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# ON THE SIGNATURE OF CERTAIN SPHERICAL REPRESENTATIONS

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**In this work we prove for all simple real groups of classical type, except for $SO^*(n)$ and $Sp(p,q)$, that if $\nu \in \mathfrak{a}^*$ does not belong to $C_\rho$, the convex hull of the Weyl group orbit of $\rho$, then the signature of the Hermitian form attached to the irreducible subquotient of the principal spherical series corresponding to $\nu$, with integral infinitesimal character, is indefinite on $K$-types in $\mathfrak{p}$.**

## 0. Introduction.

Let $G$ be a real semisimple Lie group. Let $\mathfrak{g}_0$ be the Lie algebra of $G$. We will denote the complexification of any vector space $V_0$ by $V$ and its dual by $V^*$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a fixed Cartan decomposition corresponding to the Cartan involution $\theta$ of $\mathfrak{g}_0$. Let $K$ be the corresponding maximal compact subgroup of $G$. Let us fix $T \subset K$ a maximal torus. Define $S \subset \mathfrak{t}^*$ as the set of weights of $T$ that are sums of distinct non-compact roots.

We say that $(\mu, V_\mu)$ an irreducible representation of $K$ is *unitarily small* if the weights of $\mu$ lie in the convex hull of $S$. Let us state the following:

**Salamanca-Vogan Conjecture.** *Suppose $X$ is an irreducible Hermitian $(\mathfrak{g}, K)$-module containing a $K$ type in $S$. Then:*

(1) *If $X$ is unitary, then the real part of the infinitesimal character belongs to the convex hull of the Weyl group orbit $W\cdot\rho$, where $\rho$ is the semi-sum of positive roots.*
(2) *If $X$ is not unitarizable, then the invariant hermitian form must be indefinite on unitarily small $K$-types.*

Using this conjecture the classification of unitary representation can be reduced to the unitarily small case.

If $X$ is a spherical representation, the statement (1) is true by [**HJ**]. In order to move towards (2) we may assume that real part of the infinitesimal character does not belongs to $W \cdot \rho$, and the hope is that if this holds, the invariant hermitian form is negative definite in the $K$-types in $\mathfrak{p}$. In this work we prove for all simple real groups of classical type, except for $SO^*(n)$ and $Sp(p,q)$, that if $\nu \in \mathfrak{a}^*$ does not belong to $C_\rho$, then the signature of the Hermitian form attached to the irreducible subquotient of the principal spherical series corresponding to $\nu$, with integral infinitesimal character, is indefinite on $K$-types in $\mathfrak{p}$.

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## 1. Notation and General Results.

Recall that the complexification of a Lie algebra $\mathfrak{g}_0$ will be denoted by $\mathfrak{g}$. Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$ be a maximal Abelian subspace. Let $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0) = \Sigma$ be the corresponding set of restricted roots and let $\overline{\Sigma}(\mathfrak{g}_0, \mathfrak{a}_0) = \overline{\Sigma}$ be the reduced restricted roots. If $\alpha \in \mathfrak{a}^*$, then we will denote the corresponding weight space in $\mathfrak{g}_0$ by $\mathfrak{g}_0^\alpha$ and let $m_\alpha$ the dimension of this subspace. For a choice of positive roots $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0) = \Sigma^+$ we have $\Pi(\mathfrak{g}_0, \mathfrak{a}_0) = \Pi$ the set of simple roots. Let $W \simeq N_K(\mathfrak{a}_0)\backslash Z_K(\mathfrak{a}_0)$ be the corresponding Weyl group. If $w \in W$, then we define $\Sigma^+(w) = \{\alpha \in \Sigma^+ \ : \ w\alpha \notin \Sigma^+\}$. We also denote the longest element in the Weyl group by $w_0$. If $\Omega \subseteq \Sigma^+$ then we will say that $\nu \in \mathfrak{a}^*$ is it positive (resp. *negative*) with respect to $\Omega$, if $\mathrm{Re}\langle\alpha, \nu\rangle$ is positive (resp. negative) or zero for all $\alpha \in \Omega$.

If $X$ is an admissible representation of $G$, we will also denote the corresponding $(\mathfrak{g}, K)-$module by X.

Recall that a $(\mathfrak{g}, K)-$module $(\pi, H_\pi)$ is called *spherical* if the trivial $K$-type occurs in the restriction of $(\pi, H_\pi)$ to $K$, i.e., $H_\pi$ contains a non-trivial $K$-fixed vector. Then, we have the following:

**Theorem 1.1.** *The irreducible spherical $(\mathfrak{g}, K)-$modules $(\pi, H_\pi)$ are in one-to-one correspondence with the $W$-orbits in $\mathfrak{a}^*$.*

A proof of this Theorem appears in [**K**, **BJ2**].

