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PRIMITIVE COMPACT FLAT MANIFOLDS WITH HOLONOMY GROUP $\mathbf{Z_2} \oplus \mathbf{Z_2}$

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In this paper we determine and classify all compact Riemannian flat manifolds with holonomy group isomorphic to $\mathbf{Z_2} \oplus \mathbf{Z_2}$ and first Betti number zero. Also we give explicit realizations of all of them.

Introduction.

From an important construction of Calabi (see [**Ca**], [**Wo**]), it follows that the compact Riemannian flat manifolds with first Betti number zero are the building blocks for all compact Riemannian flat manifolds. It is, therefore, of interest to construct families of such objects. These are often called primitive manifolds.

Hantzsche and Wendt (1935) constructed the only existing 3-dimensional compact Riemannian flat manifold with first Betti number zero; this manifold has holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Cobb [**Co**] constructed a family of manifolds with these properties, for all dimensions $n \geq 3$. In [**RT**] a rather larger family of primitive $(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$-manifolds was given.

The goal in this paper is the classification, up to affine equivalence, of all primitive manifolds with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. We may notice that a similar project has been carried out in [**RT2**], where a full classification of 5-dimensional Bieberbach groups with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ was given.

The classification is achieved by following a classical result of Charlap ([**Ch1**]), which reduces the problem to:

(1) classification of faithful representations $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \longrightarrow \mathrm{Gl}(n, \mathbf{Z})$, without fixed points;
(2) computation of $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \mathbf{Z}^n)$ and enumeration of *special* classes modulo some equivalence relation.

All §2 is devoted to the solution of (1), which turns out to be in general a very difficult problem. The cohomological computations in §3 are standard. In §4 we prove the classification theorem stated below, which, together with the classification of all integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ having no fixed points (see Theorem 2.7), constitutes the main result.

Theorem. *The affine equivalence classes of compact Riemannian flat manifolds with holonomy group* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and first Betti number zero are in a bijective correspondence with the* $\mathbf{Z}[\mathbf{Z}_2 \oplus \mathbf{Z}_2]$*-modules* Λ*, such that:*

(1) *As abelian group* Λ *is free and of finite rank;*
(2) $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$*;*
(3) Λ *contains a submodule equivalent to the Hantzsche-Wendt module.*

As it can be seen in the Theorem all primitive $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds are closely related to the Hantzsche-Wendt manifold. This relation will be more clear in section §5, where we construct all primitive $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds. We also identify those realizations which correspond to the classical examples of Cobb, those which correspond to the family of examples given in [**RT**] and which ones are newly found in the course of the classification. Finally, these explicit realizations allow us to compute their integral homology as well as their cohomology.

1. Preliminaries.

Let M be an n-dimensional compact Riemannian flat manifold with fundamental group Γ. Then, $M \simeq \mathbf{R}^n/\Gamma$, Γ is torsion-free and, by Bieberbach's first theorem, one has a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1, \tag{1.1}$$

where Λ is free abelian of rank n and Φ is a finite group, the holonomy group of M. This sequence induces an action of Φ on Λ that determines a structure of $\mathbf{Z}[\Phi]$-module on Λ (or an integral representation of rank n of Φ). Thus, Λ is a $\mathbf{Z}[\Phi]$-module which, moreover, is a free abelian group of finite rank. From now on we will refer to these $\mathbf{Z}[\Phi]$-modules as Φ-modules.

As indicated by Charlap in [**Ch2**], the classification up to affine equivalence of all compact Riemannian flat manifolds with holonomy group Φ can be carried out by the following steps:

(1) Find all faithful Φ-modules Λ.
(2) Find all extensions of Φ by Λ, i.e., compute $\mathrm{H}^2(\Phi;\Lambda)$.
(3) Determine which of these extensions are torsion-free.
(4) Determine which of these extensions are isomorphic to each other.

For each subgroup K of Φ, the inclusion $i : K \longrightarrow \Phi$ induces a *restriction* homomorphism $\mathrm{res}_K : \mathrm{H}^2(\Phi;\Lambda) \longrightarrow \mathrm{H}^2(K;\Lambda)$.

Definition. A class $\alpha \in \mathrm{H}^2(\Phi;\Lambda)$ is *special* if for any cyclic subgroup $K < \Phi$ of prime order, $\mathrm{res}_K(\alpha) \neq 0$.

Step (3) reduces to the determination of the special classes by virtue of the following result.

Lemma 1.1 ([**Ch1**, p. 22]). *Let Λ be a Φ-module. The extension of Φ by Λ corresponding to $\alpha \in \mathrm{H}^2(\Phi;\Lambda)$ is torsion-free if and only if α is special.*

Definition. Let Λ and Δ be Φ-modules. A *semi-linear map* from Λ to Δ is a pair (f, A), where $f : \Lambda \longrightarrow \Delta$ is a group homomorphism, $A \in \mathrm{Aut}(\Phi)$ and

$$f(\sigma \cdot \lambda) = A(\sigma) \cdot f(\lambda), \qquad \text{for } \sigma \in \Phi \text{ and } \lambda \in \Lambda.$$

The Φ-modules Λ and Δ are said to be *semi-equivalent* if f is a group isomorphism. If $A = I$, then Λ and Δ are *equivalent* via f.

Let $\mathcal{E}(\Phi)$ be the category whose objects are the *special pointed Φ-modules*, that is, pairs (Λ, α), where Λ is a faithful Φ-module and α is a special class in $\mathrm{H}^2(\Phi;\Lambda)$. The morphisms of $\mathcal{E}(\Phi)$ are the pointed semi-linear maps, that is, semi-linear maps (f, A) from (Λ, α) to (Δ, β), such that $f_*(\alpha) = A^*(\beta)$, where f_* is the morphism in cohomology induced by f and A^* is defined by $A^*(\beta)(\sigma, \tau) = \beta(A\sigma, A\tau)$ for any $(\sigma, \tau) \in \Phi \times \Phi$.

Theorem 1.2 ([**Ch1**, p. 20]). *There is a bijection between the isomorphism classes of the category $\mathcal{E}(\Phi)$ and connection preserving diffeomorphism classes of compact Riemannian flat manifolds with holonomy group Φ.*

It is well known that the first Betti number of M, where $M \simeq \mathbf{R}^n/\Gamma$ and Γ is as in (1.1) can be computed by the formula

$$\beta_1(M) = \mathrm{rk}(\Lambda^\Phi).$$

Thus, it is clear that the primitive Φ-manifolds correspond to those objects (Λ, α) in $\mathcal{E}(\Phi)$ satisfying $\Lambda^\Phi = 0$.

2. Integral Representations.

In this section we deal with the first of Charlap's steps. That is, we determine (up to equivalence) all faithful $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules Λ, such that $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$.

