal{Q} - \mathbf{1}$$

are of order $-\infty$ and $\mathcal{S}_0$ is of type μ, $\mathcal{S}_1$ of type zero. $\mathcal{Q}$ is called a *parametrix* of $\mathcal{P}$ (it is unique up to operators of order $-\infty$).

As mentioned above, every differential boundary problem $\mathcal{P} = (A, T) : C_0^\infty(E) \to C_0^\infty(F) \oplus C_0^\infty(Y)$ is a Boutet de Monvel operator. If it is elliptic in the Lopatinsky-Shapiro sense, it is also elliptic in the sense of the BdM algebra.

Definition 4.1. Equip M with a Riemannian metric. An operator $\mathcal{P} : C_0^\infty(E) \oplus C_0^\infty(X) \to C^\infty(F) \oplus C^\infty(Y)$ is called ϵ-local ($\epsilon > 0$), if

$$\operatorname{supp}(\mathcal{P}f) \subset \{x \in M;\ d(x, \operatorname{supp} f) < \epsilon\} \quad \forall f \in C_0^\infty.$$

Proposition 4.2. *Suppose M is a compact Riemannian manifold and $\epsilon > 0$ is given. Every BdM operator $\mathcal{P}$ is the sum of an ϵ-local BdM operator* (*of unchanged order and type*) *and a smoothing operator of type zero.*

Proof. Choose a finite covering of M by balls $\{U_i\}$ of radius $\epsilon/2$. Let $\{\varphi_i\}$ be a subordinate partition of unity and ψ_i cutoff functions with $\psi_i = 1$ on $\operatorname{supp} \varphi_i$ and $\operatorname{supp} \psi_i \subset U_i$. Set

$$\mathcal{P}_1 := \sum_i \varphi_i \mathcal{P} \psi_i, \quad \mathcal{P}_2 := \mathcal{P} - \mathcal{P}_1 = \sum_i \varphi_i \mathcal{P}(1 - \psi_i).$$

Then $\mathcal{P}_2$ is a smoothing BdM operator of type zero and $\mathcal{P}_1$ is ϵ-local. □

Proposition 4.3. *Let $\tilde{M} \downarrow M$ be a normal Riemannian covering of Riemannian manifolds with covering group Γ, where M is compact. Suppose the covering is trivial over balls of radius 2ϵ. Suppose*

$$\mathcal{P} : C^\infty(E) \oplus C^\infty(X) \to C^\infty(F) \oplus C^\infty(Y)$$

is an ϵ-local operator which extends to a bounded operator

$$\mathcal{P} : H^s(E) \oplus H^s(X) \to H^{s-\mu}(F) \oplus H^{s-\mu}(Y).$$

Then $\mathcal{P}$ lifts to an operator

$$\tilde{\mathcal{P}} : C^\infty(\tilde{E}) \oplus C^\infty(\tilde{X}) \to C^\infty(\tilde{F}) \oplus C^\infty(\tilde{Y}),$$

which has a bounded extension

$$\tilde{\mathcal{P}} : H^s(\tilde{E}) \oplus H^s(\tilde{X}) \to H^{s-\mu}(\tilde{F}) \oplus H^{s-\mu}(\tilde{Y}).$$

Proof. Let $\{U_i\}_{i=1,\dots,N}$ be a covering of M by balls of radius ϵ, let V_i be the corresponding balls of radius 2ϵ. Let φ_i be a subordinate covering of unity. This induces a Γ-invariant covering $\{U_{i,\gamma}\}_{\gamma\in\Gamma}$ of $\tilde{M}$ with subordinate Γ-invariant partition of unity $\varphi_{i,\gamma}$. It is clear how to lift $\mathcal{P}$. To check boundedness, let $\mathcal{F} = (F, f) \in C_0^\infty(\tilde{E}) \oplus C_0^\infty(\tilde{X})$ be given. Then (use $|a+b|^2 \le 3(|a|^2 + |b|^2)$)