This correspondence can be realized as follows: Set $\mathfrak{n}_0 = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_0^\alpha$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$. Let P=MAN the corresponding minimal parabolic subgroup of $G$. If $\nu \in \mathfrak{a}^*$, define

$$I_P^G(\nu) = \mathrm{Ind}_{MAN}^G(1 \otimes \nu \otimes 1)$$

where the right-hand-side is the space

$$\{f \in C^\infty(G, \mathbb{C}) \ : \ \forall man \in MAN \ \text{ s.t.} \quad f(gman) = a^{-\nu-\rho} f(g), f \ \text{ is } \ K-\text{finite}\}$$

and the $\mathfrak{g}$- action on this induced module is the left regular action, i.e.,:

$$(X \cdot f)(g) = \frac{d}{dt} f(\exp(-t \cdot X)g)_{|t=0}, \quad g \in G,\ X \in \mathfrak{g},\ f \in I_p^G(\nu)$$

and

$$(k \cdot f)(g) = f(k^{-1}g), \quad k \in K \text{ and } f \in I_P^G(\nu)$$

It is easy to see that $I_P^G(\nu)$ is a Harish-Chandra module of finite length. Observe that by the Iwasawa decomposition $G = KAN$ of $G$ we have

$$I_P^G(\nu)_{|_K} = K - \text{finite part of } \operatorname{Ind}_M^K(1)$$

and hence by Frobenius reciprocity

$$\dim I_P^G(\nu)^K = \dim \operatorname{Ind}_M^K(1)^K = \dim \operatorname{Hom}(\mathbb{C},\mathbb{C}) = 1. \tag{1.2}$$

So $I_P^G(\nu)$ is a spherical $(\mathfrak{g},K)$-module, and by (1.1) there exists a unique irreducible composition factor $J^G(\nu) = J_P^G(\nu)$ containing the trivial $K$-type in $I_P^G(\nu)$. It is well-known that $J^G(\nu) \simeq J^G(\nu')$ if and only if there exists $w \in W$ such that $\nu = w \cdot \nu'$, in particular $J^G(\nu)$ does not depend on the choice of the minimal parabolic subgroup containing $MA$. The $(\mathfrak{g},K)$-modules $I_P^G(\nu)$ are called *spherical principal series representations.*

We denote the set of $\nu \in \mathfrak{a}^*$ such that $J^G(\nu)$ has integral infinitesimal character by $\mathfrak{a}^*_{\text{int}}$.

Let $X$ be a $(\mathfrak{g},K)$-module. Then we say that $X$ admits an invariant Hermitian form if there exists a non-zero map $\omega = \omega_G : X \times X \longrightarrow \mathbb{C}$ such that:

(1) $\omega$ is linear in the first factor and conjugate linear in the second factor.
(2) $\omega(x,y) = \overline{\omega(y,x)}, \quad x,y \in X.$
(3) $\omega(k\cdot x, k\cdot y) = \omega(x,y), \quad x,y \in X, \quad k \in K.$
(4) $\omega(H\cdot x, y) = \omega(x, -\overline{H}\cdot y), \quad x,y \in X, \quad H \in \mathfrak{g}$, where $\overline{H}$ stand for complex conjugation of $\mathfrak{g}$ respect to $\mathfrak{g}_0$.

If $\mu, \mu'$ are two different $K$-types, then, (3) implies that $\omega(X^\mu, X^{\mu'}) = 0$. Hence $\omega$ is completely described by its restriction to the $K$-isotopic spaces on $X$.

Let $\mu \in \hat{K}$. Then $X^\mu \simeq \operatorname{Hom}_K(V_\mu, X) \otimes V_\mu$ and $\omega$ induces a Hermitian form, $\omega^\mu$, on the first factor.

**Definition 1.3.** Let $(p(\mu), q(\mu)) := (p_X(\mu), q_X(\mu)) := (p_X^G(\mu), q_X^G(\mu))$ be the *signature* of $\omega^\mu$, i.e., $p(\mu)$ (respective $q(\mu)$) is the sum of the strictly positive (respective negative) eigenspaces of $\omega^\mu$.

Let us see when there exists this Hermitian form in $J(\nu)$. This is the one being used in this article. Choose a minimal parabolic subgroup $P = MAN$ in $G$. Consider $w \in W$ and $\nu$ positive with respect to $\Sigma^+(w)$. Then there exist an intertwining operator $\Psi(w) : \mathrm{I}_P^G(\nu) \longrightarrow \mathrm{I}_P^G(w\nu)$ so that $\Psi(w)$ is an isomorphism on the trivial $K$-type. If $w = w_0$ then the image is isomorphic to $J(\nu)$.

Recall that there is a natural non-zero Hermitian paring

$$\mathrm{I}_P^G(\nu) \times \mathrm{I}_P^G(-\overline{\nu}) \xrightarrow{\langle\,,\rangle} \mathbb{C}.$$

Here $\mathrm{I}^G_{MAN}(\nu)$ is naturally identified with the $K$-finite part of $L^2(K/M)$ and with this identification, $\langle\, ,\, \rangle$ is the inner product on $L^2(K/M)$. Now suppose there exists $w \in W$ such that $-\overline{\nu} = w\nu$ and $\nu$ is positive with respect to $MAN$. Then we get the Hermitian pairing

$$\begin{aligned} \mathrm{I}^G_P(\nu) \times \mathrm{I}^G_P(\nu) &\xrightarrow{\omega_\nu} \mathbb{C} \\ (v_1\ ,\ v_2) \qquad &\longrightarrow \langle v_1, \Psi(w) v_2\rangle. \end{aligned}$$

Since $\Psi(w\nu, w_0 w^{-1}) : \mathrm{I}^G_P(w\nu) \longrightarrow \mathrm{I}^G_P(w_0\nu)$ is an isomorphism, we get that $\mathrm{I}^G_{MAN}(\nu)/\mathrm{Rad}\omega \simeq J^G(\nu)$. Hence $\omega$ induces a Hermitian form on $J(\nu)$. The same argument shows that $J(\nu)$ admits an invariant Hermitian form if and only if $-\overline{\nu}$ is $W$-conjugate to $\nu$.