Let Λ be a Φ-module. Since Λ is free abelian of finite rank, say n, then $\Lambda \simeq \mathbf{Z}^n$ as abelian groups and the $\mathbf{Z}[\Phi]$-structure on Λ induces an homomorphism $\rho : \Phi \longrightarrow \mathrm{Gl}(n, \mathbf{Z})$, i.e., an *integral representation* of Φ of rank n. Conversely, any integral representation of rank n of Φ makes $\mathbf{Z}^n$ a Φ-module. We will, then, identify Φ-modules (of rank n) with integral representations of Φ (of rank n).

Under this identification, faithful modules correspond to faithful representations (monomorphisms); equivalence of modules corresponds to equivalence of representations, that is conjugation in $\mathrm{Gl}(n, \mathbf{Z})$ by a fixed element $A \in \mathrm{Gl}(n, \mathbf{Z})$ and invariants ($\lambda \in \Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2}$), correspond to fixed points ($v \in \mathbf{Z}^n$, such that $\rho(\sigma)v = v$ for all $\sigma \in \Phi$).

Definition. An integral representation ρ of a finite group Φ is decomposable if there are integral representations ρ_1 and ρ_2 of Φ, such that ρ is equivalent to $\rho_1 \oplus \rho_2$. The representation ρ is *indecomposable* if it is not decomposable.

It follows from the previous definition that every integral representation ρ of a finite group Φ is equivalent to a direct sum of indecomposable representations. However, the indecomposable summands are in general not uniquely determined (up to order and equivalence) by ρ. That is, the Krull-Schmidt theorem does not hold for integral representations (see [**Re**]).

Since a fixed point for a representation ρ_1 is also a fixed point for the representation $\rho_1 \oplus \rho_2$ for any ρ_2, it follows that any integral representation without fixed points decomposes as a direct sum of indecomposable representations having no fixed points.

$\mathbf{Z}_2$-representations. It is well known that there are three indecomposable representations of $\mathbf{Z}_2$, up to equivalence, which are given by:

$$(1), \qquad (-1), \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Moreover, Krull-Schmidt holds in this case.

A useful invariant. We introduce now an invariant for integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, which will allow us to prove the indecomposibility of some representations and also to prove that Krull-Schmidt holds for the sub-family of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-representations without fixed points.

Proposition 2.1. *Let ρ and ρ' be two arbitrary integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and let S be any subset of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. If ρ and ρ' are equivalent via P, then P induces an isomorphism of abelian groups*

$$\frac{\cap_{g\in S} \operatorname{Ker}(\rho(g) \pm I)}{\cap_{g\in S} \operatorname{Im}(\rho(g) \mp I)} \longrightarrow \frac{\cap_{g\in S} \operatorname{Ker}(\rho'(g) \pm I)}{\cap_{g\in S} \operatorname{Im}(\rho'(g) \mp I)},$$

where the choice of signs is independent for each $g \in S$.

Proof. We have $\rho(g)^2 = I$ for all $g \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$, then $\operatorname{Ker}(\rho(g) \pm I) \supseteq \operatorname{Im}(\rho(g) \mp I)$. Since $P \in \operatorname{Gl}(n, \mathbf{Z})$, we obtain the following two equations

$$P(\operatorname{Ker}(\rho(g) \pm I))P^{-1} = \operatorname{Ker}(P\rho(g)P^{-1} \pm I) = \operatorname{Ker}(\rho'(g) \pm I)$$
$$P(\operatorname{Im}(\rho(g) \mp I))P^{-1} = \operatorname{Im}(P\rho(g)P^{-1} \mp I) = \operatorname{Im}(\rho'(g) \mp I).$$

Therefore, the restriction of P to $\bigcap_{g\in S} \operatorname{Ker}(\rho(g) \pm I)$ induces the claimed group isomorphism. □

Remark. The proposition is still valid for representations of $\mathbf{Z}_2^k$.

Notice that if ρ and ρ' are two integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, we have

$$\frac{\cap_{g\in S} \operatorname{Ker}((\rho \oplus \rho')(g) \pm I)}{\cap_{g\in S} \operatorname{Im}((\rho \oplus \rho')(g) \mp I)} \simeq \frac{\cap_{g\in S} \operatorname{Ker}(\rho(g) \pm I)}{\cap_{g\in S} \operatorname{Im}(\rho(g) \mp I)} \oplus \frac{\cap_{g\in S} \operatorname{Ker}(\rho'(g) \pm I)}{\cap_{g\in S} \operatorname{Im}(\rho'(g) \mp I)}.$$

$\mathbf{Z}_2 \oplus \mathbf{Z}_2$-representations. We adopt the following convention to describe a representation ρ of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. We write $B_1 = \rho(1,0)$, $B_2 = \rho(0,1)$ and $B_3 = B_1 B_2 = \rho(1,1)$.

Proposition 2.2. *There are three non-equivalent representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, of rank* 1 (*characters*), *without fixed points, χ_i for $i = 1,2,3$.*

There are three equivalence classes of indecomposable representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, of rank 2, *without fixed points, ρ_j for $j = 1,2,3$.*

Representatives are given by,

$$
\begin{array}{cccc}
 & B_1 & B_2 & B_3 \\
\chi_1 : & (1) & (-1) & (-1) \\
\chi_2 : & (-1) & (1) & (-1) \\
\chi_3 : & (-1) & (-1) & (1) \\
\rho_1 : & -I & J & -J \\
\rho_2 : & J & -I & -J \\
\rho_3 : & J & -J & -I
\end{array}
$$

Proof. The determination of characters is straightforward.

Given any indecomposable representation of rank 2 we may assume that one of the matrices B_i is J, otherwise the representation decomposes. From the identities $B_k B_j = B_j B_k$ and $B_j^2 = I$ it follows that $B_j = \pm I$ or $B_j = \pm J$. On the other hand, a representation for which two matrices B_i are equal to J (or $-J$) has a fixed point. Finally, by observing that J and $-J$ are conjugate by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the proof is complete. □

Let us introduce two particular representations of rank 3, that will be referred to as μ and ν. They are defined by

$$
\begin{array}{cccc}
 & B_1 & B_2 & B_3 \\
\mu : & \begin{pmatrix} -1 & 0 & 0 \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 & -1 \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -1 & 1 \\ & -1 & 0 \\ & & -1 \end{pmatrix} \\
\nu : & \begin{pmatrix} -1 & 1 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix} & \begin{pmatrix} 1 & -1 & -1 \\ & -1 & 0 \\ & & -1 \end{pmatrix} .
\end{array}
\tag{2.2}
$$

It is easy to check that both representations have no fixed points.

