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ADMISSIBLE HERMITIAN METRICS ON FAMILIES OF LINE BUNDLES OVER CERTAIN DEGENERATING RIEMANN SURFACES

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We show that a family of line bundles of degree zero over a plumbing family of Riemann surfaces with a separating (resp. non-separating) node p admits a nice (resp. almost nice) family of flat p-singular Hermitian metrics. As a consequence, we give necessary and sufficient conditions for a family of line bundles over such families of Riemann surfaces to admit an (almost) nice family of p-singular Hermitian metrics which are admissible with respect to the canonical/hyperbolic (1,1)-forms on the Riemann surfaces.

1. Introduction.

Let $\mathcal{L} = \{L_t\}$ be a family of holomorphic line bundles over a degenerating family of Riemann surfaces $\mathcal{M} = \{M_t\}$. We are interested in finding necessary and sufficient conditions for $\{L_t\}$ to admit a family of Hermitian metrics (or equivalently a Hermitian metric on $\mathcal{L}$) satisfying certain curvature conditions.

A degenerating family of Riemann surfaces $\{M_t\}$ is obtained by shrinking non-trivial closed loops of compact Riemann surfaces to form a noded Riemann surface M. This corresponds to a path in the moduli space $\mathcal{M}_q$ of compact Riemann surfaces of genus q leading to boundary points in its Deligne-Mumford compactification $\overline{\mathcal{M}}_q$. When there is only a single node, we have essentially two cases, depending on whether the node separates M. An important subclass of degenerating Riemann surfaces with a single node is obtained by means of plumbing (cf. [**Wo1**] and (2.1)). Throughout this article, we will restrict our considerations to such plumbing families of Riemann surfaces. Also we assume $q \geq 2$, and that M is stable, or equivalently, its smooth part $M^0 := M \backslash \{\text{node}\}$ admits the hyperbolic metric of constant sectional curvature -1.

This work is motivated by earlier works on the asymptotic behaviors of the hyperbolic metrics and the canonical metrics as well as those of their Green's functions on degenerating Riemann surfaces (cf. [**F**], [**H**], [**Ji**], [**JW**], [**We**], [**Wo2**]). In particular, Wolpert [**Wo2**] showed that the hyperbolic metrics glue together to form a continuous and good Hermitian metric on the vertical line bundle (induced by the tangent bundles of the fibers) over the universal curve $\overline{\mathcal{C}}_q$ over $\overline{\mathcal{M}}_q$. A notable feature is that this metric is not smooth along the noded Riemann surfaces, and it is mildly singular at the nodes.

In this article, we consider line bundles over Riemann surfaces in general. Our first main result is to give necessary and sufficient conditions for a family of line bundles of degree 0 over a plumbing family of Riemann surfaces to admit a nice/(almost nice) family of flat p-singular Hermitian metrics (cf. Theorem 1 in §2 for the precise statements and the additional necessary conditions in the separating node case; cf. also (2.2) for the definition of 'niceness' and 'almost niceness'). As applications of Theorem 1, we also give necessary and sufficient conditions for a family of line bundles over a plumbing family of Riemann surfaces to admit a nice/(almost nice) family of Hermitian metrics which are admissible with respect to the hyperbolic (resp. canonical) (1,1)-forms (cf. Theorem 2 and Theorem 3).

We sketch our approach in proving Theorem 1 as follows. First we construct a family of 'almost flat' Hermitian metrics using the flat Hermitian metric on the line bundle over the noded fiber. Then it is modified to a flat family by using the hyperbolic Green's operators. We remark that this approach is similar to [**Wo2**] in spirit, and it depends crucially on the initially constructed family of metrics being sufficiently close to a flat one. There are small but subtle differences in the construction and estimates involved in the two cases of a non-separating/separating node, and we treat the two cases separately in §3 and §4 respectively. The proofs of Theorem 2 and Theorem 3 depend on Theorem 1 as well as results of Wolpert [**Wo2**] on the continuity and goodness of the family of hyperbolic metrics and those of Wentworth [**We**] on Arakelov Green's functions respectively.

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2. Notation and statements of results.

2.1. . Throughout this article, we consider plumbing families of compact Riemann surfaces of fixed genus $q \geq 2$ degenerating into a stable singular Riemann surface M with a single separating or non-separating node p. First we recall the plumbing construction of a degenerating family of Riemann surfaces starting from M in both cases as follows (cf. e.g., [**F**], [**Wo2**]).

In the non-separating node case, the normalization $\widetilde{M}$ of M is a compact Riemann surface of genus $q-1$. In the separating node case, $\widetilde{M}$ consists of a disjoint union of two compact Riemann surfaces M_1, M_2 of genus q_1, q_2 respectively such that $q_1 + q_2 = q$. The stable condition on M implies that

q_1, $q_2 \geq 1$. In both cases, the node p corresponds to two points p_1, p_2 in $\widetilde{M}$ (with p_k in M_k, $k = 1, 2$, in the separating node case) so that $M^0 := M\backslash\{p\}$ can be identified with $\widetilde{M}\backslash\{p_1, p_2\}$. Denote the unit disc in $\mathbb{C}$ by Δ. In both cases and for $k = 1, 2$, fix a coordinate function $z_k : U_k \to \Delta$ such that $z_k(p_k) = 0$, where U_k is an open coordinate neighborhood of p_k in $\widetilde{M}$ (and thus in the separating node case, $U_k \subset M_k, k = 1, 2$). Also for each $t \in \Delta$, let $S_t := \{(z_1, z_2) \in \Delta^2 \,|\, z_1 z_2 = t\}$. Then for each $t \in \Delta$, remove the discs $|z_k| < |t|$ from $\widetilde{M}$ and glue the remaining parts of $\widetilde{M}$ with S_t via the identification $z_1 \sim (z_1, t/z_1)$ and $z_2 \sim (t/z_2, z_2)$. The resulting surfaces $\{M_t\}_{t\in\Delta}$ form an analytic family $\pi : \mathcal{M} \to \Delta$ with $M_0 = M$, where π denotes the holomorphic projection map. It is easy to see that for $t \neq 0$, each M_t is a compact Riemann surface of genus q. We remark that for fixed $t \in \Delta^*$, one may adjust the sizes of the removed discs and the plumbing collar $z_1 z_2 = t$ in the plumbing process without changing M_t. Also, away from the plumbing collars, the resulting family may be described as the Cartesian product of an open subset of $M \setminus \{p\}$ and Δ, shrinking Δ if necessary. The restriction of $\ker(d\pi)$ to $\mathcal{M}\backslash\{p\}$ forms a holomorphic line bundle $\mathcal{T}$ over $\mathcal{M}\backslash\{p\}$ such that $\mathcal{T}\big|_{M_t} = TM_t$ for $t \in \Delta^*$ and $\mathcal{T}\big|_{M^0} = TM^0$. Note that $\mathcal{T}$ extends uniquely to a holomorphic line bundle $\widetilde{\mathcal{T}}$ over $\mathcal{M}$ known as the vertical line bundle.

2.2. . Let R be a smooth compact Riemann surface or a noded Riemann surface with a node p, and let L be a holomorphic line bundle over R. We define a *p-singular Hermitian metric* h on L to be simply a C^∞ Hermitian metric on $L\big|_{R^0}$, where $R^0 := R\backslash\{p\}$ is the smooth part of R (thus in the case when R is smooth, i.e., when p is absent, such an h is simply a smooth Hermitian metric on L). Now let μ be a smooth (1,1)-form on R^0. In both the smooth and noded cases, a p-singular Hermitian metric h on L is said to be *μ-admissible* if its first Chern form satisfies $c_1(L, h) = \deg(L) \cdot \mu$ on R^0, where $\deg(L)$ denotes the degree of L over R. We remark that when R is a noded Riemann surface, $\deg(L)$ is simply the sum of the degrees of f^*L over all the components of $\widetilde{R}$, where $f : \widetilde{R} \to R$ is the normalization of R. Observe also that μ is necessarily normalized (i.e., $\int_R \mu = 1$) in the smooth case (but not necessarily so in the noded case). Also h is said to be *flat* if $c_1(L, h) \equiv 0$ on R^0.

Next we recall the definition of 'good' Hermitian metrics introduced by Mumford [**M**] in the special case of line bundles over complex manifolds. Let $\overline{X}$ be an n-dimensional complex manifold with an open subset X such that $\overline{X} - X$ is a divisor with normal crossings. Consider coordinate polydiscs $U = \Delta^n \subset \overline{X}$ such that $U \cap X = (\Delta^*)^k \times \Delta^{n-k}$ for some $1 \leq k \leq n$, and denote by $\omega_{U\cap X}$ the product metric on $U \cap X$ induced by the Poincaré metric $ds^2 = (|dz|/(|z| \log |z|))^2$ on each Δ^* and the Euclidean metric on each Δ. Now let $\overline{L}$ be a holomorphic line bundle over $\overline{X}$ and let L be the restriction of $\overline{L}$ to X. A smooth Hermitian metric h on L is said to be *good* on $\overline{X}$ if there exists a finite set of coordinate polydiscs $\{U_\alpha\}$ covering an open neighborhood of $\overline{X}-X$ in $\overline{X}$ such that for each $U_\alpha=\Delta^n$, there exists a non-vanishing holomorphic section $v\in\Gamma(U_\alpha,\overline{L}\big|_{U_\alpha})$ such that on $U_\alpha\cap X=(\Delta^*)^k\times\Delta^{n-k}$,

(i) $|h_{v\bar v}|, 1/|h_{v\bar v}|\le C_1\Big(\sum_{i=1}^{k}\log|z_i|\Big)^{2m}$ for some $C_1>0$, $m\ge 1$, and

(ii) $\partial\log h(v,v)$ and $\partial\overline{\partial}\log h(v,v)$ have Poincaré growth on $U_\alpha-U_\alpha\cap X$, i.e., there exist constants $C_2,C_3>0$ such that

$$|\partial_{t_1}\log h_{v\bar v}|^2\le C_2\omega_{U_\alpha\cap X}(t_1,t_1)\quad\text{and}$$
$$|\partial_{t_2}\overline{\partial}_{t_3}\log h_{v\bar v}|^2\le C_3\omega_{U_\alpha\cap X}(t_2,t_2)\omega_{U_\alpha\cap X}(t_3,t_3)$$

for all $t_1,t_2,t_3\in T_x(U_\alpha\cap X)$, $x\in U_\alpha\cap X$.

One easily sees that the above definition does not depend on the choice of local coordinate functions and local trivializations of $\overline{L}$ on each U_α. Moreover, it is known that given (L,h), there exists at most one extension $\overline{L}$ of L to $\overline{X}$ for which h is good (cf. [**M**, §1]).

Let $\mathcal{M}=\{M_t\}_{t\in\Delta}$ be a plumbing family of Riemann surfaces degenerating to a singular Riemann surface M with a separating or non-separating node p as described in (2.1). Also let $\mathcal{L}=\{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles over $\{M_t\}_{t\in\Delta}$, i.e., $\mathcal{L}$ is a holomorphic line bundle over $\mathcal{M}$ and $L_t=\mathcal{L}\big|_{M_t}$ for $t\in\Delta$. Now let $\{\mu_t\}_{t\in\Delta}$ be a *family of p-singular $(1,1)$-forms* on $\{M_t\}_{t\in\Delta}$, i.e., each μ_t (resp. μ_0) is a (1,1)-form on M_t (resp. $M^0=M\backslash\{p\}$) for $t\in\Delta^*$ (resp. $t=0$), and they form a continuous section of $\mathcal{T}^*\otimes\overline{\mathcal{T}^*}$ over $\mathcal{M}\backslash\{p\}$, where $\mathcal{T}$ is as in (2.1).

Definition 2.2.1. $h=\{h_t\}_{t\in\Delta}$ (with $h_t=h\big|_{L_t}$ for $t\in\Delta$) is said to be a *nice family of $\{\mu_t\}_{t\in\Delta}$-admissible* (resp. *flat*) *p-singular Hermitian metrics* on $\mathcal{L}\{L_t\}_{t\in\Delta}$ if the following conditions hold:

(i) for each $t\in\Delta$, h_t is a μ_t-admissible (resp. flat) p-singular Hermitian metric on L_t;

(ii) h is continuous on $\mathcal{M}\backslash\{p\}$, and the restriction h^* of h to $\mathcal{L}\big|^*_{\mathcal{M}}$, where $\mathcal{M}^*:=\mathcal{M}-M_0$, is smooth on $\mathcal{M}^*$; and

(iii) the Hermitian metric h^* on $\mathcal{L}\big|^*_{\mathcal{M}}$ is good on $\mathcal{M}$.

If $\{h_t\}_{t\in\Delta}$ satisfies only conditions (i) and (ii) above, then we say that $\{h_t\}_{t\in\Delta}$ is an *almost nice family of $\{\mu_t\}_{t\in\Delta}$-admissible* (resp. *flat*) *p-singular Hermitian metrics* on $\{L_t\}_{t\in\Delta}$.

Remark 2.2.2. It is well-known that for a family of line bundles $=\mathcal{L}=\{L_t\}_{t\in\Delta}$ over $\mathcal{M}=\{M_t\}_{t\in\Delta}$ as above, $\deg(L_t)$ remains the same for all $t\in$

Δ. In fact, this can easily be proved by first fixing a smooth Hermitian metric h on $\mathcal{L}$ over the manifold $\mathcal{M}$, letting $h_t = h\big|_{L_t}$ for all $t \in \Delta$, and then using the dominated convergence theorem to show that $\lim_{t\to 0}\int_{M_t} c_1(L_t, h_t) = \int_{M_0\setminus\{p\}} c_1(L_0, h_0) = \deg(L_0)$ (here the last equality holds since (f^*L_0, f^*h_0) is a smooth Hermitian metric over $\widetilde{M}$, where $f : \widetilde{M} \to M$ denotes the normalization).

Remark 2.2.3. Let $\{M_t\}_{t\in\Delta}$, $\{\mu_t\}_{t\in\Delta}$ be as above, and let $\mathcal{L} = \{L_t\}_{t\in\Delta}$, $\mathcal{L}' = \{L'_t\}_{t\in\Delta}$ be two holomorphic families of line bundles over $\{M_t\}_{t\in\Delta}$. Then one easily checks that the following statements hold:
(i) If both $\mathcal{L} = \{L_t\}_{t\in\Delta}$ and $\mathcal{L}' = \{L'_t\}_{t\in\Delta}$ admit nice (or almost nice) families of $\{\mu_t\}_{t\in\Delta}$-admissible (or flat) p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ and $h' = \{h'_t\}_{t\in\Delta}$ respectively, then so does $\mathcal{L}\otimes\mathcal{L}' = \{L_t\otimes L'_t\}_{t\in\Delta}$ (given by $h\otimes h' = \{h_t\otimes h'_t\}_{t\in\Delta}$).
(ii) If $\mathcal{L}^{\otimes m} = \{L_t^{\otimes m}\}_{t\in\Delta}$ admits a nice (or almost nice) family of $\{\mu_t\}_{t\in\Delta}$-admissible (or flat) p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ for some nonzero integer m, then so does $\mathcal{L}$ itself (given by $h'(s,s) = \big(h(s^{\otimes m}, s^{\otimes m})\big)^{\frac{1}{m}}$).

Remark 2.2.4. As an example, it follows from a result of Wolpert [**Wo2**, Theorem 5.8] that the hyperbolic metrics on $\{M_t\}_{t\in\Delta}$ glue together to form a nice family of $\{\hat{\omega}_{hyp,t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics on the family of line bundles defined by $\tilde{\mathcal{T}}$ (cf. (2.4) for the definition of $\{\hat{\omega}_{hyp,t}\}_{t\in\Delta}$).

2.3. . For a plumbing family of degenerating Riemann surfaces $\{M_t\}_{t\in\Delta}$ with singular fiber M with a node at p, we recall from (2.1) the normalization $f : \widetilde{M} \to M$ with points $p_1, p_2 \in \widetilde{M}$ corresponding to p (and recall also that $\widetilde{M} = M_1 \sqcup M_2$ with M_k of genus q_k and $p_k \in M_k, k = 1, 2$, in the separating node case). Also in the separating node case, for a family of line bundles $\mathcal{L} = \{L_t\}_{t\in\Delta}$ of degree d over $\{M_t\}_{t\in\Delta}$ (cf. Remark 2.2.2), we denote $L_{0,k} := f^*L_0\big|_{M_k}$ and denote $d_k := \deg(L_{0,k})$, $k = 1, 2$ (and thus $d_1 + d_2 = d$). Our first main result in this paper is the following:

Theorem 1. *Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be a plumbing family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a stable Riemann surface M with a single separating or non-separating node p as described in (2.1). Suppose $\mathcal{L} = \{L_t\}_{t\in\Delta}$ is a holomorphic family of line bundles of degree 0 over $\{M_t\}_{t\in\Delta}$. Then the following statements hold:*
(i) *In the case of a non-separating node, $\{L_t\}_{t\in\Delta}$ always admits an almost nice family of flat p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ such that f^*h_0 extends across p_1, p_2 to a smooth flat Hermitian metric on f^*L_0 over $\widetilde{M}$.*
(ii) *In the case of a separating node, $\{L_t\}_{t\in\Delta}$ admits a nice family of flat p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ such that f^*h_0 extends to smooth*

flat Hermitian metrics on $L_{0,1}$ over M_1 and $L_{0,2}$ over M_2 if and only if $d_1 = d_2 = 0$.

Remark 2.3.1. In both cases, it is easy to see that such $\{h_t\}_{t\in\Delta}$ is unique up to a positive continuous multiplicative function on Δ which is smooth on Δ^*.

Remark 2.3.2. In the case of a non-separating node, we do not know whether the almost nice family of flat p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ constructed in Theorem 1(i) is actually nice or not. In the case of a separating node in Theorem 1(ii), an important fact which allows us to prove the goodness of h is that we can construct flat Hermitian metrics on $L_{0,1}$ and $L_{0,2}$ agreeing at p (cf. §4). Such an ingredient is lacking in Theorem 1(i) (cf. Remark 4.2.11).

2.4. . Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$, $M^0 = M\backslash\{p\}$, be as in (2.1). The stable condition on M implies that each irreducible component of $M\backslash\{p\}$ admits the complete hyperbolic metric of constant sectional curvature -1, which will be collectively denoted by $ds^2_{\text{hyp},0}$ on M^0. For $t \neq 0$, we denote the hyperbolic metric on M_t by $ds^2_{\text{hyp},t}$. Their associated (1,1)-forms (resp. normalized (1,1)-forms) will be denoted by $\omega_{\text{hyp},t}$ (resp. $\widehat{\omega}_{\text{hyp},t}$), so that $\omega_{\text{hyp},t} = 4\pi(q-1)\widehat{\omega}_{\text{hyp},t}$ for $t \in \Delta$. By a result of Wolpert [**Wo2**, Theorem 5.8], $\{ds^2_{\text{hyp},t}\}_{t\in\Delta}$ (and thus also $\{\omega_{\text{hyp},t}\}_{t\in\Delta}$ and $\{\hat{\omega}_{\text{hyp},t}\}_{t\in\Delta}$) forms a p-singular family on $\{M_t\}_{t\in\Delta}$. We have:

Theorem 2. *Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be as in Theorem 1, and let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree d over $\{M_t\}_{t\in\Delta}$. Then the following statements hold:*
(i) *In the case of a non-separating node, $\{L_t\}_{t\in\Delta}$ always admits an almost nice family of $\{\hat{\omega}_{hyp,t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ such that the Hermitian metric f^*h_0 on $f^*L_0|_{\widetilde{M}\backslash\{p_1,p_2\}}$ is good on $\widetilde{M}$.*
(ii) *In the case of a separating node, $\{L_t\}_{t\in\Delta}$ admits a nice family of $\{\hat{\omega}_{hyp,t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ if and only if $d_1/(2q_1-1) = d_2/(2q_2-1)$. Here d_1, d_2, q_1, q_2 are as in (2.3).*

Remark 2.4.1. (i) One easily checks that the condition $d_1/(2q_1-1) = d_2/(2q_2-1)$ is satisfied by the vertical line bundle $\widetilde{\mathcal{T}}$ in (2.1). In this regard, Theorem 2 generalizes in part Wolpert's result [**Wo2**, Theorem 5.8] (cf. also Remark 2.2.4).

(ii) As exemplified by $\{ds^2_{\text{hyp},t}\}_{t\in\Delta}$ on $\mathcal{T}$, the Hermitian metrics in Theorem 2 is singular at the node p in general.

2.5. . On a smooth compact Riemann surface R of genus $q \geq 1$, the canonical (1,1)-form is given by $\omega_{\text{can}}(R) = \frac{\sqrt{-1}}{q} \sum_{i=1}^{q} \phi_i \wedge \bar{\phi}_i$, where $\{\phi_1, \phi_2, \ldots, \phi_q\}$ is an orthonormal basis of holomorphic 1-forms on R with respect to the inner product $\langle \phi, \phi' \rangle = \sqrt{-1} \int_R \phi \wedge \bar{\phi}'$. It is easy to see that $\omega_{\text{can}}(R)$ is normalized and does not depend on the choice of the orthonormal basis. Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$, $M^0 = M \backslash \{p\}$, be as in (2.1). It is known that the canonical (1,1)-forms form a p-singular family $\{\omega_{\text{can},t}\}_{t\in\Delta}$ on $\{M_t\}_{t\in\Delta}$ as follows: $\omega_{\text{can},t} = \omega_{\text{can}}(M_t)$ for $t \in \Delta^*$; in the non-separating node case, $\omega_{\text{can},0}$ on $M^0 (= \widetilde{M} \backslash \{p_1, p_2\})$ is given by $\frac{q-1}{q} \omega_{\text{can}}(\widetilde{M})\big|_{M^0}$; and in the separating node case, $\omega_{\text{can},0}$ on $M^0 (= M_1 \backslash \{p_1\} \sqcup M_2 \backslash \{p_2\})$ is given by $\frac{q_k}{q} \omega_{\text{can}}(M_k)\big|_{M_k \backslash \{p_k\}}$, $k = 1, 2$, where $\widetilde{M}$, $M_1, M_2, q, p_1, p_2, q_1, q_2$ are as in (2.1) (see Proposition 6.1.1 and Proposition 6.2.1 for more details). We have:

Theorem 3. *Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be as in Theorem 1, and let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree d over $\{M_t\}_{t\in\Delta}$. Then the following statements hold:*
(i) *In the case of a non-separating node, $\{L_t\}_{t\in\Delta}$ always admits an almost nice family of $\{\omega_{can,t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics.*
(ii) *In the case of a separating node, $\{L_t\}_{t\in\Delta}$ admits an almost nice family of $\{\omega_{can,t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics if and only if $d_1/q_1 = d_2/q_2$. Here d_1, d_2, q_1, q_2 are as in (2.3).*

Remark 2.5.1. In the non-separating node case, $\omega_{\text{can},0}$ is not normalized, and this is reflected by the fact the Hermitian metric on L_0 over M^0 is highly singular at the node p.

3. Family of flat p-singular Hermitian metrics in the non-separating node case.

3.1. . In §3, we are going to prove Theorem 1(i). As a preparation, we first describe in this subsection a procedure for the glueing of two Hermitian metrics on a holomorphic line bundle over an annulus. This procedure is known as 'grafting' in the case of Hermitian metrics on the tangent bundle of the annulus in [**Wo2**].