Now, we will define the set of K-types where we will work. So, let's define as the $\mathfrak{p}_0$-*representation of* $K$ the homomorphism

$$\begin{aligned} K &\longrightarrow GL(\mathfrak{p}_0) \\ k &\longmapsto Ad(k)_{|_{\mathfrak{p}_0}}, \end{aligned}$$

and the $\mathfrak{p}$-*representation of* $K$ the complexification of $\mathfrak{p}_0$-representation.

Recall that if $\mathfrak{g}_0$ is simple, then $\mathfrak{p}$ is either irreducible or it is a direct sum of two inequivalent irreducible representations. Consider the following set of $K$-types:

$$\mathfrak{p} = \{\mu \in \hat{K} \ :\ \mathrm{Hom}_K(\mu, \mathfrak{p}) \neq 0\}. \tag{1.4}$$

Through this work, we will consider the signature over this set of $K$-types.

Finally we will define $C_\rho$ as the convex hull of points $w\rho$ with $w \in W$. This set is also characterized in terms of positive roots by the following:

**Proposition 1.5.** *The set $C_\rho$ coincides with the set of all weights of the form*

$$r = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha \qquad \textit{where } -1/2 \leq c_\alpha \leq 1/2,$$

*and each $\alpha$ is counted with multiplicity.*

For a proof of this result we refer to [**SV**].

## 2. Problem.

Take $\nu \in \mathfrak{a}^*_{int}$ such that $J(\nu)$ has integral infinitesimal character and admits an invariant Hermitian form. Then, we will prove the following

**Theorem 2.1.** *If $\nu \notin C_\rho$ then $q_\mu(\nu) > 0$, where $\mu$ is a $K$-type in $\mathfrak{p}$ and $q$ as in Definition 1.3.*

The main problem here is that this Hermitian form has no known general expression. However, Bang-Jensen [**BJ**], proves some useful Theorems, that give conditions for an invariant Hermitian form on an irreducible spherical representation, with integral infinitesimal character, to be positive definite on the $K$-types $\mu \in \mathfrak{p}$, in terms of Langlands data, for the simple groups of classical type except $SO^*(n)$ and $Sp(p,q)$.

These Theorems will be the main tool we will use in order to prove Theorem 2.1. This will be done case by case. Bang-Jensen's results are used to get an explicit characterization of $\nu \in \mathfrak{a}^*$, that is crucial for the proof of Theorem 2.1.

## 3. Case $SL(n,\mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Let $\mu$ be a $K$-type in $\mathfrak{p} \cap [\mathfrak{g},\mathfrak{g}]$. We identify $\mathfrak{a}^*$ with $\mathbb{C}^n$ such that $\Sigma(\mathfrak{g}_0,\mathfrak{a}_0) = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$. Then $\Sigma^+(\mathfrak{g}_0,\mathfrak{a}_0) = \{(e_i - e_j) \mid 1 \leq i < j \leq n\}$. Put $r = \dim_{\mathbb{R}} \mathbb{F}$. Then, $\dim \mathfrak{g}_0^\alpha = r$. Then $\mathfrak{a}^*_{\text{int}} = \{\nu \in \mathbb{C}^n \mid \nu_i \equiv 0 \mod r\mathbb{Z},\ i = 1,\dots,n\}$. For $\nu \in \mathfrak{a}^*_{\text{int}}$ and $i \in \mathbb{Z}$ we define

$$R(i) := R_\nu(i) := \#\{\nu_j \ :\ \nu_j = r \cdot i\}.$$

Now if $\nu \in \mathfrak{a}^*_{\text{int}}$, then $J^G(\nu)$ admits an invariant Hermitian form if and only if $R_\nu(i) = R_\nu(-i)$ for all $i$.

Then we have the following:

**Theorem 3.1.** *Assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Suppose $\nu \in \mathfrak{a}^*_{\text{int}}$ and $J^G(\nu)$ admits an invariant hermitian form, $\omega$. Then $\omega$ is positive definite on $J^G(\nu)^\mu$ if and only if $R_\nu(i+1) \leq R_\nu(i)$ for all $i \geq 0$.*

*Proof.* See Theorem 5.2 in [**BJ**].

We will now prove the following proposition.