For a fixed representation ρ and each triple of integers (a_1, a_2, a_3), with $-1 \leq a_i \leq 1$, define the abelian group

$$
K^{\rho}_{(a_1,a_2,a_3)} = \frac{\bigcap_{a_i \neq 0} \operatorname{Ker}(B_i - a_i I)}{\bigcap_{a_i \neq 0} \operatorname{Im}(B_i + a_i I)} .
$$

Example. Let ρ_1 be the representation defined by $B_1 = -I$, $B_2 = J$ and $B_3 = -J$ as in Proposition 2.2. To determine the groups $K^{\rho_1}_{(-1,0,1)}$ and $K^{\rho_1}_{(0,-1,1)}$,

$$
K^{\rho_1}_{(-1,0,1)} = \frac{\operatorname{Ker}(B_1 + I) \cap \operatorname{Ker}(B_3 - I)}{\operatorname{Im}(B_1 - I) \cap \operatorname{Im}(B_3 + I)} ,
$$

$$K^{\rho_1}_{(0,-1,1)} = \frac{\operatorname{Ker}(B_2+I)\cap\operatorname{Ker}(B_3-I)}{\operatorname{Im}(B_2-I)\cap\operatorname{Im}(B_3+I)}$$

we write down the kernels and the images in the canonical basis $\{e_1, e_2\}$ of $\mathbf{Z}^2$. We have

$$\begin{aligned}
\operatorname{Ker}(B_1+I) &= \langle e_1, e_2\rangle & \operatorname{Im}(B_1-I) &= \langle 2e_1, 2e_2\rangle\\
\operatorname{Ker}(B_2+I) &= \langle e_1 - e_2\rangle & \operatorname{Im}(B_2-I) &= \langle e_1 - e_2\rangle\\
\operatorname{Ker}(B_3-I) &= \langle e_1 - e_2\rangle & \operatorname{Im}(B_3+I) &= \langle e_1 - e_2\rangle,
\end{aligned}$$

from which it follows that

$$K^{\rho_1}_{(-1,0,1)} \simeq \mathbf{Z}_2 \qquad \text{and} \qquad K^{\rho_1}_{(0,-1,1)} = 0.$$

The computation of these invariants (Proposition 2.1) for all the representations in Proposition 2.2 and for the representations μ and ν in (2.2) are as simple as those performed above. Thus, we put together the results as a lemma, omitting the details.

Lemma 2.3. *Let ρ be a representation equivalent to χ_i or ρ_i,* $(1 \le i \le 3)$ *as in Proposition 2.2 or equivalent to μ or ν as in (2.2). Then,*

(a) $$K^{\rho}_{(1,-1,-1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_1, \rho_2, \rho_3, \mu;\\ 0, & \textit{if } \rho \simeq \chi_2, \chi_3, \rho_1, \nu;\end{cases}$$

(b) $$K^{\rho}_{(-1,1,-1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_2, \rho_1, \rho_3, \mu;\\ 0, & \textit{if } \rho \simeq \chi_1, \chi_3, \rho_2, \nu;\end{cases}$$

(c) $$K^{\rho}_{(-1,-1,1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_3, \rho_1, \rho_2, \mu;\\ 0, & \textit{if } \rho \simeq \chi_1, \chi_2, \rho_3, \nu;\end{cases}$$

(d) $$K^{\rho}_{(0,1,-1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_2, \rho_3, \mu;\\ 0, & \textit{if } \rho \simeq \chi_1, \chi_3, \rho_1, \rho_2, \nu;\end{cases}$$

(e) $$K^{\rho}_{(0,-1,1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_3, \rho_2, \mu;\\ 0, & \textit{if } \rho \simeq \chi_1, \chi_2, \rho_1, \rho_3, \nu;\end{cases}$$

(f) $$K^{\rho}_{(0,-1,-1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_1, \rho_3, \mu;\\ 0, & \textit{if } \rho \simeq \chi_2, \chi_3, \rho_1, \rho_2, \nu;\end{cases}$$

(g) $$K^{\rho}_{(1,0,-1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_1, \rho_3, \mu;\\ 0, & \textit{if } \rho \simeq \chi_2, \chi_3, \rho_1, \rho_2, \nu;\end{cases}$$

(h) $$K^{\rho}_{(-1,0,1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_3, \rho_1;\\ 0, & \textit{if } \rho \simeq \chi_1, \chi_2, \rho_2, \rho_3, \mu, \nu;\end{cases}$$

$$\text{(i)} \qquad K^{\rho}_{(-1,0,-1)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_2, \rho_1, \rho_3, \mu; \\ 0, & \textit{if } \rho \simeq \chi_1, \chi_3, \rho_2, \nu; \end{cases}$$

$$\text{(j)} \qquad K^{\rho}_{(-1,1,0)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_2, \rho_1; \\ 0, & \textit{if } \rho \simeq \chi_1, \chi_3, \rho_2, \rho_3, \mu, \nu; \end{cases}$$

$$\text{(k)} \qquad K^{\rho}_{(1,-1,0)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_1, \rho_2, \mu; \\ 0, & \textit{if } \rho \simeq \chi_2, \chi_3, \rho_1, \rho_3, \nu; \end{cases}$$

$$\text{(l)} \qquad K^{\rho}_{(-1,-1,0)} = \begin{cases} \mathbf{Z}_2, & \textit{if } \rho \simeq \chi_3, \rho_1, \rho_2, \mu; \\ 0, & \textit{if } \rho \simeq \chi_1, \chi_2, \rho_3, \nu. \end{cases}$$

Corollary 2.4. *The representations μ and ν are not equivalent.*

Proof. The result follows from Proposition 2.1 and Equation (a) in the previous lemma. □

Corollary 2.5. *The representations μ and ν are indecomposable.*

Proof. Being ν of rank 3, if decomposable, it must be $\nu \simeq \chi_{j_1} \oplus \chi_{j_2} \oplus \chi_{j_3}$, for $1 \le j_1, j_2, j_3 \le 3$ or $\nu \simeq \chi_{j_1} \oplus \rho_{j_2}$, for $1 \le j_1, j_2 \le 3$. But Equations (a), (b) and (c) in Lemma 2.3 contradict both possibilities. The case of μ is similar. □

We had encountered 8 indecomposable representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ having no fixed points. The following theorem asserts that there are no more. Recall that they are given by

$$\begin{array}{cccc} & B_1 & B_2 & B_3 \\ \chi_1: & (1) & (-1) & (-1) \\ \chi_2: & (-1) & (1) & (-1) \\ \chi_3: & (-1) & (-1) & (1) \\ \rho_1: & -I & J & -J \\ \rho_2: & J & -I & -J \\ \rho_3: & J & -J & -I \\ \mu: & \begin{pmatrix} -1 & 0 & 0 \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 & -1 \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -1 & 1 \\ & -1 & 0 \\ & & -1 \end{pmatrix} \\ \nu: & \begin{pmatrix} -1 & 1 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix} & \begin{pmatrix} 1 & -1 & -1 \\ & -1 & 0 \\ & & -1 \end{pmatrix}. \end{array} \tag{2.3}$$

Theorem 2.6. *Let ρ be an indecomposable integral representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ having no fixed points. Then ρ is equivalent to one and only one of the representations in* (2.3).

The proof of Theorem 2.6 is elementary but not trivial. In order to make the paper more readable, we wrote the proof in the forthcoming subsection.