$$\left|\tilde{\mathcal{P}}\mathcal{F}\right|^2_{H^{s-\mu}} = \left|\tilde{\mathcal{P}}\sum_{i,\gamma}\varphi_{i,\gamma}\mathcal{F}\right|^2_{H^{s-\mu}} \overset{3^N}{\le} \sum_{i=1}^N \left|\tilde{\mathcal{P}}\sum_{\gamma}\varphi_{i,\gamma}\mathcal{F}\right|^2_{H^{s-\mu}}$$
$$\overset{(*)}{=} \sum_{i,\gamma}\left|\tilde{\mathcal{P}}\varphi_{i,\gamma}\mathcal{F}\right|^2_{H^{s-\mu}} \overset{\|\mathcal{P}\|^2}{\le} \sum_{i,\gamma} |\varphi_{i,\gamma}\mathcal{F}|^2_{H^s} \overset{\text{Def}}{=} |\mathcal{F}|^2_{H^s}.$$

$(*)$ holds since $\operatorname{supp}(\varphi_{i,\gamma}) \cap \operatorname{supp}(\varphi_{i,\gamma'}) = \emptyset$ if $\gamma \ne \gamma'$. □

Next we compute the trace of sufficiently regularizing BdM operators. Most important is the fact that the Γ-trace of a lift equals the trace of the operator on the base.

Theorem 4.4. *Let $\mathcal{P} : C^\infty(E) \oplus C^\infty(X) \to C^\infty(E) \oplus C^\infty(X)$ be a BdM operator of order $-\mu < -m = \dim M$ and type d. For $s > d - 1/2$, $\mathcal{P}$ extends to a bounded trace class operator*

$$\mathcal{P} : H^s(E) \oplus H^s(X) \to H^s(E) \oplus H^s(X).$$

The value of the trace is independent of s.

If $\mathcal{P}$ is ϵ-local then its lift $\tilde{\mathcal{P}} : H^s(\tilde{E}) \oplus H^s(\tilde{X}) \to H^s(\tilde{E}) \oplus H^s(\tilde{X})$ (defined for $s > d - 1/2$) is Γ-tr and

$$\operatorname{Sp}_\Gamma(\tilde{\mathcal{P}}) = \operatorname{Sp}(\mathcal{P}).$$

If $-\mu = -\infty$ and $\mathcal{P}$ has integral kernels as on page 431 then explicitly

$$\operatorname{Sp}(\mathcal{P}) = \int_M \operatorname{Sp}_{E_x} a(x,x)dx + \int_{\partial M} \operatorname{Sp}_{X'_x} p(x',x')dx'$$
$$+ \int_M \operatorname{Sp}_{E_x} g_0(x,x)dx + \sum_{i=1}^d \int_{\partial M} \operatorname{Sp}_{E_{x'}} \partial^{i-1}_{\nu,x} p_i(x,y)|_{x=x'=y} dx'$$

(Sp_F denotes the trace on the finite dimensional vector space F; ∂_ν is differentiation in normal direction).

Proof. The inclusion $H^{s+\mu} \hookrightarrow H^s$ is of trace class by Theorem 3.4. Therefore $\mathcal{P} : H^s \xrightarrow{\mathcal{P}} H^{s+\mu} \hookrightarrow H^s$ is of trace class, being the composition of a bounded operator and a trace class operator. If $\mu - m > s' - s > 0$ then

$$\operatorname{Sp}(\mathcal{P} : H^{s'} \to H^{s'}) = \operatorname{Sp}(H^{s'} \hookrightarrow H^s \xrightarrow{\mathcal{P}} H^{s+\mu} \hookrightarrow H^{s'})$$
$$= \operatorname{Sp}(H^s \xrightarrow{\mathcal{P}} H^{s+\mu} \hookrightarrow H^{s'} \hookrightarrow H^s) = \operatorname{Sp}(\mathcal{P} : H^s \to H^s).$$

Here we used the trace property, noting that $H^{s+\mu} \hookrightarrow H^s$ is trace class. Inductively, the trace is independent of s for arbitrary s.