First we fix a smooth function $\eta : (0,1) \to \mathbb{R}$ such that

$$\eta(s) = \begin{cases} 0, & \text{if } 0 < s < \frac{1}{4}, \\ 1, & \text{if } \frac{3}{4} < s < 1, \end{cases} \quad \text{and} \qquad (3.1.1)$$

$$0 \leq \eta(s) \leq 1 \quad \text{for all } 0 < s < 1.$$

Given two real numbers $a_o, b_o > 0$, we consider the Euclidean region

$$T(a_o) := \{\xi = a + ib \in \mathbb{C} \,|\, 0 < a < a_o\} \qquad (3.1.2)$$

under the action of $\mathbb{Z}$ given by $\xi \to \xi + inb_o$, $n \in \mathbb{Z}$. The quotient space is an annulus which we denote by $A(a_o; b_o)$. Now let L be a holomorphic line bundle over $A(a_o; b_o)$, and let h_1, h_2 be two smooth Hermitian metrics on L. Then we construct a new smooth Hermitian metric $\tilde{h} = \tilde{h}(\eta, h_1, h_2)$ on L given by

$$\tilde{h}(e,e) := h_1(e,e)^{\eta(a/a_o)} h_2(e,e)^{1-\eta(a/a_o)} \tag{3.1.3}$$

for any $[\xi] \in A(a_o; b_o)$ (with $\xi = a + ib$, $0 < a < a_o$) and $e \in L_{[\xi]}$. It is easy to see that $\tilde{h}$ is well-defined on $A(a_o; b_o)$ (i.e., the definition does not depend on the choice of ξ for $[\xi]$). Following [**Wo2**], we call $\tilde{h}$ the *grafting* of h_1 and h_2 on $A(a_o; b_o)$ relative to η.

To state our next proposition, we need to make some more definitions. Let $T(a_o)$, $A(a_o; b_o)$, L, h_1, h_2, $\tilde{h}$ be as above. We = consider the Euclidean region

$$T'(a_o) := \left\{ \xi = a + ib \in \mathbb{C} \;\middle|\; \frac{a_o}{8} < a < \frac{7a_o}{8} \right\} \subset T(a_o), \tag{3.1.4}$$

and let $A'(a_o; b_o)$ be the sub-annulus of $A(a_o; b_o)$ corresponding to the subset $T'(a_o)$ of $T(a_o)$. Since h_1 and h_2 are two Hermitian metrics on the same line bundle L, $\log(h_1/h_2)$ is a well-defined smooth real-valued function on $A(a_o; b_o)$. Then we define

$$\Phi(h_1, h_2) := \sup_{[\xi] \in A'(a_o; b_o)} \left| \log \left(\frac{h_1}{h_2} \right) ([\xi]) \right|. \tag{3.1.5}$$

Next we define a smooth function associated to $\tilde{h}$ given by

$$\psi_{\tilde{h}} := \frac{2\pi c_1(L, \tilde{h})}{(i/2) d\xi \wedge d\bar{\xi}} \quad \text{on } A(a_o; b_o). \tag{3.1.6}$$

Here, in terms of the Euclidean coordinate ξ on $T(a_o)$, $(i/2) d\xi \wedge d\bar{\xi}$ denotes also the flat $(1,1)$-form on $A(a_o; b_o)$ descended from the Euclidean (1,1)-form on $T(a_o)$. We remark also that ratio of (1,1)-forms makes sense here since $A(a_o; b_o)$ is a 1-dimensional complex manifold.

In the case of the grafting of two flat Hermitian metrics, we have:

Proposition 3.1.1. *Let η be as in (3.1.1). Then there exists a constant $C = C(\eta) > 0$ such that for any flat Hermitian metrics h_1, h_2 on any holomorphic line bundle L over $A(a_o; b_o)$, the grafting $\tilde{h} = \tilde{h}(\eta, h_1, h_2)$ of h_1 and h_2 on $A(a_o; b_o)$ relative to η satisfies*

$$\int_{A(a_o; b_o)} |\psi_{\tilde{h}}|^2 \frac{i}{2} d\xi \wedge d\bar{\xi} \le C \cdot (\Phi(h_1, h_2))^2 \cdot \frac{b_o}{a_o^3}, \tag{3.1.7}$$

where $\Phi(h_1, h_2)$ and $\psi_{\tilde{h}}$ are as in (3.1.5) and (3.1.6) respectively.

Proof. Given $a_o, b_o > 0$, one easily sees that

$$\xi = a + ib, \quad 0 < a < a_o,\ 0 \le b < b_o, \tag{3.1.8}$$

gives a parametrization of $A(a_o; b_o)$. By assumption on the flatness of h_1, h_2, we have

$$c_1(L, h_1) = c_1(L, h_2) = 0 \quad \text{on} \ \ A(a_o; b_o). \tag{3.1.9}$$

This implies that $\log(h_1/h_2)$ is a harmonic function on $A(a_o; b_o)$ (cf. (3.1.5)). Together with (3.1.3), it is easy to check that

$$\begin{aligned} 2\pi c_1(L, \widetilde{h}) = -\sqrt{-1}\big((\partial_\xi \partial_{\bar\xi} \eta(a/a_o)) \cdot \log(h_1/h_2) \\ + \partial_\xi \eta(a/a_o) \cdot \partial_{\bar\xi} \log(h_1/h_2) \\ + \partial_{\bar\xi} \eta(a/a_o) \cdot \partial_\xi \log(h_1/h_2)\big) d\xi \wedge d\bar\xi. \end{aligned} \tag{3.1.10}$$

Here we denote $\partial_\xi \eta := \partial \eta / \partial \xi$, etc. By the chain rule, one has

(3.1.11)
$$\partial_\xi \eta \left(\frac{a}{a_o}\right) = \partial_{\bar\xi} \eta \left(\frac{a}{a_o}\right) = \frac{1}{2a_o} \eta' \left(\frac{a}{a_o}\right), \ \text{ and } \ \partial_\xi \partial_{\bar\xi} \eta \left(\frac{a}{a_o}\right) = \frac{1}{4a_o^2} \eta'' \left(\frac{a}{a_o}\right).$$

As in (3.1.4), we define

$$T''(a_o) := \left\{ \xi = a + ib \in \mathbb{C} \,\middle|\, \frac{a_o}{4} < a < \frac{3a_o}{4} \right\} \subset T(a_o), \tag{3.1.12}$$

and let $A''(a_o; b_o)$ denote the sub-annulus of $A(a_o; b_o)$ corresponding to $T''(a_o)$ (so that $T''(a_o) \subset\subset T'(a_o) \subset\subset T(a_o)$, and $A''(a_o; b_o) \subset\subset A'(a_o; b_o) \subset\subset A(a_o; b_o)$). From (3.1.1), (3.1.10), (3.1.11) and (3.1.12), one sees that $\operatorname{supp}(2\pi c_1(L, \widetilde{h})) \subset A''(a_o; b_o)$ and hence $\operatorname{supp}(\psi_{\tilde h}) \subset A''(a_o; b_o)$. By lifting the function $\log(h_1/h_2)$ via the covering map from $T(a_o)$ to $A(a_o; b_o)$, we get a harmonic function on $T(a_o)$, which we denote by the same symbol. Now for any point $\xi \in T''(a_o)$, one easily checks that the circle $C(\xi; a_o/8)$ centered at ξ and with radius $a_o/8$ lies inside $T'(a_o)$. By differentiating (with respect to ξ and $\bar\xi$) the Poisson integral formula for $\log(h_1/h_2)$ over $C(\xi; a_o/8)$, one easily sees from (3.1.5) that

$$\left| \partial_\xi \log\left(\frac{h_1}{h_2}\right) \right|, \ \left| \partial_{\bar\xi} \log\left(\frac{h_1}{h_2}\right) \right| \le \frac{16}{a_o} \Phi(h_1, h_2) \tag{3.1.13}$$

on $T''(a_o)$, and hence same estimate also holds on $A''(a_o; b_o)$. Combining (3.1.5), (3.1.6), (3.1.10), (3.1.11) and (3.1.13), one sees that there exists a constant $C_1 = C_1(\eta) > 0$ such that

$$|\psi_{\tilde h}(\xi)| \le \frac{C_1}{a_o^2} \Phi(h_1, h_2) \quad \text{for all } \xi \in A''(a_o; b_o). \tag{3.1.14}$$

Since $\operatorname{supp}(\psi_{\tilde{h}}) \subset A''(a_o; b_o)$, we have, with C_1 as in (3.1.14),

$$
\begin{aligned}
(3.1.15) \qquad & \int_{A(a_o;b_o)} |\psi_{\tilde{h}}(\xi)|^2 \frac{i}{2} d\xi \wedge d\bar{\xi} \\
&= \int_{A''(a_o;b_o)} |\psi_{\tilde{h}}(\xi)|^2 \frac{i}{2} d\xi \wedge d\bar{\xi} \\
&\leq \int_0^{b_o} \int_{a_o/4}^{3a_o/4} \left| \frac{C_1}{a_o^2} \Phi(h_1, h_2) \right|^2 \, dadb \quad (\text{by } (3.1.12), (3.1.14)) \\
&= C_1^2 \cdot (\Phi(h_1, h_2))^2 \cdot \frac{b_o}{2a_o^3},
\end{aligned}
$$

which leads to (3.1.7) (with $C = C_1^2/2 (> 0)$ depending only on η). Thus we have finished the proof of Proposition 3.1.1. □

3.2. . Throughout the rest of §3, unless otherwise stated, we let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be a plumbing family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a stable Riemann surface M with a non-separating node p as in (2.1). Also we let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree 0 over $\{M_t\}_{t\in\Delta}$ as in Theorem 1(i).

First we recall from (2.1) the normalization $f : \widetilde{M} \to M$ with points $p_1, p_2 \in \widetilde{M}$ corresponding to p, the coordinate functions $z_k : U_k \to \Delta$, $k = 1, 2$, with $U_k \subset \widetilde{M}$, and the coordinate neighborhood $U_1 \times U_2 \simeq \Delta^2$ of p in $\mathcal{M}$ such that $M_t \cap \Delta^2 = \{(z_1, z_2) \in \Delta^2 \,|\, z_1 z_2 = t\}$ for $t \in \Delta$. Fix a small number $\delta > 0$. We define, for $t \in \Delta^*$,

$$
\begin{aligned}
(3.2.1) \qquad \mathrm{I}_t :&= \left\{ (z_1, t/z_1) \in \Delta^2 \;\middle|\; |t|^{\frac{1}{2}+2\delta} < |z_1| < |t|^{\frac{1}{2}-2\delta} \right\} \\
&= \left\{ (t/z_2, z_2) \in \Delta^2 \;\middle|\; |t|^{\frac{1}{2}+2\delta} < |z_2| < |t|^{\frac{1}{2}-2\delta} \right\} \subset M_t.
\end{aligned}
$$

Let $t_0 := \frac{1}{4^{1-4\delta}}$, and let $\pi : \mathcal{M} \to \Delta$ be as in (2.1). We fix an open subset of M given by

$$
(3.2.2) \qquad N_0 := M \setminus \sqcup_{k=1,2} \left\{ |z_k| \leq \frac{1}{2} \right\}.
$$

From the description in (2.1), one easily sees that there is an associated open subset $N := N_0 \times \Delta(t_o) \subset \mathcal{M}$ such that $\pi\big|_N$ is given by the projection onto the second factor. Here $\Delta(t_o) := \{t \in \Delta \,\big|\, |t| < t_o\}$. We denote

$$
(3.2.3) \qquad N_t := N_0 \times \{t\} \subset M_t \quad \text{for } \; t \in \Delta(t_o).
$$

It is easy to check that $\mathrm{I}_t \cap N_t = \emptyset$ for any $t \in \Delta^*(t_o) := \Delta(t_o) \setminus \{0\}$. Also we define the following open subsets of $\mathcal{M}$ given by

$$
(3.2.4) \qquad \mathrm{I} := \cup_{t\in\Delta^*(t_o)} \mathrm{I}_t, \quad V := U_1 \times U_2 \cap \pi^{-1}(\Delta(t_o)).
$$

First we have:

Lemma 3.2.1. *The line bundles $\mathcal{L}\big|_V$ and $\mathcal{L}\big|_N$ are holomorphically trivial.*

Proof. Since $U_1 \times U_2$ is Stein and contractible, it follows from the Oka principle that $\mathcal{L}\big|_{U_1\times U_2}$ is holomorphically trivial, and thus so is its restriction $\mathcal{L}\big|_V$. By construction, we have $N \simeq N_0 \times \Delta(t_o)$. Moreover the projection map onto the first factor is a deformation retract from N onto N_0, which induces an isomorphism between $H^1(N, \mathcal{C}^*)$ and $H^1(N_0, \mathcal{C}^*)$, where $\mathcal{C}^*$ denotes the sheaf of germs of non-vanishing continuous functions. Under this isomorphism, the class $[\mathcal{L}\big|_N]$ corresponds to $[\mathcal{L}\big|_{N_0}]$. Being a line bundle over an open Riemann surface, $\mathcal{L}\big|_{N_0}$ is holomorphically (and thus topologically) trivial. Hence $\mathcal{L}\big|_N$ is also topologically trivial. Since N is Stein, it follows from the Oka principle that $\mathcal{L}\big|_N$ is holomorphically trivial.

Lemma 3.2.2. *Notation as in Theorem 1*(i). *There exists a real-analytic flat Hermitian metric h_0 on $L_0\big|_{M\setminus\{p\}}$ such that f^*h_0 extends across p_1, p_2 to a real-analytic flat Hermitian metric on f^*L_0 over $\widetilde{M}$.*

Proof. By Remark 2.2.2, we have $\deg(f^*L_0) = 0$ on $\widetilde{M}$, and thus there exists a flat Hermitian metric $\widetilde{h}_0$ on f^*L_0 over $\widetilde{M}$, which is easily seen to be real-analytic from the equation $c_1(f^*L_0, \widetilde{h}_0) = 0$. Then Lemma 3.2.2 follows readily by identifying $L_0\big|_{M\setminus\{p\}}$ with $f^*L_0\big|_{\widetilde{M}\setminus\{p_1,p_2\}}$ and letting h_0 to be the restriction of $\widetilde{h}_0$ to $L_0\big|_{M\setminus\{p\}}$.

Next for $|t| < t_0$, we define four open subsets of M_t given by

$$V_{1,t} := \left\{(z_1, t/z_1) \in \Delta^2 \,\middle|\, |t|^{\frac{1}{2}+2\delta} < |z_1| < 1\right\},$$
$$V_{2,t} := \left\{(t/z_2, z_2) \in \Delta^2 \,\middle|\, |t|^{\frac{1}{2}+2\delta} < |z_2| < 1\right\},$$
$$R_{1,t} := \left\{(z_1, t/z_1) \in \Delta^2 \,\middle|\, \frac{1}{2} < |z_1| < 1\right\},$$
$$R_{2,t} := \left\{(t/z_2, z_2) \in \Delta^2 \,\middle|\, \frac{1}{2} < |z_2| < 1\right\}.$$

Also we let

$$V_k := \cup_{|t|<t_0} V_{k,t}, \quad R_k := \cup_{|t|<t_0} R_{k,t}, \quad k = 1, 2, \tag{3.2.5}$$

which are easily seen to be open subsets of $\mathcal{M}$. Moreover, one = easily sees from (3.2.1), (3.2.3), (3.2.5) that for $0 < |t| < t_0$,

(3.2.6)
$$V_{1,t} \cup V_{2,t} \cup N_t = M_t, \quad V_{1,t} \cap V_{2,t} = \mathrm{I}_t, \quad \text{and} \quad V_{k,t} \cap N_t = R_{k,t}, \quad k = 1, 2.$$

With notations as in (3.1), one easily checks that for $0 < |t| < t_0$, the multivalent map

$$(3.2.7) \qquad (z_1, t/z_1) \in \mathrm{I}_t \to \xi = \frac{\log z_1}{|\log |t||} - \frac{1}{2} + 2\delta \in T(4\delta)$$

descends to a biholomorphism between I_t and the annulus $A(4\delta; 2\pi/|\log |t||)$. Similarly, for $0 < |t| < t_0$ and $k = 1, 2$, the multivalent map

$$(3.2.8) \qquad (z_1, z_2) \in R_{k,t} \text{ (with } z_1 z_2 = t) \to \xi_k = \log z_k + \log 2 \in T(\log 2)$$

descends to a biholomorphism between $R_{k,t}$ and $A(\log 2; 2\pi)$. From now on, we will fix non-vanishing holomorphic sections

$$(3.2.9) \qquad e_V \in \Gamma(V, \mathcal{L}\big|_V) \quad \text{and} \quad e_N \in \Gamma(N, \mathcal{L}\big|_N),$$

which provide holomorphic trivializations of $\mathcal{L}\big|_V$ and $\mathcal{L}\big|_N$ respectively (cf. Lemma 3.2.1). Also we fix a flat Hermitian metric h_0 on $L_0\big|_{M\setminus\{p\}}$ as given in Lemma 3.2.2. Next we consider three smooth Hermitian metrics $(\mathcal{L}\big|_V, h_{V,k})$, $k = 1, 2$, and $(\mathcal{L}\big|_N, h_N)$ given by

$$(3.2.10) \qquad \begin{aligned} h_{V,1}(e_V, e_V)(z) &:= h_0(e_V, e_V)(z_1, 0) \quad \text{for } z = (z_1, z_2) \in V, \\ h_{V,2}(e_V, e_V)(z) &:= h_0(e_V, e_V)(0, z_2) \quad \text{for } z = (z_1, z_2) \in V, \\ h_N(e_N, e_N)(z, t) &:= h_0(e_N, e_N)(z, 0) \quad \text{for } (z, t) \in N = N_0 \times \Delta(t_o) \end{aligned}$$

respectively (cf. (3.2)). For $k = 1, 2$, since $V_k \subset V$ (cf. 3.2.5), we obtain a smooth Hermitian metric $(\mathcal{L}\big|_{V_k}, h_{V_k})$ given by

$$(3.2.11) \qquad h_{V_k} := h_{V,k}\big|_{V_k}.$$

Then we obtain smooth families of Hermitian metrics $\{(L_t\big|_{V_{k,t}}, h_{V_{k,t}})\}_{|t|<t_0}$, $k = 1, 2$, and $\{(L_t\big|_{N_t}, h_{N_t})\}_{|t|<t_0}$ given by

$$(3.2.12) \qquad h_{V_{k,t}} := h_{V_k}\big|_{V_{k,t}}, \ k = 1, 2, \quad \text{and} \quad h_{N_t} := h_N\big|_{N_t}$$

for $|t| < t_0$. For each $0 < |t| < t_0$, the Hermitian metrics $h_{V_{1,t}}$, $h_{V_{2,t}}$, h_{N_t} may not agree on the overlaps I_t, $R_{1,t}$, $R_{2,t}$ (cf. (3.2.7)). Let η be as in (3.1.1). By grafting $h_{V_{1,t}}$, $h_{V_{2,t}}$ on I_t ($\cong A(4\delta; 2\pi/|\log |t||)$ as given in (3.2.8)) relative to η and grafting $h_{V_{k,t}}$, h_{N_t} on $R_{k,t}$ ($\cong A(\log 2; 2\pi)$ as given in (3.2.9)) relative to η, $k = 1, 2$ (cf. (3.1)), we obtain a smooth Hermitian metric $\tilde{h}_t$ on L_t. At $t = 0$, we let $\tilde{h}_0 := h_0$ on $L_0\big|_{M\setminus\{p\}}$. Then we consider the family of p-singular Hermitian metrics $\tilde{h} = \{\tilde{h}_t\}_{|t|<t_0}$ on $\{L_t\}_{|t|<t_0}$ with $\tilde{h}\big|_{L_t} = \tilde{h}_t$ for $|t| < t_0$. We have

Proposition 3.2.3. *$\tilde{h}$ is a smooth Hermitian metric on $\mathcal{L}\big|_{\mathcal{M}\cap\pi^{-1}\{|t|<t_0\}\setminus\{p\}}$.*

Proof. Obviously $\tilde{h}$ is equal to h_{V_1}, h_{V_2}, h_N on $V_1 \backslash (\mathrm{I} \cup R_1)$, $V_2 \backslash (\mathrm{I} \cup R_2)$, $N \backslash (R_1 \cup R_2)$ respectively. Also one easily checks from (3.1.3) and (3.2.8) that for $z = (z_1, z_2) \in \mathrm{I}$,

$$\text{(3.2.13)} \quad \tilde{h}(z) = h_{V_1}(z)^{\eta((\log|z_1|/\log|t| - (1/2) + 2\delta)/4\delta)} \cdot h_{V_2}(z)^{1-\eta((\log|z_1|/\log|t| - (1/2)+2\delta)/4\delta)}.$$

Similarly one easily checks from (3.1.3) and (3.2.9) that for $k = 1, 2$ and $z = (z_1, z_2) \in R_k$,

$$\text{(3.2.14)} \quad \tilde{h}(z) = h_N(z)^{\eta((\log|z_k| + \log 2)/\log 2)} \cdot h_{V_k}(z)^{1-\eta((\log|z_k|+\log 2)/\log 2)}.$$

At $t = 0$, one also sees from (3.2.11) that h_{V_1}, h_{V_2}, h_N are all equal to h_0 on $V_{1,0}, V_{2,0}, N_0$ respectively. Together with (3.2.13), (3.2.14) and the smoothness of h_{V_1}, h_{V_2}, h_N, the smoothness of $\tilde{h}$ follows immediately. Thus we have finished the proof of Proposition 3.2.3.

Finally we extend $\widetilde{h}$ arbitrarily (across $\{M_t\}_{|t| \geq t_0}$) to a smooth Hermitian metric on $\mathcal{L}\big|_{\mathcal{M}\backslash\{p\}}$, and we denote the extension by the same symbol. Thus we get a family of p-singular Hermitian metrics $\{\widetilde{h}_t\}_{t\in\Delta}$ on $\{L_t\}_{t\in\Delta}$, where $\tilde{h}_t := \tilde{h}\big|_{L_t}$ for all $t \in \Delta$.

3.3. . Notation as in (3.2). We are going to obtain some estimates which will be needed to prove Theorem 1(i). Recall from (2.1) the coordinate neighborhood Δ^2 of p in $\mathcal{M}$ such that $M_t \cap \Delta^2 = \{(z_1, z_2) \in \Delta^2 \,\big|\, z_1 z_2 = t\}$ for $t \in \Delta$. For $0 < r_o < 1$, define $\Delta^2(r_o) := \{(z_1, z_2) \in \Delta^2 \,|\, |z_1|, |z_2| < r_o\}$. Let $ds^2_{\mathrm{hyp},t}$, $\omega_{\mathrm{hyp},t}$ be as in (2.4). First we recall the following result of Wolpert:

Proposition 3.3.1 ([**Wo2**, Expansion 4.2])**.** *There exist constants $C_1, C_2 > 0$ such that for all $t \in \Delta^*$ and $k = 1, 2$, we have, on $M_t \cap \Delta^2(\frac{3}{4})$,*

$$C_1 \left(\frac{\pi}{\log|t|} \csc \frac{\pi \log|z_k|}{\log|t|} \frac{|dz_k|}{|z_k|} \right)^2 \leq ds^2_{\mathrm{hyp},t} \leq C_2 \left(\frac{\pi}{\log|t|} \csc \frac{\pi \log|z_k|}{\log|t|} \frac{|dz_k|}{|z_k|} \right)^2.$$

Remark. Proposition 3.3.1 also holds in the case of a separating node.

