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Let k be an imaginary quadratic number field and k^1 the Hilbert 2-class field of k. We give a characterization of those k with $\mathrm{Cl}_2(k) \simeq (2, 2^m)$ such that $\mathrm{Cl}_2(k^1)$ has 2 generators.

1. Introduction.

Let k be an algebraic number field with $\mathrm{Cl}_2(k)$, the Sylow 2-subgroup of its ideal class group, $\mathrm{Cl}(k)$. Denote by k^1 the Hilbert 2-class field of k (in the wide sense). Also let k^n (for n a nonnegative integer) be defined inductively as: $k^0 = k$ and $k^{n+1} = (k^n)^1$. Then

$$k^0 \subseteq k^1 \subseteq k^2 \subseteq \cdots \subseteq k^n \subseteq \cdots$$

is called the 2-class field tower of k. If n is the minimal integer such that $k^n = k^{n+1}$, then n is called the length of the tower. If no such n exists, then the tower is said to be of infinite length.

At present there is no known decision procedure to determine whether or not the (2-)class field tower of a given field k is infinite. However, it is known by group theoretic results (see [2]) that if $\operatorname{rank} \mathrm{Cl}_2(k^1) \leq 2$, then the tower is finite, in fact of length at most 3. (Here the rank means minimal number of generators.) On the other hand, until now (see Table 1 and the penultimate paragraph of this introduction) all examples in the mathematical literature of imaginary quadratic fields with $\operatorname{rank} \mathrm{Cl}_2(k^1) \geq 3$ (let us mention in particular Schmithals [13]) have infinite 2-class field tower. Nevertheless, if we are interested in developing a decision procedure for determining if the 2-class field tower of a field is infinite, then a good starting point would be to find a procedure for sieving out those fields with $\operatorname{rank} \mathrm{Cl}_2(k^1) \leq 2$. We have already started this program for imaginary quadratic number fields k. In [1] we classified all imaginary quadratic fields whose 2-class field k^1 has cyclic 2-class group. In this paper we determine when $\mathrm{Cl}_2(k^1)$ has rank 2 for imaginary quadratic fields k with $\mathrm{Cl}_2(k)$ of type $(2, 2^m)$. (The notation $(2, 2^m)$ means the direct sum of a group of order 2 and a cyclic group of order 2^m.) The group theoretic results mentioned above also show that such fields have 2-class field tower of length 2.

From a classification of imaginary quadratic number fields k with $\mathrm{Cl}_2(k) \simeq (2, 2^m)$ and our results from [1] we see that it suffices to consider discriminants $d = d_1 d_2 d_3$ with prime discriminants $d_1, d_2 > 0$, $d_3 < 0$ such that exactly one of the (d_i/p_j) equals -1 (we let p_j denote the prime dividing d_j); thus there are only two cases:

A) $(d_1/p_2) = (d_1/p_3) = +1$, $(d_2/p_3) = -1$;

B) $(d_1/p_3) = (d_2/p_3) = +1$, $(d_1/p_2) = -1$.

The C_4-factorization corresponding to the nontrivial 4-part of $\mathrm{Cl}_2(k)$ is $d = d_1 \cdot d_2 d_3$ in case A) and $d = d_1 d_2 \cdot d_3$ in case B). Note that, by our results from [1], some of these fields have cyclic $\mathrm{Cl}_2(k^1)$; however, we do not exclude them right from the start since there is no extra work involved and since it provides a welcome check on our earlier work.

The main result of the paper is that $\operatorname{rank} \mathrm{Cl}_2(k^1) = 2$ only occurs for fields of type B); more precisely, we prove the following:

Theorem 1. *Let k be a complex quadratic number field with $\mathrm{Cl}_2(k) \simeq (2, 2^m)$, and let k^1 be its 2-class field. Then $\operatorname{rank} \mathrm{Cl}_2(k^1) = 2$ if and only if $\operatorname{disc} k = d_1 d_2 d_3$ is the product of three prime discriminants $d_1, d_2 > 0$ and $-4 \neq d_3 < 0$ such that $(d_1/p_3) = (d_2/p_3) = +1$, $(d_1/p_2) = -1$, and $h_2(K) = 2$, where K is a nonnormal quartic subfield of one of the two unramified cyclic quartic extensions of k such that $\mathbb{Q}(\sqrt{d_1 d_2}\,) \subset K$.*

This result is the first step in the classification of imaginary quadratic number fields k with $\operatorname{rank} \mathrm{Cl}_2(k^1) = 2$; it remains to solve these problems for fields with $\operatorname{rank} \mathrm{Cl}_2(k) = 3$ and those with $\mathrm{Cl}_2(k) \supseteq (4, 4)$ since we know that $\operatorname{rank} \mathrm{Cl}_2(k^1) \geq 5$ whenever $\operatorname{rank} \mathrm{Cl}_2(k) \geq 4$ (using Schur multipliers as in [1]).

As a demonstration of the utility of our results, we give in Table 1 below a list of the first 12 imaginary quadratic fields k, arranged by decreasing value of their discriminants, with $\operatorname{rank} \mathrm{Cl}_2(k) = 2$ and noncyclic $\mathrm{Cl}_2(k^1)$.

Here f denotes a generating polynomial for a field K as in Theorem 1, r denotes the rank of $\mathrm{Cl}_2(k^1)$. The cases where $r = 3$ follow from our theorem combined with Blackburn's upper bound for the number of generators of derived groups (it implies that finite 2-groups G with $G/G' \simeq (2, 4)$ satisfy $\operatorname{rank} G' \leq 3$), see [3].

In order to verify that $\mathrm{Cl}_2(k^1)$ has rank at least 3 for $k = \mathbb{Q}(\sqrt{-2379}\,)$ it is sufficient to show that its genus class field k_{gen} has class group $(4, 4, 8)$: In fact, $\mathrm{Cl}_2(k^1)$ then contains a quotient of $(4, 4, 8)$ by $(2, 2) \simeq \mathrm{Gal}(k^1/k_{\mathrm{gen}})$, and the claim follows.

We mention one last feature gleaned from the table. It follows from conditional Odlyzko bounds (assuming the Generalized Riemann Hypothesis) that those quadratic fields with $\operatorname{rank} \mathrm{Cl}_2(k^1) \geq 3$ and discriminant $0 > d > -2000$ have finite class field tower; unconditional proofs are not known. Hence, conditionally, we conclude that those k with discriminants $-1015, -1595$ and

Table 1.

disc k	factors	$\mathrm{Cl}_2(k)$	type	f	$\mathrm{Cl}_2(K)$	r	$\mathrm{Cl}_2(k_{\mathrm{gen}})$
-1015	$-7\cdot5\cdot29$	$(2,8)$	A	x^4-22x^2+261	(4)	≥ 3	$(2,2,8)$
-1240	$-31\cdot8\cdot\ \ 5$	$(2,4)$	B	x^4-6x^2-31	(2)	2	$(2,2,8)$
-1443	$-3\cdot13\cdot37$	$(2,4)$	B	x^4-86x^2-75	(2)	2	$(2,2,8)$
-1595	$-11\cdot5\cdot29$	$(2,8)$	A	x^4+26x^2+1445	(4)	≥ 3	$(2,2,8)$
-1615	$-19\cdot5\cdot17$	$(2,4)$	B	x^4+26x^2-171	(2)	2	$(2,2,8)$
-1624	$-7\cdot8\cdot29$	$(2,8)$	B	x^4-30x^2-7	(2)	2	$(2,2,8)$
-1780	$-4\cdot5\cdot89$	$(2,4)$	A	x^4+6x^2+89	(4)	3	$(2,2,4)$
-2035	$-11\cdot5\cdot37$	$(2,4)$	B	x^4-54x^2-11	$(2,2)$	3	$(2,2,16)$
-2067	$-3\cdot13\cdot53$	$(2,4)$	A	x^4+x^2+637	$(2,2)$	3	$(2,2,4)$
-2072	$-7\cdot8\cdot37$	$(2,8)$	B	x^4+34x^2-7	$(2,2)$	≥ 3	$(2,2,8)$
-2379	$-3\cdot13\cdot61$	$(4,4)$				≥ 3	$(4,4,8)$
-2392	$-23\cdot8\cdot13$	$(2,4)$	B	x^4+18x^2-23	(2)	2	$(2,2,8)$

-1780 have finite (2-)class field tower even though $\operatorname{rank}\mathrm{Cl}_2(k^1)\geq 3$. Of course, it would be interesting to determine the length of their towers.