**Proposition 3.2.** *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If $R_\nu(i+1) \leq R_\nu(i)$ for all $i \geq 0$ then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or $0$.*

*Proof.* We will give the proof only for $Sl(n,\mathbb{R})$. The case $Sl(n,\mathbb{C})$ follows immediately from the definition of $\mathfrak{a}^*_{\text{int}}$ and the fact that for this group, each root has multiplicity 2. Take $\nu \in \mathfrak{a}^*_{\text{int}}$ such that:

(1) $R_\nu(i+1) \leq R_\nu(i)$ for all $i \geq 0$,
(2) $R_\nu(i) = R_\nu(-i)$ for all $i$;

thus, we are assuming that $\omega$ is positive definite on $J^G(\nu)^\mu$. By definition, $C_\rho$ is stable by Weyl group action, hence by (1) and (2) we can consider,

(3.3)

$$\nu = (k,\dots,k,k-1,\cdots,k-1,\cdots\cdots,1,\cdots,1,0,\cdots,0,-1,\cdots,-1,\cdots \\ \cdots\cdots,-(k-1),\cdots,-(k-1),-k,\cdots,-k)$$

where $k \leq [\frac{n}{2}] - 1$.

In order to complete this proof, we will need the following:

**Lemma 3.4.** *Take $\nu$ as above and consider $R(0) \geq 1$ (otherwise, $\nu \equiv 0$ by (1)). Then*

$$2(k-i) - 2 < n - \left( 2 \sum_{j=0}^{i} R(k-j) \right) + 1 \qquad \text{where } i = 0, \dots, k-1. \tag{3.5}$$

*Proof.* Since $n = 2\sum_{j=0}^{k-1} R(k-j) + R(0)$ we have

$$\begin{aligned} 2(k-i) - 2 \leq 2 \sum_{j=i+1}^{k-1} R(k-j) + R(0) &= n - 2\sum_{j=0}^{i} R(k-j) \\ &< n - 2\sum_{j=0}^{i} R(k-j) + 1. \end{aligned}$$

Then, (3.5) has been proved.

*Proof of Proposition* 3.2 (Continuation). Let define

$$\mathcal{T}(m) = \begin{cases} \sum_{t=0}^{m-1} R(k-t), & \text{if } m > 0 \\ 0, & \text{if } m = 0 \end{cases}. \tag{3.6}$$

Now, it is easy to check, that Lemma 3.4 allows us to rewrite $\nu$ as follows:

$$\begin{aligned} \nu = &\frac{1}{2} \sum_{m=0}^{k-1} \sum_{j=1}^{R(k-m)} \Bigg\{ (e_{\mathcal{T}(m)+j} - e_{n-\mathcal{T}(m)-j+1}) \\ &+ \sum_{s=\mathcal{T}(m)+j}^{2(k-m)-2+\mathcal{T}(m)+j} [(e_{\mathcal{T}(m)+j} - e_{s+1}) + (e_{s+1} - e_{n-\mathcal{T}(m)-j+1})] \Bigg\}. \end{aligned}$$

And so, Proposition 3.2 is proved. □

*Proof of Theorem* 2.1 for $Sl(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$. This is immediate from Proposition 3.2, Theorem 3.1 and Proposition 1.5. □

## 4. Case $Sp(2n, \mathbb{F})$, $n > 2$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Now, assume that $G = Sp(2n, \mathbb{R})$ or $G = Sp(2n, \mathbb{C})$. In this case, $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ is of type $C_n$, and identifying $\mathfrak{a} \simeq \mathbb{C}^n$ we have that the restricted roots are $\{\pm e_i \pm e_j, \pm 2e_l\}$. Put $r = \dim_{\mathbb{R}} \mathbb{F}$.

Here, $J^G(\nu)$ has integral infinitesimal character if and only if $\nu_i \in r\mathbb{Z}$. Take $\nu \in \mathfrak{a}^*$, and replace it by a Weyl group conjugate, then we may assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0 \tag{4.1}$$

and with this assumption define

$$R(i) = R_\nu(i) = \#\{j \ : \ \nu_j = ri\}, \quad i \geq 0.$$

Then we can state the following:

**Theorem 4.2.** *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

(1) *$R(i+1) \leq R(i) + 1$, for $i \geq 1$;*
(2) *If $R(i+1) = R(i) + 1$, for $i \geq 1$, then $R(i)$ is odd;*
(3) *$R(1) \leq 2R(0) + 2$.*

*Proof.* See Theorem 8.4 in [**BJ**].

Now, we can prove the following:

**Proposition 4.3.** *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(3) above are satisfied then $\nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or $0$.*

*Proof.* Let us assume that $G = Sp(2n, \mathbb{R})$ and so $r = 1$. It follows by Condition (2) in Theorem 4.2, that if $R(j) = 0$ for any $j \geq 1$ then $R(j+1) = 0$. This implies that if $R(1) \neq 0$ then

$$\nu = (k, \ldots, k, k-1, \ldots, k-1, \ldots\ldots, 1, \ldots, 1, 0, \ldots, 0)$$

or

$$\nu = (k, \ldots, k, k-1, \ldots, k-1, \ldots\ldots, 1)$$

where $k \leq n$. Now, since $n = \sum_{j=0}^{k} R(k-j)$, we have that

$$k - j \leq n - \sum_{i=0}^{j} R(k-i) \tag{4.4}$$

with $j = 0, \ldots, k-2$. Let us define $\mathcal{T}(m)$ as in (3.6), and Inequality 4.4 allows us to write $\nu$ as follows

$$\nu = \frac{1}{2} \sum_{m=0}^{k-2} \sum_{j=1}^{R(k-m)} \Bigg\{ 2e_{\mathcal{T}(m)+j} + \sum_{s=\mathcal{T}(m)+j+1}^{(k-m)-1+\mathcal{T}(m)+j} \big[\, (e_{\mathcal{T}(m)+j} - e_s) + (e_{\mathcal{T}(m)+j} + e_s) \,\big] \Bigg\} + \frac{1}{2} \sum_{j=1}^{R(1)} 2e_{\mathcal{T}(k-1)+j}.$$