Identical arguments apply to the lift $\tilde{\mathcal{P}}$, replacing trace by Γ-trace and using Theorem 3.4.

Now we come to the explicit computation, and $\mu = -\infty$. Observe (with $\mathcal{P}$ in the usual matrix form) $\mathrm{Sp}(\mathcal{P}) = \mathrm{Sp}(A) + \mathrm{Sp}(G) + \mathrm{Sp}(p)$. Note that A and p are actually defined on L^2. The above argument applies to show that $\mathrm{Sp}(A : H^s \to H^s) = \mathrm{Sp}(A : L^2 \to L^2)$. A is an integral operator with a smooth kernel and therefore with trace

$$\mathrm{Sp}(A) = \int_M \mathrm{Sp}_{E_x} a(x,x)dx.$$

Similarly $\mathrm{Sp}(p) = \int_{\partial M} \mathrm{Sp}_{X_{x'}} p(x',x')dx'$. For the obvious splitting $G = G_0 + G_1 + \cdots + G_d$, note that each summand is trace class. G_0 behaves exactly as A does. For $i > 0$, the operator G_i is a composition

$$H^s(E) \stackrel{\partial_\nu^{i-1}}{\to} H^{s-i+1}(E) \stackrel{\mathrm{res}}{\to} H^{s-i+1/2}(E|_{\partial M}) \stackrel{K_i}{\to} H^\infty(E) \stackrel{i}{\hookrightarrow} H^s(E).$$

Each of the operators is bounded and the inclusion is trace class (res denotes the restriction to the boundary and K_i is the obvious integral operator with smooth kernel from $E|_{\partial M} \to E$). Using the trace property and the fact that inclusions of Sobolev spaces commute with differentiation and restriction to the boundary, we see

$$\mathrm{Sp}(G_i) = \mathrm{Sp}(i \circ \underbrace{\mathrm{res} \circ \partial_\nu^{i-1} \circ K_i}_{=:P_i}).$$

Now P_i is an integral operator with smooth kernel on ∂M, namely

$$P_i f(x') = \int_{\partial M} (\partial_{\nu,x}^{i-1} g_i)(x',y')f(y')dy'.$$

Therefore it extends to a trace class operator on $L^2(E|_{\partial M})$ with

$$\mathrm{Sp}(G_i) = \mathrm{Sp}(P_i) = \int_{\partial M} \mathrm{Sp}_{E_{x'}}(\partial_{\nu,x}^{i-1} g_i(x,y))|_{x=x'=y} dx'.$$

This establishes the formula for $\mathrm{Sp}(\mathcal{P})$.

Identical arguments apply to the lift $\tilde{\mathcal{P}}$ as far as follows:

$$\mathrm{Sp}_\Gamma(\tilde{\mathcal{P}}) = \mathrm{Sp}_\Gamma(\tilde{A}) + \mathrm{Sp}_\Gamma(\tilde{p}) + \mathrm{Sp}_\Gamma(\tilde{G}_0) + \sum_{i=1}^d \mathrm{Sp}_\Gamma(\tilde{P}_i),$$

where each summand is the lift of an integral operator with smooth kernel on $L^2(E)$, $L^2(X)$ and $L^2(E|_{\partial M})$, respectively.

Therefore, it remains to show that for an ϵ-local trace class operator R on L^2 the Γ-trace of the lift coincides with the trace on the base.

Let s_i be an orthonormal basis of $L^2(E)$ such that the support of each s_i is contained in a set over which $\tilde{M}| \to M$ and $E| \to M$ are trivial. Choose for each s_i one lift $\tilde{s}_i \in L^2(\tilde{E})$. Then we have the standard formula for trace and Γ-trace (2.1)

$$\mathrm{Sp}\,(R) = \sum_i (Rs_i, s_i)_{L^2\Omega^*(X,V)}; \qquad \mathrm{Sp}_\Gamma(\tilde{R}) = \sum_i (\tilde{R}\tilde{s}_i, \tilde{s}_i)_{L^2\Omega^*(\tilde{X},\tilde{V})}.$$