The structure of this paper is as follows: We use results from group theory developed in Section 2 to pull down the condition $\operatorname{rank}\mathrm{Cl}_2(k^1)=2$ from the field k^1 with degree 2^{m+2} to a subfield L of k^1 with degree 8. Using the arithmetic of dihedral fields from Section 4 we then go down to the field K of degree 4 occurring in Theorem 1.

2. Group Theoretic Preliminaries.

Let G be a group. If $x,y\in G$, then we let $[x,y]=x^{-1}y^{-1}xy$ denote the commutator of x and y. If A and B are nonempty subsets of G, then $[A,B]$ denotes the subgroup of G generated by the set $\{[a,b]:a\in A,b\in B\}$. The lower central series $\{G_j\}$ of G is defined inductively by: $G_1=G$ and $G_{j+1}=[G,G_j]$ for $j\geq 1$. The derived series $\{G^{(n)}\}$ is defined inductively by: $G^{(0)}=G$ and $G^{(n+1)}=[G^{(n)},G^{(n)}]$ for $n\geq 0$. Notice that $G^{(1)}=G_2=[G,G]$ the commutator subgroup, G', of G.

Throughout this section, we assume that G is a finite, nonmetacyclic, 2-group such that its abelianization $G^{\mathrm{ab}}=G/G'$ is of type $(2,2^m)$ for some positive integer m (necessarily ≥ 2). Let $G=\langle a,b\rangle$, where $a^2\equiv b^{2^m}\equiv 1 \bmod G_2$ (actually $\bmod\, G_3$ since G is nonmetacyclic, cf. [**1**]); $c_2=[a,b]$ and $c_{j+1}=[b,c_j]$ for $j\geq 2$.

Lemma 1. *Let G be as above (but not necessarily metabelian). Suppose that $d(G')=n$ where $d(G')$ denotes the minimal number of generators of the derived group $G'=G_2$ of G. Then*

$$G'=\langle c_2,c_3,\dots,c_{n+1}\rangle;$$

moreover,

$$G_2/G_2^2 \simeq \langle c_2G_2^2\rangle \oplus \cdots \oplus \langle c_{n+1}G_2^2\rangle.$$

Proof. By the Burnside Basis Theorem, $d(G_2) = d(G_2/\Phi(G))$, where $\Phi(G)$ is the Frattini subgroup of G, i.e., the intersection of all maximal subgroups of G, see [**5**]. But in the case of a 2-group, $\Phi(G) = G^2$, see [**8**]. By Blackburn, [**3**], since G/G_2^2 has elementary derived group, we know that $G_2/G_2^2 \simeq \langle c_2G_2^2\rangle \oplus \cdots \oplus \langle c_{n+1}G_2^2\rangle$. Again, by the Burnside Basis Theorem, $G_2 = \langle c_2, \dots , c_{n+1}\rangle$. □

Lemma 2. *Let G be as above. Moreover, assume G is metabelian. Let H be a maximal subgroup of G such that H/G' is cyclic, and denote the index $(G' : H')$ by 2^κ. Then G' contains an element of order 2^κ.*

Proof. Without loss of generality, let $H = \langle b, G'\rangle$. Notice that $G' = \langle c_2, c_3, \dots\rangle$ and by our presentation of H, $H' = \langle c_3, c_4, \dots\rangle$. Thus, $G'/H' = \langle c_2H'\rangle$. But since $(G' : H') = 2^\kappa$, the order of c_2 is $\ge 2^\kappa$. This establishes the lemma. □

Lemma 3. *Let G be as above and again assume G is metabelian. Let H be a maximal subgroup of G such that H/G' is cyclic, and assume that $(G' : H') \equiv 0 \bmod 4$. If $d(G') = 2$, then $G_2 = \langle c_2, c_3\rangle$ and $G_j = \langle c_2^{2^{j-2}}, c_3^{2^{j-3}}\rangle$ for $j > 2$.*

Proof. Assume that $d(G') = 2$. By Lemma 1, $G_2 = \langle c_2, c_3\rangle$ and hence $c_4 \in \langle c_2, c_3\rangle$. Write $c_4 = c_2^x c_3^y$ where x, y are positive integers. Without loss of generality, let $H = \langle b, c_2, c_3\rangle$ and write $(G' : H') = 2^\kappa$ for some $\kappa \ge 2$. Since $c_3, c_4 \in H'$ we have, $c_2^x \equiv 1 \bmod H'$. By the proof of Lemma 2, this implies that $x \equiv 0 \bmod 2^\kappa$. Write $x = 2^\kappa x_1$ for some positive integer x_1. On the other hand, since $c_4, c_2^{2^\kappa x_1} \in G_4$, we see that $c_3^y \equiv 1 \bmod G_4$. If y were odd, then $c_3 \in G_4$. This, however, implies that $G_2 = \langle c_2\rangle$, contrary to our assumptions. Thus y is even, say $y = 2y_1$. From all of this we see that $c_4 = c_2^{2^\kappa x_1} c_3^{2y_1}$. Consequently, by induction we have $c_j \in \langle c_2^{2^{j-2}}, c_3^{2^{j-3}}\rangle$ for all $j \ge 4$. Since $G_j = \langle c_2^{2^{j-2}}, c_3^{2^{j-3}}, \dots , c_{j-1}^2, c_j, c_{j+1}, \dots\rangle$, cf. [**1**], we obtain the lemma. □

Let us translate the above into the field-theoretic language. Let k be an imaginary quadratic number field of type A) or B) (see the Introduction), and let M/k be one of the two quadratic subextensions of k^1/k over which k^1 is cyclic. If $h_2(M) = 2^{m+\kappa}$ and $\mathrm{Cl}_2(k) = (2, 2^m)$, then Lemma 2 implies that $\mathrm{Cl}_2(k^1)$ contains an element of order 2^κ. Table 2 contains the relevant information for the fields occurring in Table 1. An application of the class number formula to $M/\mathbb{Q}$ (see e.g., Proposition 3 below) shows immediately that $h_2(M) = 2^{m+\kappa}$, where 2^κ is the class number of the quadratic subfield $\mathbb{Q}(\sqrt{d_id_j}\,)$ of M, where $(d_i/p_j) = +1$; in particular, we always have $\kappa \ge 2$, and the assumption $(G' : H') \geq 4$ is always satisfied for the fields that we consider.

Table 2.