The case $G = Sp(2n, \mathbb{C})$ follows from this using that $r = 2$ and the multipilcities of the positive roots. Hence, in this way we have proved Proposition 4.3. □

Now, we can give the:

*Proof of Theorem* 2.1 for $Sp(2n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$. It follows from Proposition 4.3, Theorem 4.2 and Proposition 1.5. □

## 5. Case $SO(p, q)$, $p > q + 1$ and $SU(p, q)$, $p > q$.

Let us assume that $G = SO(p, q)$, $p > q + 1$ or $G = SU(p, q)$, $p > q$; and as usual identify $\mathfrak{a} \simeq \mathbb{C}^q$. Then the restricted roots become $\{\pm e_i \pm e_j, \pm e_l\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(p, q) \\ 2, & \text{if } G = SU(p, q). \end{cases}$$

Define, $\epsilon = 0, 1$ by $\epsilon \equiv p - q - 1 + r \mod 2\mathbb{Z}$. In these cases, $J(\nu)$ has integral infinitesimal character if and only if

$$2\nu_i \equiv r\epsilon \mod 2r\mathbb{Z}, \qquad i = 1, \ldots, q.$$

We can replace $\nu$ by a Weyl group conjugate, and assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0. \tag{5.1}$$

With this assumption we define

$$R(i) = R_\nu(i) = \#\left\{j \; : \; \nu_j = r\left(i + \frac{\epsilon}{2}\right)\right\}, \quad i \geq 1 \tag{5.2}$$

and

$$R(0) = R_\nu(0) = (2 - \epsilon)\#\left\{j \; : \; \nu_j = r\left(\frac{\epsilon}{2}\right)\right\}. \tag{5.3}$$

Take $s = \frac{p-q-1+r-\epsilon}{2}$. Now, we can state the following:

**Theorem 5.4.** *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

1. *$R(i + 1) \leq R(i) + 1$, for $i \geq 0$, $i \neq s - 1$;*
2. $$R(i + 1) = R(i) + 1, \;\; \textit{for } i \geq 0, i \neq s - 1 \textit{ then } \begin{cases} R(i) \textit{ is even}, & \textit{if } i < s \\ R(i) \textit{ is odd}, & \textit{if } i > s; \end{cases}$$
3. *$R(s) \leq R(s - 1) + 2$;*
4. *$R(s) = R(s - 1) + 2$, then $R(s - 1)$ is even.*

*Proof.* Cf. Theorem 6.2 in [**BJ**].

Now, we can prove the following:

**Proposition 5.5.** *Consider $\nu \in \mathfrak{a}^*_{int}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(4) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or $0$.*

*Proof.* Assume that $r = 1$ and $\epsilon = 0$, in other words that $G = SO(p, q)$ with $p - q$ even. So $s = \frac{p-q}{2} \in \mathbb{Z}$. In this setting $J(\nu)$ has integral infinitesimal character if and only if $\nu_i \in \mathbb{Z}$ and

$$R(i) = \#\{n \,:\, \nu_n = i\} \quad i \geq 1$$

and

$$R(0) = 2\#\{n \,:\, \nu_n = 0\}.$$

Then by conditions (1)-(4), Formula (5.1) and assuming $R(s) \neq 0$ and there exists $j < s$ such that $R(j) \neq 0$ we can consider $\nu$ as follows

$$\nu = (s+k, \ldots, s+k, s+(k-1), \ldots, s+(k-1), \ldots\ldots, s, \ldots, s, \nu_j, \ldots, \nu_j, \ldots\ldots, \nu_1, \ldots \nu_1) \tag{5.6}$$

with $0 \leq \nu_i < s$, $i = 1, \ldots, j$.

In order to complete the proof of this proposition, we will need the following:

**Lemma 5.7.** *Take $\nu$ as above. Then*

$$k - i + 1 \leq q - \sum_{r=0}^{i} R(s + k - r).$$

*Proof.* Since $R(s) \neq 0$ we have, by condition (2) in Theorem 5.4 $R(s+m) \neq 0$, for $m = 1, \ldots, k$. Hence, using that $q = \sum_{r=0}^{k} R(s+k-r) + \sum_{p=0}^{j-1} R(\nu_{j-p})$ we have

$$\begin{aligned} k - i + 1 &\leq \sum_{r=i+1}^{k} R(s + k - r) + \sum_{p=0}^{j-1} R(\nu_{j-p}) \\ &= q - \sum_{r=0}^{i} R(s + k - r). \end{aligned}$$