The fact that $\tilde{R}$ is the lift of R and $\tilde{s}_i$ the lift of s_i implies that the two expressions coincide, i.e.,

$$\mathrm{Sp}_\Gamma(\tilde{R}) = \mathrm{Sp}\,(R).$$

This applies to all the above operators and completes the proof. □

5. Proof of the L^2-index theorem.

Situation 5.1. Let $\tilde{M}\downarrow M$ be a normal covering of a compact manifold with boundary with deck transformation group Γ. Let $\mathcal{P} = (A, T) : C_0^\infty(E) \to C_0^\infty(F) \oplus C_0^\infty(Y)$ be an elliptic differential boundary problem on M. Denote its lift to $\tilde{M}$ with $\tilde{\mathcal{P}} : C_0^\infty(\tilde{E}) \to C_0^\infty(\tilde{F}) \oplus C_0^\infty(\tilde{Y})$. Suppose $\mathcal{P}$ has order $\mu \geq 0$ and type $d \leq \mu$.

We have the extension $\mathcal{P} : H^\mu(E) \to L^2(F) \oplus L^2(Y)$.

Let $H_0 : L^2(E) \to \ker(\mathcal{P})$ be the orthogonal projection onto the kernel, $H_1 : L^2(F) \oplus L^2(Y) \to \mathrm{im}(\mathcal{P})^\perp$ the orthogonal projection onto the cokernel of $\mathcal{P}$. Similarly, let $\bar{H}_0$ and $\bar{H}_1$ be the projections onto kernel and cokernel of $\tilde{\mathcal{P}}$.

We want to prove the L^2-index Theorem 1.2 for ∂-manifolds:

Theorem 5.2. $\dim_\Gamma \ker \tilde{\mathcal{P}} = \mathrm{Sp}_\Gamma(\bar{H}_0)$ *and* $\dim_\Gamma \operatorname{coker} \tilde{\mathcal{P}} = \mathrm{Sp}_\Gamma(\bar{H}_1)$ *are finite, and*

$$\mathrm{ind}_\Gamma(\tilde{\mathcal{P}}) := \mathrm{Sp}_\Gamma(\bar{H}_0) - \mathrm{Sp}_\Gamma(\bar{H}_1) = \mathrm{ind}(\mathcal{P}) = \mathrm{Sp}(H_0) - \mathrm{Sp}(H_1).$$

The idea of the proof is the following: H_i and $\bar{H}_i$ have in general nothing to do with each other. But suppose we could find a bounded liftable "inverse" $\mathcal{Q}$ to $\mathcal{P}$. Then the equations

$$\mathcal{P}\mathcal{Q} = \mathbf{1} - H_1 \quad \text{and} \quad \mathcal{Q}\mathcal{P} = \mathbf{1} - H_0$$

could be lifted and we could compare the trace of H_i and $\bar{H}_i$ directly. This is not possible. We use a parametrix instead:

Let $\mathcal{Q}$ be an ϵ-local parametrix of $\mathcal{P}$ (use Proposition 4.2) so that

$$\begin{aligned} &\mathcal{P}\mathcal{Q} = \mathbf{1} - \mathcal{S}_1, \quad \mathcal{Q}\mathcal{P} = \mathbf{1} - \mathcal{S}_0 \\ \Longrightarrow \quad &\tilde{\mathcal{P}}\tilde{\mathcal{Q}} = \mathbf{1} - \tilde{\mathcal{S}}_1, \quad \tilde{\mathcal{Q}}\tilde{\mathcal{P}} = \mathbf{1} - \tilde{\mathcal{S}}_0. \end{aligned} \tag{5.3}$$

Automatically, $\mathcal{S}_0 = \mathbf{1} - \mathcal{Q}\mathcal{P}$ and $\mathcal{S}_1$ are ϵ-local since the right hand side is. Note that $\mathcal{S}_0$ and $\tilde{\mathcal{S}}_0$ are operators of order $-\infty$ and type μ, whereas $\mathcal{S}_1$ and $\tilde{\mathcal{S}}_1$ have order $-\infty$ and type zero.