M_1	$\mathrm{Cl}_2(M_1)$	M_2	$\mathrm{Cl}_2(M_2)$
$\mathbb{Q}(\sqrt{5}, \sqrt{-7\cdot 29})$	$(2,16)$	$\mathbb{Q}(\sqrt{5\cdot 29}, \sqrt{-7})$	$(2,16)$
$\mathbb{Q}(\sqrt{2}, \sqrt{-5\cdot 31})$	$(4,4)$	$\mathbb{Q}(\sqrt{5}, \sqrt{-2\cdot 31})$	$(2,16)$
$\mathbb{Q}(\sqrt{13}, \sqrt{-3\cdot 37})$	$(2,16)$	$\mathbb{Q}(\sqrt{37}, \sqrt{-3\cdot 13})$	$(2,16)$
$\mathbb{Q}(\sqrt{-11}, \sqrt{5\cdot 29})$	$(2,16)$	$\mathbb{Q}(\sqrt{29}, \sqrt{-5\cdot 11})$	$(2,16)$
$\mathbb{Q}(\sqrt{5}, \sqrt{-17\cdot 19})$	$(4,4)$	$\mathbb{Q}(\sqrt{17}, \sqrt{-5\cdot 19})$	$(2,16)$
$\mathbb{Q}(\sqrt{29}, \sqrt{-2\cdot 7})$	$(2,16)$	$\mathbb{Q}(\sqrt{2}, \sqrt{-7\cdot 29})$	$(2,16)$
$\mathbb{Q}(\sqrt{5\cdot 89}, \sqrt{-1})$	$(4,4)$	$\mathbb{Q}(\sqrt{5}, \sqrt{-89})$	$(2,8)$
$\mathbb{Q}(\sqrt{37}, \sqrt{-5\cdot 11})$	$(4,4)$	$\mathbb{Q}(\sqrt{5}, \sqrt{-37\cdot 11})$	$(2,32)$
$\mathbb{Q}(\sqrt{53}, \sqrt{-3\cdot 13})$	$(4,4)$	$\mathbb{Q}(\sqrt{13\cdot 53}, \sqrt{-3})$	$(2,2,4)$
$\mathbb{Q}(\sqrt{37}, \sqrt{-2\cdot 7})$	$(2,16)$	$\mathbb{Q}(\sqrt{2}, \sqrt{-7\cdot 37})$	$(2,16)$
$\mathbb{Q}(\sqrt{13}, \sqrt{-2\cdot 23})$	$(4,4)$	$\mathbb{Q}(\sqrt{2}, \sqrt{-13\cdot 23})$	$(2,16)$

We now use the above results to prove the following useful proposition.

Proposition 1. *Let G be a nonmetacyclic 2-group such that $G/G' \simeq (2, 2^m)$; (hence $m > 1$). Let H and K be the two maximal subgroups of G such that H/G' and K/G' are cyclic. Moreover, assume that $(G' : H') \equiv 0 \bmod 4$. Finally, assume that N is a subgroup of index 4 in G not contained in H or K. Then*

$$(N : N') \begin{cases} = & 2^m \quad \text{if } d(G') = 1 \\ = & 2^{m+1} \quad \text{if } d(G') = 2 \\ \geq & 2^{m+2} \quad \text{if } d(G') \geq 3 \end{cases}.$$

Proof. Without loss of generality we assume that G is metabelian. Let $G = \langle a, b\rangle$, where $a^2 \equiv b^{2^m} \equiv 1 \bmod G_3$. Also let $H = \langle b, G'\rangle$ and $K = \langle ab, G'\rangle$ (without loss of generality). Then $N = \langle ab^2, G'\rangle$ or $N = \langle a, b^4, G'\rangle$.

Suppose that $N = \langle ab^2, G'\rangle$.

First assume $d(G') = 1$. Then $G' = \langle c_2\rangle$ and thus $N' = \langle [ab^2, c_2]\rangle$. But $[ab^2, c_2] = c_2^2\eta_4$ for some $\eta_4 \in G_4 = \langle c_2^4\rangle$ (cf. Lemma 1 of [1]). Hence, $N' = \langle c_2^2\rangle$, and so $(G' : N') = 2$. Since $(N : G') = 2^{m-1}$, we get $(N : N') = 2^m$ as desired.

Next, assume that $d(G') = 2$. Then $N = \langle ab^2, c_2, c_3\rangle$ by Lemma 1. Notice that $[ab^2, c_2] = c_2^2\eta_4$ and $[ab^2, c_3] = c_3^2\eta_5$ where $\eta_j \in G_j$ for $j = 4, 5$. Hence $N' = \langle c_2^2\eta_4, c_3^2\eta_5, N_3\rangle$ and so $\langle c_2^2\eta_4, c_3^2\eta_5\rangle \subseteq N'$. But then $N'G_5 \supseteq \langle c_2^4, c_3^2\rangle = G_4$ by Lemma 3. Therefore, by [5], $N' \supseteq G_4$. But notice that $N_3 \subseteq G_4$. Thus $N' = \langle c_2^2, c_3^2\rangle$ and so $(G' : N') = 4$ which in turn implies that $(N : N') = 2^{m+1}$, as desired.

Finally, assume $d(G') \geq 3$. Then $d(G'/G_5) = 3$. Moreover there exists an exact sequence

$$N/N' \longrightarrow (N/G_5)/(N/G_5)' \longrightarrow 1,$$

and thus $\#N^{\rm ab} \geq \#(N/G_5)^{\rm ab}$. Hence it suffices to prove the result for $G_5 = 1$ which we now assume. $N = \langle ab^2, c_2, c_3, c_4\rangle$ and so, arguing as above, we have $N' = \langle c_2^2\eta_4, c_3^2\eta_5, c_4^2\eta_6, N_3\rangle = \langle c_2^2\eta_4, c_3^2, N_3\rangle$, where $\eta_j \in G_j$. But $N_3 = \langle [ab^2, c_2^2\eta_4]\rangle = \langle c_2^4\rangle$. Therefore, $N' = \langle c_2^2\eta_4, c_3^2\rangle$. From this we see that $(G' : N') = 8$ and thus $(N : N') = 2^{m+2}$ as desired.

Now suppose that $N = \langle a, b^4, G'\rangle$. Then the proof is essentially the same as above once we notice that $[a, b^4] \equiv {c_3}^2{c_2}^{-4} \bmod G_5$.

This establishes the proposition. □

3. Number Theoretic Preliminaries.

Proposition 2. *Let K/k be a quadratic extension, and assume that the class number of k, $h(k)$, is odd. If K has an unramified cyclic extension M of order 4, then M/k is normal and* $\mathrm{Gal}(M/k) \simeq D_4$.

Proof. Rédei and Reichardt [12] proved this for $k = \mathbb{Q}$; the general case is analogous. □

We shall make extensive use of the class number formula for extensions of type $(2, 2)$:

Proposition 3. *Let K/k be a normal quartic extension with Galois group of type $(2, 2)$, and let k_j $(j = 1, 2, 3)$ denote the quadratic subextensions. Then*

$$h(K) = 2^{d-\kappa-2-\upsilon} q(K)h(k_1)h(k_2)h(k_3)/h(k)^2, \qquad (1)$$

where $q(K) = (E_K : E_1E_2E_3)$ denotes the unit index of K/k (E_j is the unit group of k_j), d is the number of infinite primes in k that ramify in K/k, κ is the $\mathbb{Z}$-rank of the unit group E_k of k, and $\upsilon = 0$ except when $K \subseteq k(\sqrt{E_k}\,)$, where $\upsilon = 1$.

Proof. See [10]. □

Another important result is the ambiguous class number formula. For cyclic extensions K/k, let $\mathrm{Am}(K/k)$ denote the group of ideal classes in K fixed by $\mathrm{Gal}(K/k)$, i.e., the ambiguous ideal class group of K, and Am_2 its 2-Sylow subgroup.

Proposition 4. *Let K/k be a cyclic extension of prime degree p; then the number of ambiguous ideal classes is given by*

$$\#\,\mathrm{Am}(K/k) = h(k)\,\frac{p^{t-1}}{(E:H)},$$

where t is the number of primes (including those at ∞) of k that ramify in K/k, E is the unit group of k, and H is its subgroup consisting of norms of elements from $K^\times$. Moreover, $\mathrm{Cl}_p(K)$ is trivial if and only if $p \nmid \#\,\mathrm{Am}(K/k)$.