For $t \in \Delta^*$, we define the smooth function given by

$$\text{(3.3.1)} \qquad \phi_t := \frac{2\pi c_1(L_t, \widetilde{h}_t)}{= \omega_{\mathrm{hyp},t}} \quad \text{on } M_t,$$

where $\widetilde{h}_t$ is as in (3.2) (cf. (3.1.6)). Let I_t be as in (3.2.1). We have:

Proposition 3.3.2. $\displaystyle\int_{I_t} \phi_t^2 \omega_{\mathrm{hyp},t} = O\left(\frac{1}{|\log|t||}\right)$ *as* $t \to 0$.

Proof. First we recall from (3.2.8) the biholomorphism $\mathrm{I}_t \cong A(4\delta; 2\pi/|\log|t||)$ for $t \in \Delta^*$. By (3.2.1), (3.2.8) and Proposition 3.3.1, one easily checks that there exist constants $C_3, C_4 > 0$ such that for $t \in \Delta^*$,

$$C_3 \,\frac{i}{2} d\xi \wedge d\bar{\xi} \leq \omega_{\mathrm{hyp},t} \leq C_4 \,\frac{i}{2} d\xi \wedge d\bar{\xi} \quad \text{on } \mathrm{I}_t, \tag{3.3.2}$$

where ξ is as in (3.2.8). Let $t_0, \delta, V, \mathrm{I}, V_{1,t}, V_{2,t}$ be as in (3.2), and let h_0 be as in Lemma 3.2.2. Recall from (3.2) that on I_t, $\tilde{h}_t$ is obtained by grafting $h_{V_{1,t}}$ and $h_{V_{2,t}}$ relative to η, where η is as in (3.1.1). From (3.2.1) and (3.2.4), one sees that $|z_1|, |z_2| \leq |t_0|^{\frac{1}{2}-2\delta} < 1$ for all $z = (z_1, z_2) \in \mathrm{I}$. Together with (3.2.11) and the extension property of h_0 as given in Lemma 3.2.2, one sees that there exist constants C_5, $C_6 > 0$ such that for $k = 1, 2$,

$$C_5 \leq |h_{V,k}(e_V, e_V)(z)| \leq C_6 \quad \text{for all } z \in \mathrm{I}, \tag{3.3.3}$$

where e_V is as in (3.2.10). Together with (3.2.12), (3.2.13) and using the notation in (3.1.5), it follows that there exists a constant $C_7 > 0$ such that for $0 < |t| < t_0$, one has

$$\Phi(h_{V_{1,t}}, h_{V_{2,t}}) \leq C_7 \tag{3.3.4}$$

with respect to the annulus $\mathrm{I}_t \cong= A(4\delta; 2\pi/|\log|t||)$. With $\psi_{\tilde{h}_t}$ as defined in (3.1.6), one sees from (3.3.1) and (3.3.2) that

$$\psi_{\tilde{h}_t}(\xi) \geq C_3 \cdot \phi_t(\xi) \quad \text{for } \xi \in \mathrm{I}_t,\ 0 < |t| < t_0. \tag{3.3.5}$$

From (3.2.11), (3.2.13) and the flatness of h_0, it follows that for $0 < |t| < t_0$, $h_{V_{1,t}}$ and $h_{V_{2,t}}$ are also flat Hermitian metrics. Let $C = C(\eta) > 0$ be as in Proposition 3.1.1. Then we have

(3.3.6)

$$\begin{aligned}\int_{\mathrm{I}_t} \phi_t^2 \omega_{\mathrm{hyp},t} &\leq \frac{C_4}{C_3^2} \int_{\mathrm{I}_t} \phi_t^2 \,\frac{i}{2} d\xi \wedge d\bar{\xi} \quad \text{(by (3.3.2), (3.3.5))} \\ &\leq \frac{C_4}{C_3^2} \cdot C \cdot C_7^2 \cdot \frac{1}{(4\delta)^3} \cdot \frac{2\pi}{|\log|t||} \quad \text{(by Proposition 3.1.1, (3.3.4))} \\ &= O\left(\frac{1}{|\log|t||}\right) \quad \text{as } t \to 0.\end{aligned}$$

Thus we have finished the proof of Proposition 3.3.2.

Next for $|t| < t_0$ and $k = 1, 2$, we let $R_{k,t}$ be as in (3.2.5), and let R_k be as in (3.2.6). We have:

Proposition 3.3.3. *For $k = 1, 2$, $\int_{R_{k,t}} \phi_t^2 \omega_{\mathrm{hyp},t} = O(|t|^2)$ as $t \to 0$.*

Proof. First we recall from (3.2.9) the biholomorphism $R_{k,t} \cong A(\log 2; 2\pi)$ for $t \in \Delta^*$. By (3.2.1), (3.2.9) and Proposition 3.3.1, one easily checks that there exist constants $C_3, C_4 > 0$ such that for $t \in \Delta^*$,

$$C_3 \frac{i}{2} d\xi_k \wedge d\bar{\xi}_k \leq \omega_{\text{hyp},t} \leq C_4 \frac{i}{2} d\xi_k \wedge d\bar{\xi}_k \quad \text{on } R_{k,t}, \tag{3.3.7}$$

where ξ_k is as in (3.2.9). Let $t_0, N, N_t, V_k, V_{k,t}$, $k = 1, 2$, be as in (3.2), and let h_0 be as in Lemma 3.2.2. Recall from (3.2) that on $R_{k,t}$, $\tilde{h}_t$ is obtained by grafting h_{N_t} and $h_{V_{k,t}}$ relative to η, where η is as in (3.1.1). As in Lemma 3.2.1, $\mathcal{L}\big|_{R_k}$, $k = 1, 2$, are holomorphically trivial, and we fix non-vanishing holomorphic sections $e_k \in \Gamma(R_k, \mathcal{L}\big|_{R_k})$, $k = 1, 2$. Let e_N, e_V be as in (3.2.10), and for $k = 1, 2$, we let $f_k(z)$, $g_k(z)$ be the non-vanishing holomorphic functions on R_k satisfying

$$e_V(z) = f_k(z) e_k(z) \quad \text{and} \quad e_N(z) = g_k(z) e_k(z) \quad \text{for } z \in R_k. \tag{3.3.8}$$

For simplicity, we denote $[z_k, \cdot]$ to be $(z_1, \cdot)$ or $(\cdot, z_2)$ according as $k = 1$ or $k = 2$. Then for $k = 1, 2$, $|t| < t_0$ and $z = [z_k, t/z_k] \in R_{k,t}$ (with $\frac{1}{2} < |z_k| < 1$), one has

$$\begin{aligned} \frac{h_{N,t}}{h_{V_{k,t}}}(z) &= \frac{h_N(e_k, e_k)(z)}{h_{V,k}(e_k, e_k)(z)} \quad \text{(cf. (3.2.12) and (3.2.13))} \\ &= \frac{h_N(e_N, e_N)(z)}{h_{V,k}(e_V, e_V)(z)} \cdot \frac{|f_k(z)|^2}{|g_k(z)|^2} \quad \text{(by (3.3.8))} \\ &= \frac{h_0(e_N, e_N)[z_k, 0]}{h_0(e_V, e_V)[z_k, 0]} \cdot \frac{|f_k(z)|^2}{|g_k(z)|^2} \quad \text{(by (3.2.11))} \\ &= \frac{|g_k[z_k, 0]|^2}{|f_k[z_k, 0]|^2} \cdot \frac{|f_k[z_k, t/z_k]|^2}{|g_k[z_k, t/z_k]|^2} \quad \text{(by (3.3.8))}. \end{aligned} \tag{3.3.9}$$

Using the Cauchy integral formula and shrinking t_0 if necessary, one easily checks that for any $0 < \epsilon < \frac{1}{2}$, there exists a constant $C_5 = C_5(\epsilon) > 0$ such that for $k = 1, 2$, $0 < |t| < t_0$ and $\frac{1}{2} + \epsilon < |z_k| < 1 - \epsilon$, one has

$$1 - C_5|t| \leq \frac{|f_k[z_k, t/z_k]|^2}{|f_k[z_k, 0]|^2}, \frac{|g_k[z_k, t/z_k]|^2}{|g_k[z_k, 0]|^2} \leq 1 + C_5|t|. \tag{3.3.10}$$

Using the notation in (3.1.5), it follows that there exists a constant $C_6 > 0$ such that for $k = 1, 2$ and $0 < |t| < t_0$, one has

$$\Phi(h_{N_t}, h_{V_{k,t}}) \leq C_6|t| \tag{3.3.11}$$

with respect to the annulus $R_{k,t} \cong A(\log 2; 2\pi)$. With $\psi_{\tilde{h}_t}$ as defined in (3.1.6), one sees from (3.3.1) and (3.3.7) that

$$\psi_{\tilde{h}_t}(\xi_k) \geq C_3 \cdot \phi_t(\xi_k) \quad \text{for } \xi_k \in R_{k,t},\ 0 < |t| < t_0. \tag{3.3.12}$$

As in Proposition 3.3.2, since $h_{V_{k,t}}$ and h_{N_t} are flat Hermitian metrics (cf. (3.2.11), (3.2.13)), we have, with $= C = C(\eta) > 0$ as in Proposition 3.1.1,

$$\begin{aligned}(3.3.13)\quad \int_{R_{k,t}} \phi_t^2 \omega_{\text{hyp},t} &\leq \frac{C_4}{C_3^2} \int_{R_{k,t}} \phi_t^2 \, \frac{i}{2} d\xi_k \wedge d\bar{\xi}_k \quad \text{(by (3.3.7), (3.3.12))} \\ &\leq \frac{C_4}{C_3^2} \cdot C \cdot C_6^2 |t|^2 \cdot \frac{2\pi}{(\log 2)^3} \quad \text{(by Proposition 3.1.1)} \\ &= O(|t|^2) \quad \text{as} \ \ t \to 0.\end{aligned}$$

Thus we have finished the proof of Proposition 3.3.3.

For $t \in \Delta$, we denote by $\| \ \|_2$ the L^2-norm on M_t with respect to $\omega_{\text{hyp},t}$.

Proposition 3.3.4. $\|\phi_t\|_2^2 = O\left(\frac{1}{|\log |t||}\right)$ *as* $t \to 0$.

Proof. From the construction of $\tilde{h}_t$, it is easy to check that for $0 < |t| < t_0$, $c_1(L_t, \widetilde{h}_t) = 0$ on $M_t \backslash (\mathrm{I}_t \sqcup R_{1,t} \sqcup R_{2,t})$. Then by Proposition 3.3.2 and Proposition 3.3.3, we have

$$\begin{aligned}(3.3.14)\quad \int_{M_t} \phi_t^2 \omega_{\text{hyp},t} &= \int_{\mathrm{I}_t} \phi_t^2 \omega_{\text{hyp},t} + \int_{R_{1,t}} \phi_t^2 \omega_{\text{hyp},t} + \int_{R_{2,t}} \phi_t^2 \omega_{\text{hyp},t} \\ &= O\left(\frac{1}{|\log |t||}\right) + O(|t|^2) + O(|t|^2) \\ &= O\left(\frac{1}{|\log |t||}\right) \quad \text{as } t \to 0.\end{aligned}$$

Thus we have finished the proof of Proposition 3.3.4.

3.4. . Before we go on, we give a lemma which will be needed in subsequent discussion.

Lemma 3.4.1. *Let X be a smooth compact Riemann surface of genus $q \geq 2$ and endowed with the hyperbolic metric, and let $v \in C^\infty(X)$. Then there exist constants C_1, $C_2 > 0$ (which do not depend on X or v) such that the following statements hold:*

(i) *For any $x \in X$ and any real number r satisfying $0 < r \leq \rho_x$, where ρ_x denotes the injectivity radius at x, we have*

$$|v(x)| \leq \frac{C_1}{\tanh r} \sqrt{\int_{B(x,r)} |v|^2 \omega_{\text{hyp}}} + C_2 \sinh r \sqrt{\int_{B(x,r)} |\Delta v|^2 \omega_{\text{hyp}}}.$$

Here $B(x,r)$ denotes the geodesic ball centered at x and of radius r, ω_{hyp} denotes the hyperbolic volume form of X, and Δ denotes the hyperbolic Laplacian.

(ii) *In particular, we have*

$$\sup_{x\in X}|v(x)| \le \frac{C_1}{4\pi(q-1)\tanh\rho_X}\left|\int_X v\omega_{\text{hyp}}\right| + \left(\frac{C_1}{\lambda_{1,X}\tanh\rho_X} + C_2\sinh\rho_X\right)\|\Delta v\|_2.$$

Here ρ_X denotes the injectivity radius of X, and $\lambda_{1,X}$ denotes the first non-zero eigenvalue of X.

Proof. Write $\Delta(R) := \{z \in \mathbb{C} \,\big|\, |z| < R\}$ for $R > 0$ (so that $\Delta(1) = \Delta$). The universal cover of (X, ω_{hyp}) is $(\Delta, idz \wedge d\bar{z}/(1-|z|^2)^2)$. Denote the hyperbolic distance on Δ by $d(\cdot,\cdot)$. It is well-known that $|z| = \tanh(d(0,z))$ for $z \in \Delta$. For any $x \in X$ and $0 < r \le \rho_x$, we may identify $B(x,r)$ with $\Delta(\tanh\rho_t)$ via the covering map from Δ to X sending the origin to x. In terms of such identification, we have

$$\partial_z\partial_{\bar{z}}v = -\frac{1}{(1-|z|^2)^2}\Delta v \quad \text{on } \Delta(\tanh r). \tag{3.4.1}$$

Next we make a change of variable given by $z' := z/(\tanh\rho_t)$, and denote $v'(z') := v(z)$, so that $v' \in C^\infty(\Delta)$. Using Nash-Moser iteration technique (cf. e.g., [**GT**, Theorem 8.24] and observe that the left hand side of the trivial equation $\partial_z\partial_{\bar{z}}v = \partial_z\partial_{\bar{z}}v$ is of constant coefficients, which implies that the conditions in (8.5) and (8.6) of [**GT**] are satisfied), one deduces that there exist constants $C_1, C_2 > 0$ such that

(3.4.2)

$$\begin{aligned}
|v(0)| &= |v'(0)| \\
&\le C_1\sqrt{\int_\Delta |v'(z')|^2\frac{i}{2}dz'\wedge d\bar{z}'} + c_2\sqrt{\int_\Delta |\partial_{z'}\partial_{\bar{z}'}v'(z')|^2\frac{i}{2}dz'\wedge d\bar{z}'} \\
&\le \frac{C_1}{\tanh r}\sqrt{\int_{\Delta(\tanh r)}|v(z)|^2\frac{i}{2}dz\wedge d\bar{z}} \\
&\quad + C_2\tanh r\sqrt{\int_{\Delta(\tanh r)}|\partial_z\partial_{\bar{z}}v(z)|^2\frac{i}{2}dz\wedge d\bar{z}} \\
&\le \frac{C_1}{\tanh r}\sqrt{\int_{\Delta(\tanh r)}|v(z)|^2\omega_{\text{hyp}}} + C_2\sinh r\sqrt{\int_{\Delta(\tanh r)}|\Delta v|^2\omega_{\text{hyp}}},
\end{aligned}$$

where the last inequality follows from the inequality that $\frac{i}{2}dz\wedge d\bar{z} \le \omega_{\text{hyp}}$ on Δ, (3.4.1) and the fact that $\dfrac{1}{1-|z|^2} < \cosh^2 r$ for $|z| < \tanh r$. This finishes the proof of Lemma 3.4.1(i). To prove Lemma 3.4.1(ii), we recall from standard spectral theory for elliptic operators that

$$(3.4.3)\qquad \|v\|_2^2 \le \left(\frac{1}{4\pi(q-1)}\int_X v\omega_{\rm hyp}\right)^2 + \frac{1}{\lambda_{1,X}^2}\|\Delta v\|_2^2, \quad \text{and thus}$$
$$\|v\|_2 \le \frac{1}{4\pi(q-1)}\left|\int_X v\omega_{\rm hyp}\right| + \frac{1}{\lambda_{1,X}}\|\Delta v\|_2.$$

Then by letting $r = \rho_X$ in Lemma 3.4.1(i), we have, at any $x \in X$,

$$(3.4.4)\qquad \begin{aligned} |v(x)| &\le \frac{C_1}{\tanh\rho_X}\|v\|_2 + C_2\sinh\rho_X\|\Delta v\|_2 \\ &\le \frac{C_1}{4\pi(q-1)\tanh\rho_X}\left|\int_X v\omega_{\rm hyp}\right| \\ &\quad + \left(\frac{C_1}{\lambda_{1,X}\tanh\rho_X} + C_2\sinh\rho_X\right)\|\Delta v\|_2 \quad (\text{by } (3.4.3)), \end{aligned}$$

which easily leads to Lemma 3.4.1(ii). Thus we have finished the proof of Lemma 3.4.1.

3.5. . Notation as in (3.1) to (3.3). Let $\{M_t\}_{t\in\Delta}$ be as in Theorem 1(i). For $t \in \Delta^*$, we denote by $\lambda_{1,t}$ the first non-zero eigenvalue of the Laplacian Δ_t with respect to $ds^2_{{\rm hyp},t}$ on M_t. We shall need the following:

Lemma 3.5.1. *Let $\{M_t\}_{t\in\Delta}$ be as in Theorem 1(i) with a non-separating node p. Then there exists a constant $\alpha > 0$ such that $\lambda_{1,t} \ge \alpha$ for all $t \in \Delta^*$.*

Proof. The above lemma is well-known and follows from results of [**SWY**] and [**H**] (see e.g., [**Ji**, Corollary 3.4]).

For $t \in \Delta^*$, we define the smooth function

$$(3.5.1)\qquad u_t := G_t\phi_t \qquad \text{on } M_t,$$

where ϕ_t is as in (3.3.1), and G_t is the Green's operator with respect to $ds^2_{{\rm hyp},t}$ on M_t, i.e., u_t is the (unique) smooth function on M_t satisfying

$$(3.5.2)\qquad \Delta_t u_t = \phi_t, \quad \text{and} \quad \int_{M_t} u_t\,\omega_{{\rm hyp},t} = 0$$

(cf. e.g., [**GH**, p. 84] for the definition of the Green's operator).

Proposition 3.5.2. *Let $\{M_t\}_{t\in\Delta}$ be as in Theorem 1(i) with a non-separating node p, and let u_t be as in (3.5.1). Then for any continuous section $\{z_t\}_{t\in\Delta} \in \{M_t\}_{t\in\Delta}$ with $z_t \in M_t$ and $z_0 \ne p$, we have $u_t(z_t) \to 0$ as $t \to 0$.*

Proof. Since each L_t is of degree 0, it follows from (3.3.1) that $\int_{M_t} \phi_t \omega_{\mathrm{hyp},t} = 0$ for $t \in \Delta^*$. = Together with (3.5.2), it follows from standard properties of Green's operator that

$$\|u_t\|_2 \leq \frac{1}{\lambda_{1,t}} \|\phi_t\|_2 \quad \text{for } t \in \Delta^*. \tag{3.5.3}$$

Since $z_0 \neq p$, it is easy to see that there exist constants = $\rho_1, \rho_2 > 0$ such that $\rho_1 \leq \rho_{z_t} \leq \rho_2$ for all = $t \in \Delta^*$, where ρ_{z_t} denotes the injectivity radius of z_t in M_t. Then by Lemma 3.4.1(i) and (3.5.2), we have

(3.5.4)

$$\begin{aligned} |u_t(z_t)| &\leq \frac{C_1}{\tanh \rho_1} \|v_t\|_2 + C_2 \sinh \rho_2 \|\phi_t\|_2 \\ &\leq \left(\frac{C_1}{\alpha \tanh \rho_1} + C_2 \sinh \rho_2 \right) \|\phi_t\|_2 \quad \text{(by Lemma 3.5.1 and (3.5.3))} \\ &\to 0 \quad \text{as } t \to 0 \text{ (by Proposition 3.3.4).} \end{aligned}$$

Here C_1, C_2 are as in Lemma 3.4.1, and α is as in Lemma 3.5.1. Thus we have finished the proof of Proposition 3.5.2.

Next we define a family of p-singular Hermitian metrics = $h = \{h_t\}_{t\in\Delta}$ on $\{L_t\}_{t\in\Delta}$ by letting

$$h_t := e^{-u_t} \widetilde{h}_t \quad \text{for } t \in \Delta^*, \tag{3.5.5}$$

and letting h_0 be as given in Lemma 3.2.2. We are ready to give the proof of Theorem 1(i) as follows.

Proof of Theorem 1(i). Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ (with a non-separating node $p \in M$), $\mathcal{L} = \{L_t\}_{t\in\Delta}$ and $f : \widetilde{M} \to M$ be as in Theorem 1(i). Also let $h = \{h_t\}_{t\in\Delta}$ be as in (3.5.5). By (3.5.5), we have

$$\begin{aligned} c_1(L_t, h_t) =& \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_t + c_1(L_t, \widetilde{h}_t) \\ &= -\frac{1}{2\pi} \Delta_t u_t \cdot \omega_{\mathrm{hyp},t} + c_1(L_t, \widetilde{h}_t) \\ &= 0 \quad \text{on } M_t \quad \text{(by (3.3.1), (3.5.2)).} \end{aligned} \tag{3.5.6}$$

Together with Lemma 3.2.2 (for h_0), it follows that for each $t \in \Delta$, h_t is a flat p-singular Hermitian metric on L_t, and this verifies condition (i) of Definition 2.2.1 for $\{h_t\}_{t\in\Delta}$. From (3.3.1) and the construction of $\tilde{h} = \{\tilde{h}_t\}_{t\in\Delta}$ in (3.2), it is easy to see that $\{\phi_t\}_{t\in\Delta^*}$ glue together to form a smooth function on $\mathcal{M}\backslash M_0$. Also, it is obvious that for each $t_* \in \Delta^*$, there exist $\lambda_* = \lambda_*(t_*) > 0$ and $\epsilon = \epsilon(t_*) > 0$ such that $\lambda_{1,t} \geq \lambda_*$ for all $|t - t_*| < \epsilon$ (cf. [**KS**, Theorem 2] or Lemma 3.5.1). Thus by [**KS**, Theorem 3] and (3.5.1), $\{u_t\}_{t\in\Delta^*}$ glue together to form a smooth function on $\mathcal{M}\backslash M_0$. Then the smoothness of h on $\mathcal{M}\backslash M_0$ follows from those of

$\{u_t\}_{t\in\Delta^*}$ and $\tilde{h}$. Moreover, by letting u_0 to be the constant zero function on $M\backslash\{p\}$, if follows from Proposition 3.5.2 that $\{u_t\}_{t\in\Delta}$ glue together to form a continuous function on $\mathcal{M}\backslash\{p\}$ (with (3.5.5) also satisfied at $t=0$). Then the continuity of h on $\mathcal{M}\backslash\{p\}$ follows from those of $\{u_t\}_{t\in\Delta}$ and $\tilde{h}$, and this verifies condition (ii) of Definition 2.2.1 for $\{h_t\}_{t\in\Delta}$. Thus $\{h_t\}_{t\in\Delta}$ forms an almost nice family of flat p-singular Hermitian metrics. By Lemma 3.2.2, f^*h_0 extends to a smooth flat Hermitian metric on f^*L_0 over $\widetilde{M}$. Thus we have completed the proof of Theorem 1(i).