By assuming Theorem 2.6 one can skip the following subsection without losing the understanding of the whole paper.

Theorem 2.7. *Let ρ be an integral representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ having no fixed points. Let χ_i and ρ_i, for $1 \leq i \leq 3$, μ and ν be as in (2.3).*

Then, there exist unique non-negative integers m_i and k_i, for $1 \leq i \leq 3$, s and t, such that

$$\rho \simeq m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu.$$

Proof. From Theorem 2.6 –and previous considerations– it follows that ρ admits such a decomposition. So, it is clear that the groups $K^{\rho}_{(a_1,a_2,a_3)}$ will be all isomorphic to a power of $\mathbf{Z}_2$. Define $c(a_1, a_2, a_3) =$ exponent of $K^{\rho}_{(a_1,a_2,a_3)}$. One can compute, from Lemma 2.3, the 12 numbers $c(a_1, a_2, a_3)$, in particular

$$c(1, -1, -1) = m_1 + k_2 + k_3 + s,$$
$$c(-1, 1, -1) = m_2 + k_1 + k_3 + s,$$
$$\vdots$$

All parameters but t appear in these equations. It is not difficult to see that the linear system formed by those 12 equations and 7 unknowns has rank 7. Therefore, all but t are uniquely determined by ρ. Finally, if ρ has rank n we have an extra equation,

$$n = m_1 + m_2 + m_3 + 2k_1 + 2k_2 + 2k_3 + 3s + 3t,$$

from which follows that also t is uniquely determined by ρ. □

Remark. Theorem 2.7 says that the Krull-Schmidt theorem is valid for the sub-family of integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ without fixed points. Moreover, it follows from its proof that the multiplicities of the indecomposable summands can be effectively computed. This could be done by writing down the linear system in the proof and by computing the exponents $c(a_1, a_2, a_3)$ for the given ρ.

Indecomposable representations without fixed points of rank $n \geq 3$. As we said before, this subsection is devoted to the proof of Theorem 2.6.

While elementary, the proof is not trivial and since it is almost all technical, the reader can skip this part and continue with §3. In order to make it not too long not every single point will be explained. However, full details may be found in [**T**].

From now on, we follow some of the main ideas in [**Na**]. Unfortunately, as noted by Charlap ([**Ch2**, p. 135]), that paper "lacks complete proofs and is extremely laconic". Moreover, low rank representations are not included.

We start with a pair of matrices $A, B \in \mathrm{Gl}(n, \mathbf{Z})$ $(n \geq 3)$, such that

$$A^2 = I = B^2 \quad \text{and} \quad AB = BA. \tag{2.4}$$

Notice that from such a pair one can define several integral representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (as many as $|\mathrm{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)| = 6$), possibly non-equivalent. It is clear that all of them are decomposable or all of them are indecomposable simultaneously. By the representation *defined by A and B* we will refer to the representation ρ defined by

$$\rho(1,0) = A \quad \text{and} \quad \rho(0,1) = B.$$

In addition to (2.4) we assume that the representation defined by A and B is indecomposable and has no fixed points.

To achieve the result we will go through the following steps.

(1) Show that A and B have some special canonical type (see Lemma 2.8).
(2) Show that conjugating, in $\mathrm{Gl}(n,\mathbf{Z})$, the special type of (1) is equivalent to performing some elementary operations on rows and columns (see Lemma 2.9).
(3) Reduce matrices A and B, according to (2), until it is clear whether the representation decomposes or not.

From (2.1) and the assumption $n \geq 3$ it follows that $A \neq B$ and $A \neq -B$. Otherwise, the representation defined by A and B decomposes. Observe that, in particular, the representation is faithful.

It is not difficult to show that for any integral matrix A, of rank n, the sub-lattice $\Lambda = \operatorname{Ker} A$ of $\mathbf{Z}^n \subseteq \mathbf{R}^n$ admits a direct complement Λ^c. That is, always there exists Λ^c, such that $\mathbf{Z}^n = \Lambda \oplus \Lambda^c$.

Let $\Lambda = \operatorname{Ker}(A - B)$. We have $0 \subsetneq \Lambda \subsetneq \mathbf{Z}^n$ and we can write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Since $A - B|_\Lambda = 0$, then $A_{11} = B_{11}$ and $A_{21} = B_{21}$. Moreover, from the equation $(A-B)(A+B) = A^2 - B^2 = 0$ we get $2A_{21} = 0$ and $A_{22} = -B_{22}$. Hence, we may assume that A and B are of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} A_{11} & B_{12} \\ & -A_{22} \end{pmatrix}.$$

The identity $A^2 = I$ implies that $A_{11}^2 = I = A_{22}^2$. Therefore, both decompose as a direct sum of three blocks, an identity and a minus identity block and a matrix K, being K a direct sum of matrices J (see (2.1)). Moreover, the condition of having no fixed points, for the representation defined by A and B, forces A_{11} to be $-I$. Finally, we get the following form for A and B,

$$A = \begin{pmatrix} -I & A_1 & 0 & A_3 \\ & I & & \\ & & -I & \\ & & & K \end{pmatrix}, \qquad B = \begin{pmatrix} -I & 0 & B_2 & B_3 \\ & -I & & \\ & & I & \\ & & & -K \end{pmatrix}, \tag{2.5}$$

where $A_3 K = A_3$ and $B_3 K = -B_3$.

Remark. Not all three blocks I, $-I$ and K must be present in the decomposition of A_{22}.

Notation: By $A = [\alpha_1 \dots \alpha_n]$ we indicate the matrix with columns $\alpha_1, \dots,$ α_n, being α_1 the first one from the left.

It follows from the identities below (2.5) that $A_3 = [\alpha_1 \alpha_1 \alpha_2 \alpha_2 \dots]$ and that $B_3 = [\alpha_1(-\alpha_1)\alpha_2(-\alpha_2)\dots]$. We can then consider $B_3^\wedge = [\alpha_1 \alpha_2 \dots]$, having half of the number of columns of B_3. Conversely, for any matrix $A = [\alpha_1 \alpha_2 \dots]$ one can consider $A^\vee = [\alpha_1(-\alpha_1)\alpha_2(-\alpha_2)\dots]$. Obviously, $B_3^{\wedge\vee} = B_3$.

Lemma 2.8 (Canonical type)**.** *Let A and B be as in (2.4). Suppose the representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ defined by A and B is indecomposable and has no fixed points. Then, we may assume that A and B have the following type,*

$$A = \begin{pmatrix} -I & A_1 & 0 & 0 \\ & I & & \\ & & -I & \\ & & & K \end{pmatrix}, \qquad B = \begin{pmatrix} -I & 0 & B_2 & B_3^\vee \\ & -I & & \\ & & I & \\ & & & -K \end{pmatrix},$$

where A_1, B_2 and B_3 have entries in the ring $\mathbf{Z}_2\, (= \{0,1\})$.