Proof. See Lang [**9**, part II] for the formula. For a proof of the second assertion (see e.g., Moriya [**11**]), note that $\mathrm{Am}(K/k)$ is defined by the exact sequence

$$1 \longrightarrow \mathrm{Am}(K/k) \longrightarrow \mathrm{Cl}(K) \longrightarrow \mathrm{Cl}(K)^{1-\sigma} \longrightarrow 1,$$

where σ generates $\mathrm{Gal}(K/k)$. Taking p-parts we see that $p \nmid \#\,\mathrm{Am}(K/k)$ is equivalent to $\mathrm{Cl}_p(K) = \mathrm{Cl}_p(K)^{1-\sigma}$. By induction we get $\mathrm{Cl}_p(K) = \mathrm{Cl}_p(K)^{(1-\sigma)^p}$, but since $(1-\sigma)^p \equiv 0 \bmod p$ in the group ring $\mathbb{Z}[G]$, this implies $\mathrm{Cl}_p(K) \subseteq \mathrm{Cl}_p(K)^p$. But then $\mathrm{Cl}_p(K)$ must be trivial. □

We make one further remark concerning the ambiguous class number formula that will be useful below. If the class number $h(k)$ is odd, then it is known that $\#\,\mathrm{Am}_2(K/k) = 2^r$ where $r = \mathrm{rank}\,\mathrm{Cl}_2(K)$.

We also need a result essentially due to G. Gras [**4**]:

Proposition 5. *Let K/k be a quadratic extension of number fields and assume that $h_2(k) = \#\,\mathrm{Am}_2(K/k) = 2$. Then K/k is ramified and*

$$\mathrm{Cl}_2(K) \simeq \begin{cases} (2,2) \text{ or } \mathbb{Z}/2^n\mathbb{Z}\ (n \ge 3) & \text{if } \#\kappa_{K/k} = 1, \\ \mathbb{Z}/2^n\mathbb{Z}\ (n \ge 1) & \text{if } \#\kappa_{K/k} = 2, \end{cases}$$

where $\kappa_{K/k}$ denotes the set of ideal classes of k that become principal (capitulate) in K.

Proof. We first notice that K/k is ramified. If the extension were unramified, then K would be the 2-class field of k, and since $\mathrm{Cl}_2(k)$ is cyclic, it would follow that $\mathrm{Cl}_2(K) = 1$, contrary to assumption.

Before we start with the rest of the proof, we cite the results of Gras that we need (we could also give a slightly longer direct proof without referring to his results). Let K/k be a cyclic extension of prime power order p^r, and let σ be a generator of $G = \mathrm{Gal}(K/k)$. For any p-group M on which G acts we put $M_i = \{m \in M : m^{(1-\sigma)^i} = 1\}$. Moreover, let ν be the algebraic norm, that is, exponentiation by $1 + \sigma + \sigma^2 + \ldots + \sigma^{p^r-1}$. Then [**4**, Cor. 4.3] reads:

Lemma 4. *Suppose that* $M^\nu = 1$*; let* n *be the smallest positive integer such that* $M_n = M$ *and write* $n = a(p-1)+b$ *with integers* $a \ge 0$ *and* $0 \le b \le p-2$*. If* $\# M_{i+1}/M_i = p$ *for* $i = 0, 1, \ldots, n-1$*, then* $M \simeq (\mathbb{Z}/p^{a+1}\mathbb{Z})^b \times (\mathbb{Z}/p^a\mathbb{Z})^{p-1-b}$*.*

We claim that if $\kappa_{K/k} = 2$, then $M = \mathrm{Cl}_2(K)$ satisfies the assumptions of Lemma 4: In fact, let $j = j_{k\to K}$ denote the transfer of ideal classes. Then $c^{1+\sigma} = j(N_{K/k}c)$ for any ideal class $c \in \mathrm{Cl}_2(K)$, hence $M^\nu = j(\mathrm{Cl}_2(k)) = 1$. Moreover, $M_1 = \mathrm{Am}_2(K/k)$ in our case, hence M_1/M_0 has order 2. Since the orders of M_{i+1}/M_i decrease towards 1 as i grows (Gras [**4**, Prop. 4.1.ii)]), we conclude that $\# M_{i+1}/M_i = 2$ for all $i < n$. Since $a = n$ and $b = 0$ when $p = 2$, Lemma 4 now implies that $\mathrm{Cl}_2(K) \simeq \mathbb{Z}/2^n\mathbb{Z}$, that is, the 2-class group is cyclic.

The second result of Gras that we need is [**4**, Prop. 4.3]:

Lemma 5. *Suppose that* $M^\nu \ne 1$ *but assume the other conditions in Lemma 4. Then* $n \ge 2$ *and*

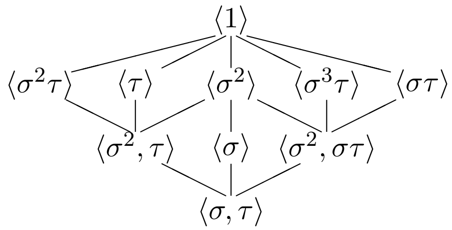
$$M \simeq \begin{cases} (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^{n-2} & \text{if } n < p; \\ (\mathbb{Z}/p\mathbb{Z})^p \text{ or } (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^{n-2} & \text{if } n = p; \\ (\mathbb{Z}/p^{a+1}\mathbb{Z})^b \times (\mathbb{Z}/p^a\mathbb{Z})^{p-1-b} & \text{if } n > p. \end{cases}$$

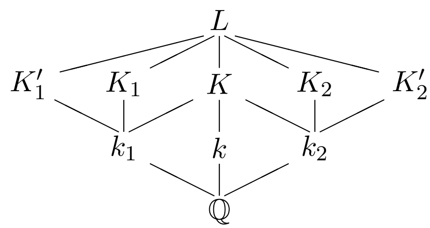
If $\kappa_{K/k} = 1$, then this lemma shows that $\mathrm{Cl}_2(K)$ is either cyclic of order ≥ 4 or of type $(2,2)$. (Notice that the hypothesis of the lemma is satisfied since K/k is ramified implying that the norm $N_{K/k} : \mathrm{Cl}_2(K) \longrightarrow \mathrm{Cl}_2(k)$ is onto; and so the argument above this lemma applies.) It remains to show that the case $\mathrm{Cl}_2(K) \simeq \mathbb{Z}/4\mathbb{Z}$ cannot occur here.

Now assume that $\mathrm{Cl}_2(K) = \langle C \rangle \simeq \mathbb{Z}/4\mathbb{Z}$; since K/k is ramified, the norm $N_{K/k} : \mathrm{Cl}_2(K) \longrightarrow \mathrm{Cl}_2(k)$ is onto, and using $\kappa_{K/k} = 1$ once more we find $C^{1+\sigma} = c$, where c is the nontrivial ideal class from $\mathrm{Cl}_2(k)$. On the other hand, $c \in \mathrm{Cl}_2(k)$ still has order 2 in $\mathrm{Cl}_2(K)$, hence we must also have $C^2 = C^{1+\sigma}$. But this implies that $C^\sigma = C$, i.e., that each ideal class in K is ambiguous, contradicting our assumption that $\# \mathrm{Am}_2(K/k) = 2$. □

4. Arithmetic of some Dihedral Extensions.

In this section we study the arithmetic of some dihedral extensions $L/\mathbb{Q}$, that is, normal extensions L of $\mathbb{Q}$ with Galois group $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_4$, the dihedral group of order 8. Hence D_4 may be presented as $\langle \tau, \sigma | \tau^2 = \sigma^4 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$. Now consider the following diagrams (Galois correspondence):





In this situation, we let $q_1 = (E_L : E_1E_1'E_K)$ and $q_2 = (E_L : E_2E_2'E_K)$ denote the unit indices of the bicyclic extensions L/k_1 and L/k_2, where E_i and E_i' are the unit groups in K_i and K_i', respectively. Finally, let κ_i denote the kernel of the transfer of ideal classes $j_{k_i\to K_i} : \mathrm{Cl}_2(k_i) \longrightarrow \mathrm{Cl}_2(K_i)$ for $i = 1, 2$.