We know already that $\mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_i = \mathrm{Sp}\mathcal{S}_i$ (Theorem 4.4). It remains to show that we can compute the index also in terms of the $\mathcal{S}_i$, namely

$$\mathrm{Sp}\mathcal{S}_0 - \mathrm{Sp}\mathcal{S}_1 = \mathrm{Sp}H_0 - \mathrm{Sp}H_1 \tag{5.4}$$

(and similarly on $\tilde{M}$). This will be achieved using Proposition 2.6. We start with:

Proposition 5.5. *The image of the projection $H_0 : L^2(E) \to L^2(E)$ (i.e., the kernel of $\mathcal{P}$) is contained in $H^\infty(E)$ and H_0 restricts to a bounded operator $H_0 : H^s(E) \to H^{s+t}(E)$ for arbitrary $s, t \geq 0$. Especially $H_0 : H^s \to H^s$ is trace class for every $s \geq 0$ and the trace is independent of s.*

The same holds for $\bar{H}_0$ if we replace tr *by* tr_Γ*.*

Proof. Elliptic regularity and the corresponding a priori estimates (the theory works as in the compact case, compare [**12**, 4.14] for a generalization) imply that the kernels of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are contained in every Sobolev space $H^s(E)$ and $H^s(\tilde{E})$ respectively, and that the Sobolev norms on this subspace are equivalent to the L^2-norm. This implies everything if we consider H_0 as composition of the bounded operator $H_0 : H^s \to H^{s+m+1}$ with the trace class operator $i : H^{s+m+1} \hookrightarrow H^s$ (and similarly for $\bar{H}_0$). □

Now we can prove Equation 5.4. The following computations are formulated only for the lifted operators. They are valid also on the base with the obvious changes.

Multiplying the equations in (5.3) with $\bar{H}_1$ from the left and with $\bar{H}_0$ from the right, we get

$$\bar{H}_1 = \bar{H}_1\tilde{\mathcal{S}}_1 \qquad \bar{H}_0 = \tilde{\mathcal{S}}_0\bar{H}_0, \tag{5.6}$$

where the equation for $\bar{H}_0$ is valid on H^μ and the one for $\bar{H}_1$ is valid on all of L^2 . By multiplication of (5.3) with $\tilde{\mathcal{P}}$ we get on H^μ

$$\tilde{\mathcal{P}}\tilde{\mathcal{S}}_0 = \tilde{\mathcal{S}}_1\tilde{\mathcal{P}}.$$

Following Atiyah [**1**] we now define

$$\bar{T}_i := (1 - \bar{H}_i)\tilde{\mathcal{S}}_i(1 - \bar{H}_i) \quad (i = 0, 1).$$

Because of Theorem 2.3 (3) $\bar{T}_0$ is a Γ-tr operator on the Hilbert $\mathcal{N}(\Gamma)$-module H^μ and $\bar{T}_1$ is a Γ-tr operator on the Hilbert $\mathcal{N}(\Gamma)$-module L^2. Since $\bar{H}_i$ are projectors

$$\begin{aligned} \mathrm{Sp}_\Gamma\bar{T}_0 &= \mathrm{Sp}_\Gamma(\tilde{\mathcal{S}}_0(1 - \bar{H}_0)) = \mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_0 - \mathrm{Sp}_\Gamma\bar{H}_0 \qquad &(\text{use } (5.6)),\\ \mathrm{Sp}_\Gamma\bar{T}_1 &= \mathrm{Sp}_\Gamma((1 - \bar{H}_1)\tilde{\mathcal{S}}_1) = \mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_1 - \mathrm{Sp}_\Gamma\bar{H}_1 \qquad &(\text{use } (5.6)).\end{aligned}$$