So, the proof of the lemma is complete. □

*Proof Proposition* 5.5. (Cont.) Recalling that each root $e_i \pm e_j$ has multiplicity 1 and $e_l$ has multiplicity $p - q = 2s$, and defining

$$\mathcal{T}(m) = \begin{cases} \sum_{i=0}^{m-1} R(s + k - i), & \text{if } m \geq 1 \\ 0, & \text{if } m = 0 \end{cases}$$

and

$$\mathcal{S}(n) = \begin{cases} \sum_{i=0}^{n-1} R(\nu_{j-i}), & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases}$$

we can, since is easy to check that $k-m \le q - \sum_{j=0}^{m} R(s+k-j)$ for $m = 0, \ldots, k-1$ and $\nu_i < s$ for $i = 1, \ldots, j$, rewrite $\nu$ as follows

$$\nu = \left( \sum_{m=0}^{k-1} \sum_{r=1}^{R(s+k-m)} \frac{1}{2} \left\{ \sum_{t=1}^{k-m} [(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})] \right\} + s\, e_{\mathcal{T}(m)+r} \right) + \sum_{r=1}^{R(s)} s e_{\mathcal{T}(s-1)+r}$$
$$+ \sum_{n=0}^{j-1} \sum_{p=1}^{R(\nu_{j-n})} \nu_{j-n} e_{\mathcal{T}(k+1)+\mathcal{S}(n)+p}.$$

Now let us keep $r = 1$ and consider $\epsilon = 1$, i.e., $G = SO(p,q)$, but now, $p - q$ is odd. Here $s = \frac{p-q-1}{2}$ and $J(\nu)$ has integral infinitesimal character if and only if $\nu_n = \frac{2l_n+1}{2}$ with $l_n \in \mathbb{Z}_+$ and $n = 1, \ldots, q$. Then, we have

$$R(i) = \# \left\{ n \,:\, \nu_n = \frac{2i+1}{2} \right\} \quad i \ge 1$$

and

$$R(0) = \# \left\{ n \,:\, \nu_n = \frac{1}{2} \right\}.$$

Again, as before, we can assume that

$$\nu = \left( \frac{2(s+k)+1}{2}, \ldots, \frac{2(s+k)+1}{2}, \frac{2(s+(k-1))+1}{2}, \ldots, \frac{2(s+(k-1))+1}{2}, \ldots, \frac{2s+1}{2}, \ldots, \frac{2s+1}{2}, \frac{2l_j+1}{2}, \ldots, \frac{2l_j+1}{2}, \ldots\ldots, \frac{2l_1+1}{2}, \ldots, \frac{2l_1+1}{2} \right) \tag{5.8}$$

with $0 \le l_i < s$, $i = 1, \ldots, j$. By Lemma 5.7 and since $l_i + \frac{1}{2} < s$, we can rewrite $\nu$ as follows

$$\nu = \left( \sum_{m=0}^{k} \sum_{r=1}^{R(s+k-m)} \frac{1}{2} \sum_{t=0}^{k-m} [(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t+1}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t+1})] + \left(s - \frac{1}{2}\right) e_{\mathcal{T}(m)+r} \right)$$
$$+ \sum_{n=0}^{j} \sum_{u=1}^{R(l_{j-n})} \left( l_{j-n} + \frac{1}{2} \right) e_{\mathcal{T}(k+1)+\mathcal{S}(n)+u}.$$

When $R(s) = 0$, condition (2) in Theorem 5.4 implies that $R(s+j) = 0$ for all $j$. Then this case or when $R(\nu_i) = 0$ for all $i$, can be easily deduced from the cases above, and hence, we have completed this proof for $SO(p,q)$

and $r = 1$ The cases corresponding to $SU(p,q)$ follows almost immediately from this cases above using, as before, the multiplicities of the positive roots for this group. □

Then we can give the:

*Proof of Theorem* 2.1 for $SO(p,q)$, $p > q+1$ and $SU(p,q)$, $p > q$. Is immediate from Proposition 5.5, Theorem 5.4 and Proposition 1.5. □

## 6. Case $SU(n,n)$, $n \geq 2$.

In this case, identifying $\mathfrak{a} \simeq \mathbb{C}^n$ we have that the restricted roots are $\{\pm e_i \pm e_j, \pm 2e_l\}$. $J^G(\nu)$ has integral infinitesimal character if and only if $\nu_i \equiv \epsilon \mod 2\mathbb{Z}$, $\epsilon = 0, 1$. Take $\nu \in \mathfrak{a}^*$. Again, we may assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0. \tag{6.1}$$

Hence we define

$$R(i) = R_\nu(i) = \#\{j \ : \ \nu_j = 2i + \epsilon\}, \quad i \geq 1 \tag{6.2}$$

and

$$R(0) = (2\epsilon)\#\{j \ : \ \nu_j = \epsilon\}. \tag{6.3}$$

Now, we can state the following:

**Theorem 6.4.** *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

1. *$R(i+1) \leq R(i) + 1$, for $i \geq 0$;*
2. *$R(i+1) = R(i) + 1$, then $R(i)$ is odd.*

*Proof.* See Theorem 7.1 in [**BJ**].