The following remark will be used several times: If $K_1 = k_1(\sqrt{\alpha}\,)$ for some $\alpha \in k_1$, then $k_2 = \mathbb{Q}(\sqrt{a}\,)$, where $a = \alpha\alpha'$ is the norm of α. To see this, let $\gamma = \sqrt{\alpha}$; then $\gamma^\tau = \gamma$, since $\gamma \in K_1$. Clearly $\gamma^{1+\sigma} = \sqrt{a} \in K$ and hence fixed by σ^2. Furthermore,

$$(\gamma^{1+\sigma})^{\sigma\tau} = \gamma^{\sigma\tau+\sigma^2\tau} = \gamma^{\tau\sigma^3+\tau\sigma^2} = (\gamma^\tau)^{\sigma^3+\sigma^2} = \gamma^{\sigma^3+\sigma^2} = \gamma^{(1+\sigma)\sigma^2} = \gamma^{1+\sigma},$$

implying that $\sqrt{a} \in k_2$. Finally notice that $\sqrt{a} \notin \mathbb{Q}$, since otherwise $\sqrt{\alpha'} = \sqrt{a}/\sqrt{\alpha} \in K_1$ implying that $K_1/\mathbb{Q}$ is normal, which is not the case.

Recall that a quadratic extension $K = k(\sqrt{\alpha}\,)$ is called *essentially ramified* if $\alpha\mathcal{O}_k$ is not an ideal square. This definition is independent of the choice of α.

Proposition 6. *Let $L/\mathbb{Q}$ be a non-CM totally complex dihedral extension not containing $\sqrt{-1}$, and assume that L/K_1 and L/K_2 are essentially ramified. If the fundamental unit of the real quadratic subfield of K has norm -1, then $q_1q_2 = 2$.*

Proof. Notice first that k cannot be real (in fact, K is not totally real by assumption, and since L/k is a cyclic quartic extension, no infinite prime can ramify in K/k); thus exactly one of k_1, k_2 is real, and the other is complex. Multiplying the class number formulas, Proposition 3, for L/k_1 and L/k_2 (note that $\upsilon = 0$ since both L/K_1 and L/K_2 are essentially ramified) we find that $2q_1q_2$ is a square. If we can prove that $q_1, q_2 \le 2$, then $2q_1q_2$ is a square between 2 and 8, which implies that we must have $2q_1q_2 = 4$ and $q_1q_2 = 2$ as claimed.

We start by remarking that if $\zeta\eta$ becomes a square in L, where ζ is a root of unity in L, then so does one of $\pm\eta$. This follows from the fact that the only nontrivial roots of unity that can be in L are the sixth roots of unity $\langle\zeta_6\rangle$, and here $\zeta_6 = -\zeta_3^2$.

Now we prove that $q_1 \le 2$ under the assumptions we made; the claim $q_2 \le 2$ will then follow by symmetry. Assume first that k_1 is real and let ε be the fundamental unit of k_1. We claim that $\sqrt{\pm\varepsilon} \notin L$. Suppose otherwise; then $k_1(\sqrt{\pm\varepsilon}\,)$ is one of K_1, K_1' or K. If $k_1(\sqrt{\pm\varepsilon}\,) = K_1$, then $K_1' = k_1(\sqrt{\pm\varepsilon'}\,)$ and $K = k_1(\sqrt{\varepsilon\varepsilon'}\,)$. (Here and below $x' = x^\sigma$.) This however cannot occur since by assumption $\varepsilon\varepsilon' = -1$ implying that $\sqrt{-1} \in L$, a contradiction. Similarly, if $k_1(\sqrt{\pm\varepsilon}\,) = K$, then again $\sqrt{-1} \in L$.

Thus $\sqrt{\pm\varepsilon} \notin L$, and $E_1 = \langle -1, \varepsilon, \eta\rangle$ for some unit $\eta \in E_1$. Suppose that $\sqrt{u\eta} \in L$ for some unit $u \in k_1$. Then $L = K_1(\sqrt{u\eta}\,)$, contradicting our assumption that L/K_1 is essentially ramified. The same argument shows that $\sqrt{u\eta'} \notin L$, hence either $E_L = \langle \zeta, \varepsilon, \eta, \eta'\rangle$ and $q_1 = 1$ or $E_L = \langle \zeta, \varepsilon, \eta, \sqrt{u\eta\eta'}\,\rangle$ for some unit $u \in k_1$ and $q_1 = 2$. Here ζ is a root of unity generating the torsion subgroup W_L of E_L.

Next consider the case where k_1 is complex, and let ε denote the fundamental unit of k_2. Then $\pm\varepsilon$ stays fundamental in L by the argument above.

Let η be a fundamental unit in K_1. If $\pm\eta$ became a square in L, then clearly L/K_1 could not be essentially ramified. Thus if we have $q_1 \ge 4$, then $\pm\varepsilon\eta = \alpha^2$ is a square in L. Applying τ to this relation we find that $-1 = \varepsilon\varepsilon'$ is a square in L, contradicting the assumption that L does not contain $\sqrt{-1}$. □

Proposition 7. *Suppose that $q_2 = 1$. Then K_2/k_2 is essentially ramified if and only if $\kappa_2 = 1$; if K_2/k_2 is not essentially ramified, then $\kappa_2 = \langle[\mathfrak{b}]\rangle$, where $K_2 = k_2(\sqrt{\beta}\,)$ and $(\beta) = \mathfrak{b}^2$.*

Proof. First notice that if K_2/k_2 is not essentially ramified, then $\kappa_2 \ne 1$: In fact, in this case we have $(\beta) = \mathfrak{b}^2$, and if we had $\kappa_2 = 1$, then $\mathfrak{b}$ would have to be principal, say $\mathfrak{b} = (\gamma)$. This implies that $\beta = \varepsilon\gamma^2$ for some unit $\varepsilon \in k_2$, which in view of $q_2 = 1$ implies that ε must be a square. But then β would be a square, and this is impossible.

Conversely, suppose $\kappa_2 \ne 1$. Let $\mathfrak{a}$ be a nonprincipal ideal in k_2 of absolute norm a, and assume that $\mathfrak{a} = (\alpha)$ in K_2. Then $\alpha^{1-\sigma^2} = \eta$ for some unit $\eta \in E_2$, and similarly $\alpha^{\sigma-\sigma^3} = \eta'$, where η' is a unit in E_2'. But then $\eta\eta' = \alpha^{1+\sigma-\sigma^2-\sigma^3} \overset{2}{=} N_{L/k}\alpha = \pm N_{L/k}\mathfrak{a} = \pm a^2 \overset{2}{=} \pm 1$ in $L^\times$, where $\overset{2}{=}$ means equal up to a square in $L^\times$. Thus $\pm\eta\eta'$ is a square in L, so our assumption that $q_2 = 1$ implies that $\pm\eta\eta'$ must be a square in k_2. The same argument show that $\pm\eta/\eta'$ is a square in k_2, hence we find $\eta \in k_2$. Thus $\alpha^{1-\sigma^2}$ is fixed by σ^2 and so $\beta := \alpha^2 \in k_2$. This gives $K_2 = k_2(\sqrt{\beta}\,)$, hence K_2/k_2 is not essentially ramified, and moreover, $\mathfrak{a} \sim \mathfrak{b}$. □

From now on assume that k is one of the imaginary quadratic fields of type A) or B) as explained in the Introduction. Let

$k_1 = \mathbb{Q}(\sqrt{d_1}\,)$ and $k_2 = \mathbb{Q}(\sqrt{d_2 d_3}\,)$ in case A), and

$k_1 = \mathbb{Q}(\sqrt{d_3}\,)$ and $k_2 = \mathbb{Q}(\sqrt{d_1 d_2}\,)$ in case B).