4. Family of flat p-singular Hermitian metrics in the separating node case.

4.1. . We are going to prove Theorem 1(ii) in §4. In (4.1), we will first prove that under the conditions of Theorem 1(ii), $\{L_t\}_{t\in\Delta}$ admits an almost nice family of flat p-singular Hermitian metrics $h=\{h_t\}_{t\in\Delta}$. To streamline our discussion, we will keep the notation as close to §3 as possible and simply refer to §3 when the arguments and calculations in §3 also prevail verbatim in the present case of a separating node. The goodness of h will be proved in (4.2).

Throughout §4, unless otherwise stated, we let $\mathcal{M}=\{M_t\}_{t\in\Delta}$ be a plumbing family of compact Riemann surfaces of genus $q\geq 2$ degenerating to a stable singular Riemann surface M with a single separating node p as in (2.1). Recall also from (2.1) the normalization $f:\widetilde{M}(=M_1\sqcup M_2)\to M$ with points $p_k\in M_k$, where M_k is a smooth compact Riemann surface of genus q_k, $k=1,2$ (so that $q_1,q_2\geq 1$ and $q_1+q_2=q$). We also denote the two components of $M\backslash\{p\}$ by M_1^0 and M_2^0, so that via f, we have the identifications

$$M_1^0\simeq M_1\backslash\{p_1\},\quad M_2^0\simeq M_2\backslash\{p_2\}. \tag{4.1.1}$$

Let $\mathcal{L}=\{L_t\}_{t\in\Delta},\ L_{0,k}=f^*L_0\big|_{M_k},\ k=1,2$, be as in Theorem 1(ii), so that $d_k=\deg(L_{0,k})=0,\ k=1,2$. Via f, we have an identification of the fibers

$$L_0\big|_{\{p\}}\simeq L_{0,1}\big|_{\{p_1\}}\simeq L_{0,2}\big|_{\{p_2\}}. \tag{4.1.2}$$

Fix a small number $\delta>0$, and let $t_o:=1/4^{1-4\delta}$, $z_k:U_k\to\Delta$, V_k, R_k, V, N, I_t, N_t, $t\in\Delta^*$, $k=1,2$, be similarly defined as in (3.2). Note that in the present case of a separating node, N and the N_t's consist of two components. Using the arguments in Lemma 3.2.1, one easily sees that $\mathcal{L}\big|_V$ and $\mathcal{L}\big|_N$ are all holomorphically trivial, and as in (3.2), we will thus fix non-vanishing holomorphic sections

$$e_V\in\Gamma(V,\mathcal{L}\big|_V),\quad e_N\in\Gamma(N,\mathcal{L}\big|_N) \tag{4.1.3}$$

throughout the remaining section.

Lemma 4.1.1. *For $k = 1, 2$, there exists a real-analytic Hermitian metric $h_{0,k}$ on $L_0\big|_{M_k^0}$ such that $f^*h_{0,k}$ extends across p_k to a real-analytic flat Hermitian metric $\widetilde{f^*h_{0,k}}$ on $L_{0,k}$ over M_k; moreover, under the identification in (4.1.2), we have*

$$(L_{0,1}, \widetilde{f^*h_{0,1}})\big|_{\{p_1\}} \simeq (L_{0,2}, \widetilde{f^*h_{0,2}})\big|_{\{p_2\}}. \tag{4.1.4}$$

Proof. As in Lemma 3.2.2, the existence of such real-analytic flat $h_{0,k}$ (minus the condition (4.1.4)) follows easily from the degree condition $\deg(L_{0,k}) = 0$, $k = 1, 2$. Then (4.1.4) can easily be attained by multiplying $h_{0,1}$ by a suitable positive constant, if necessary.

By means of grafting (relative to an η as given in (3.1.1)) as in (3.2) and with h_0 replaced by $h_{0,k}$ on M_k, $k = 1, 2$, we obtain a smooth family of p-singular Hermitian metrics $\tilde{h} = \{\tilde{h}_t\}_{|t|<t_0}$ on $\{L_t\}_{t\in\Delta}$ with $\tilde{h}\big|_{L_t} = \tilde{h}_t$ for $|t| < t_0$. Also, we let h_{V_1}, h_{V_2}, h_N, $h_{V_{1,t}}$, $h_{V_{2,t}}$, h_{N_t} be the intermediate Hermitian metrics involved in the construction of $\tilde{h}$ as in (3.2). As in (3.3.1), we let

$$\phi_t := \frac{2\pi c_1(L_t, \widetilde{h}_t)}{\omega_{\mathrm{hyp},t}} \quad \text{for } t \in \Delta^*. \tag{4.1.5}$$

We shall need the following stronger version of Proposition 3.3.2 in the case of a separating node:

Proposition 4.1.2. *There exists a constant $\gamma > 0$ such that*

$$\int_{\mathrm{I}_t} \phi_t^2 \omega_{\mathrm{hyp},t} = O(|t|^\gamma) \quad \textit{as } t \to 0. \tag{4.1.6}$$

Proof. We are going to prove Proposition 4.1.2 by modifying the proof of Proposition 3.3.2 as follows. From Lemma 4.1.1 and (3.2.11), one sees that the Hermitian metrics $h_{V,1}$ and $h_{V,2}$ on $\mathcal{L}\big|_V$ are real-analytic, and they agree with each other at p. Together with (3.2.12) and (3.2.13), it follows that for $0 < |t| < t_0$ and $z = (z_1, z_2) \in \mathrm{I}_t$ (with $z_1 z_2 = t$), one has

$$\begin{aligned} \left|\frac{h_{V_1}}{h_{V_2}}(z)\right|, \ \left|\frac{h_{V_2}}{h_{V_1}}(z)\right| &\le 1 + C_1|z_1| + C_2|z_2| \\ &\le 1 + C_3|t|^{\frac{1}{2}-2\delta} \quad (\text{cf. } (3.2.1)) \end{aligned} \tag{4.1.7}$$

for some positive constants C_1, C_2, C_3 independent of t. As in (3.3.4) and using the notation in (3.1.5), it follows that there exists a constant $C_4 > 0$ such that for $0 < |t| < t_0$, one has

$$\Phi(h_{V_{1,t}}, h_{V_{2,t}}) \le C_4|t|^{\frac{1}{2}-2\delta} \tag{4.1.8}$$

with respect to the annulus $\mathrm{I}_t \cong A(4\delta; 2\pi/|\log|t||)$. Then (4.1.6) follows easily from a calculation similar to (3.3.6) with (3.3.4) replaced by (4.1.8), which gives Proposition 4.1.2.

Proposition 4.1.3. *For* $k = 1, 2$, $\int_{R_{k,t}} \phi_t^2 \omega_{\mathrm{hyp},t} = O(|t|^2)$ *as* $t \to 0$.

Proof. The proof of Proposition 4.1.3 (for the separating node case) can easily be obtained by adapting that of Proposition 3.3.3 (for the non-separating node case).

Analogous to Lemma 3.5.1, we have:

Lemma 4.1.4. *Let* $\{M_t\}_{t\in\Delta}$ *be as in Theorem* 1(ii) *with a separating node.*

(i) *Then there exists a constant* $\beta > 0$ *such that* $\lambda_{1,t} > \beta/\big|\log|t|\big|$ *for all* $t \in \Delta^*$.

(ii) *There exist constants* c_1, $c_2 > 0$ *such that*
$$\frac{c_1}{|\log|t||} \le \rho_t \le \frac{c_2}{|\log|t||}$$
for all $t \in \Delta^*$, *where* ρ_t *denotes the injectivity radius of* M_t.

Proof. By [**SWY**], there exists a constant $C_1 > 0$ such that for all $t \in \Delta^*$, $\lambda_{1,t} > C_1\, l_t$, where l_t is the minimum of the lengths of simple closed geodesics (with respect to $ds^2_{\mathrm{hyp},t}$) on M_t which separate M_t into two components. In our separating node case, it is well-known that there exist constants $C_2, C_3 > 0$ such that for all $t \in \Delta^*$,

$$\frac{C_2}{|\log|t||} \le l_t \le \frac{C_3}{|\log|t||} \tag{4.1.9}$$

(cf. e.g., [**Wo2**, Example 4.3]), which leads to Lemma 4.1.4(i). Lemma 4.1.4(ii) follows from (4.1.9) and the simple fact that $\rho_t = l_t/2$.

We shall need the following stronger version of Proposition 3.3.4 in the separating node case:

Proposition 4.1.5. *There exists a constant* $\mu > 0$ *such that* $\|\phi_t\|_2 = O(|t|^\mu)$ *as* $t \to 0$.

Proof. As in Proposition 3.3.4, it follows from Proposition 4.1.2 and Proposition 4.1.3 that

$$\begin{aligned} \int_{M_t} \phi_t^2 \omega_{\mathrm{hyp},t} &= \int_{\mathrm{I}_t} \phi_t^2 \omega_{\mathrm{hyp},t} + \int_{R_{1,t}} \phi_t^2 \omega_{\mathrm{hyp},t} + \int_{R_{2,t}} \phi_t^2 \omega_{\mathrm{hyp},t} \\ &= O(|t|^\gamma) + O(|t|^2) + O(|t|^2) \\ &= O(|t|^{\mu'}) \quad \text{as } t \to 0, \end{aligned} \tag{4.1.10}$$

where γ is as in Proposition 4.1.2 and $\mu' = \min\{\gamma, 2\} > 0$, and this leads to Proposition 4.1.5 with $\mu = \mu'/2$.

As in (3.5.1) and (3.5.2), we let $u_t := G_t\phi_t$ on M_t. Then = similar to Proposition 3.5.2, we have:

Proposition 4.1.6. (i) *There exists a constant $\mu > 0$ such that*

$$\|u_t\|_2 = O(|t|^\mu) \quad \text{and} \quad \|\partial u_t\|_2 = \|\bar\partial u_t\|_2 = O(|t|^\mu) \quad \text{as } t \to 0.$$

(ii) *For any continuous section $\{z_t\}_{t\in\Delta}$ in $\{M_t\}_{t\in\Delta}$ with $z_t \in M_t$ and $z_0 \neq p$, we have $u_t(z_t) \to 0$ as $t \to 0$.*

Proof. As in (3.5.3), we have

$$\begin{aligned} (4.1.11) \quad \|u_t\|_2 &\le \frac{1}{\lambda_{1,t}}\|\phi_t\|_2 \\ &\le \frac{1}{\beta/|\log|t||}|t|^\mu \quad (\text{by Lemma 4.1.4 and Proposition 4.1.5}) \\ &= O(|t|^{\mu/2}) \quad \text{as } t \to 0, \end{aligned}$$

where β and $\mu > 0$ are as in Lemma 4.1.4 and Proposition 4.1.5 respectively, and we have used the simple fact that $\lim_{x\to 0+} x^\alpha \log x = 0$ for any $\alpha > 0$. Replacing μ by 2μ in (4.1.11), we get the first estimate of Proposition 4.1.6(i). Since u_t is real-valued, we have

(4.1.12)

$$\begin{aligned} \|\partial u_t\|_2^2 = \|\bar\partial u_t\|_2^2 &= \int_{M_t} \langle \bar\partial u_t, \bar\partial u_t\rangle \omega_{\mathrm{hyp},t} \\ &= \int_{M_t} \langle \Delta_t u_t, u_t\rangle \omega_{\mathrm{hyp},t} \\ &= \int_{M_t} \langle \phi_t, u_t\rangle \omega_{\mathrm{hyp},t} \\ &\le \|\phi_t\|_2\|u_t\|_2 \\ &= O(|t|^\mu) \quad \text{as } t \to 0 \quad (\text{by Proposition 4.1.5 and (4.1.11)}) \end{aligned}$$

for some $\mu > 0$. This finishes the proof of Proposition 4.1.6(i). Let $\{z_t\}$ be as in Proposition 4.1.6(ii). Using the arguments in the proof of Proposition 3.5.2, one easily sees that there exist constants $C, \mu > 0$ such that, as in (3.5.4),

$$\begin{aligned} (4.1.13) \quad |u_t(z_t)| &\le C\big(\|u_t\|_2 + \|\phi_t\|_2\big) \\ &= O(|t|^\mu) \quad \text{as } t \to 0 \\ &\qquad (\text{by Proposition 4.1.5 and Proposition 4.1.6(i)}), \end{aligned}$$

which gives Proposition 4.1.6(ii).

We remark that Proposition 4.1.6(ii) will be strengthened later in Proposition 4.2.4, and it is proved here for the sake of exposition. To summarize our results at this point, we have:

Proposition 4.1.7. *Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ (with a separating node $p \in M$), M_1, M_2, $\mathcal{L} = \{L_t\}_{t\in\Delta}$ and $f : \widetilde{M} \to M$, $L_{0,1}$, $L_{0,2}$, d_1 and d_2 be as in Theorem 1(ii). Then $\{L_t\}_{t\in\Delta}$ admits an almost nice family of flat p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ such that f^*h_0 extends to smooth flat Hermitian metrics on $L_{0,1}$ over M_1 and $L_{0,2}$ over M_2 if and only if $d_1 = d_2 = 0$.*

Proof. The 'if' part of Proposition 4.1.7 can be proved by using the arguments in the proof of Theorem 1(i) in (3.4) with Proposition 3.5.2 and Lemma 3.2.2 replaced by Proposition 4.1.6(ii) and Lemma 4.1.1 respectively (and with each h_t constructed as in (3.5.5)). The 'only if' part of Proposition 4.1.7 simply follows from the calculation that

$$d_k = \int_{M_k} c_1(L_{0,k}, h_{0,k}) = 0,$$

where $h_{0,k}$ denotes the smooth flat Hermitian metric on $L_{0,k}$ obtained by extending f^*h_0 on $L_{0,k}\big|_{M_k\setminus\{p_k\}}$, $k = 1, 2$.

4.2. . Notation as in (4.1). Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ (with a separating node $p \in M$) and $\mathcal{L} = \{L_t\}_{t\in\Delta}$ (with $d_1 = d_2 = 0$) be as in the 'if' part of Theorem 1(ii). Also let $h = \{h_t\}_{t\in\Delta}$ be the almost nice family of flat p-singular Hermitian metrics on $\mathcal{L} = \{L_t\}_{t\in\Delta}$ constructed in (4.1) in the proof of the 'if' part of Proposition 4.1.7. In this section, we are going to complete the proof of Theorem 1(ii) by showing that the restriction $h\Big|_{\mathcal{M}\setminus M_0}$ is good on $\mathcal{M}$ (cf. (2.2)).

First we recall the family of p-singular Hermitian metrics $\tilde{h} = \{\tilde{h}_t\}_{t\in\Delta}$ on $\mathcal{L} = \{L_t\}_{t\in\Delta}$ constructed in (4.1).

Proposition 4.2.1. $\tilde{h}\Big|_{\mathcal{M}\setminus M_0}$ *is good on $\mathcal{M}$.*

Proof. First we recall from (4.1) that $\tilde{h}$ is smooth on $\mathcal{M}\setminus\{p\}$. Thus, to prove the goodness of $\tilde{h}$, it suffices to consider a coordinate open neighborhood of p in $\mathcal{M}$. Let $V \subset U_1 \times U_2 = \{z = (z_1, z_2) \,\big|\, |z_1|, |z_2| < 1\}$ be as in (4.1). It is easy to see that one can choose a sufficiently small $r_0 > 0$ such that the subset $\Delta^{*2}(r_0) := \{z = (z_1, z_2) \,\big|\, 0 < |z_1|, |z_2| < r_0\} \subset U_1 \times U_2$ satisfies $\Delta^{*2}(r_0) \subset V$ and $\Delta^{*2}(r_0) \cap R_k = \emptyset$, where R_k is as in (3.2.6), $k = 1, 2$. Let η, δ and e_V be as in (3.1.1), (3.2.1) and (4.1.3) respectively. Then as in (3.2.14), one sees from (3.1) and (4.1) that for $z = (z_1, z_2) \in \Delta^{*2}(r_0)$,

(4.2.1)

$$\tilde{h}(e_V, e_V)(z) = \big(h_{0,1}(e_V, e_V)(z_1, 0)\big)^{\eta(\tau)} \cdot \big(h_{0,2}(e_V, e_V)(0, z_2)\big)^{1-\eta(\tau)}, \quad \text{where}$$

$$\tau := \frac{1}{4\delta}\left(\frac{\log|z_1|}{\log|z_1| + \log|z_2|} - \frac{1}{2} + 2\delta\right).$$

From Lemma 4.1.1, one sees that there exists a constant $C_1 > 0$ such that for $k, \ell, m, n = 1, 2$ and $[z_\ell, 0]$ with $|z_\ell| < r_0$ (here, as in (3.3.9), $[z_\ell, 0] = (z_1, 0)$ or $(0, z_2)$ according as $\ell = 1$ or 2), we have

$$\begin{aligned} \frac{1}{C_1} \le h_{0,k}(e_V, e_V)[z_\ell, 0] \le C_1, \\ \big|\partial_{z_m} \log h_{0,k}(e_V, e_V)[z_\ell, 0]\big| \le C_1, \quad \text{and} \\ \big|\partial_{z_m}\partial_{\overline{z_n}} \log h_{0,k}(e_V, e_V)[z_\ell, 0]\big| \le C_1. \end{aligned} \tag{4.2.2}$$

First we see from (3.1.1), (4.1.1) and (4.2.2) that there exists a constant $C_2 > 0$ such that for $z \in \Delta^{*2}(r_0)$,

$$\frac{1}{C_2} \le \tilde{h}(e_V, e_V)(z) \le C_2. \tag{4.2.3}$$

For $k, \ell = 1, 2$ and $z = (z_1, z_2) \in \Delta^{*2}(r_0)$, one computes directly from (4.2.1) that

$$\begin{aligned} \big|\partial_{z_1}\eta(\tau)\big| &= \left|\eta'(\tau) \cdot \frac{1}{8\delta z_1} \cdot \frac{\log|z_2|}{(\log|z_1| + \log|z_2|)^2}\right| \\ &\le C_3 \frac{1}{|z_1|\big|\log|z_1|\big|} \end{aligned} \tag{4.2.4}$$

for some constant $C_3 > 0$. Similarly, one can verify that there exists a constant $C_4 > 0$ such that for $k, \ell = 1, 2$ and $z = (z_1, z_2) \in \Delta^{*2}(r_0)$,

$$\begin{aligned} \big|\partial_{z_2}\eta(\tau)\big| \le C_4 \cdot \frac{1}{|z_2|\big|\log|z_2|\big|}, \quad \text{and} \\ \big|\partial_{z_k}\partial_{\overline{z_\ell}}\eta(\tau)\big| \le C_4 \cdot \frac{1}{|z_k|\big|\log|z_k|\big|} \cdot \frac{1}{|z_\ell|\big|\log|z_\ell|\big|}. \end{aligned} \tag{4.2.5}$$

Finally by differentiating $\log \tilde{h}(e_V, e_V)(z)$ using the product rule, one sees from (3.1.1), (4.2.1), (4.2.3) and (4.2.5) that there exists a constant $C_5 > 0$ such that for $k, \ell = 1, 2$ and $z = (z_1, z_2) \in \Delta^{*2}(r_0)$,

$$\begin{aligned} \big|\partial_{z_k} \log \tilde{h}(e_V, e_V)(z)\big| \le C_5 \cdot \frac{1}{|z_k|\big|\log|z_k|\big|}, \quad \text{and} \\ \big|\partial_{z_k}\partial_{\overline{z_\ell}} \log \tilde{h}(e_V, e_V)(z)\big| \le C_5 \cdot \frac{1}{|z_k|\big|\log|z_k|\big|} \cdot \frac{1}{|z_\ell|\big|\log|z_\ell|\big|}, \end{aligned} \tag{4.2.6}$$

which, together with (4.2.3), lead to the goodness of $\tilde{h}$, and this finishes the proof of Proposition 4.2.1.

To facilitate ensuing discussion, we recall briefly the family of diffeomorphisms $\{\Phi_t : M_{t_o} \to M_t\}_{0<|t|<|t_o|}$ constructed in [**Wo2**, 5.4T]. Recall from (4.1) the coordinate neighborhood $U_1 \times U_2 = \{z = (z_1, z_2) \,\big|\, |z_1|, |z_2| < 1\}$ of

p in $\mathcal{M}$, so that $V_t := (U_1 \times U_2) \cap M_t$ is defined by the equation $z_1 z_2 = t$, $t \in \Delta^*$. Let $V^* := \cup_{0<|t|<|t_o|} V_t$. Fix a number t_o such that $0 < |t_o| < 1$. Let

$$\nu := \frac{\log z_1}{\log |t_o|}, \quad \nu' := \frac{\log z_2}{\log |t_o|}, \quad \epsilon := \frac{\log t}{\log |t_o|} - 1, \tag{4.2.7}$$

so that the equation $z_1 z_2 = t$ becomes $\nu + \nu' = 1 + \epsilon$, and $t = |t_o|$ corresponds to $\epsilon = 0$. The universal cover of each annulus $V_t = V_{t(\epsilon)}$ is given by $=$ $H_\epsilon := \{\zeta \in \mathbb{C} \,\big|\, 0 < \operatorname{Re} \zeta < 1 + \operatorname{Re} \epsilon\}$ with deck transformations

$$\zeta \to \zeta + 2n\pi i / \log |t_o|, \quad n \in \mathbb{Z}. \tag{4.2.8}$$

The inclusion $V_t \subset (U_1 \setminus \{|z_1| \leq |t|\}) \times (U_2 \setminus \{|z_2| \leq |t|\})$ induces an inclusion on their universal covers given by

$$\zeta \in H_\epsilon \hookrightarrow (\nu, \nu') = (\zeta, 1 + \epsilon - \zeta) \in H_\epsilon \times H_\epsilon. \tag{4.2.9}$$

Fix a small constant $0 < \delta_1 < \frac{1}{2}$ and fix a smooth increasing function $\varphi(x),\ 0 \leq x \leq 1$, such that

$$\varphi(0) = 0, \quad \varphi(1) = 1 \quad \text{and} \quad \operatorname{supp}(\varphi') \subset [\delta_1, 1 - \delta_1]. \tag{4.2.10}$$

To distinguish the ζ coordinate on H_0 from that on a general H_ϵ, we denote it by $\hat{\zeta}$. Now we define a mapping $f : H_0 \times \{\epsilon \in \mathbb{C} \,\big|\, \operatorname{Re} \epsilon > 0\} \to \mathbb{C}$ given by

$$f(\hat{\zeta}, \epsilon) := \hat{\zeta} + \epsilon \varphi(\operatorname{Re} \hat{\zeta}). \tag{4.2.11}$$

It is easy to see that for each ϵ, $f(\cdot, \epsilon)$ is a diffeomorphism from H_0 to H_ϵ. In addition, each $f(\cdot, \epsilon)$ descends to a diffeomorphism between V_{t_o} and $V_{t(\epsilon)}$, which we denote by the same symbol. Recall from the plumbing construction in (2.1) that each $M_t \setminus V_t$ is canonically biholomorphic to $M_0 \setminus (U_1 \cup U_2)$, so that each $M_t \setminus V_t$ is canonically biholomorphic to $M_{t_o} \setminus V_{t_o}$. Finally we define $\Phi_t : M_{t_o} \to M_t$ to be $f(\cdot, \epsilon)$ on V_t and the inverse of the above canonical biholomorphism on $M_{t_o} \setminus V_{t_o}$. Then one easily sees that each Φ_t is a diffeomorphism from M_{t_o} to M_t, and $\{\Phi_t\}_{0<|t|<|t_o|}$ forms a smooth family of diffeomorphisms.