With this, we can prove the following:

**Proposition 6.5.** *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(2) above are satisfied then $\nu = \sum_{\alpha \in \Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or $0$.*

*Proof.* Let us first assume that $\epsilon = 0$. By (6.3) we have that $R(0) = 0$. Then it follows from condition (2) in Theorem 6.4 that $\nu \equiv 0$. So we can suppose that $\epsilon = 1$ and again, by condition (2), we can consider that

$$\nu = (2k+1, \ldots, 2k+1, \ldots\ldots, 1, \ldots, 1) \tag{6.6}$$

with $R(0) \geq 2$. Since $n = \sum_{i=0}^{k-1} R(k-i) + \frac{1}{2}R(0)$, we have

$$k - m < n - \sum_{i=0}^{m} R(k-i), \qquad m = 0, \ldots, k-1$$

and this formula plus the fact that each root $e_i \pm e_j$ has multiplicity 2 and each $2e_l$ has multiplicity one, allows us to say that

$$\nu = \left( \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \sum_{t=1}^{k-m} [(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})] + e_{\mathcal{T}(m)+r} \right) + \sum_{r=1}^{\frac{1}{2}R(0)} e_{\mathcal{T}(k)+r}$$

and in this way, we have completed this proposition. □

Hence, we can give the:

*Proof of Theorem* 2.1 for $SU(n,n)$, $n > 2$. It is immediate from Proposition 6.5, Theorem 6.4 and Proposition 1.5. □

## 7. Case $SO(n+1,n)$ and $SO(2n+1,\mathbb{C})$, $n \geq 2$.

Let us assume that $G = SO(n+1,n)$ or $G = SO(2n+1,\mathbb{C})$, and as usual identify $\mathfrak{a} \simeq \mathbb{C}^n$. Then the restricted roots become $\{\pm e_i \pm e_j, \pm e_l\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(n+1,n) \\ 2, & \text{if } G = SO(2n+1,\mathbb{C}). \end{cases}$$

In these cases, $J(\nu)$ integral infinitesimal character if and only if

$$\nu_i \in r\mathbb{Z} + r\frac{\epsilon}{2}, \text{ with } \epsilon = 0, 1.$$

We can replace $\nu$ by a Weyl group conjugate, and assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0 \tag{7.1}$$

and so, we define

$$R(i) = R_\nu(i) = \#\left\{ j \ : \ \nu_j = r\left(i + \frac{\epsilon}{2}\right)\right\}, \quad i \geq 1 \tag{7.2}$$

and

$$R(0) = R_\nu(0) = (2-\epsilon)\#\left\{ j \ : \ \nu_j = r\frac{\epsilon}{2}\right\}. \tag{7.3}$$

Let us see the following:

**Theorem 7.4.** *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

1. $R(i+1) \leq R(i) + 1$, *for* $i \geq 0$;
2. *if* $R(i+1) = R(i) + 1$, *for* $i \geq 1$, *then* $R(i)$ *is odd;*
3. $R(0) > 2 - \epsilon$.

*Proof.* Cf. Theorem 9.4 in [**BJ**]. The condition $R(0) > 2 - \epsilon$ does not appear in [**BJ**]. However, it is easy to see by inspection of the proof that this condition is needed.

Now, we can prove the following:

**Proposition 7.5.** *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(3) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or 0.*

*Proof.* Let us assume that r=1. The other case, r=2, follows from this one. And also consider $\epsilon = 0$, so here we have that $J(\nu)$ admits integral infinitesimal character if and only if $\nu_i \in \mathbb{Z}$. Hence by condition (2) in Theorem 7.4 we have that

$$\nu = (k, \dots, k, k-1, \dots, k-1, \dots\dots, 1, \dots, 1, 0, \dots, 0). \tag{7.6}$$

As before, since $R(0) \geq 2$, we can prove that $k - i \leq n - \sum_{j=0}^{i} R(k-j)$, and this inequality allows us to write down

$$\nu = \frac{1}{2}\sum_{m=0}^{k-1}\sum_{r=1}^{R(k-m)}\sum_{t=1}^{k-m}[(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})]$$

where $\mathcal{T}(m)$ is defined in (3.6). The case $\epsilon = 1$ follows from this one, using that

$$\nu = \nu_1 + \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$$

with $\nu_1$ as in (7.6). Since we do not use $e_i$'s in the case above, we can put

$$\nu = \nu_1 + \frac{1}{2}\sum_{i=1}^{n} e_i$$

and we have completed the proof of this proposition. □

Then we have:

*Proof of Theorem* 2.1 for $SO(n+1, n)$ and $SO(2n+1, \mathbb{C})$, $n \geq 2$. It follows from Proposition 7.5, Theorem 7.4 and Proposition 1.5. □

## 8. Case $SO(n, n)$ and $SO(2n, \mathbb{C})$, $n \geq 4$.

Let us assume that $G = SO(n, n)$ or $G = SO(2n, \mathbb{C})$. In this case $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ is of type $D_n$ and, if as usual we identify $\mathfrak{a} \simeq \mathbb{C}^n$, then the restricted roots become $\{\pm e_i \pm e_j\}$. Define

$$r = \begin{cases} 1, & \text{if } G = SO(n, n) \\ 2, & \text{if } G = SO(2n, \mathbb{C}). \end{cases}$$