Proof. We may first assume that A and B are as in (2.5). If $A_3 = [\alpha_1 \alpha_1 \alpha_2 \alpha_2 \dots]$, take $C_3 = [(-\alpha_1)0(-\alpha_2)0\dots]$. By conjugating A and B by

$$C = \begin{pmatrix} I & 0 & 0 & C_3 \\ & I & & \\ & & I & \\ & & & I \end{pmatrix},$$

one eliminates A_3.

Denote by $\overline{P}$ the matrix obtained from P by the canonical projection $\mathbf{Z} \longrightarrow \mathbf{Z}_2$. Let $\widetilde{B_3} = \overline{B_3^\wedge}^\vee$. Now the lemma follows by conjugating A and B by

$$C = \begin{pmatrix} I & \frac{\overline{A_1} - A_1}{2} & \frac{\overline{B_2} - B_2}{2} & \frac{\widetilde{B_3} - B_3^\vee}{2} \\ & I & & \\ & & I & \\ & & & I \end{pmatrix}.$$

□

Lemma 2.9. *Let A and B be of the canonical type as in Lemma 2.8. If A' and B' are obtained from A and B by any of the following elementary row and column operations* I-IV*, then the representations given by A, B and A', B' are equivalent.*

I. *Column elementary operations on A_1.*
II. *Column elementary operations on B_2.*
III. *Column elementary operations on B_3.*
IV. *Simultaneous row elementary operations on A_1, B_2 and B_3.*

Proof. Let $C = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & C_3 & \\ & & & C_4 \end{pmatrix}$ be unimodular. Recall that any elementary row (column) operation performed on an integral matrix P can be realized by multiplying P on the left (right) by an adequate unimodular matrix. It is straightforward to see that if $C_4K = KC_4$, then $A' = CAC^{-1}$ and $B' = CBC^{-1}$ are of the canonical type.

We notice that the matrix, which by right multiplication interchanges columns $2i+1$ and $2j+1$ and that simultaneously interchanges columns $2i+2$ and $2j+2$, commutes with K. Also the matrix, which replaces column $2i+1$ by the sum of columns $2i+1$ and $2j+1$ and that simultaneously replaces column $2i+2$ by the sum of columns $2i+2$ and $2j+2$, commutes with K.

It is clear that performing on $B_3^\vee$ the latest operations is equivalent to performing on B_3 any elementary column operation. □

From now on we will concentrate on the sub-matrices A_1, B_2 and B_3. Recall that we can think of these as matrices with entries in the ring $\mathbf{Z}_2$.

Lemma 2.10. *Let A and B be as in (2.5). If $A_{22} = \begin{pmatrix} I & -I & \\ & & K \end{pmatrix}$, then the representation given by A and B is decomposable.*

Proof. Suppose the representation given by A and B (see (2.5)) is indecomposable. We may assume that A and B are of the canonical type. According to Lemma 2.9 we can reduce A_1 by row and column operations. Since a zero column in any of A_1, B_2 or B_3 would decompose the representation, we have

$$A_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & \mathbf{0} & \end{bmatrix}$$

if A_1 is not square.

Consider the lower parts of B_2 and B_3 that correspond to the lower part of A_1 (the last zero rows). That one of B_2 can be reduced by row and column operations. It turns out that all these rows must be linearly independent. Otherwise one could obtain the following shapes for A_1, B_2 and B_3,

$$A_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & \mathbf{0} & \end{bmatrix}, \quad B_2 = \begin{bmatrix} & * & \\ 0 & \cdots & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ 0 & \cdots & 1 \end{bmatrix},$$

from which it is clear that the representation decomposes.

Now is not difficult to see that one can obtain the following shape for A_1, B_2 and B_3,

$$A_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & \mathbf{0} & \end{bmatrix}, \quad B_2 = \begin{bmatrix} \mathbf{0} & * \\ \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & \mathbf{0} \end{bmatrix}, \quad B_3 = \begin{bmatrix} * & * \\ \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & \mathbf{0} \end{bmatrix}.$$

To continue, we first reduce the $*$ block of B_2. If in this block the rows are linearly dependent, then we get a new partition of the upper part of B_3 and, after operating on B_3, we have

$$A_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & \mathbf{0} & \end{bmatrix}, \quad B_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad B_3 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

By writing down the corresponding matrices A and B it is clear that the representation decomposes.

Hence, it should be

$$A_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & \mathbf{0} & \end{bmatrix}, \quad B_2 = \begin{bmatrix} \mathbf{0} & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad B_3 = \begin{bmatrix} \mathbf{0} & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \\ \mathbf{I} & \mathbf{0} \end{bmatrix},$$

which again gives a decomposable representation.

Therefore, A_1 must be square and in that case we obtain

$$A_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

The corresponding representation, given by A and B, is clearly decomposable if any of A_1, B_2 or B_3 is of rank $m \geq 2$. So, it remains possibly indecomposable the representation defined by

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 1 & 1 & -1 \\ & -1 & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & -1 & 0 \end{pmatrix}.$$

However, the unimodular matrix

$$P = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ & -1 & 0 & 1 & 1 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

satisfies

$$PA = \begin{pmatrix} -1 & 1 & 0 & & \\ & 1 & & & \\ & & -1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} P \qquad \text{and} \qquad PB = \begin{pmatrix} -1 & 0 & 1 & & \\ & -1 & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & -1 & 0 \end{pmatrix} P.$$

□

Lemma 2.11. *Let A and B be as in (2.5). Suppose the lower block A_{22} of A decomposes as the sum of at most 2 blocks from among I, $-I$ and K. If the representation, defined by A and B, is indecomposable, then we may assume that A and B have one of the following forms:*

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} -1 & 1 & -1 \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -1 & 1 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & 0 & 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix}.$$

Notice that the representations defined by the pairs A_i, B_i $(i = 1, 2)$ in Lemma 2.11 are exactly the representations μ and ν introduced in (2.2).

The proof of Lemma 2.11 is similar and easier than that of Lemma 2.10, even if there are several cases to be considered. For the details see [**T**].

The following proposition completes this sub-section.

Proposition 2.12. *Let μ and ν be the representations defined respectively by the pairs of matrices A_1, B_1 and A_2, B_2 in Lemma 2.11. If $\sigma \in \operatorname{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, then $\mu \circ \sigma \sim \mu$ and $\nu \circ \sigma \sim \nu$.*

Proof. Denote by I_Q the conjugation by Q, i.e., $I_Q(A) = QAQ^{-1}$. Considering the first pair, A_1 and B_1, the matrix $P_1 = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & -1 \end{pmatrix}$ satisfies

$$I_{P_1}(A_1) = B_1, \quad I_{P_1}(B_1) = A_1, \quad I_{P_1}(A_1B_1) = A_1B_1.$$

On the other hand, the matrix $P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ satisfies

$$I_{P_2}(A_2) = A_2B_2, \quad I_{P_2}(A_2B_2) = A_2, \quad I_{P_2}(B_2) = B_2.$$

Since $\operatorname{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2) \simeq S_3$, the result follows for the first pair.