Throughout the remaining discussion in (4.2), we will fix a coordinate open cover of the total space of $\{M_t\}_{0<|t|<|t_o|}$ as follows. First we fix coordinate open subsets $\{(U_\alpha, \zeta_\alpha)\}_{\alpha \in A}$ of M_{t_o} covering $M_{t_o} \setminus V_{t_o}(1 - \delta_2)$ for some $\delta_2 > 0$ and such that

$$U_\alpha \cap V_{t_o} \subset \{(z_1, t_o/z_1) \,\big|\, |z_1| > |t_o|^{\delta_1/2} \text{ or } |z_1| < |t_o|^{1-\delta_1/2}\} \tag{4.2.12}$$

for all $\alpha \in A$, shrinking δ_2 if necessary (such choice of δ_2 ensures that $\operatorname{supp}(\hat{\varphi}) \cap U_\alpha = \emptyset$ for all $\alpha \in A$, where $\hat{\varphi}$ is as in (4.2.18) below). Here ζ_α denotes the coordinate function on U_α, and δ_1 is as in (4.2.10). Via the maps $i_{k,t}^{-1} \circ i_{k,t_o}$, $k = 1, 2$, as above, one sees that

$$\{V^*(1 - \delta_2)\} \cup \{(U_\alpha \times \Delta^*(|t_o|), (\zeta_\alpha, t))\}_{\alpha \in A} \tag{4.2.13}$$

forms a coordinate open cover of $\{M_t\}_{0<|t|<|t_o|}$, where $V^*(1-\delta_2) := V^* \cap (U_1^*(1-\delta_2)\times U_2^*(1-\delta_2))$, so that $V^*(1-\delta_2) = \cup_{0<|t|<|t_o|} V_t(1-\delta_2)$ (cf. (4.2.15) below). Shrinking δ_2 if necessary, it is easy to see that we may assume that supp $(\phi_t) \cap (U_\alpha \times \{t\}) = \emptyset$ for $0 < |t| < |t_o|$ and each $\alpha \in A$. By (4.2.7), we may write $\frac{\partial}{\partial \epsilon} = \log|t_0| \cdot = t\frac{\partial}{\partial t}$ on Δ^*. Associated to $\{\Phi_t\}_{0<|t|<|t_o|}$ is a lifting of the vector field $\partial/\partial\epsilon$ to a smooth vector field $\partial/\partial\sigma$ on the total space of $\{M_t\}_{0<|t|<|t_o|}$ as follows. First observe that on V^* and in terms of (ζ,ϵ), we have $(\zeta,\epsilon) = (f(\hat{\zeta},\epsilon),\epsilon)$, and $\partial/\partial\sigma$ is given by

$$\frac{\partial}{\partial\sigma} = \frac{\partial}{\partial\epsilon} + \frac{\partial f}{\partial\epsilon}\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial\epsilon} + \varphi\frac{\partial}{\partial\zeta} \quad \text{on } V^*. \tag{4.2.14}$$

Also for each $\alpha \in A$, $\partial/\partial\sigma$ is simply given by $\partial/\partial\epsilon (= \log|t_o| \cdot t(\partial/\partial t))$ on $U_\alpha \times \Delta^*(|t_o|)$. Using (4.2.12), one can check that $\partial/\partial\sigma$ forms a smooth global vector field on the total space of $\{M_t\}_{0<|t|<|t_o|}$ such that $\pi_*(\partial/\partial\sigma)(z) = \partial/\partial\epsilon$ at any point $z \in \{M_t\}_{0<|t|<|t_o|}$, where $\pi : \mathcal{M} \to \Delta$ denotes the projection map. Notice that there is a slight abuse of notation here, as $\partial/\partial\sigma$ is not a coordinate vector field. Observe also that by (4.2.8), $\partial/\partial\zeta$ descends to a non-vanishing tangent vector field on each M_t, $0 < |t| < |t_o|$, and $\{\partial/\partial\sigma, \partial/\partial\zeta\}$ forms a basis of $T_z V^*$ at each $z \in V*$.

Write $U_1^* \times U_2^* := (U_1 \setminus \{0\}) \times (U_2 \setminus \{0\})$. For $0 < r < 1$, we define $U_1^*(r) \times U_2^*(r) := \{z = (z_1, z_2) \,\big|\, 0 < |z_1|, |z_2| < r\} \subset U_1 \times U_2$, and we define

$$V_t(r) := V_t \cap (U_1^*(r) \times U_2^*(r)) \subset M_t. \tag{4.2.15}$$

For a tangent vector v on $U_1^* \times U_2^*$, we denote by $\|v\|_{U_1^*\times U_2^*}$ the norm of v with respect to the product metric induced by the Poincaré metrics $|dz_k|^2/|z_k|^2\big|\log|z_k|\big|^2$ on U_k^*, $k = 1, 2$.

Lemma 4.2.2. *Let t_o, $\frac{\partial}{\partial\zeta}$, $\frac{\partial}{\partial\sigma}$ be as above. Then for any constant $0 < \kappa < 1$, there exist constants $C_1 = C_1(\kappa), C_2 = C_2(\kappa) > 0$ such that*

$$\frac{C_1}{\big|\log|t|\big|} \leq \Big\|\frac{\partial}{\partial\zeta}(z)\Big\|_{U_1^*\times U_2^*}, \Big\|\frac{\partial}{\partial\sigma}(z)\Big\|_{U_1^*\times U_2^*} \leq C_2$$

for any $0 < |t| < |t_o|$ and $z \in V_t(\kappa)$.

Proof. Using the chain rule and the correspondence in (4.2.9), one easily computes that on $V^* \subset U_1^* \times U_2^*$, we have

$$\begin{pmatrix} \frac{\partial}{\partial\nu} \\ \frac{\partial}{\partial\nu'} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial\zeta} \\ \frac{\partial}{\partial\epsilon} \end{pmatrix} = \begin{pmatrix} 1-\varphi & 1 \\ -\varphi & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial\zeta} \\ \frac{\partial}{\partial\sigma} \end{pmatrix}, \tag{4.2.16}$$

where the second equality follows from (4.2.14). Solving (4.2.16), we get

$$\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial\nu} - \frac{\partial}{\partial\nu'} = \log|t_o|\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right), \tag{4.2.17}$$

$$\frac{\partial}{\partial\sigma} = \varphi\frac{\partial}{\partial\nu} + (1-\varphi)\frac{\partial}{\partial\nu'} = \log|t_o|\left(\varphi z_1\frac{\partial}{\partial z_1} + (1-\varphi)z_2\frac{\partial}{\partial z_2}\right),$$

where the second inequality on each line of (4.2.17) follows from (4.2.7). Together with the boundedness of φ and the fact that $\|z_k\partial/\partial z_k\|_{U_1^*\times U_2^*} \sim 1/|z_k|$ on $U_1^*(\kappa)\times U_2^*(\kappa)$, one obtains Lemma 4.2.2 immediately.

From (4.2.9), one sees that (ζ,ϵ) provides coordinate system for V^*. Let φ be as in (4.2.11). Define

$$\hat{\varphi}(\zeta,\epsilon) := \varphi(\hat{\zeta}) \quad \text{for } (\zeta,\epsilon)\in V^*, \tag{4.2.18}$$

where $\hat{\zeta} = \hat{\zeta}(\zeta,\epsilon)$ is defined implicitly by (4.2.11), i.e., $\zeta = \hat{\zeta} + \epsilon\varphi(\mathrm{Re}\,\hat{\zeta})$. Denote the Lie bracket of two vector fields X, Y by $[X,Y]$, and denote $\hat{\varphi}_\zeta := \partial\hat{\varphi}/\partial\zeta$, etc. Also we denote $\hat{\varphi}_{;\sigma} := \frac{\partial}{\partial\sigma}\hat{\varphi}$, etc., so that $\hat{\varphi}_{;\zeta} = \hat{\varphi}_\zeta$, $\hat{\varphi}_{;\sigma\bar{\zeta}} = \frac{\partial}{\partial\bar{\zeta}}\big(\frac{\partial}{\partial\sigma}\big)\hat{\varphi}$, etc.

Lemma 4.2.3. (i) *On V^*, we have*

$$\Big[\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\sigma}\Big] = \hat{\varphi}_\zeta\frac{\partial}{\partial\zeta}, \quad \Big[\frac{\partial}{\partial\bar{\zeta}},\frac{\partial}{\partial\sigma}\Big] = \hat{\varphi}_{\bar{\zeta}}\frac{\partial}{\partial\zeta},$$

$$\Big[\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\bar{\sigma}}\Big] = \hat{\varphi}_\zeta\frac{\partial}{\partial\bar{\zeta}}, \quad \Big[\frac{\partial}{\partial\bar{\zeta}},\frac{\partial}{\partial\bar{\sigma}}\Big] = \hat{\varphi}_{\bar{\zeta}}\frac{\partial}{\partial\bar{\zeta}}, \quad \textit{and}$$

$$\Big[\frac{\partial}{\partial\sigma},\frac{\partial}{\partial\bar{\sigma}}\Big] = (\hat{\varphi}_\epsilon + \hat{\varphi}\hat{\varphi}_\zeta)\frac{\partial}{\partial\bar{\zeta}} - (\hat{\varphi}_{\bar{\epsilon}} + \hat{\varphi}\hat{\varphi}_{\bar{\zeta}})\frac{\partial}{\partial\zeta}.$$

(ii) *Let $(U_\alpha\times\Delta^*(|t_o|),(\zeta_\alpha,t))$, $\alpha\in A$, be as in (4.2.13). We have*

$$\Big[\frac{\partial}{\partial\zeta_\alpha},\frac{\partial}{\partial\sigma}\Big] = \Big[\frac{\partial}{\partial\bar{\zeta}_\alpha},\frac{\partial}{\partial\sigma}\Big] = \Big[\frac{\partial}{\partial\zeta_\alpha},\frac{\partial}{\partial\bar{\sigma}}\Big] = \Big[\frac{\partial}{\partial\bar{\zeta}_\alpha},\frac{\partial}{\partial\bar{\sigma}}\Big] = 0 \quad \textit{on } U_\alpha\times\Delta^*(|t_o|).$$

(iii) *There exists a constant $C>0$ such that $\hat{\varphi}$ and its partial derivatives satisfy*

$$|\hat{\varphi}|,\ |\hat{\varphi}_*|,\ |\hat{\varphi}_{**}|,\ |\hat{\varphi}_{***}| \le C \quad \textit{on } V^*,$$

where each subscript $*$ can be $\zeta,\bar{\zeta},\epsilon$ or $\bar{\epsilon}$.

(iv) *There exists a constant $C'>0$ such that*

$$|\hat{\varphi}|,\ |\hat{\varphi}_{;*}|,\ |\hat{\varphi}_{;**}|,\ |\hat{\varphi}_{;***}| \le C' \quad \textit{on } V^*,$$

where each subscript $*$ can be $\zeta,\bar{\zeta},\sigma$ or $\bar{\sigma}$.

Proof. Lemma 4.2.3(i) follows easily from (4.2.14) and the fact that $\hat{\varphi}$ is a real-valued function. The commutation relations in Lemma 4.2.3(ii) are trivial since (ζ_α,σ) provide smooth coordinates on $U_\alpha\times\Delta^*(|t_o|)$. Lemma 4.2.3(iii) follows easily from the chain rule, the boundedness of φ and its derivatives and that of $\frac{\partial\hat{\zeta}}{\partial\zeta}$ and $\frac{\partial\hat{\zeta}}{\partial\epsilon}$. Finally Lemma 4.2.3(iv) follows easily from Lemma 4.2.3(iii) and (4.2.14), and this finishes the proof of Lemma 4.2.3.

Let $t_o, \mathrm{I}_t, R_{k,t}$, $k = 1, 2$, be as in Lemma 4.2.2, (3.2.1) and (3.2.6) respectively. It is clear from (4.1.10) and the plumbing construction in (2.1) that shrinking t_o if necessary, there exists $r_o > 0$ such that

$$\mathrm{I}_t \subset U_1^*(r_o) \times U_2^*(r_o) \quad \text{and} \quad B_t(z, \rho_t) \subset V_t\left(\frac{1}{2}\right) \tag{4.2.19}$$

for $z \in U_1^*(r_o) \times U_2^*(r_o)$, $0 < |t| < |t_o|$. Let ϕ_t be as in (4.1.5), and let u_t be as in Proposition 4.1.6 (so that (3.5.1) and (3.5.2) hold). First we strengthen Proposition 4.1.6(i) in the following:

Proposition 4.2.4. *There exists a constant $\mu > 0$ such that*

$$\sup_{z\in M_t} |u_t(z)| = O(|t|^\mu) \quad \textit{as } t \to 0.$$

Proof. Let $\lambda_{1,t}$ and ρ_t be as in Lemma 4.1.4. By Lemma 3.4.1(ii), (3.5.2) and (4.1.5), there exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}\sup_{z\in M_t} |u_t(z)| &\leq \Big(\frac{C_1}{\lambda_{1,t}\tanh\rho_t} + C_2 \sinh\rho_t\Big)\|\phi_t\|_2 \\ &\leq \big(C_3\big|\log|t|\big|^2 + C_4\big|\log|t|\big|\big)\|\phi_t\|_2,\end{aligned}$$

where the last inequality follows from Lemma 4.1.4 and the identities $\lim_{x\to 0}\tanh x/x = \lim_{x\to 0}\sinh x/x = 1$. Together with Proposition 4.1.5 and the identity $\lim_{x\to 0+} x^\alpha \log x = 0$ for any $\alpha > 0$, Proposition 4.2.4 follows immediately.

Denote

$$u := \{u_t\}_{t\in\Delta^*}. \tag{4.2.20}$$

Then u forms a smooth function on $\{M_t\}_{t\in\Delta^*}$ (cf. Proof of Theorem 1(i)). Recall that $\partial/\partial\zeta$ is only defined on $U_1^* \times U_1^*$ and is tangential to each fiber $V_t \subset M_t$, while $\partial/\partial\sigma$ is a vector field on the total space of $\{M_t\}_{0<|t|<|t_o|}$ but not tangential to any M_t. With slight abuse of notation, we denote, for $t \in \Delta^*$,

$$u_{t;\zeta} := \frac{\partial}{\partial\zeta}(u), \quad u_{t;\sigma\bar{\sigma}} := \frac{\partial}{\partial\bar{\sigma}}\Big(\frac{\partial}{\partial\sigma}(u)\Big) \quad \text{on } M_t, \quad \text{etc.}$$

Notice that $u_{t;\zeta} = \frac{\partial}{\partial\zeta}(u_t)$, while for fixed t, $\frac{\partial}{\partial\sigma}(u_t)$ does not make sense.

Lemma 4.2.5. (i) *There exists a constant $\mu > 0$ such that*

$$\sup_{z\in V_t} |u_{t;\zeta\bar{\zeta}*}| = O(|t|^\mu) \quad \textit{and} \quad \sup_{z\in=V_t} |u_{t;\zeta\bar{\zeta}**}| = O(|t|^\mu) \quad \textit{as } t \to 0,$$

where each subscript $*$ *may be* $\zeta, \bar\zeta, \sigma$ *or* $\bar\sigma$*.*

(ii) *For* $0<|t|<|t_o|$ *and* $\alpha\in A$*, we have* $u_{t;\zeta_\alpha\bar\zeta_\alpha}=0$ *on* $U_\alpha\times\{t\}\subset M_t$*.*

Proof. To prove Lemma 4.2.5(i), we first recall that

$$u_{t;\zeta\bar\zeta} = -g_{t,\zeta\bar\zeta}\phi_t \quad \text{on } V_t \tag{4.2.21}$$

(cf. (3.5.2)). First we recall that $\mathrm{supp}(\phi_t)\subset \mathrm{I}_t\cup R_{1,t}\cup R_{2,t}$ (cf. e.g., Proposition 3.3.4), and thus by (4.2.21), we also have $\mathrm{supp}(u_{t;\zeta\bar\zeta})$, $\mathrm{supp}(u_{t;\zeta\bar\zeta*})$, $\mathrm{supp}(u_{t;\zeta\bar\zeta**})\subset \mathrm{I}_t\cup R_{1,t}\cup R_{2,t}$. From (4.1.5), (4.2.1), (4.2.7) and (4.2.9), one sees that on I_t,

$$\begin{aligned} u_{t;\zeta\bar\zeta} &= \partial_\zeta\partial_{\bar\zeta}\log\tilde h(e_V,e_V) \\ &= \partial_\zeta\partial_{\bar\zeta}\big(\hat\eta\cdot\log\hat h\big), \quad \text{where} \\ \hat\eta(\zeta,\epsilon) &:= \eta\left(\frac{1}{4\delta}\cdot\frac{\mathrm{Re}\,\zeta}{1+\mathrm{Re}\,\epsilon}-\frac{1}{8\delta}+\frac12\right), \quad \text{and} \\ \hat h(z) &:= \frac{h_{0,1}(e_V,e_V)(z_1,0)}{h_{0,2}(e_V,e_V)(0,z_2)} \quad \text{for } z=(z_1,z_2)\in \mathrm{I}_t. \end{aligned} \tag{4.2.22}$$

Here $\eta, h_{0,1}, h_{0,2}, e_V, \delta$ are as in (4.2.1). One easily = checks that

$$\hat\eta_\zeta = \eta'\cdot\frac{1}{8\delta(1+\mathrm{Re}\,\epsilon)}, \quad \hat\eta_\epsilon = \eta'\cdot\frac{-\mathrm{Re}\,\zeta}{8\delta(1+\mathrm{Re}\,\epsilon)^2}. \tag{4.2.23}$$

Since $\mathrm{Re}\,\epsilon>0$ and $\left|\frac{\mathrm{Re}\,\zeta}{1+\mathrm{Re}\,\epsilon}\right| = \frac{\log|z_1|}{\log|t|} < \frac12+2\delta$ on I_t (cf. (3.2.1) and (4.2.7)), it follows that $\hat\eta_\zeta$, $\hat\eta_\epsilon$ are uniformly bounded on I_t for $0<|t|<|t_o|$ (and hence so is $\hat\eta_{;\sigma}:=(\partial/\partial\sigma)\hat\eta$ by (4.2.14) and Lemma 4.2.3(iv)). Similarly, one can show that there exists a constant $C_1>0$ such that for $0<|t|<|t_o|$,

$$|\hat\eta_{;****}|\le C_1 \quad \text{on } \mathrm{I}_t, \tag{4.2.24}$$

where each subscript $*$ may be $\zeta,\bar\zeta,\sigma$, $\bar\sigma$ or empty (so that $|\hat\eta_{;\sigma\bar\zeta}|\le C_1$, etc.). By Lemma 4.1.1, $\log\hat h$ is a real analytic function and is equal to 0 at $z=(z_1,z_2)=(0,0)$. Thus we may write

$$\log\hat h = c_1z_1+c_2z_2+\bar c_1\bar z_1+\bar c_2\bar z_2+\mathcal{A}^{(\ge2)}(z_1,z_2) \tag{4.2.25}$$

for some constants c_1, c_2, where $\mathcal{A}^{(\ge2)}(z_1,z_2)$ denotes a (convergent) power series in z_1, $\bar z_1$, z_2, $\bar z_2$ such that each term is of degree ≥ 2. Together with (4.2.17) and the fact that $|z_k| = O(|t|^{\frac12-2\delta})$ on I_t, one can easily show that there exists a constant $\mu_1>0$ such that

$$\sup_{z\in \mathrm{I}_t}|(\log\hat h)_{;***}(z)|,\ \sup_{z\in \mathrm{I}_t}|(\log\hat h)_{;****}(z)| = O(|t|^{\mu_1}) \tag{4.2.26}$$

as $t\to0$, where each subscript $*$ can be ζ, $\bar\zeta$, σ or $\bar\sigma$. Combining (4.2.22), (4.2.24), (4.2.26) and Lemma 4.2.3(iv), one sees that

$$\sup_{z\in \mathrm{I}_t}|u_{t;\zeta\bar\zeta*}(z)|,\ \sup_{z\in \mathrm{I}_t}|u_{t;\zeta\bar\zeta**}(z)| = O(|t|^{\mu_1}) \tag{4.2.27}$$

as $t \to 0$, where each subscript $*$ can be ζ, $\bar{\zeta}$, σ or $\bar{\sigma}$. Similarly, one can show that there exists $\mu_2 > 0$ such that for $k = 1, 2$,

$$\sup_{z \in R_{k,t}} |u_{t;\zeta\bar{\zeta}*}(z)|, \sup_{z \in R_{k,t}} |u_{t;\zeta\bar{\zeta}**}(z)| = O(|t|^{\mu_2}) \tag{4.2.28}$$

as $t \to 0$, where each subscript $*$ can be ζ, $\bar{\zeta}$, σ or $\bar{\sigma}$. By combining (4.2.27) and (4.2.28), one obtains Lemma 4.2.5(i) immediately. Lemma 4.2.5(ii) follows easily from the equality $u_{t;\zeta_\alpha\bar{\zeta}_\alpha} = -g_{t,\zeta_\alpha\bar{\zeta}_\alpha}\phi_t$ on U_α and the fact that $\mathrm{supp}(\phi_t) \cap U_\alpha = \emptyset$ for each $\alpha \in A$, and this finishes the proof of Lemma 4.2.5.