Therefore,

$$\mathrm{Sp}_\Gamma\bar{T}_0 = \mathrm{Sp}_\Gamma\bar{T}_1 \quad \iff \quad \mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_0 - \mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_1 = \mathrm{Sp}_\Gamma\bar{H}_0 - \mathrm{Sp}_\Gamma\bar{H}_1.$$

Next observe

$$\ker\tilde{\mathcal{P}} \subset \ker\bar{T}_0; \qquad \ker\tilde{\mathcal{P}}^* \subset \ker\bar{T}_1^*;$$

$$\tilde{\mathcal{P}}\bar{T}_0 = \tilde{\mathcal{P}}\tilde{\mathcal{S}}_0 - \underbrace{\tilde{\mathcal{P}}\bar{H}_0}_{=0}\tilde{\mathcal{S}}_0 - \tilde{\mathcal{P}}\underbrace{\tilde{\mathcal{S}}_0\bar{H}_0}_{=\bar{H}_0} + \underbrace{\tilde{\mathcal{P}}\bar{H}_0}_{=0}\tilde{\mathcal{S}}_0\bar{H}_0 = \tilde{\mathcal{S}}_1\tilde{\mathcal{P}} = \cdots = \bar{T}_1\tilde{\mathcal{P}}.$$

Application of Proposition 2.6 with $V = H^\mu$, $W = L^2$ (then $\mathcal{P} : V \to W$ is bounded) yields $\mathrm{Sp}_\Gamma\bar{T}_0 = \mathrm{Sp}_\Gamma\bar{T}_1$, i.e., $\mathrm{ind}_\Gamma\tilde{\mathcal{P}} = \mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_0 - \mathrm{Sp}_\Gamma\tilde{\mathcal{S}}_1$. Similarly, $\mathrm{ind}\,\mathcal{P} = \mathrm{Sp}\mathcal{S}_0 - \mathrm{Sp}\mathcal{S}_1$. Now Theorem 4.4 appplied to the ϵ-local smoothing operators $\mathcal{S}_0$, $\mathcal{S}_1$ finishes the proof of Theorem 1.2.

6. Index and adjoint boundary value problems.

The purpose of this section is to simplify the index formula by replacing the cokernel with the kernel of the adjoint.

Theorem 6.1. *Let $E, F \downarrow M$, $X, Y \downarrow \partial M$ be Riemannian vector bundles, $\mathcal{P} := (A, p) : C_0^\infty(E) \to C_0^\infty(F) \oplus C_0^\infty(Y)$ an elliptic differential boundary problem. Suppose the differential boundary problem $\mathcal{Q} := (B, q) : C_0^\infty(F) \to C_0^\infty(E) \oplus C_0^\infty(X)$ is adjoint to (A, p) with respect to the Greenian formula*

$$(Ae, f)_{L^2(F)} - (e, Bf)_{L^2(E)} = (pe, sf)_{L^2(Y)} - (te, qf)_{L^2(X)}. \tag{6.2}$$

(Here t, s are auxiliary boundary differential operators, and adjointness means that the formula holds $\forall e \in C_0^\infty(E), \forall f \in C_0^\infty(F)$.) Then

$$L^2(F) \oplus L^2(Y) \supset \qquad \mathrm{im}(\mathcal{P})^\perp \xrightarrow{p_1} L^2(F) : (f, y) \mapsto f$$

is an isomorphism onto $\ker(\mathcal{Q})$ *with inverse*

$$\alpha : \ker(\mathcal{Q}) \to \mathrm{im}(\mathcal{P})^\perp : f \mapsto (f, -sf).$$

Proof. First, we have to prove that the maps have range as stated. Take $(f, y) \in \mathrm{im}(\mathcal{P})^\perp$. In particular, $f \perp A(\{e;\ pe = 0\})$. Choosing e which are supported in the interior of M (these are dense in L^2) (6.2) implies $Bf = 0$. [**12**, Lemma 4.7] yields that the set $\{te|\ e \in C_0^\infty(E)$ and $pe = 0\}$ is dense in $L^2(Y)$ (observe that ellipticity implies that (p, t) is a Dirichlet system in the notion of [**12**, Lemma 4.7]). Then (6.2) also implies $qf = 0$. That α has the correct image follows immediately from the Greenian formula.