Then there exist two unramified cyclic quartic extensions of k which are D_4 over $\mathbb{Q}$ (see Proposition 2). Let us say a few words about their construction. Consider e.g., case B); by Rédei's theory (see [**12**]), the C_4-factorization $d = d_1d_2 \cdot d_3$ implies that unramified cyclic quartic extensions of $k = \mathbb{Q}(\sqrt{d}\,)$ are constructed by choosing a "primitive" solution (x, y, z) of $d_1d_2X^2 + d_3Y^2 = Z^2$ and putting $L = k(\sqrt{d_1d_2}, \sqrt{\alpha}\,)$ with $\alpha = z + x\sqrt{d_1d_2}$ (primitive here means that α should not be divisible by rational integers); the other unramified cyclic quartic extension is then $\widetilde{L} = k(\sqrt{d_1d_2}, \sqrt{d_1\alpha}\,)$. Since $4\alpha\beta = (x\sqrt{d_1d_2} + y\sqrt{d_3} + z)^2$ for $\beta = \frac{1}{2}(z + y\sqrt{d_3}\,)$, we also have $L = k(\sqrt{d_3}, \sqrt{\beta}\,)$ etc. If $d_3 = -4$, then it is easy to see that we may choose β as the fundamental unit of k_2; if $d_3 \neq -4$, then genus theory says that a) the class number h of k_2 is twice an odd number u; and b) the prime ideal $\mathfrak{p}_3$ above d_3 in k_2 is in the principal genus, so $\mathfrak{p}_3^u = (\pi_3)$ is principal. Again it can be checked that $\beta = \pm\pi_3$ for a suitable choice of the sign.

Example. Consider the case $d = -31 \cdot 5 \cdot 8$; here $\pi_3 = \pm(3 + 2\sqrt{10}\,)$, and the positive sign is correct since $3 + 2\sqrt{10} \equiv (1 + \sqrt{10}\,)^2 \bmod 4$ is primary. The minimal polynomial of $\sqrt{\pi_3}$ is $f(x) = x^4 - 6x^2 - 31$: Compare Table 1.

The fields $K_2 = k_2(\sqrt{\alpha}\,)$ and $\widetilde{K}_2 = k_2(\sqrt{d_2\alpha}\,)$ will play a dominant role in the proof below; they are both contained in $M = F(\sqrt{\alpha}\,)$ for $F = k_2(\sqrt{d_2}\,)$, and it is the ambiguous class group $\mathrm{Am}(M/F)$ that contains the information we are interested in.

Lemma 6. *The field F has odd class number (even in the strict sense), and we have $\#\,\mathrm{Am}(M/F) \mid 2$. In particular, $\mathrm{Cl}_2(M)$ is cyclic (though possibly trivial).*

Proof. The class group in the strict sense of k_2 is cyclic of order 2 by Rédei's theory [**12**] (since $(d_2/p_3) = (d_3/p_2) = -1$ in case A) and $(d_1/p_2) = (d_2/p_1) = -1$ in case B)). Since F is the Hilbert class field of k_2 in the strict sense, its class number in the strict sense is odd.

Next we apply the ambiguous class number formula. In case A), F is complex, and exactly the two primes above d_3 ramify in M/F. Note that $M = F(\sqrt{\alpha}\,)$ with α primary of norm d_3y^2; there are four primes above d_3 in F, and exactly two of them divide α to an odd power, so $t = 2$ by the decomposition law in quadratic Kummer extensions. By Proposition 4 and the remarks following it, $\#\,\mathrm{Am}_2(M/F) = 2/(E : H) \le 2$, and $\mathrm{Cl}_2(M)$ is cyclic.

In case B), however, F is real; since $\alpha \in k_2$ has norm $d_3y^2 < 0$, it has mixed signature, hence there are exactly two infinite primes that ramify in M/F. As in case A), there are two finite primes above d_3 that ramify in M/F, so we get $\#\,\mathrm{Am}_2(M/F) = 8/(E : H)$. Since F has odd class number in the strict sense, F has units of independent signs. This implies that the group of units that are positive at the two ramified infinite primes has $\mathbb{Z}$-rank 2, i.e., $(E : H) \ge 4$ by consideration of the infinite primes alone. In particular, $\# \operatorname{Am}_2(M/F) \le 2$ in case B). □

Next we derive some relations between the class groups of K_2 and $\widetilde{K}_2$; these relations will allow us to use each of them as our field K in Theorem 1.

Proposition 8. *Let L and $\widetilde{L}$ be the two unramified cyclic quartic extensions of k, and let K_2 and $\widetilde{K}_2$ be two quadratic extensions of k_2 in L and $\widetilde{L}$, respectively, which are not normal over $\mathbb{Q}$.*

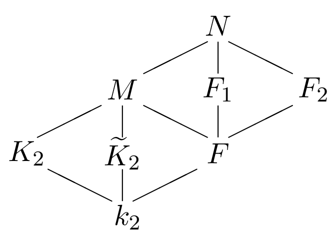
a) *We have $4 \mid h(K_2)$ if and only if $4 \mid h(\widetilde{K}_2)$;*
b) *If $4 \mid h(K_2)$, then one of $\operatorname{Cl}_2(K_2)$ or $\operatorname{Cl}_2(\widetilde{K}_2)$ has type $(2,2)$, whereas the other is cyclic of order ≥ 4.*

Proof. Notice that the prime dividing $\operatorname{disc}(k_1)$ splits in k_2. Throughout this proof, let $\mathfrak{p}$ be one of the primes of k_2 dividing $\operatorname{disc}(k_1)$.

If we write $K_2 = k_2(\sqrt{\alpha}\,)$ for some $\alpha \in k_2$, then $\widetilde{K}_2 = k_2(\sqrt{d_2\alpha}\,)$. In fact, K_2 and $\widetilde{K}_2$ are the only extensions F/k_2 of k_2 with the properties
1. F/k_2 is a quadratic extension unramified outside $\mathfrak{p}$;
2. kF/k is a cyclic extension.
Therefore it suffices to observe that if $k_2(\sqrt{\alpha}\,)$ has these properties, then so does $k_2(\sqrt{d_2\alpha}\,)$. But this is elementary.

In particular, the compositum $M = K_2\widetilde{K}_2 = k_2(\sqrt{d_2}, \sqrt{\alpha}\,)$ is an extension of type $(2,2)$ over k_2 with subextensions K_2, $\widetilde{K}_2$ and $F = k_2(\sqrt{d_2}\,)$. Clearly F is the unramified quadratic extension of k_2, so both M/K_2 and $M/\widetilde{K}_2$ are unramified. If K_2 had 2-class number 2, then M would have odd class number, and M would also be the 2-class field of $\widetilde{K}_2$. Thus $2 \parallel h(K_2)$ implies that $2 \parallel h(\widetilde{K}_2)$. This proves part a) of the proposition.