The second case is analogous, we take Q_1 and Q_2 instead of P_1 and P_2, with

$$Q_1 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

3. Cohomology Computations.

In this section we shall compute the second cohomology groups $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ for any $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module Λ satisfying $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$. In order to determine special classes in $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$, we also investigate the restriction functions $\operatorname{res}_K : \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \longrightarrow \mathrm{H}^2(K; \Lambda)$ for each subgroup K of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ of order 2.

Regard $\mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ as the homology of the standard complex of functions $\{(\mathcal{F}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda); \partial^n\}_{n \geq 0}$ and recall that, in particular, we have

$$\partial^2 f(x, y, z) = x \cdot f(y, z) - f(xy, z) + f(yz, x) - f(x, y);$$

$$\partial^1 g(x, y) = x \cdot g(y) - g(xy) + g(x).$$

As is well known, we may assume that every (2-)cocycle h is normalized, that is $h(x, I) = h(I, x) = 0$, for any $x \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

If Λ and Δ are two $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules, then

$$\mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda \oplus \Delta) \simeq \mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \oplus \mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Delta). \tag{3.1}$$

If Λ and Δ are semi-equivalent via the semi-linear map (F, σ), then the map defined on the cocycles for Λ by

$$f(g_1, \ldots, g_n) \mapsto Ff(\sigma g_1, \ldots, \sigma g_n)$$

induces an isomorphism

$$\mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \simeq \mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Delta). \tag{3.2}$$

We will make use of the cohomology long exact sequence (3.3) induced by a short exact sequence of Φ-modules as $0 \longrightarrow \Lambda_1 \xrightarrow{j} \Lambda \xrightarrow{\pi} \Lambda_2 \longrightarrow 0$.

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^1(\Phi; \Lambda) \xrightarrow{\pi'} \mathrm{H}^1(\Phi; \Lambda_2) \xrightarrow{\delta^1} \mathrm{H}^2(\Phi; \Lambda_1) \xrightarrow{j'} \\ \mathrm{H}^2(\Phi; \Lambda) \xrightarrow{\pi'} \mathrm{H}^2(\Phi; \Lambda_2) \xrightarrow{\delta^2} \mathrm{H}^3(\Phi; \Lambda_1) \xrightarrow{j'} \cdots \end{aligned} \tag{3.3}$$

Recall that $j'[f] = [j \circ f]$ and $\pi'[g] = [\pi \circ g]$ for f and g cocycles. Also recall that $\delta^n : \mathrm{H}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \longrightarrow \mathrm{H}^{n+1}(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ is defined by $\delta^n[f] = [h]$, for any cocycle f if h satisfies $j'h = \partial^n g$, where g is any element in $\mathcal{F}^n(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ for which $\pi' g = f$.

We come now to the computations.

Let Λ be a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module, such that $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$. By (3.1) we can assume that Λ is indecomposable. Therefore, by Theorem 2.6 we may restrict to the case Λ is one of the modules in (2.3). Moreover, since clearly χ_1, χ_2 and χ_3 are all semi-equivalent as ρ_1, ρ_2 and ρ_3 are all semi-equivalent, by (3.2) there remain 4 cases to be considered. Precisely those given by

$$\begin{array}{llccc} & & B_1 & B_2 & B_3 \\ \text{I.} & \chi_1 : & (1) & (-1) & (-1) \\ \text{II.} & \rho_1 : & -I & J & -J \\ \text{III.} & \mu : & \begin{pmatrix} -1 & 0 & 0 \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 & -1 \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -1 & 1 \\ & -1 & 0 \\ & & -1 \end{pmatrix} \\ \text{IV.} & \nu : & \begin{pmatrix} -1 & 1 & 0 \\ & 1 & 0 \\ & & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix} & \begin{pmatrix} 1 & -1 & -1 \\ & -1 & 0 \\ & & -1 \end{pmatrix}. \end{array} \tag{3.4}$$

Since the computations that follow are standard, we will only indicate how to get the results. However, full details may be found in [**T**].

Case I. In this case one can first determine all the (normalized) cocycles, that is, those functions $f \in \mathcal{F}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ for which $\partial^2 f = 0$. Consider the linear system of 27 equations and 9 unknowns given by $\partial^2 f = 0$. It is not difficult to check that this system is equivalent to the following linearly independent set of equations:

$$\begin{aligned}
&h(B_2, B_2) = 0,\\
&h(B_3, B_3) = 0,\\
&h(B_2, B_3) = h(B_2, B_1),\\
&h(B_3, B_2) = h(B_3, B_1),\\
&h(B_2, B_3) = -h(B_1, B_3),\\
&h(B_3, B_2) = -h(B_1, B_2),\\
&h(B_1, B_1) = h(B_1, B_2) + h(B_1, B_3).
\end{aligned}$$

Thus, it is clear that a general cocycle is of the form

h	B_1	B_2	B_3
B_1	$-(\alpha+\beta)$	$-\beta$	$-\alpha$
B_2	α	0	α
B_3	β	β	0

for some integers α and β. If we let h_α (resp. h_β) be the cocycle obtained by setting $\alpha = 1$ and $\beta = 0$ (resp. $\alpha = 0$ and $\beta = 1$), it is immediate that h_α and h_β generates the group $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ we are considering. Now it is not difficult to see that $h_\alpha \sim h_\beta$, $h_\alpha \not\sim 0$ and $2h_\alpha \sim 0$. Thus,

$$\mathrm{H}^2(\langle B_1, B_2\rangle; \Lambda) \simeq \mathbf{Z}_2 \simeq \langle [h_\alpha]\rangle \simeq \langle [h_\beta]\rangle. \tag{3.5}$$

Case II. Consider the submodule $\Lambda_1 = \langle e_1 + e_2\rangle$. One can see that B_2 acts by (1) and B_1 by (-1) on Λ_1. Let Λ_2 be the quotient Λ/Λ_1. On Λ_2, B_1 and B_2 both act as (-1). Now we have the long exact sequence (3.3). It follows from (3.5) and (3.2) that $\mathrm{H}^2(\Phi; \Lambda_1) \simeq \mathbf{Z}_2$, as well as $\mathrm{H}^2(\Phi; \Lambda_2) \simeq \mathbf{Z}_2$. Since we have explicit generators, one can show that the map j' is the zero map. On the other hand, one can also show that the map δ^2 is injective. Hence, it follows that

$$\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) = 0. \tag{3.6}$$

Case III. Let Λ_1 be the submodule $\langle e_1\rangle$ and, as before, let Λ_2 be the quotient Λ/Λ_1. Then, Λ_1 is semi-equivalent to the module in Case I, while Λ_2 is semi-equivalent the module in Case II. Since one can show that, in the corresponding long exact sequence j' is injective, then it follows from (3.6) that

$$j' : \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_1) \longrightarrow \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \tag{3.7}$$

is an isomorphism.