Proposition 4.2.6. *There exist constants $C > 0$ and $m \geq 1$ such that for any $0 < |t| < |t_o|$ and $z \in V_t$,*

(i) $|g_{t,\zeta\bar{\zeta}}(z)|,\ |g_t^{\zeta\bar{\zeta}}(z)| \leq C\big|\log|t|\big|^{2m}$, *and*
(ii) *for any constant $0 < \kappa < 1$, there exists a constant $C = C(\kappa) > 0$ such that for all $0 < |t| < |t_o|$,*

$$\big|(\log g_{t,\zeta\bar{\zeta}})_{;\zeta}\big|,\ \big|(\log g_{t,\zeta\bar{\zeta}})_{;\sigma}\big|,\ \big|(\log g_{t,\zeta\bar{\zeta}})_{;\sigma\bar{\zeta}}\big|,\ \big|(\log g_{t,\zeta\bar{\zeta}})_{;\sigma\bar{\sigma}}\big| \leq C' \quad \text{on } V_t(\kappa). \tag{4.2.29}$$

Proof. From (4.2.17), one sees that $\partial/\partial\zeta$ is a local non-vanishing holomorphic section of the vertical line bundle $\tilde{T}$ near the node p. Then Proposition 4.2.6(i) and the first two inequalities of Proposition 4.2.6(ii) follow from Wolpert's result [**Wo2**, Theorem 5.8] that $\{\omega_{\mathrm{hyp},t}\}$ is good (cf. (2.2)) and the fact that for $\log|z_1| + \log|z_2| = \log|t|$ for $= z = (z_1, z_2) \in V_t$. Since the verification of the third and fourth inequalities of Proposition 4.2.6(ii) are similar, we will only prove the latter. The goodness of $\{\omega_{\mathrm{hyp},t}\}$ also implies that on $U_1^* \times U_2^*$, one has

$$\left|(\bar{\partial}\partial\log g_{t,\zeta\bar{\zeta}})\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar{\sigma}}\right)\right| \leq C_1\left\|\frac{\partial}{\partial\sigma}\right\|^2_{U_1^*\times U_2^*} \tag{4.2.30}$$

for some constant $C_1 > 0$. By expanding $d(\partial\log g_{t,\zeta\bar{\zeta}})$, one has, for $0 < |t| < |t_o|$ and $0 < \kappa < 1$,

$$\begin{aligned}\big|(\log g_{t,\zeta\bar{\zeta}})_{;\sigma\bar{\sigma}}\big| &= \left|-(\bar{\partial}\partial\log g_{t,\zeta\bar{\zeta}})\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar{\sigma}}\right) - (\partial\log g_{t,\zeta\bar{\zeta}})\left(\left[\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar{\sigma}}\right]\right)\right| \\ &\leq \left|(\bar{\partial}\partial\log g_{t,\zeta\bar{\zeta}})\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar{\sigma}}\right)\right| + \big|(\hat{\varphi}_\epsilon + \hat{\varphi}\hat{\varphi}_\zeta)(\partial\log g_{t,\zeta\bar{\zeta}})_{;\zeta}\big| \\ &\leq C_2 \quad \text{on } V_t(\kappa),\end{aligned} \tag{4.2.31}$$

where the last inequality follows from (4.2.30), Lemma 4.2.3(iii) and the first inequality of (4.2.29). This finishes the proof of Proposition 4.2.6.

Proposition 4.2.7. *Let r_o be as in (4.2.19). There exists a constant $\mu > 0$ such that*

$$\sup_{z \in V_t(r_o)} |u_{t;\zeta}(z)| = O(|t|^{\mu}) \quad \textit{as } t \to 0.$$

Proof. Similar to the proofs of Proposition 3.5.2 and Proposition 4.2.4, it follows from Lemma 3.4.1(i) and Lemma 4.1.4(ii) that there exist constants $C_1, C_2 > 0$ such that for $0 < |t| < \frac{1}{2}$ and $z \in V_t(r_o)$,

$$|u_{t;\zeta}(z)| \leq C_1 \big| \log |t| \big| \sqrt{\int_{B_t(z,\rho_t)} |u_{t;\zeta}|^2 \omega_{\text{hyp},t}} + \frac{C_2}{\big|\log |t|\big|} \sqrt{\int_{B_t(z,\rho_t)} |\Delta_t(u_{t;\zeta})|^2 \omega_{\text{hyp},t}}. \tag{4.2.32}$$

First we have

(4.2.33)

$$\begin{aligned} \int_{B_t(z,\rho_t)} |u_{t;\zeta}|^2 \omega_{\text{hyp},t} &= \int_{B_t(z,\rho_t)} \|\partial u_t\|^2 \Big\| \frac{\partial}{\partial \zeta} \Big\|^2 \omega_{\text{hyp},t} \\ &\leq C_3 \big| \log |t| \big|^{2m} \|\partial u_t\|_2^2 \quad \text{(by Proposition 4.2.6(i))} \\ &= O(|t|^{\mu_1}) \quad \text{(by Proposition 4.1.6(i))} \end{aligned}$$

as $t \to 0$, where $C_3, \mu_1 > 0$ are constants independent of t, and $m \geq 1$ is as in Proposition 4.2.6(i). Since $B_t(z, \rho_t) \subset V_t(\frac{1}{2}) \subset V_t$ (cf. (4.2.19)), one has

$$\begin{aligned} &\int_{B_t(z,\rho_t)} |\Delta_t(u_{t;\zeta})|^2 \omega_{\text{hyp},t} \\ &\leq \int_{V_t(\frac{1}{2})} |g_t^{\zeta\bar{\zeta}} u_{t;\zeta\bar{\zeta}\zeta}|^2 \omega_{\text{hyp},t} \\ &\leq C_4 \big| \log |t| \big|^{4m} \cdot \sup_{z \in V_t} |u_{t;\zeta\bar{\zeta}\zeta}|^2 \quad \text{(by Proposition 4.2.6(i))} \\ &= O(|t|^{\mu_2}) \quad \text{as } t \to 0 \quad \text{(by Lemma 4.2.5)}, \end{aligned} \tag{4.2.34}$$

where $\mu_2 > 0$ is some constant. Finally by combining (4.2.32), (4.2.33) and (4.2.34), one obtains Proposition 4.2.7 immediately.

Proposition 4.2.8. *There exists a constant $\mu > 0$ such that*

$$\sup_{z \in M_t} |u_{t;\sigma}(z)| = O(|t|^{\mu}) \quad \textit{as } t \to 0.$$

Proof. Similar to the proof of Proposition 4.2.4, it follows from Lemma 3.4.1(ii) and Lemma 4.1.4 that there exist constants $C_1, C_2 > 0$ such that for $0 < |t| < \frac{1}{2}$, one has

(4.2.35)
$$\sup_{z\in M_t} |u_{t;\sigma}(z)| \le C_1 \big|\log|t|\big| \left|\int_{M_t} u_{t;\sigma}\omega_{\mathrm{hyp},t}\right| + C_2 \big|\log|t|\big|^2 \|\Delta_t(u_{t;\sigma})\|_2.$$

First by (3.5.2), we have $\int_{M_t} u_t\omega_{\mathrm{hyp},t} = 0$ for all $t = t(\epsilon) \neq 0$, which implies that

$$\frac{\partial}{\partial\epsilon}\int_{M_t} u_t\omega_{\mathrm{hyp},t} = \int_{M_t} u_{t;\sigma}\omega_{\mathrm{hyp},t} + \int_{M_t} u_t L_{\partial/\partial\sigma}(\omega_{\mathrm{hyp},t}) = 0, \tag{4.2.36}$$

where $L_{\partial/\partial\sigma}$ denotes the Lie derivative with respect to $\partial/\partial\sigma$. For each $0 < |t| < |t_o|$, it follows from (4.2.13) that $\{V_t(1-\delta_2)\} \cup \{(U_\alpha \times \{t\}, \zeta_\alpha)\}_{\alpha\in A}$ forms an open cover of M_t. On $V_t(1-\delta_2)$, the $\zeta\bar{\zeta}$-component of the tensor $L_{\partial/\partial\sigma}(\omega_{\mathrm{hyp},t})$ is given by

(4.2.37)
$$\begin{aligned}\left|\big(L_{\partial/\partial\sigma}(\omega_{\mathrm{hyp},t})\big)_{;\zeta\bar{\zeta}}\right| &= \left|\frac{\partial}{\partial\sigma}\big(g_{t,\zeta\bar{\zeta}}\big) + \hat{\varphi}_\zeta g_{t,\zeta\bar{\zeta}}\right| \\ &= |g_{t,\zeta\bar{\zeta}}\big(\log g_{t,\zeta\bar{\zeta}}\big)_{;\sigma} + \hat{\varphi}_\zeta g_{t,\zeta\bar{\zeta}}| \\ &\le C_3 g_{t,\zeta\bar{\zeta}} \quad \text{(by Lemma 4.2.3(iii), Proposition 4.2.6)}\end{aligned}$$

for some constant $C_3 > 0$. On each U_α, one has

(4.2.38)
$$\begin{aligned}\left|\Big(L_{\partial/\partial\sigma}(\omega_{\mathrm{hyp},t})\Big)_{;\zeta_\alpha\bar{\zeta}_\alpha}\right| &= \left|\frac{\partial}{\partial\epsilon}\big(g_{t,\zeta_\alpha\bar{\zeta}_\alpha}\big)\right| \\ &= \left|\log|t_o| \cdot t\frac{\partial}{\partial t}\big(g_{t,\zeta_\alpha\bar{\zeta}_\alpha}\big)\right| \\ &\le C_\alpha |t| \cdot g_{t,\zeta_\alpha\bar{\zeta}_\alpha} \quad \text{(cf. [\textbf{Wo2}, Expansion 4.2])}\end{aligned}$$

for some constant $C_\alpha > 0$. Thus we have

$$\begin{aligned}\left|\int_{M_t} u_{t;\sigma}\,\omega_{\mathrm{hyp},t}\right| &= \left|\int_{M_t} u_t L_{\partial/\partial\sigma}(\omega_{\mathrm{hyp},t})\right| \quad \text{(by (4.2.36))} \\ &\le C_4 \int_{M_t} |u_t|\,\omega_{\mathrm{hyp},t} \quad \text{(by (4.2.37), (4.2.38))} \\ &\le C_4 \cdot 4\pi(q-1)\sup_{z\in M_t}|u_t(z)| \\ &= O(|t|^{\mu_1}) \quad \text{as } t \to 0 \quad \text{(by Proposition 4.2.4)},\end{aligned} \tag{4.2.39}$$

where $C_4, \mu_1 > 0$ are some constants. On $V_t(1-\delta_2)$, we have

(4.2.40)
$$\begin{aligned}\Delta_t(u_{t;\sigma}) &= -g_t^{\zeta\bar\zeta} u_{t;\sigma\bar\zeta\zeta} \\ &= -g_t^{\zeta\bar\zeta}(u_{t;\bar\zeta\sigma\zeta} + (\hat\varphi_{\bar\zeta} u_{t;\zeta})_{;\zeta}) \quad \text{(by Lemma 4.2.3(i))} \\ &= -g_t^{\zeta\bar\zeta}(u_{t;\bar\zeta\zeta\sigma} + \hat\varphi_\zeta u_{t;\bar\zeta\zeta} + \hat\varphi_{\bar\zeta} u_{t;\zeta\zeta} + \hat\varphi_{\bar\zeta\zeta} u_{t;\zeta}) \quad \text{(by Lemma 4.2.3(i))}.\end{aligned}$$

Using the identity $(\nabla(\partial u_t))_{\zeta\zeta} = u_{t;\zeta\zeta} - (\log g_{t,\zeta\bar\zeta})_{;\zeta} u_{t;\zeta}$, one has, on $V_t(1-\delta_2)$,

(4.2.41)
$$\begin{aligned}|g_t^{\zeta\zeta} u_{t;\zeta\zeta}|^2 &\le 2\|\nabla(\partial u_t)\|^2 + 2|g_t^{\zeta\bar\zeta}(\log g_{t,\zeta\bar\zeta})_{;\zeta} u_{t;\zeta}|^2 \\ &\le 2\|\nabla(\partial u_t)\|^2 + C_5\big|\log|t|\big|^{2m}\|\partial u_t\|^2 \quad \text{(by Proposition 4.2.6)}\end{aligned}$$

for some constant $C_5 > 0$ and with $m \ge 1$ as in Proposition 4.2.6(i). Also,

(4.2.42)
$$\begin{aligned}&\|\nabla(\partial u_t)\|_2^2 \\ &= \int_{M_t} \langle \nabla(\partial u_t), \nabla(\partial u_t)\rangle\, \omega_{\mathrm{hyp},t} \\ &= \int_{M_t} \langle \nabla^*\nabla(\partial u_t), \partial u_t\rangle\, \omega_{\mathrm{hyp},t} \\ &= \int_{M_t} \langle \bar\partial^*\bar\partial\partial u_t + \partial u_t, \partial u_t\rangle\, \omega_{\mathrm{hyp},t} \quad \text{(cf. e.g., [S, p. 63, Equation (1.3.4)])} \\ &= \int_{M_t} \langle \bar\partial\partial u_t, \bar\partial\partial u_t\rangle\, \omega_{\mathrm{hyp},t} + \int_{M_t} \langle \partial u_t, \partial u_t\rangle\, \omega_{\mathrm{hyp},t} \\ &= \|\Delta_t u_t\|_2^2 + \|\partial u_t\|_2^2.\end{aligned}$$

Combining (4.2.41) and (4.2.42), we have

$$\begin{aligned}\int_{V_t(1-\delta_2)} |g_t^{\zeta\bar\zeta} u_{t;\zeta\zeta}|^2 \omega_{\mathrm{hyp},t} &\le 2\|\Delta_t u_t\|_2^2 + (2 + C_5\big|\log|t|\big|^{2m})\|\partial u_t\|_2^2 \\ &= 2\|\phi_t\|_2^2 + (2 + C_5\big|\log|t|\big|^{2m})\|\partial u_t\|_2^2 \\ &= O(|t|^{\mu_2}) \quad \text{as } t \to 0,\end{aligned} \tag{4.2.43}$$

where $\mu_2 > 0$ is some constant, and the last line follows from Proposition 4.1.5 and Proposition 4.1.6(i). For $0 < |t| < |t_o|$ and $\alpha \in A$, one sees from Lemma 4.2.3(ii) that on $U_\alpha \times \{t\}$,

(4.2.44)
$$\Delta_t(u_{t;\sigma}) = -g_t^{\zeta_\alpha\bar\zeta_\alpha} u_{t;\sigma\zeta_\alpha\bar\zeta_\alpha} = -g_t^{\zeta_\alpha\bar\zeta_\alpha} u_{t;\zeta_\alpha\bar\zeta_\alpha\sigma} = 0 \quad \text{(by Lemma 4.2.5(ii))}.$$

Thus we have

(4.2.45)

$$\begin{aligned}
&\|\Delta_t(u_{t;\sigma})\|_2^2 \\
&= \int_{V_t(1-\delta_2)} |\Delta_t(u_{t;\sigma})|^2 \omega_{\mathrm{hyp},t} \quad (\text{by } (4.2.44)) \\
&\le 4 \int_{V_t(1-\delta_2)} |g_t^{\zeta\bar\zeta}|^2 \big(|u_{t;\bar\zeta\zeta\sigma}|^2 + |\hat\varphi_\zeta u_{t;\bar\zeta\zeta}|^2 + |\hat\varphi_{\bar\zeta} u_{t;\zeta\zeta}|^2 + |\hat\varphi_{\bar\zeta\zeta} u_{t;\zeta}|^2 \big) \, \omega_{\mathrm{hyp},t} \\
&\qquad (\text{by } (4.2.40)) \\
&\le C_5 \big| \log |t| \big|^{4m} \int_{V_t(1-\delta_2)} |u_{t;\bar\zeta\zeta\sigma}|^2 \omega_{\mathrm{hyp},t} \\
&\quad + C_6 \|\phi_t\|_2^2 + C_7 \int_{V_t(1-\delta_2)} |g_t^{\zeta\bar\zeta} u_{t;\zeta\zeta}|^2 \omega_{\mathrm{hyp},t} \\
&\qquad + C_8 \big| \log |t| \big|^{2m} \|\partial u_t\|_2^2 \quad (\text{by Proposition 4.2.6(i), Lemma 4.2.3(iii)}) \\
&= O(|t|^{\mu_2}) \quad \text{as } t \to 0,
\end{aligned}$$

where C_5, C_6, C_7, C_8, $\mu_2 > 0$ are some constants, and the last line follows from Lemma 4.2.5, Proposition 4.1.5, (4.2.43) and Proposition 4.1.6(i). By combining (4.2.35), (4.2.39) and (4.2.45), one obtains Proposition 4.2.8 immediately.

Similar to Proposition 4.2.7 and Proposition 4.2.8, we have:

Proposition 4.2.9. *Let r_o be as in (4.2.19). There exists a constant $\mu > 0$ such that*

$$\sup_{z \in V_t(r_o)} |u_{t;\sigma\bar\zeta}(z)| = O(|t|^\mu), \quad \textit{and} \tag{4.2.46}$$

$$\sup_{z \in M_t} |u_{t;\sigma\bar\sigma}(z)| = O(|t|^\mu) \quad \textit{as } t \to 0. \tag{4.2.47}$$

Proof. The proofs of (4.2.46) and (4.2.47) are similar to those of Proposition 4.2.7 and Proposition 4.2.8 respectively, and we will leave their verifications to the reader.

Summarizing our discussion in this section, we have:

Proposition 4.2.10. *Let r_o be as in (4.2.19), and let $u = \{u_t\}_{t \in \Delta^*}$ be as in (4.2.20). Then there exist constants C_1, C_2, $C_3 > 0$ such that*

(i) $|u(z)| \le C_1$,
(ii) $|\partial_{t_1} u(z)| \le C_2 \|t_1\|_{U_1^*(r_o) \times U_2^*(r_o)}$, *and*
(iii) $|\partial_{t_2} \bar\partial_{t_3} u(z)| \le C_3 \|t_2\|_{U_1^*(r_o) \times U_2^*(r_o)} \|t_3\|_{U_1^*(r_o) \times U_2^*(r_o)}$

for all $z \in U_1^*(r_o) \times U_2^*(r_o)$ and t_1, t_2, $t_3 \in T_z(U_1^*(r_o) \times U_2^*(r_o))$.

Proof. First Proposition 4.2.10(i) follows immediately from Proposition 4.2.4. Recall that at each $z \in V_t$ with $0 < |t| < |t_o|$, $\{\partial/\partial\zeta, \partial/\partial\sigma\}$ forms a basis of $T_z(U_1^* \times U_2^*)$. Thus, to prove Proposition 4.2.10(ii) and (iii), it suffices to consider the case when $t_1, t_2, t_3 \in \{\partial/\partial\zeta, \partial/\partial\sigma\}$. Then Proposition 4.2.10(ii) follows immediately from Lemma 4.2.2, Proposition 4.2.7 and Proposition 4.2.8.

To prove Proposition 4.2.10(iii), we need to consider the four expressions

(4.2.48)
$$\bar\partial\partial u\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\bar\zeta}\right),\ \bar\partial\partial u\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\zeta}\right),\ \bar\partial\partial u\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\bar\sigma}\right),\ \bar\partial\partial u\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\sigma}\right).$$

First, from (3.5.5), (3.5.6) and Proposition 4.2.1, one sees that

(4.2.49)
$$\left|\bar\partial\partial u\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\bar\zeta}\right)\right| = \left|(\bar\partial\partial\log\tilde h(e_V, e_V))\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\bar\zeta}\right)\right| \le C_1\left\|\frac{\partial}{\partial\zeta}\right\|^2_{U_1^*(r_o)\times U_2^*(r_o)},$$

where e_V is as in (4.1.3). As in (4.2.31), one has for $z \in V_t(r_o)$,

$$\begin{aligned}(4.2.50)\qquad \left|\bar\partial\partial u\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\zeta}\right)\right| &= \left|-u_{t;\sigma\bar\zeta} - \partial u\left(\left[\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\zeta}\right]\right)\right| \\ &\le \left|u_{t;\sigma\bar\zeta}\right| + |\hat\varphi_{\bar\zeta}||u_{t;\zeta}| \quad \text{(by Lemma 4.2.3(i))} \\ &\le C_2\left\|\frac{\partial}{\partial\sigma}\right\|_{U_1^*(r_o)\times U_2^*(r_o)}\left\|\frac{\partial}{\partial\zeta}\right\|_{U_1^*(r_o)\times U_2^*(r_o)},\end{aligned}$$

where the last inequality follows from Lemma 4.2.2, Lemma 4.2.3(iii), Proposition 4.2.7 and Proposition 4.2.9. Since we obviously have $|\bar\partial\partial u(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\bar\sigma})| = |\bar\partial\partial u(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\zeta})|$, the desired estimate for the third expression follows from that for the second one. Similarly, we have

(4.2.51)
$$\begin{aligned}\left|\bar\partial\partial u\left(\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\sigma}\right)\right| &= \left|-u_{t;\sigma\bar\sigma} - \partial u\left(\left[\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial\bar\sigma}\right]\right)\right| \\ &\le \left|u_{t;\sigma\bar\sigma}\right| + |\hat\varphi_{\bar\epsilon} + \hat\varphi\hat\varphi_{\bar\zeta}||u_{t;\zeta}| \quad \text{(by Lemma 4.2.3(i))} \\ &\le C_3\left\|\frac{\partial}{\partial\sigma}\right\|^2_{U_1^*(r_o)\times U_2^*(r_o)},\end{aligned}$$

where the last inequality follows from Lemma 4.2.2, Lemma 4.2.3(iii), Proposition 4.2.7 and Proposition 4.2.9. Combining (4.2.49), (4.2.50) and (4.2.51), one obtains Proposition 4.2.10(iii). Thus we have finished the proof of Proposition 4.2.10.

Now we are ready to give the following:

Proof of Theorem 1(ii). In light of Proposition 4.1.7, it remains to prove that the restriction (to $\mathcal{M} \setminus M_0$) of the almost nice family of flat p-singular

Hermitian metrics $h = \{h_t\}_{t\in\Delta}\Big|_{\mathcal{M}\setminus M_0}$ constructed in the 'if' part of Proposition 4.1.7 is good on $\mathcal{M}$. First we consider the coordinate open neighborhood $U_1(r_o) \times U_2(r_o)$ of the node p, where r_o is as in (4.2.19). Let $\tilde{h}$ be as in Proposition 4.2.1 and let u be as in (4.2.20). Then from (3.5.5), we have $h = e^{-u}\tilde{h}$, which implies $\partial \log h(e_V, e_V) = -\partial u + \partial \log \tilde{h}(e_V, e_V)$ and $\bar{\partial}\partial \log h(e_V, e_V) = -\bar{\partial}\partial u + \bar{\partial}\partial \log \tilde{h}(e_V, e_V)$ on $U_1^*(r_o) \times U_2^*(r_o)$, where e_V is as in (4.1.3). Then by combining Proposition 4.2.1 and Proposition 4.2.10, one easily checks that $h\big|_{U_1^*(r_o)\times U_2^*(r_o)}$ is good on $U_1(r_o) \times U_2(r_o)$. Similarly one can prove the goodness of h on other coordinate open subsets of $\mathcal{M}$ intersecting M_0. This finishes the proof of Theorem 1(ii).

Remark 4.2.11. We remark that if one attempts to use the above approach to prove the goodness of $h = \{h_t\}_{t\in\Delta}$ in the non-separating node case in Theorem 1(i), then in terms of the above notation and those in Proposition 4.2.10, one will only obtain estimates of the form $|\partial_{t_1} \log h(e_V, e_V)| \leq C\big|\log|t|\big|^n \|t_1\|_{U_1^*(r_o)\times U_2^*(r_o)}$ on $(U_1^*(r_o) \times U_2^*(r_o)) \cap M_t$ for some constants C, $n > 0$, etc., which are slightly weaker than what is desired.