Before we go on, we give a Hasse diagram for the fields occurring in this proof:



Now assume that $4 \mid h(K_2)$. Since $\operatorname{Cl}_2(M)$ is cyclic by Lemma 6, there is a unique quadratic unramified extension N/M, and the uniqueness implies at once that N/k_2 is normal. Hence $G = \operatorname{Gal}(N/k_2)$ is a group of order 8 containing a subgroup of type $(2,2) \simeq \operatorname{Gal}(N/F)$: In fact, if $\operatorname{Gal}(N/F)$

were cyclic, then the primes ramifying in M/F would also ramify in N/M contradicting the fact that N/M is unramified. There are three groups satisfying these conditions: $G = (2,4)$, $G = (2,2,2)$ and $G = D_4$. We claim that G is nonabelian; once we have proved this, it follows that exactly one of the groups $\mathrm{Gal}(N/K_2)$ and $\mathrm{Gal}(N/\widetilde{K}_2)$ is cyclic, and that the other is not, which is what we want to prove.

So assume that G is abelian. Then M/F is ramified at two finite primes $\mathfrak{q}$ and $\mathfrak{q}'$ of F dividing $\mathfrak{p}$ (in k_2); if F_1 and F_2 denote the quadratic subextensions of N/F different from M then F_1/F and F_2/F must be ramified at a finite prime (since F has odd class number in the strict sense: See Lemma 6); since both F_1 and F_2 are normal (even abelian) over k_2, ramification at $\mathfrak{q}$ implies ramification at the conjugated ideal $\mathfrak{q}'$. Hence both $\mathfrak{q}$ and $\mathfrak{q}'$ ramify in F_1/F and F_2/F, and since they also ramify in M/F, they must ramify completely in N/F, again contradicting the fact that N/M is unramified.

We have proved that $\mathrm{Cl}_2(K_2)$ and $\mathrm{Cl}_2(\widetilde{K}_2)$ contain subgroups of type (4) and $(2,2)$, respectively. Now we wish to apply Proposition 5. But we have to compute $\#\,\mathrm{Am}_2(\widetilde{K}_2/k_2)$. Since the class number of $\widetilde{K}_2$ is even, it is sufficient to show that $\#\,\mathrm{Am}_2(\widetilde{K}_2/k_2) \le 2$. In case A), there is exactly one ramified prime (it divides d_1), hence $\#\,\mathrm{Am}_2(\widetilde{K}_2/k_2) = 2/(E:H) \le 2$. In case B), there are two ramified primes (one is infinite, the other divides d_3), hence $\#\,\mathrm{Am}_2(\widetilde{K}_2/k_2) = 4/(E:H)$; but -1 is not a norm residue at the ramified infinite prime, hence $(E:H) \ge 2$ and $\#\,\mathrm{Am}_2(\widetilde{K}_2/k_2) \le 2$ as claimed.

Now Proposition 5 implies that $\mathrm{Cl}_2(K_2)$ is cyclic of order ≥ 4, and that $\mathrm{Cl}_2(\widetilde{K}_2) \simeq (2,2)$. This concludes our proof. □

Proposition 9. *Assume that k is one of the imaginary quadratic fields of type A) or B) as explained in the Introduction. Then there exist two unramified cyclic quartic extensions of k. Let L be one of them, and write*

$k_1 = \mathbb{Q}(\sqrt{d_1}\,)$ *and* $k_2 = \mathbb{Q}(\sqrt{d_2 d_3}\,)$ *in case* A), *and*
$k_1 = \mathbb{Q}(\sqrt{d_3}\,)$ *and* $k_2 = \mathbb{Q}(\sqrt{d_1 d_2}\,)$ *in case* B).

Then $h_2(L) = \frac{1}{4} h_2(k) h_2(K_1) h_2(K_2)$ *unless possibly when* $d_3 = -4$ *in case* B).

Proof. Observe that $\upsilon = 0$ in case A) and B); Kuroda's class number formulas for L/k_1 and L/k_2 gives

$$h_2(L) = \frac{q_1 h_2(K_1)^2 h_2(K)}{2h_2(k_1)^2} = \frac{q_2 h_2(K_2)^2 h_2(K)}{4h_2(k_2)^2}$$

in case A) and

$$h_2(L) = \frac{q_1 h_2(K_1)^2 h_2(K)}{4h_2(k_1)^2} = \frac{q_2 h_2(K_2)^2 h_2(K)}{2h_2(k_2)^2}$$

in case B)). Multiplying them together and plugging in the class number formula for $K/\mathbb{Q}$ yields

$$h_2(L)^2 = \frac{q_1\, q_2}{8} \frac{h_2(K_1)^2\, h_2(K_2)^2\, h_2(k)^2}{h_2(k_1)^2\, h_2(k_2)^2}.$$

Now $h_2(k_1) = 1$, $h_2(k_2) = 2$ and $q_1 q_2 = 2$ (by Proposition 6), and taking the square root we find $h_2(L) = \frac{1}{4} h_2(k) h_2(K_1) h_2(K_2)$ as claimed. □

5. Classification.

In this section we apply the results obtained in the last few sections to give a proof for Theorem 1.

Proof of Theorem 1. Let L be one of the two cyclic quartic unramified extensions of k, and let N be the subgroup of $\mathrm{Gal}(k^2/k)$ fixing L. Then N satisfies the assumptions of Proposition 1, thus there are only the following possibilities:

$d(G')$	$h_2(L)$	$h_2(K_1)h_2(K_2)$
1	2^m	2
2	2^{m+1}	4
≥ 3	$\geq 2^{m+2}$	≥ 8

Here, the first two columns follow from Proposition 1, the last (which we do not claim to hold if $d_3 = -4$ in case B)) is a consequence of the class number formula of Proposition 9. In particular, we have $d(G') \geq 3$ if one of the class numbers $h_2(K_1)$ or $h_2(K_2)$ is at least 8. Therefore it suffices to examine the cases $h_2(K_2) = 2$ and $h_2(K_2) = 4$ (recall from above that $h_2(K_2)$ is always even).

We start by considering case A); it is sufficient to show that $h_2(K_1)h_2(K_2) \neq 4$. We now apply Proposition 5; notice that we may do so by the proof of Proposition 8.

a) If $h_2(K_2) = 2$, then $\#\kappa_2 = 2$ by Proposition 5, hence $q_2 = 2$ by Proposition 7 and then $q_1 = 1$ by Proposition 6. The class number formulas in the proof of Proposition 9 now give $h_2(K_1) = 1$ and $h_2(L) = 2^m$.

It can be shown using the ambiguous class number formula that $\mathrm{Cl}_2(K_1)$ is trivial if and only if ε_1 is a quadratic nonresidue modulo the prime ideal over d_2 in k_1; by Scholz's reciprocity law, this is equivalent to $(d_1/d_2)_4(d_2/d_1)_4 = 1$, and this agrees with the criterion given in [**1**].

b) If $h_2(K_2) = 4$, we may assume that $\mathrm{Cl}_2(K_2) = (4)$ from Proposition 8.b). Then $\#\kappa_2 = 2$ by Proposition 5, $q_2 = 2$ by Proposition 7 and $q_1 = 1$ by Proposition 6. Using the class number formula we get $h_2(K_1) = 2$ and $h_2(L) = 2^{m+2}$.

Thus in both cases we have $h_2(K_1)h_2(K_2) \neq 4$, and by the table at the beginning of this proof this implies that $\operatorname{rank} \mathrm{Cl}_2(k^1) \neq 2$ in case A).

Next we consider case B); here we have to distinguish between $d_3 \neq -4$ (case B_1) and $d_3 = -4$ (case B_2).