Here, $J(\nu)$ integral infinitesimal character if and only if

$$\nu_i \in r\mathbb{Z} + r\frac{\epsilon}{2}, \text{ with } \epsilon = 0, 1.$$

Again, we can assume that

$$\nu_1 \geq \nu_2 \geq \cdots \geq |\nu_n| \geq 0 \tag{8.1}$$

and define

$$R(i) = R_\nu(i) = \#\left\{j \;:\; \nu_j = r\left(i + \frac{\epsilon}{2}\right)\right\}, \quad i \geq 1 \tag{8.2}$$

and

$$R(0) = R_\nu(0) = (2-\epsilon)\#\left\{j \;:\; \nu_j = r\frac{\epsilon}{2}\right\} + (1-\epsilon). \tag{8.3}$$

Take $\nu \in \mathfrak{a}^*_{\text{int}}$, then $J(\nu)$ admits an invariant Hermitian form if and only if $n$ is even, or $n$ is odd, $\epsilon = 0$ and $R(0) > 1$. Let us see the following:

**Theorem 8.4.** *If $J(\nu)$ has integral infinitesimal character, then an invariant Hermitian form is positive definite on $J(\nu)^\mu$, $\mu \in \mathfrak{p}$, if and only if the following conditions are satisfied:*

(1) *$R(i+1) \leq R(i) + 1$, for $i \geq 0$;*
(2) *$R(i+1) = R(i) + 1$, for $i \geq 1$, then $R(i)$ is odd;*
(3) *$R(0)$ is odd;*
(4) *$R(0) > 1$.*

*Proof.* See Theorem 10.3 in [**BJ**]. The condition $R(0) > 1$ does not appear in the statement of this theorem in [**BJ**], but it is easy to see, checking the proof, that, otherwise, $q_\mu(\nu) > 0$.

Now, we can prove the following:

**Proposition 8.5.** *Consider $\nu \in \mathfrak{a}^*_{\text{int}}$ such that $J(\nu)$ has integral infinitesimal character. If conditions (1)-(4) above are satisfied then $\nu = \sum_{\alpha\in\Sigma^+} c_\alpha \alpha$ with $c_\alpha = 1/2$ or $0$.*

*Proof.* Let us assume that $r = 1$, $\epsilon = 0$. Since the Weyl group is the group of permutation and sign changes involving an even number of signs of the set of $n$ elements, by condition (2) and $R(0) > 1$, we can suppose that $\nu_i \geq 0$ and

$$\nu = (k, \ldots, k, k-1, \ldots, k-1, \ldots\ldots, 1, \ldots, 1, 0, \ldots, 0).$$

Since $R(0) > 1$, we have $k - i \leq n - \sum_{j=0}^{i} R(k-j)$, and thus

$$\nu = \frac{1}{2}\sum_{m=0}^{k-1}\sum_{r=1}^{R(k-m)}\sum_{t=1}^{k-m}[(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})],$$

where $\mathcal{T}(m)$ was defined in (3.6). Now, let us assume that $\epsilon = 1$ and $n$ even. So, by a conjugation by the Weyl group we can assume that $\nu_i \geq 0$ for $i = 0, \ldots, n-1$ and moreover

$$\nu = \Big(k + \frac{1}{2}, \ldots, k + \frac{1}{2}, k - 1 + \frac{1}{2}, \ldots, k - 1 + \frac{1}{2}, \\ \ldots\ldots, 1 + \frac{1}{2}, \ldots, 1 + \frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2}\Big).$$

Since $R(0) > 1$ and by condition (3) in Theorem 8.4 we can prove that $k - i + 1 \leq n - \sum_{j=0}^{i} R(k-j)$, and defining $\delta = \begin{cases} 1, & \text{if } \nu_n = \frac{1}{2} \\ 0, & \text{if } \nu_n = -\frac{1}{2}. \end{cases}$ we can rewrite $\nu$ as follows

$$\nu = \sum_{m=0}^{k-1} \sum_{r=1}^{R(k-m)} \sum_{t=1}^{k-m} \frac{1}{2}\Big\{[(e_{\mathcal{T}(m)+r} - e_{\mathcal{T}(m)+r+t}) + (e_{\mathcal{T}(m)+r} + e_{\mathcal{T}(m)+r+t})] \\ + (e_{\mathcal{T}(m)+r} + (-1)^{\mathcal{T}(m)+r+\delta} e_n)\Big\} + \sum_{j=1}^{R(0)-1} \frac{1}{2}(e_{\mathcal{T}(k)+j} + (-1)^j e_n)$$

where $\mathcal{T}(m)$ was defined in (3.6). And, since $n$ is even and $R(0)$ is odd, we have completed our proof. The case $SO(2n, \mathbb{C})$ follows as above using multiplicities of positive roots for this particular case. □

Then we can give the:

*Proof of Theorem* 2.1 for $SO(n,n)$ and $SO(2n, \mathbb{C})$, $n \geq 4$. It is immediate from Proposition 8.5, Theorem 8.4 and Proposition 1.5. □

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