Case IV. If we let Λ_1 be the submodule $\langle e_1 + 2e_2 \rangle$ and Λ_2 as usual, it is not difficult to see that Λ_1 is given by the character χ_1 and that Λ_2 decomposes as the direct sum of modules given by the characters χ_2 and χ_3.

In this case we need more information on the long exact sequence (3.3). We compute explicitly, as in Case I, the groups $\mathrm{H}^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ and $\mathrm{H}^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2)$. We find that $\mathrm{H}^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \simeq \mathbf{Z}_4 \oplus \mathbf{Z}_2$ and $\mathrm{H}^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, moreover we find that the map $\pi' : \mathrm{H}^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \longrightarrow \mathrm{H}^1(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_2)$ is defined by $\pi'(1,0) = \pi'(0,1) = (1,1)$. Thus, we have the exact sequence

$$\cdots \longrightarrow \mathbf{Z}_4 \oplus \mathbf{Z}_2 \xrightarrow{\pi'} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{\delta^1} \mathbf{Z}_2 \xrightarrow{j'}$$
$$\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \xrightarrow{\pi'} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{\delta^2} \mathrm{H}^3(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda_1) \longrightarrow \cdots$$

It can be shown that δ^2 is injective and therefore, j' is onto. Since $\operatorname{Im} \pi' = \langle (1,1) \rangle = \operatorname{Ker} \delta^1$, it follows that δ^1 is also onto, from which it is immediate that j' is the zero map. Hence,

$$\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) = 0. \tag{3.8}$$

Putting together (3.5)-(3.8) we get the following.

Proposition 3.1. *Let Λ be a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module, such that $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$. If Λ is equivalent to the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module given by the representation $\rho = m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu$, then*

$$\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \simeq \mathbf{Z}_2^{m_1} \oplus \mathbf{Z}_2^{m_2} \oplus \mathbf{Z}_2^{m_3} \oplus \mathbf{Z}_2^{s}.$$

Restriction functions. We now investigate the restriction functions $\operatorname{res}_K : \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda) \longrightarrow \mathrm{H}^2(K; \Lambda)$, for any of the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules Λ in which we are interested and where K is any non-trivial subgroup of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. It is clear that it suffices to assume that Λ is one of the modules in Cases I-IV in (3.4).

Set $K_i = \langle B_i \rangle$. Recall that there are three indecomposable $\mathbf{Z}_2$-modules (see (2.1)). It is well known that $\mathrm{H}^2(\mathbf{Z}_2; \Lambda) \simeq \mathbf{Z}_2 = \langle [f] \rangle$ if Λ is the trivial module of rank 1, where $f : \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow \Lambda$ is defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x = y = 1; \\ 0, & \text{if } x = 0 \text{ o } y = 0; \end{cases}$$

and that $\mathrm{H}^2(\mathbf{Z}_2; \Lambda) = 0$ for the other two indecomposable modules.

We now come to a case by case analysis following the cases in (3.4).

Case I. Since $\mathrm{H}^2(K_i; \Lambda) = 0$ for $i = 2$ or $i = 3$, we only consider the case $i = 1$. We have (see (3.5))

$$h_\alpha \restriction_{\langle B_1 \rangle \times \langle B_1 \rangle} (x,y) = \begin{cases} -1, & \text{if } (x,y) = (B_1, B_1); \\ 0, & \text{if } (x,y) \neq (B_1, B_1). \end{cases}$$

Notice that $h_\alpha \restriction_{\langle B_1\rangle\times\langle B_1\rangle} \sim f$, then

$$\operatorname{res}_{\langle B_i\rangle} = \begin{cases} \mathrm{id}_{\mathbf{Z}_2}, & \text{if } B_i = B_1; \\ 0, & \text{if } B_i = B_2, B_3. \end{cases}$$

Case III. We observe that each B_i $(1 \le i \le 3)$ decomposes as $\left(\begin{smallmatrix} -1 & \\ & J \end{smallmatrix}\right)$. In fact, there exist unimodular matrices P_i such that $P_i B_i = \left(\begin{smallmatrix} -1 & \\ & J \end{smallmatrix}\right) P_i$. Precisely,

$$P_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & -1 \end{pmatrix}; \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Thus, we have $\mathrm{H}^2(K_i;\Lambda) = 0$ and therefore, all the restriction functions are zero.

Cases II and IV. Since $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2;\Lambda) = 0$ (see (3.5) and (3.7)), there is nothing to be done.

4. Classification.

It is straightforward to deduce from the Preliminaries that the classification, up to affine equivalence, of all primitive $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds can be achived by

(i) determining the semi-equivalence classes of faithful $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules Λ such that $\Lambda^{\mathbf{Z}_2\oplus\mathbf{Z}_2} = 0$;

(ii) determining, for each $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module Λ in (i), the equivalence classes of special cohomology classes in $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2;\Lambda)$.

Recall that α and β in $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2;\Lambda)$ are equivalent $(\alpha \sim \beta)$ if and only if there exist a semi-linear map $(f,\phi) : \Lambda \longrightarrow \Lambda$, such that $f_*\alpha = \phi^*\beta$.

We start dealing with (ii), postponing (i).

Let Λ be a fixed $\mathbf{Z}_2\oplus\mathbf{Z}_2$-module, such that $\Lambda^{\mathbf{Z}_2\oplus\mathbf{Z}_2} = 0$. Since equivalence implies semi-equivalence, we may assume that Λ is given by

$$\rho = m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu,$$

(see Theorem 2.7). Then, by Proposition 3.1, we have

$$\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2;\Lambda) \simeq \mathbf{Z}_2^{m_1} \oplus \mathbf{Z}_2^{m_2} \oplus \mathbf{Z}_2^{m_3} \oplus \mathbf{Z}_2^{s}.$$

According to this decomposition, we may express a class $\alpha \in \mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda)$ as a 4-tuple, $\alpha = (v_1, v_2, v_3, v_4)$, where $v_i \in \mathbf{Z}_2^{m_i}$ for $i = 1, 2, 3$ and $v_4 \in \mathbf{Z}_2^{s}$.

Proposition 4.1. *Let $\alpha \in \mathrm{H}^2(\mathbf{Z}_2\oplus\mathbf{Z}_2;\Lambda)$, $\alpha = (v_1, v_2, v_3, v_4)$ and let $\delta_i = 1$ if $v_i \neq 0$ or $\delta_i = 0$ if $v_i = 0$. Then*

$$\alpha \sim (\overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4}),$$

where $\overline{\delta_i} = (\delta_i, 0, \ldots, 0) \in \mathbf{Z}_2^{m_i}$ $(i = 1, 2, 3)$ or $\overline{\delta_4} \in \mathbf{Z}_2^{s}$.

Before proving this proposition, we state the following general lemma.