5. Family of admissible Hermitian metrics with respect to hyperbolic (1,1)-forms.

5.1. . Before we prove Theorem 2, we first make some remarks on good Hermitian metrics on holomorphic line bundles over Riemann surfaces (cf. (2.2)). Let R be a compact Riemann surface with points $x_1, x_2, \ldots, x_m \in R$, and let $R_0 := R \setminus \{x_1, x_2, \ldots, x_m\}$.

Remark 5.1.1. (i) For $k = 1, 2$, let L_k be a holomorphic line bundle over R_0 and let h_k be a smooth Hermitian metric on L_k which is good on R. Then it is easy to check that the Hermitian metric $h_1 \otimes h_2$ on $L_1 \otimes L_2$ is also good on R.

(ii) Let L be a holomorphic line bundle over R_0. Suppose that for some non-zero integer m, a Hermitian metric h' on $L^{\otimes m}$ is good on R. Then the Hermitian metric h on L given by $h(s,s) = \big(h'(s^{\otimes m}, s^{\otimes m})\big)^{\frac{1}{m}}$ is also good on R.

We shall need the following well-known fact:

Proposition 5.1.2. *Let $R_0 = R\setminus\{x_1, x_2, \ldots, x_m\}$ be as above. Suppose that R_0 admits the complete hyperbolic metric ds^2_{hyp} of constant sectional curvature -1. Then the Hermitian metric ds^2_{hyp} on TR_0 is good on R, and the corresponding line bundle extension is $TR(\log) := TR \otimes [x_1]^{-1} \otimes \cdots \otimes [x_m]^{-1}$. Here $[x_k]$ denotes the divisor line bundle over R associated to $x_k, 1 \leq k \leq m$.*

Proof. It is well-known that for $1 \le k \le m$, there always exists a coordinate unit disc Δ centered at x_k such that $ds^2_{\rm hyp}$ is given near x_k by

$$ds^2_{\rm hyp} = \frac{dz \otimes d\bar{z}}{|z|^2(\log|z|)^2} \quad \text{on } \Delta^*.$$

A local non-vanishing holomorphic section of $TR(\log)$ near x_k is given by $v = z\frac{\partial}{\partial z} \in \Gamma(\Delta, TR(\log)|_\Delta)$, so that $ds^2_{\rm hyp}(v,v) = 1/(\log|z|)^2$ on Δ^*, from which the goodness of $ds^2_{\rm hyp}$ follows easily.

5.2. . Now we have:

Proof of Theorem 2(i). Let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree d over $\mathcal{M} = \{M_t\}_{t\in\Delta}$ with a non-separating node $p \in M$, and let $f : \widetilde{M} \to M$, p_1, p_2, q be as in Theorem 2(i). Recall from (2.1) the vertical line bundle $\widetilde{\mathcal{T}} = \{T_t\}_{t\in\Delta}$ over $\mathcal{M}$ such that $T_t = TM_t$ for $t \neq 0$. One easily sees that $\{T_t\}_{t\in\Delta}$ forms a holomorphic family of line bundles of degree $2-2q$, and $f^*T_0 = T\widetilde{M}(\log) = T\widetilde{M} \otimes [p_1]^{-1} \otimes [p_2]^{-1}$ (cf. e.g., [**Wo2**, §1]). Next we define the holomorphic line bundle over $\mathcal{M}$ given by

$$\mathcal{L}' := \widetilde{\mathcal{T}}^{\otimes d} \otimes \mathcal{L}^{\otimes(2q-2)}. \tag{5.2.1}$$

Then one easily checks that $\mathcal{L}' = \{L'_t\}_{t\in\Delta}$ with $L'_t = T_t^{\otimes d} \otimes L_t^{\otimes(2q-2)}$ forms a holomorphic family of line bundles of degree 0 over $\{M_t\}_{t\in\Delta}$. By Theorem 1(i), there exists an almost nice family of flat p-singular Hermitian metrics $h' = \{h'_t\}_{t\in\Delta}$ on $\{L'_t\}_{t\in\Delta}$ such that $f^*h'_0$ extends across p_1, p_2 to a smooth flat Hermitian metric on $f^*L'_0$. This implies easily that $h' = \{h'_t\}_{t\in\Delta}$ forms an almost nice family of $\{\hat\omega_{{\rm hyp},t}\}$-admissible p-singular Hermitian metrics on $\{L'_t\}_{t\in\Delta}$ such that the Hermitian metric $f^*h'_0$ on $f^*L'_0|_{\widetilde{M}\setminus\{p_1,p_2\}}$ is good on $\widetilde{M}$. By [**Wo2**, Theorem 5.8], $\{ds^2_{{\rm hyp},t}\}_{t\in\Delta}$ forms an almost nice family of ($\{\hat\omega_{{\rm hyp},t}\}_{t\in\Delta}$-admissible) p-singular Hermitian metrics on $\widetilde{\mathcal{T}}$, and by Proposition 5.1.2, the Hermitian $ds^2_{{\rm hyp},0}$ on $= f^*T_0|_{\widetilde{M}\setminus\{p_1,p_2\}}$ is good on $\widetilde{M}$. Together with Remark 2.2.2(i) and Remark 5.1.1(i), it follows from (5.2.1) that one obtains an almost nice family of $\{\hat\omega_{{\rm hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics $h'' = \{h''_t\}_{t\in\Delta}$ on $\mathcal{L}^{\otimes(2q-2)} = \{L_t^{\otimes(2q-2)}\}_{t\in\Delta}$ given by $h''_t = h'_t \otimes (ds^2_{{\rm hyp},t})^{\otimes(-d)}$, $t \in \Delta$, such that the Hermitian metric $f^*h''_0$ on $f^*L_0^{\otimes(2q-2)}|_{\widetilde{M}\setminus\{p_1,p_2\}}$ is good on $\widetilde{M}$. Together with Remark 2.2.2(ii) and Remark 5.1.1(i), one finally obtains the desired almost nice family of $\{\hat\omega_{{\rm hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$ on $\{L_t\}_{t\in\Delta}$ given by $h_t(s,s) = \big(h''_t(s^{\otimes(2q-2)}, s^{\otimes(2q-2)})\big)^{\frac{1}{2q-2}}$, and this finishes the proof of Theorem 2(i).

5.3. . For a smooth compact Riemann surface X of genus ≥ 2, we denote the hyperbolic Green's function on X by $g_{\text{hyp}}(\cdot,\cdot) \in C^\infty(X \times X \setminus \{\text{diagonal}\})$. It is known that $g_{\text{hyp}}(x,y) = g_{\text{hyp}}(y,x)$ for all $x \neq y \in X$. Also for a fixed point $x \in X$, it is known that in terms of local holomorphic coordinates z near x, $g_{\text{hyp}}(x,\cdot) \in C^\infty(X \setminus \{x\})$ satisfies

$$g_{\text{hyp}}(x,z) = -\log|z-x|^2 + \alpha(z) \tag{5.3.1}$$

for some smooth function $\alpha(z)$ near x, and

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} g_{\text{hyp}}(x,\cdot) = \omega_{\text{hyp}}(X) \quad \text{on } X \setminus \{x\} \tag{5.3.2}$$

(see e.g., [**Ji**] for the definition and above properties of $g_{\text{hyp}}(\cdot,\cdot)$).

Proposition 5.3.1. *Let $\{M_t\}_{t\in\Delta}$ be as in (2.1) with a separating node $p \in M$. Let $\mathcal{X} = \{x_t\}_{t\in\Delta}$ be a holomorphic section of $\{M_t\}_{t\in\Delta}$ such that $x_0 \in M_1 \setminus \{p\}$ or $x_0 \in M_2 \setminus \{p\}$. Then the family of divisor line bundles $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$ does not admit any almost nice family of $\{\omega_{\text{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics.*

Proof. For simplicity, we only consider the case when $x_0 \in M_1 \setminus \{p\}$, and we suppose that $\{[x_t]\}_{t\in\Delta}$ admits an almost nice family of $\{\omega_{\text{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics $h = \{h_t\}_{t\in\Delta}$. Fix a (holomorphic) canonical section $s_{\mathcal{X}} = \{s_{x_t}\}_{t\in\Delta}$ of $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$ (i.e., $s_{\mathcal{X}}$ vanishes only along $\mathcal{X}$ with vanishing order equal to one). Then for $t \in \Delta^*$, we define the function given by

$$\phi_t(z) = g_{\text{hyp},t}(x_t, z) + \log h_t(s_{x_t}, s_{x_t})(z) \quad \text{for } z \in M_t \setminus \{x_t\}. \tag{5.3.3}$$

Since both $g_{\text{hyp},t}(x_t,z)$ and $-\log h_t(s_{x_t}, s_{x_t})(z)$ are of the form given in (5.3.1) for z near x_t, it follows that ϕ_t extends uniquely across x_t to a smooth function on M_t which we denote by the same symbol. By (5.3.2), it is easy to see that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi_t = \omega_{\text{hyp},t} - \omega_{\text{hyp},t} = 0 \quad \text{on } M_t \setminus \{x_t\}.$$

Thus ϕ_t is harmonic on $M_t \setminus \{x_t\}$ and hence also harmonic on M_t, which implies that $\phi_t(z) \equiv c_t$ for some constant c_t. Now fix two continuous sections $\{y_t\}_{t\in\Delta}$, $\{z_t\}_{t\in\Delta}$ of $\{M_t\}_{t\in\Delta}$ such that $y_0 \in M_1 \setminus \{p, x_0\}$ and $z_0 \in M_2 \setminus \{p\}$. Then we have $h_0(s_{x_0}, s_{x_0})(y_0) > 0$ and $h_0(s_{x_0}, s_{x_0})(z_0) > 0$. Also, since $x_0, y_0 \in M_1 \setminus \{p\}$ and $z_0 \in= M_2 \setminus \{p\}$, it follows from a result of [**Ji**, Theorem 1.1, part 2] that

$$\lim_{t\to 0} g_{\text{hyp},t}(x_t, y_t) = +\infty, \quad \text{and} \quad \lim_{t\to 0} g_{\text{hyp},t}(x_t, z_t) = -\infty. \tag{5.3.4}$$

Together with the continuity of $\{h_t\}_{t\in\Delta}$ (off p), we have

$$\begin{aligned} c_t = \phi_t(y_t) &= g_{\mathrm{hyp},t}(x_t, y_t) + \log h_t(s_{x_t}, s_{x_t})(y_t) \\ &\to +\infty + \log h_0(s_{x_0}, s_{x_0})(y_0) \\ &= +\infty \quad \text{as } \ t \to 0. \end{aligned} \tag{5.3.5}$$

On the other hand, we also have

$$\begin{aligned} c_t = \phi_t(z_t) &= g_{\mathrm{hyp},t}(x_t, z_t) + \log h_t(s_{x_t}, s_{x_t})(z_t) \\ &\to -\infty + \log h_0(s_{x_0}, s_{x_0})(z_0) \\ &= -\infty \quad \text{as } t \to 0, \end{aligned}$$

which contradicts (5.3.5). Therefore, $\{[x_t]\}_{t\in\Delta}$ cannot admit any almost nice family of $\{\omega_{\mathrm{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics, and this finishes the proof of Proposition 5.3.1.

Now we are ready to give the following:

Proof of Theorem 2(ii). Let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree d over $\mathcal{M} = \{M_t\}_{t\in\Delta}$ with a separating node $p \in M$ as in Theorem 2(i). Also we let $f : \widetilde{M} \to M$, $\widetilde{M} = M_1 \sqcup M_2, L_{0,1}, L_{0,2}, p_1, p_2, q_1, q_2, q, d_1, d_2$ be as in Theorem 2(ii). Denote by $\hat{\omega}^{(k)}_{\mathrm{hyp}}$ the normalized complete hyperbolic (1,1)-form on $M_k\backslash\{p_k\}, k = 1, 2$. Then it is easy to check that at $t = 0$,

$$f^*\hat{\omega}_{\mathrm{hyp},0} = \frac{2q_k - 1}{2q - 2}\hat{\omega}^{(k)}_{\mathrm{hyp}} \quad \text{on } M_k\backslash\{p_k\}, \ k = 1, 2. \tag{5.3.6}$$

Recall also that the vertical line bundle $\widetilde{\mathcal{T}} = \{T_t\}_{t\in\Delta}$ forms a holomorphic family of line bundles of degree $2 - 2q$ such that $T_t = TM_t$ for $t \neq 0$ and $f^*T_0|_{M_k} = TM_k \otimes [p_k]^{-1}$, so that $\deg(f^*T_0|_{M_k}) = 1 - 2q_k, \ k = 1, 2$.

To prove the 'if' part of Theorem 2(ii), we assume that $d_1/(2q_1 - 1) = d_2/(2q_2 - 1)$. Define $\mathcal{L}' := \widetilde{\mathcal{T}}^{\otimes d} \otimes \mathcal{L}^{\otimes(2q-2)}$ so that $\mathcal{L}' = \{L'_t\}_{t\in\Delta}$ (with $L'_t = T_t^{\otimes d} \otimes L_t^{\otimes(2q-2)}, t \in \Delta$) forms a holomorphic family of line bundles of degree 0 over $\{M_t\}_{t\in\Delta}$. Moreover, one easily sees that $\deg(f^*L'_0|_{M_k}) = 0, k = 1, 2$. Thus by Theorem 1(ii), there exists a nice family of flat p-singular Hermitian metrics $h' = \{h'_t\}_{t\in\Delta}$ on $\{L'_t\}_{t\in\Delta}$ such that the $f^*h'_0$ extends to smooth flat Hermitian metrics on $f^*L'_0|_{M_k}, k = 1, 2$. Then one can construct the desired nice family of $\{\hat{\omega}_{\mathrm{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics on $\{L_t\}_{t\in\Delta}$ from $\{h'_t\}_{t\in\Delta}$ on $\{L'_t\}_{t\in\Delta}$ and $\{ds^2_{\mathrm{hyp},t}\}_{t\in\Delta}$ on $\widetilde{\mathcal{T}} = \{T_t\}_{t\in\Delta}$ as in Theorem 2(i), and this finishes the proof of the 'if' part.

To prove the 'only if' part, we assume that $\mathcal{L} = \{L_t\}_{t\in\Delta}$ admits an almost nice family of $\{\omega_{\mathrm{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics.

We define

$$(5.3.7)\quad \mathcal{L}'' := \mathcal{L}^{\otimes(2q_2-1)} \otimes \big([\mathcal{X}]^{\otimes(2q_1-1)} \otimes [\mathcal{Y}]^{\otimes(2q_2-1)}\big)^{\otimes(-d_2)} \otimes [\mathcal{X}]^{\otimes(d_2(2q_1-1)-d_1(2q_2-1))}$$

over $\mathcal{M}$, and write $\mathcal{L}'' = \{L''_t\}_{t\in\Delta}$, where $L''_t = \mathcal{L}''|_{M_t}$ for $t \in \Delta$. Then it is easy to check that $\{L''_t\}_{t\in\Delta}$ forms a family of line bundles of degree 0 such that $\deg(f^*L''_0|_{M_k}) = 0,\ k = 1, 2$. Thus by Theorem 1(ii), $\{L''_t\}_{t\in\Delta}$ admits a nice family of flat p-singular Hermitian metrics. Together with the assumption and the 'if' part of Theorem 2(ii), it follows that $\mathcal{L}'', \mathcal{L}^{\otimes q_2}, \big([\mathcal{X}]^{\otimes(2q_1-1)} \otimes [\mathcal{Y}]^{\otimes(2q_2-1)}\big)^{\otimes(-d_2)}$ all admit almost nice families of $\{\omega_{\mathrm{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics. Hence so does $[\mathcal{X}]^{\otimes(d_2(2q_1-1)-d_1(2q_2-1))}$ by (5.3.7) and Remark 2.2.2(i). If $d_2(2q_1 - 1) - d_1(2q_2 - 1) \neq 0$, then by Remark 2.2.2(ii), $[\mathcal{X}]$ itself also admits an almost nice family of $\{\omega_{\mathrm{hyp},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics, which contradicts Proposition 5.3.1. Hence we must have $d_1/(2q_1 - 1) = d_2/(2q_2 - 1)$, and we have finished the proof of Theorem 2(ii).

6. Family of admissible Hermitian metrics with respect to canonical (1,1)-forms.

6.1. . Let R be a smooth compact Riemann surface of genus $q \geq 1$, and recall from (2.5) the canonical (1,1)-form $\omega_{\mathrm{can}}(R)$ on R. Let $\{A_i, B_i\}_{1\leq i\leq q}$ be a standard symplectic homology basis of $H_1(R, \mathbb{Z})$ (so that the intersection pairings satisfy $\#[A_i, A_j] = 0,\ \#[A_i, B_j] = \delta_{ij},\ \#[B_i, B_j] = 0,\ 1 \leq i, j \leq q$). In terms of its associated normalized basis of abelian differentials $\{\omega_1, \ldots, \omega_q\}$ (so that $\int_{A_i} \omega_j = \delta_{ij},\ 1 \leq i, j \leq q$) and period matrix $\Omega = (\Omega_{ij})_{1\leq i,j\leq q}$ with $\Omega_{ij} = \int_{B_j} \omega_i$, it is well-known that

$$(6.1.1)\qquad \omega_{\mathrm{can}}(R) = \frac{\sqrt{-1}}{2q} \sum_{1\leq i,j\leq q} (\operatorname{Im}\Omega)^{-1}_{ij} \omega_i \wedge \bar{\omega}_j,$$

where $\operatorname{Im}\Omega$ denotes the imaginary part of Ω.

Now let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be a plumbing family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a Riemann surface M with a non-separating node p, and let $f : \widetilde{M} \to M$, p_1, p_2 be as in (2.1). Throughout (6.1), we will make the identification $\widetilde{M}\backslash\{p_1, p_2\} \simeq M\backslash\{p\}$ via f as in (2.5).

Proposition 6.1.1. *Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be as above with a non-separating node $p \in M$. Then there exists a family of p-singular $(1, 1)$-forms $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$ on $\{M_t\}_{t\in\Delta}$ (cf. (2.2)) such that $\omega_{\mathrm{can},t} = \omega_{\mathrm{can}}(M_t)$ for $t \in \Delta^*$, and*

$$(6.1.2)\qquad \omega_{\mathrm{can},0} = \frac{q-1}{q}\omega_{\mathrm{can}}(\widetilde{M})|_{M\backslash\{p\}} \quad \text{on } M\backslash\{p\}.$$

Proof. By [**F**, p. 51] or [**Y**, p. 135], one can find $2q$ continuous families of closed loops $\{A_{i,t}, B_{i,t}\}_{1\leq i\leq q, t\in\Delta}$ on $\{M_t\}_{t\in\Delta}$ such that for $t\in\Delta^*$, $\{A_{i,t}, B_{i,t}\}_{1\leq i\leq q}$ form a standard symplectic homology basis of M_t, and at $t=0$, $\{A_{i,0}, B_{i,0}\}_{1\leq i\leq q-1}$ do not meet p and form a standard symplectic homology basis of $\widetilde{M}$ (under the identification $\widetilde{M}\backslash\{p_1,p_2\}\simeq M\backslash\{p\}$); moreover, the associated period matrices $\Omega(t)$ of $M_t, t\in\Delta^*$, are of the form

$$\Omega(t)=\begin{pmatrix}\Omega_{ij} & a_i\\ a_j & \frac{1}{2\pi i}\log t+c_0\end{pmatrix}+O(t), \tag{6.1.3}$$

where $\Omega=(\Omega_{ij})_{1\leq i,j\leq q-1}$ is the associated period matrix of $\widetilde{M}$, and $a_1,\ldots,a_{q-1}$, c_0 are constants independent of t, so that

$$\lim_{t\to 0}(\operatorname{Im}\Omega(t))^{-1}=\begin{pmatrix}(\operatorname{Im}\Omega)^{-1}_{ij} & 0\\ 0 & 0\end{pmatrix}. \tag{6.1.4}$$

Moreover by [**F**, p. 51] or [**Y**, p. 135], on $\{M_t\}_{t\in\Delta}$, there exist q holomorphic families of 1-forms $\{\omega_{1,t},\ldots,\omega_{q,t}\}_{t\in\Delta}$ such that for $t\in\Delta^*$, $\{\omega_{1,t},\ldots,\omega_{q,t}\}$ form the normalized basis of abelian differentials associated to $\{A_{i,t},B_{i,t}\}_{1\leq i\leq q}$, and at $t=0$, $\{\omega_{1,0},\ldots,\omega_{q-1,0}\}$ is the restriction to $M\backslash\{p\}$ of the normalized basis of abelian differentials associated to $\{A_{i,0},B_{i,0}\}_{1\leq i\leq q-1}$ on $\widetilde{M}$. Together with (6.1.1) and (6.1.4), by letting

$$\omega_{\mathrm{can},t}=(\sqrt{-1}/2q)\sum_{1\leq i,j\leq q}(\operatorname{Im}\Omega(t))^{-1}_{ij}\omega_{i,t}\wedge\overline{\omega}_{j,t}$$

for all $t\in\Delta$, one obtains Proposition 6.1.1 easily.