Let us start with case B_1).

a) If $h_2(K_2) = 2$, then $\#\kappa_2 = 2$, $q_2 = 2$ and $q_1 = 1$ as above. The class number formula gives $h_2(K_1) = 2$ and $h_2(L) = 2^{m+1}$.

b) If $\mathrm{Cl}_2(K_2) = (4)$ (which we may assume without loss of generality by Proposition 8.b)) then $\#\kappa_2 = 2$, $q_2 = 2$ and $q_1 = 1$, again exactly as above. This implies $h_2(K_1) = 4$ and $h_2(L) = 2^{m+3}$.

Finally, consider case B_2).

Here we apply Kuroda's class number formula (see [**10**]) to L/k_1, and since $h_2(k_1) = 1$ and $h_2(K_1) = h_2(K_1')$, we get $h_2(L) = \frac{1}{2} q_1 h_2(K_1)^2 h_2(k) = 2^m q_1 h_2(K_1)^2$. From $K_2 = k_2(\sqrt{\varepsilon}\,)$ (for a suitable choice of L; the other possibility is $\widetilde{K}_2 = k_2(\sqrt{d_2\varepsilon}\,))$, where ε is the fundamental unit of k_2, we deduce that the unit ε, which still is fundamental in k, becomes a square in L, and this implies that $q_1 \geq 2$. Moreover, we have $K_1 = k_1(\sqrt{\pi\lambda}\,)$, where $\pi, \lambda \equiv 1 \bmod 4$ are prime factors of d_1 and d_2 in $k_1 = \mathbb{Q}(i)$, respectively. This shows that K_1 has even class number, because $K_1(\sqrt{\pi}\,)/K_1$ is easily seen to be unramified.

Thus $2 \mid q_1$, $2 \mid h_2(K_1)$, and so we find that $h_2(L)$ is divisible by $2^m \cdot 2 \cdot 4 = 2^{m+3}$. In particular, we always have $d(G') \geq 3$ in this case.

This concludes the proof. □

The referee (whom we'd like to thank for a couple of helpful remarks) asked whether $h_2(K) = 2$ and $h_2(K) > 2$ infinitely often. Let us show how to prove that both possibilities occur with equal density in case B_1).

Before we can do this, we have to study the quadratic extensions K_1 and $\widetilde{K}_1$ of k_1 more closely. We assume that $d_2 = p$ and $d_3 = r$ are odd primes in the following, and then say how to modify the arguments in the case $d_2 = 8$ or $d_3 = -8$. The primes p and r split in k_1 as $p\mathcal{O}_1 = \mathfrak{p}\mathfrak{p}'$ and $r\mathcal{O}_1 = \mathfrak{r}\mathfrak{r}'$. Let h denote the odd class number of k_1 and write $\mathfrak{p}^h = (\pi)$ and $\mathfrak{r}^h = (\rho)$ for primary elements π and ρ (this is can easily be proved directly, but it is also a very special case of Hilbert's first supplementary law for quadratic reciprocity in fields K with odd class number h (see [**7**]): If $\mathfrak{a}^h = \alpha\mathcal{O}_K$ for an ideal $\mathfrak{a}$ with odd norm, then α can be chosen primary (i.e., congruent to a square mod $4\mathcal{O}_K$) if and only if $\mathfrak{a}$ is primary (i.e., $[\varepsilon/\mathfrak{a}] = +1$ for all units $\varepsilon \in \mathcal{O}_K^\times$, where $[\,\cdot\,/\,\cdot\,]$ denotes the quadratic residue symbol in K)). Let $[\,\cdot\,/\,\cdot\,]$ denote the quadratic residue symbol in k_1. Then $[\pi/\rho][\pi'/\rho] = [p/\rho] = (p/r) = -1$, so we may choose the conjugates in such a way that $[\pi/\rho] = +1$ and $[\pi'/\rho] = [\pi/\rho'] = -1$.

Put $K_1 = k_1(\sqrt{\pi\rho})$ and $\widetilde{K}_1 = k_1(\sqrt{\pi\rho'})$; we claim that $h_2(\widetilde{K}_1) = 2$. This is equivalent to $h_2(\widetilde{L}_1) = 1$, where $\widetilde{L}_1 = k_1(\sqrt{\pi}, \sqrt{\rho'})$ is a quadratic unramified extension of $\widetilde{K}_1$. Put $\widetilde{F}_1 = k_1(\sqrt{\pi})$ and apply the ambiguous class number formula to $\widetilde{F}_1/k_1$ and $\widetilde{L}_1/\widetilde{F}_1$: Since there is only one ramified prime in each of these two extensions, we find $\mathrm{Am}(\widetilde{F}_1/k_1) = \mathrm{Am}(\widetilde{L}_1/\widetilde{F}_1) = 1$; note that we have used the assumption that $[\pi/\rho'] = -1$ in deducing that $\mathfrak{r}'$ is inert in $\widetilde{F}_1/k_1$.

In our proof of Theorem 1 we have seen that there are the following possibilities when $h_2(K_2) \mid 4$:

q_2	$\mathrm{Cl}_2(K_2)$	q_1	$h_2(K_1)$	$\widetilde{q}_2$	$\mathrm{Cl}_2(\widetilde{K}_2)$	$h_2(L)$
2	(2)	1	2	2	(2)	2^{m+1}
2	(4)	1	4	?	(2, 2)	2^{m+3}

In order to decide whether $\widetilde{q}_2 = 1$ or $\widetilde{q}_2 = 2$, recall that we have $h_2(K_1) = 4$; thus $\widetilde{K}_1$ must be the field with 2-class number 2, and this implies $h_2(\widetilde{L}) = 2^{m+2}$ and $\widetilde{q}_2 = 1$. In particular we see that $4 \mid h_2(K_2)$ if and only if $4 \mid h_2(K_1)$ as long as $K_1 = k_1(\sqrt{\pi\rho})$ with $[\pi/\rho] = +1$.

The ambiguous class number formula shows that $\mathrm{Cl}_2(K_1)$ is cyclic, thus $4 \mid h_2(K_1)$ if and only if $2 \mid h_2(L_1)$, where $L_1 = K_1(\sqrt{\pi})$ is the quadratic unramified extension of K_1. Applying the ambiguous class number formula to L_1/F_1, where $F_1 = k_1(\sqrt{\pi})$, we see that $2 \mid h_2(L_1)$ if and only if $(E : H) = 1$. Now E is generated by a root of unity (which always is a norm residue at primes dividing $r \equiv 1 \bmod 4$) and a fundamental unit ε. Therefore $(E : H) = 1$ if and only if $\{\varepsilon/\mathfrak{R}_1\} = \{\varepsilon/\mathfrak{R}_2\} = +1$, where $\mathfrak{r}\mathcal{O}_{F_1} = \mathfrak{R}_1\mathfrak{R}_2$ and where $\{\cdot/\cdot\}$ denotes the quadratic residue symbol in F_1. Since $\{\varepsilon/\mathfrak{R}_1\}\{\varepsilon/\mathfrak{R}_2\} = [\varepsilon/\mathfrak{r}] = +1$, we have proved that $4 \mid h_2(K_1)$ if and only if the prime ideal $\mathfrak{R}_1$ above $\mathfrak{r}$ splits in the quadratic extension $F_1(\sqrt{\varepsilon})$. But if we fix p and q, this happens for exactly half of the values of r satisfying $(p/r) = -1$, $(q/r) = +1$.

If $d_2 = 8$ and $p = 2$, then $2\mathcal{O}_{k_1} = \mathfrak{z}\mathfrak{z}'$, and we have to choose $\mathfrak{z}^h = (\pi)$ in such a way that $k_1(\sqrt{\pi})/k_1$ is unramified outside $\mathfrak{p}$. The residue symbols $[\alpha/\mathfrak{z}]$ are defined as Kronecker symbols via the splitting of $\mathfrak{z}$ in the quadratic extension $k_1(\sqrt{\alpha})/k_1$. With these modifications, the above arguments remain valid.

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