It remains to check $\alpha \circ p_1 = \mathbf{1}_{\mathrm{im}\,\mathcal{P}^\perp}$: If $(f, y) \in \mathrm{im}(\mathcal{P})^\perp$, then for arbitrary $e \in C_0^\infty(E)$

$$\begin{aligned}(pe, y) &\overset{(f,y)\perp\mathrm{im}(\mathcal{P})}{=} -(Ae, f)\\ &\overset{(6.2)}{=} -(e, Bf) - (pe, sf) + (te, qf) \overset{f\in\ker\mathcal{Q}}{=} -(pe, sf).\end{aligned}$$

Again, [**12**, Lemma 4.7] implies that $\mathrm{im}(p)$ is dense in $L^2(Y)$ and therefore $y = -sf$. ∎

Being in the situation of the L^2-index Theorem 1.2, the isomorphism of Theorem 6.1 is equivariant under the group operation and $\operatorname{coker}(\tilde{\mathcal{P}})$ is Γ-isomorphic to $\ker(\tilde{\mathcal{Q}})$. Therefore the index theorem can be stated as follows:

Theorem 6.3. *Suppose M is a compact boundary manifold with normal covering $\tilde{M}$ and covering group Γ. Let $\mathcal{P} := (A, T)$ be an elliptic differential boundary problem on M with lift $\tilde{\mathcal{P}}$. Let $\mathcal{Q} := (B, S)$ be an adjoint with lift $\tilde{\mathcal{Q}}$. Then*

$$\operatorname{ind}(\mathcal{P}) = \quad \operatorname{ind}_\Gamma(\tilde{\mathcal{P}}) = \dim_\Gamma(\ker \tilde{\mathcal{P}}) - \dim_\Gamma(\ker \tilde{\mathcal{Q}}).$$

We apply this to compute the Euler characteristic of a ∂-manifold. Lott/Lück [7] get the same result with other methods.

Theorem 6.4. *Suppose M is a compact manifold with boundary $\partial M = M_1 \amalg M_2$. Let $\tilde{M}$ be a normal covering of M with covering group Γ. Then*

$$\chi(M, M_1) = \sum_p (-1)^p \dim_\Gamma \mathcal{H}^p_{(2)}(\tilde{M}, \tilde{M}_1)$$

with $\mathcal{H}^p_{(2)}(\tilde{M}, \tilde{M}_1) = \{\omega \in C^\infty(\Lambda^p T\tilde{M});\ |\omega|_{L^2} < \infty, d\omega = 0 = \delta\omega, b_1^*(\omega) = 0 = b_2^*(*\omega)\}$*.* ($b_i : \tilde{M}_i \hookrightarrow \tilde{M}$ *are the inclusions.*)

Proof. To keep notation simple suppose $M_1 = \emptyset$. We known $\chi(M) = \operatorname{ind}(\mathcal{P}^{ev})$, where $\mathcal{P}^{ev/odd}$ are the boundary problems

$$(d + \delta, b_2^* \circ *) : C^\infty(\Lambda^{ev/odd} TM) \to C^\infty(\Lambda^{odd/ev} TM) \oplus C^\infty(\Lambda^* T\partial M).$$

We have the following Greenian formula

$$\begin{aligned}((d+\delta)\omega, \eta)_{L^2(M)} \\ = (\omega, (\delta + d)\eta)_{L^2(M)} \pm \int_{\partial M} b^*\omega \wedge b^*(*\eta) \pm \int_{\partial M} b^*(\eta) \wedge b^*(*\omega).\end{aligned}$$

Theorems 6.1 and 6.3 yield then

$$\chi(M) = \operatorname{ind}(\tilde{\mathcal{P}}^{ev}) = \dim\ker(\tilde{\mathcal{P}}^{ev}) - \dim\ker(\tilde{\mathcal{P}}^{odd}).$$

In view of elliptic regularity this is just the claim. □

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