For a smooth compact Riemann surface X, we denote the Arakelov Green's function on X by $g_{\mathrm{can}}(\cdot,\cdot)\in C^\infty(X\times X\backslash\{\text{diagonal}\})$. It is known that $g_{\mathrm{can}}(x,y)=g_{\mathrm{can}}(y,x)$ for all $x\neq y\in X$. Also for a fixed point $x\in X$, it is known that $g_{\mathrm{can}}(x,\cdot)\in C^\infty(X\backslash\{x\})$ satisfies an identity analogous to (5.3.1), and

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial g_{\mathrm{can}}(x,\cdot)=\omega_{\mathrm{can}}(X)\quad\text{on } X\backslash\{x\} \tag{6.1.5}$$

(see [**We**, p. 432] for the definition and above properties of $g_{\mathrm{can}}(\cdot,\cdot)$). Let $\mathcal{M}=\{M_t\}_{t\in\Delta}$ be as in (2.1). For $t\in\Delta^*$, we simply denote the Arakelov Green's function on M_t by $g_{\mathrm{can},t}$. At $t=0$, by abuse of notation, we denote by $g_{\mathrm{can},0}$ the Arakelov Green's function on $\widetilde{M}$ as well as its restriction to $M\backslash\{p\}$. We recall the following result of Wentworth:

Proposition 6.1.2 ([**We**, Theorem 7.2])**.** *Let $\{M_t\}_{t\in\Delta}$ be as in (2.1) with a non-separating node $p\in M$, and let $q\,(q\geq 2)$, p_1,p_2 be as in (2.1). Let $\{x_t\}_{t\in\Delta},\{y_t\}_{t\in\Delta}$ be smooth sections of $\{M_t\}_{t\in\Delta}$ (i.e., $x_t,y_t\in M_t$ for all*

$t \in \Delta$) such that $x_t \neq y_t$ for each $t \in \Delta$ and $x_0, y_0 \neq p$. Then

$$\begin{aligned}\lim_{t\to 0}\left[g_{\mathrm{can},t}(x_t,y_t) - \frac{1}{12q^2}\log|t|\right] &\\ = g_{\mathrm{can},0}(x_0,y_0) + \frac{5}{6q^2}g_{\mathrm{can},0}(p_1,p_2) &\\ - \frac{1}{2q}\sum_{\alpha=1,2}\Big[g_{\mathrm{can},0}(x_0,p_\alpha) + g_{\mathrm{can},0}(y_0,p_\alpha)\Big]. &\end{aligned} \tag{6.1.6}$$

Next we have:

Proposition 6.1.3. *Let $\{M_t\}_{t\in\Delta}$ be as in (2.1) with a non-separating node $p \in M$. Let $\mathcal{X} = \{x_t\}_{t\in\Delta}$ be a holomorphic section of $\{M_t\}_{t\in\Delta}$ with $x_0 \neq p$. Then there exists an almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics on the family of divisor line bundles $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$.*

Proof. For $t \in \Delta^*$, we define

$$g'_{\mathrm{can},t}(z) := g_{\mathrm{can},t}(x_t,z) - \frac{1}{12q^2}\log|t| \quad \text{for } z \in M_t\backslash\{x_t\}. \tag{6.1.7}$$

Also, for $t = 0$, we define

$$\begin{aligned}g'_{\mathrm{can},0}(z) :=& g_{\mathrm{can},0}(x_0,z) + \frac{5}{6q^2}g_{\mathrm{can},0}(p_1,p_2)\\ &- \frac{1}{2q}\sum_{\alpha=1,2}\Big[g_{\mathrm{can},0}(x_0,p_\alpha) + g_{\mathrm{can},0}(z,p_\alpha)\Big],\end{aligned} \tag{6.1.8}$$

for $z \in M\backslash\{p, x_0\}$. Then one easily checks that $g'_{\mathrm{can},t} \in C^\infty(M_t\backslash\{x_t\})$ for $t \in \Delta^*$ and $g'_{\mathrm{can},0} \in C^\infty(M\backslash\{p,x_0\})$, and $g'_{\mathrm{can},t}$ satisfies (6.1.5) for all $t \in \Delta$. By (6.1.5) and (6.1.7), we have, for $t \in \Delta^*$,

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial g'_{\mathrm{can},t} = \frac{\sqrt{-1}}{2\pi}g'_{\mathrm{can},t}(x_t,\cdot) = \omega_{\mathrm{can},t} \quad \text{on } M_t\backslash\{x_t\}. \tag{6.1.9}$$

Similarly by (6.1.5) and (6.1.8), we have

(6.1.10)

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial g'_{\mathrm{can},0} &= \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\left(g_{\mathrm{can},0}(x_0,\cdot) - \frac{1}{2q}g_{\mathrm{can},0}(\cdot,p_1) - \frac{1}{2q}g_{\mathrm{can},0}(\cdot,p_2)\right)\\ &= \omega_{\mathrm{can}}(\widetilde{M}) - \frac{1}{2q}\omega_{\mathrm{can}}(\widetilde{M}) - \frac{1}{2q}\omega_{\mathrm{can}}(\widetilde{M})\\ &= \omega_{\mathrm{can},0} \quad \text{on } M\backslash\{p,x_0\} \text{ (by (6.1.2))}.\end{aligned}$$

Fix a canonical holomorphic section $s_{\mathcal{X}}$ of $[\mathcal{X}]$ (i.e., $s_{\mathcal{X}}$ vanishes only along $\mathcal{X}$ with vanishing order equal to one), which then determines a holomorphic family of canonical sections $\{s_{x_t}\}_{t\in\Delta}$ of $\{[x_t]\}_{t\in\Delta}$ with $s_{x_t} = s_{\mathcal{X}}|_{[x_t]}$ for all $t \in \Delta$. Then from (6.1.7), (6.1.8), (6.1.9), (6.1.10), one easily sees that for each $t \in \Delta$, $g'_{\mathrm{can},t}$ and s_{x_t} defines uniquely a smooth $\omega_{\mathrm{can},t}$-admissible

Hermitian metric h_t on $[x_t]$ over M_t (over $M\backslash\{p\}$ when $t=0$) satisfying $h_t(s_{x_t}, s_{x_t})(z) = \exp(-g'_{\mathrm{can},t}(z))$ for $z \in M_t \setminus \{x_t\}$ (for $z \in M\backslash\{p, x_0\}$ when $t=0$). Moreover, by Proposition 6.1.2, $\{g'_{\mathrm{can},t}\}_{t\in\Delta}$ forms a continuous family of smooth functions on $\{M_t\backslash\{x_t\}\}_{t\in\Delta}\backslash\{p\}$, and this implies easily that $\{h_t\}_{t\in\Delta}$ also form an almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics. Thus we have finished the proof of Proposition 6.1.3.

Now we are ready to give the following:

Proof of Theorem 3(i). Let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree d over $\mathcal{M} = \{M_t\}_{t\in\Delta}$ with a non-separating node $p \in M$ as in Theorem 3(i). From the plumbing construction in (2.1), it is easy to see that one can fix a holomorphic section $\mathcal{X} = \{x_t\}_{t\in\Delta}$ of $\{M_t\}_{t\in\Delta}$ such that $x_0 \neq p$. Define the holomorphic line bundle $\mathcal{L}' := \mathcal{L} \otimes [\mathcal{X}]^{\otimes(-d)}$ over $\mathcal{M}$. Then $\mathcal{L}' = \{L'_t\}_{t\in\Delta}$ with $L'_t = L_t \otimes [x_t]^{\otimes(-d)}$ forms a holomorphic family of line bundles of degree 0 over $\{M_t\}_{t\in\Delta}$. Thus by Theorem 1(i), there exists an almost nice family of flat (and thus $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible) p-singular Hermitian metrics $h' = \{h'_t\}_{t\in\Delta}$ on $\{L'_t\}_{t\in\Delta}$. Also by Proposition 6.1.3, $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$ admits an almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics $h'' = \{h''_t\}_{t\in\Delta}$. Then by Remark 2.2.2(i), $h := h' \otimes h''^{\otimes d} = \{h'_t \otimes h''^{\otimes d}_t\}_{t\in\Delta}$ is an almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics on $\{L_t\}_{t\in\Delta} = \{L'_t \otimes [x_t]^{\otimes d}\}_{t\in\Delta}$, and we have finished the proof of Theorem 3(i).

6.2. . Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be a plumbing family of compact Riemann surfaces of genus $q \geq 2$ degenerating to a stable singular Riemann surface M with a separating node p as in (2.1). Also let $f : \widetilde{M} = M_1 \sqcup M_2, p_1, p_2, q_1, q_2$ be as in (2.1). We will also make the identification $M\backslash\{p\} \simeq M_1\backslash\{p_1\} \sqcup M_2\backslash\{p_2\}$ via f as in (2.5).

Proposition 6.2.1. *Let $\mathcal{M} = \{M_t\}_{t\in\Delta}$ be as above with a separating node $p \in M$. Then there exists a family of p-singular $(1,1)$-forms $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$ on $\{M_t\}_{t\in\Delta}$ such that $\omega_{\mathrm{can},t} = \omega_{\mathrm{can}}(M_t)$ for $t \in \Delta^*$ and*

$$\omega_{\mathrm{can},0} = \frac{q_k}{q}\omega_{\mathrm{can}}(M_k)\big|_{M_k\backslash\{p_k\}} \quad \text{on } M_k\backslash\{p_k\},\ k=1,2. \tag{6.2.1}$$

Proof. In the separating node case, it is easy to find $2q$ continuous families of closed loops $\{A_{i,t}, B_{i,t}\}_{1\leq i\leq q, t\in\Delta}$ such that for $t \in \Delta^*$, $\{A_{i,t}, B_{i,t}\}_{1\leq i\leq q}$ form a standard symplectic basis of M_t, and at $t=0$, $\{A_{i,0}, B_{i,0}\}_{1\leq i\leq q_1}$ (resp. $\{A_{i,0}, B_{i,0}\}_{q_1+1\leq i\leq q}$) form a standard symplectic homology basis of M_1 (resp. M_2). Then the associated period matrices $\Omega(t)$ of M_t, $t \in \Delta^*$, satisfy

$$\lim_{t\to 0}\Omega(t) = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}, \tag{6.2.2}$$

where Ω_1, Ω_2 are the associated period matrices of M_1, M_2 respectively (cf. e.g., [**We**, §3]). Moreover by [**F**, p. 38] or [**Y**, p. 129], there exist q holomorphic families of 1-forms $\{\omega_{1,t}, \ldots, \omega_{q,t}\}_{t\in\Delta}$ on $\{M_t\}_{t\in\Delta}$ such that for $t \in \Delta^*$, $\{\omega_{1,t}, \ldots, \omega_{q,t}\}$ form the associated normalized basis of abelian differentials on M_t, and at $t = 0$, $\{\omega_{1,0}, \ldots, \omega_{q_1,0}\}$ (resp. $\{\omega_{q_1+1,0}, \ldots, \omega_{q,0}\}$) on $M_1\backslash\{p_1\}$ (resp. $M_2\backslash\{p_2\}$) are given by the restriction of the associated normalized basis of abelian differentials on M_1 (resp. M_2), and $\{\omega_{1,0}, \ldots, \omega_{q_1,0}\}$ (resp. $\{\omega_{q_1+1,0}, \ldots, \omega_{q,0}\}$) vanish identically on $M_2\backslash\{p_2\}$ (resp. $M_1\backslash\{p_1\}$). Together with (6.1.1) and (6.2.2), one sees that Proposition 6.2.1 can be obtained easily by letting $\omega_{\mathrm{can},t} = (\sqrt{-1}/2q)\sum_{1\le i,j\le q}(\operatorname{Im}\Omega(t))^{-1}_{ij}\omega_{i,t} \wedge \overline{\omega}_{j,t}$ for all $t \in \Delta$.

At $t = 0$, we simply denote $g^{(k)}_{\mathrm{can},0} := g_{\mathrm{can}}(M_k)$ as well as its restriction to $M_k\backslash\{p_k\}$, $k = 1, 2$ (recalling that $M\backslash\{p\} \simeq M_1\backslash\{p_1\} \sqcup M_2\backslash\{p_2\}$). We will need the following result of Wentworth:

Proposition 6.2.2 ([**We**, Theorem 6.10])**.** *Let $\{M_t\}_{t\in\Delta}$ be as in (2.1) with a separating node $p \in M$. Let $\{x_t\}_{t\in\Delta}, \{y_t\}_{t\in\Delta}$ be continuous sections of $\{M_t\}_{t\in\Delta}$ such that $x_t \neq y_t$ for all $t \in \Delta$. Then the following statements hold:*

(i) *If $x_0, y_0 \in M_1\backslash\{p_1\}$, then*

$$\lim_{t\to 0}\left[g_{\mathrm{can},t}(x_t, y_t) - \frac{q_2^2}{q^2}\log|t|\right] = g^{(1)}_{\mathrm{can},0}(x_0, y_0) - \frac{q_2}{q}g^{(1)}_{\mathrm{can},0}(x_0, p_1) - \frac{q_2}{q}g^{(1)}_{\mathrm{can},0}(y_0, p_1).$$

Also similar statement holds for the case when $x_0, y_0 \in M_2\backslash\{p_2\}$.

(ii) *If $x_0 \in M_1\backslash\{p_1\}$ and $y_0 \in M_2\backslash\{p_2\}$, then*

$$\lim_{t\to 0}\left[g_{\mathrm{can},t}(x_t, y_t) + \frac{q_1q_2}{q^2}\log|t|\right] = \frac{q_1}{q}g^{(1)}_{\mathrm{can},0}(x_0, p_1) + \frac{q_2}{q}g^{(2)}_{\mathrm{can},0}(y_0, p_2).$$

Next we have:

Proposition 6.2.3. *Let $\{M_t\}_{t\in\Delta}$ be as in (2.1) with a separating node $p \in M$. Let $\mathcal{X} = \{x_t\}_{t\in\Delta}, \mathcal{Y} = \{y_t\}_{t\in\Delta}$ be two holomorphic sections of $\{M_t\}_{t\in\Delta}$ such that $x_0 \in M_1\backslash\{p_1\}$ and $y_0 \in M_2\backslash\{p_2\}$. Then the holomorphic family of divisor line bundles $[\mathcal{X}]^{\otimes q_1} \otimes [\mathcal{Y}]^{\otimes q_2} = \{[x_t]^{\otimes q_1} \otimes [y_t]^{\otimes q_2}\}_{t\in\Delta}$ admits an almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics.*

Proof. First we fix two (holomorphic) canonical sections $s_{\mathcal{X}} = \{s_{x_t}\}_{t\in\Delta}$ of $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$ and $s_{\mathcal{Y}} = \{s_{y_t}\}_{t\in\Delta}$ of $[\mathcal{Y}] = \{[y_t]\}_{t\in\Delta}$. For $t \in \Delta^*$, we define

$$g'_{\mathrm{can},t}(z) := q_1 g_{\mathrm{can},t}(x_t, z) + q_2 g_{\mathrm{can},t}(y_t, z) \quad \text{for } z \in M_t\backslash\{x_t, y_t\}. \tag{6.2.3}$$

At $t = 0$, we define

(6.2.4)
$$g'_{\mathrm{can},0}(z) := \begin{cases} q_1 g^{(1)}_{\mathrm{can},0}(x_0, z) - \frac{q_1 q_2}{q} g^{(1)}_{\mathrm{can},0}(x_0, p_1) \\ \qquad + \frac{q_2^2}{q} g^{(2)}_{\mathrm{can},0}(y_0, p_2), & \text{if } z \in M_1 \backslash \{p_1, x_0\}, \\ q_2 g^{(2)}_{\mathrm{can},0}(y_0, z) - \frac{q_1 q_2}{q} g^{(2)}_{\mathrm{can},0}(y_0, p_2) \\ \qquad + \frac{q_1^2}{q} g^{(1)}_{\mathrm{can},0}(x_0, p_1), & \text{if } z \in M_2 \backslash \{p_2, y_0\}. \end{cases}$$

Then for $t \in \Delta^*$, it follows easily from (6.1.5) and (6.2.3) that

$$\text{(6.2.5)} \qquad \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} g'_{\mathrm{can},t} = (q_1 + q_2)\omega_{\mathrm{can},t} = q\omega_{\mathrm{can},t} \quad \text{on } M_t \backslash \{x_t, y_t\}.$$

By (6.1.5) and (6.2.4), we also have, at $t = 0$,

$$\text{(6.2.6)} \qquad \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} g'_{\mathrm{can},0} = \begin{cases} q_1 \omega_{\mathrm{can}}(M_1) & \text{on } M_1 \backslash \{p_1, x_0\}, \\ q_2 \omega_{\mathrm{can}}(M_2) & \text{on } M_2 \backslash \{p_2, y_0\}, \end{cases}$$
$$= q\omega_{\mathrm{can},0} \quad \text{on } M \backslash \{p, x_0, y_0\} \quad \text{(by (6.2.1))}.$$

Also, for all $t \in \Delta$, it is clear that $g'_{\mathrm{can},t}(z) = q_1 \log|z|^2 + O(1)$ in terms of holomorphic coordinates z near x_t with $z(x_t) = 0$. Similarly, $g'_{\mathrm{can},t} = q_2 \log|z|^2 + O(1)$ for z near x_t with $z(y_t) = 0$. Together with (6.2.5), (6.2.6), one easily sees that for each $t \in \Delta$, $g'_{\mathrm{can},t}$ and $s_{x_t}^{\otimes q_1} \otimes s_{y_t}^{\otimes q_2}$ define uniquely a $\omega_{\mathrm{can},t}$-admissible Hermitian metric h_t on $[x_t]^{\otimes q_1} \otimes [y_t]^{\otimes q_2}$ over M_t (over $M\backslash\{p\}$ when $t = 0$) satisfying $h_t(s_{x_t}^{\otimes q_1} \otimes s_{y_t}^{\otimes q_2}, s_{x_t}^{\otimes q_1} \otimes s_{y_t}^{\otimes q_2})(z) = \exp(-g'_{\mathrm{can},t}(z))$ for $z \in M_t \backslash \{x_t, y_t\}$ (for $z \in M\backslash\{p, x_0, y_0\}$ when $t = 0$). Now for any continuous section $\{z_t\}_{t\in\Delta}$ of $\{M_t\}_{t\in\Delta}$ such that $z_0 \in M_1 \backslash \{p_1, x_0\}$, we have

$$\begin{aligned} &\lim_{t\to 0} g'_{\mathrm{can},t}(z_t) \\ &= \lim_{t\to 0} \Big[q_1 g_{\mathrm{can},t}(x_t, z_t) + q_2 g_{\mathrm{can},t}(y_t, z_t) \Big] \quad \text{(by (6.2.3))} \\ &= \lim_{t\to 0} \left[q_1 \left(g_{\mathrm{can},t}(x_t, z_t) - \frac{q_2^2}{q^2} \log|t| \right) + q_2 \left(g_{\mathrm{can},t}(y_t, z_t) + \frac{q_1 q_2}{q^2} \log|t| \right) \right] \\ &= q_1 \left(g^{(1)}_{\mathrm{can},0}(x_0, z_0) - \frac{q_2}{q} g^{(1)}_{\mathrm{can},0}(x_0, p_1) - \frac{q_2}{q} g^{(1)}_{\mathrm{can},0}(z_0, p_1) \right) \\ &\quad + q_2 \left(\frac{q_1}{q} g^{(1)}_{\mathrm{can},0}(z_0, p_1) + \frac{q_2}{q} g^{(2)}_{\mathrm{can},0}(y_0, p_2) \right) \quad \text{(by Proposition 6.2.2)} \\ &= g'_{\mathrm{can},0}(z_0) \quad \text{(by (6.2.4))}. \end{aligned}$$

Similarly, one can check that $\lim_{t\to 0} g'_{\mathrm{can},t}(z_t) = g'_{\mathrm{can},0}(z_0)$ for any continuous section $\{z_t\}_{t\in\Delta}$ of $\{M_t\}_{t\in\Delta}$ such that $z_0 \in M_2 \backslash \{p_2, y_0\}$. Hence $\{g'_{\mathrm{can},t}\}_{t\in\Delta}$ form a continuous family of smooth functions on $\{M_t \backslash \{x_t, y_t\}\}_{t\in\Delta} \backslash \{p\}$, and this implies easily that $\{h_t\}_{t\in\Delta}$ also form an almost nice family of

$\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics. Thus we have finished the proof of Proposition 6.2.3.

Similar to Proposition 5.3.1, we have:

Proposition 6.2.4. *Let $\{M_t\}_{t\in\Delta}$ be as in (2.1) with a separating node $p \in M$. Let $\mathcal{X} = \{x_t\}_{t\in\Delta}$ be a holomorphic section of $\{M_t\}_{t\in\Delta}$ such that $x_0 \in M_1\backslash\{p_1\}$ or $x_0 \in M_2\backslash\{p_2\}$. Then the family of divisor line bundles $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$ does not admit any almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics.*

Proof. From Proposition 6.2.2, we have

$$\lim_{t\to 0} g_{\mathrm{can},t}(x_t, y_t) = -\infty, \quad \text{and} \quad \lim_{t\to 0} g_{\mathrm{can},t}(x_t, z_t) = +\infty \tag{6.2.7}$$

for any continuous sections $\{x_t\}_{t\in\Delta}$, $\{y_t\}_{t\in\Delta}$ and $\{z_t\}_{t\in\Delta}$ of $\{M_t\}_{t\in\Delta}$ such that $x_0 \neq y_0 \in M_1\backslash\{p_1\}$ and $z_0 \in M_2\backslash\{p_2\}$. Then Proposition 6.2.4 can be proved by using the arguments in the proof of Proposition 5.3.1 with $g_{\mathrm{can},t}$ replacing $g_{\mathrm{hyp},t}$ and with (6.2.7) replacing (5.3.4).

Finally we are ready to give the following:

Proof of Theorem 3(ii). Let $\mathcal{L} = \{L_t\}_{t\in\Delta}$ be a holomorphic family of line bundles of degree d over $\{M_t\}_{t\in\Delta}$ with a separating node $p \in M$ as in Theorem 3(ii). Also let $f : \widetilde{M} = M_1 \sqcup M_2 \to M$, $p_1, p_2, q_1, q_2, d_1, d_2$ be as in Theorem 3(ii). We fix two holomorphic sections $\mathcal{X} = \{x_t\}_{t\in\Delta}$ and $\mathcal{Y} = \{y_t\}_{t\in\Delta}$ of $\{M_t\}_{t\in\Delta}$ such that $x_0 \in M_1\backslash\{p_1\}$ and $y_0 \in M_2\backslash\{p_2\}$. Also we fix two (holomorphic) canonical sections $s_{\mathcal{X}} = \{s_{x_t}\}_{t\in\Delta}$, $s_{\mathcal{Y}} = \{s_{y_t}\}_{t\in\Delta}$ of $[\mathcal{X}] = \{[x_t]\}_{t\in\Delta}$ and $[\mathcal{Y}] = \{[y_t]\}_{t\in\Delta}$ respectively.

To prove the 'if' part, we assume that

$$\frac{d_1}{q_1} = \frac{d_2}{q_2} = \frac{d}{q}, \quad \text{since } d_1 + d_2 = d,\ q_1 + q_2 = q. \tag{6.2.8}$$

Then we define

$$\mathcal{L}' := \mathcal{L}^{\otimes q} \otimes \left([\mathcal{X}]^{\otimes q_1} \otimes [\mathcal{Y}]^{\otimes q_2}\right)^{\otimes(-d)} \quad \text{over } \mathcal{M}, \tag{6.2.9}$$

and write $\mathcal{L}' = \{L'_t\}_{t\in\Delta}$, where $L'_t = \mathcal{L}'|_{M_t}$ for $t \in \Delta$. Using (6.2.8), it is easy to check that $\{L'_t\}_{t\in\Delta}$ forms a holomorphic family of line bundles of degree 0 such that $\deg(f^*L'_0|_{M_k}) = qd_k - dq_k = 0, k = 1, 2$. Then by Theorem 1(ii), $\{L'_t\}_{t\in\Delta}$ admits a nice family of flat (and thus $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible) p-singular Hermitian metrics. By Proposition 6.2.3 and Remark 2.2.2(i), $\left([\mathcal{X}]^{\otimes q_1} \otimes [\mathcal{Y}]^{\otimes q_2}\right)^{\otimes d}$ also admits an almost nice family of $\{\omega_{\mathrm{can},t}\}_{t\in\Delta}$-admissible p-singular Hermitian metrics, and hence so does $\mathcal{L}^{\otimes q}$ by Remark 2.2.2(i) and (6.2.9). Then so does $\mathcal{L}$ itself by Remark 2.2.2(ii), and this finishes the proof of the 'if' part.

Finally the 'only if' part of Theorem 3(ii) can be proved by using the arguments in the proof of Theorem 2(ii) with the line bundle in (5.3.7) replaced by

$$\mathcal{L}'' := \mathcal{L}^{\otimes q_2} \otimes \left([\mathcal{X}]^{\otimes q_1} \otimes [\mathcal{Y}]^{\otimes q_2}\right)^{\otimes(-d_2)} \otimes [\mathcal{X}]^{\otimes(d_2 q_1 - d_1 q_2)}$$

and with Proposition 5.3.1 replaced by Proposition 6.2.4. This finishes the proof of Theorem 3(ii).

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