Lemma 4.2. *Let* Λ_1 *be a* Φ*-module and let* $\alpha \in \mathrm{H}^n(\Phi;\Lambda_1)$. *If* $\Lambda = \underbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}_{r}$ *, then for each* $\beta_{(i_2,\ldots,i_r)} = (\alpha, i_2\alpha, \ldots, i_r\alpha) \in \mathrm{H}^n(\Phi;\Lambda)$*, with* $i_j = 0, 1$*, there exists a* Φ*-morphism* $f : \Lambda \longrightarrow \Lambda$*, such that*

$$f_*(\alpha, 0, \ldots, 0) = \beta.$$

Furthermore, for each $1 \le j \le r$*, there exists a* Φ*-morphism* $g : \Lambda \longrightarrow \Lambda$*, such that*

$$g_*(\alpha, 0, \ldots, 0) = (0, \ldots, 0, \underset{j}{\alpha}, 0, \ldots, 0).$$

Proof. Given $\beta_{(i_2,\ldots,i_r)}$, we define $f : \Lambda \longrightarrow \Lambda$ by the block matrix

$$A = \begin{pmatrix} I & & & \\ i_2 I & I & & \\ \vdots & & I & \\ i_r I & & & I \end{pmatrix},$$

where each block corresponds to a Λ_1 summand.

If $\phi \in \Phi$, ϕ acts on Λ by

$$B = \begin{pmatrix} B_1 & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_1 \end{pmatrix}.$$

By checking that AB=BA, one shows that f is a Φ-morphism.

Suppose $\alpha_1 : \Phi \times \cdots \times \Phi \longrightarrow \Lambda_1$ is a cocycle representing α. Thus, the cocycle $\tilde{\alpha} : \Phi \times \cdots \times \Phi \longrightarrow \Lambda$ defined by $p_1(\tilde{\alpha}) = \alpha_1$ and $p_i(\tilde{\alpha}) = 0$, for $2 \le i \le r$, where $p_i : \Lambda \longrightarrow (\Lambda_1)_i$ is the i-th projection, is a representative of $(\alpha, 0, \ldots, 0)$.

Therefore, $f_*(\alpha, 0, \ldots, 0) = [f \circ \tilde{\alpha}]$. It is not difficult to see that the composition $f \circ \tilde{\alpha} : \Phi \times \cdots \times \Phi \longrightarrow \Lambda_1 \oplus \cdots \oplus \Lambda_r$ satifies $p_1(f \circ \tilde{\alpha}) = \alpha_1$ and $p_{i_j}(f \circ \tilde{\alpha}) = i_j\alpha_1$, therefore $f_*(\alpha, 0, \ldots, 0) = \beta_{(i_2,\ldots,i_r)}$.

Finally, we define g by the matrix

$$\begin{matrix} \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{matrix} \begin{pmatrix} I & & & & \\ & 0 & \cdots & I & \\ & \vdots & I & \vdots & \\ & I & \cdots & 0 & \\ & & & & I \end{pmatrix},$$

and by arguing in a similar way as above, the lemma is proved. □

Proof of Proposition 4.1. We shall define an adequate semi-linear map (f, I). We define f on each of its indecomposable submodules. Since $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \chi_i) \simeq \mathbf{Z}_2$ and $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \mu) \simeq \mathbf{Z}_2$, the result follows by applying Lemma 4.2 to the submodules corresponding to representations $m_i\chi_i$ and $s\mu$. In the other submodules define f as the identity. □

In light of the characterization in Proposition 4.1, it is not difficult to decide when a class $\alpha \in \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$ is special.

Proposition 4.3. *The class* $\alpha = (\overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \overline{\delta_4})$, *where* $\delta_i = 0$ *or* $\delta_i = 1$, *is special if and only if* $\delta_i = 1$ *for* $1 \leq i \leq 3$.

Proof. It suffices to consider the classes $(\delta_1\alpha_1, \delta_2\alpha_2, \delta_3\alpha_3, \delta_4\alpha_4)$ with α_i the chosen generators of $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \chi_i)$ (see (3.5) and (3.2)), for $1 \leq i \leq 3$, and α_4 the generator of $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \mu)$ given by (3.7).

From the computations of the restriction functions it follows directly that

$$\mathrm{res}_{\langle B_i \rangle} = \begin{cases} \delta_1, & \text{if } i = 1; \\ \delta_2, & \text{if } i = 2; \\ \delta_3, & \text{if } i = 3; \end{cases}$$

proving the proposition. □

Thus, it is clear that in $\mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Lambda)$, there are at most two equivalence classes of special classes. Representatives for each of them are $\alpha_1 = (\overline{1}, \overline{1}, \overline{1}, 0)$ and $\alpha_2 = (\overline{1}, \overline{1}, \overline{1}, \overline{1})$.

Lemma 4.4. *The classes* α_1 *and* α_2 *are equivalent.*

Proof. Let $\Lambda = \langle e_1, \dots, e_n \rangle$. The action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is given by χ_1 in $\langle e_i \rangle$ if $1 \leq i \leq m_1$, by χ_2 in $\langle e_j \rangle$ if $m_1 + 1 \leq j \leq m_1 + m_2$, etc. In particular $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ acts by μ in $\Delta_s = \langle e_{m+2k+1}, e_{m+2k+2}, e_{m+2k+3} \rangle$.

We shall define an additive $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-morphism $f : \Lambda \longrightarrow \Lambda$ (i.e., a $\mathbf{Z}$-isomorphism), such that $f_*\alpha_1 = \alpha_2$. Let Δ be the submodule $\Delta = \langle e_{m_1+m_2+1}, \Delta_s \rangle$. Define $f : \Delta \longrightarrow \Delta$ by $f = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and notice that the matrix of f is unimodular. The action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ on Δ is given by

$$B_1 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & & & \\ & -1 & 1 & -1 \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}.$$

We verify that $f : \Delta \longrightarrow \Delta$ is in fact a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-morphism computing,

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ -1 & -1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & -1 & 1 & -1 \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ -1 & -1 & 1 & -1 \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & -1 & 1 & -1 \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

For $\alpha' \in \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Delta)(\simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2)$, $\alpha' = (1, 0)$ we compute $f_*\alpha'$. If $c : (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \times (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \longrightarrow \Delta$ is a cocycle representing α', then $f_*\alpha' = f_*[c] = [f \circ c]$. It is clear from the previous section that we can choose the last three coordinates in c to be zero and in the first one

$(c)_1$	B_1	B_2	B_3
B_1	0	0	0
B_2	1	0	1
B_3	-1	0	-1

The last two coordinates of the cocycle $f \circ c$ are zero, while the first two have values

$f \circ c$	B_1	B_2	B_3
B_1	$\binom{0}{0}$	$\binom{0}{0}$	$\binom{0}{0}$
B_2	$\binom{1}{1}$	$\binom{0}{0}$	$\binom{1}{1}$
B_3	$\binom{-1}{-1}$	$\binom{0}{0}$	$\binom{-1}{-1}$

Now, it is clear that $[(f \circ c)_1] \in \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \langle e_{m_1+m_2+1} \rangle)$ does not vanish and that $[(f \circ c) \restriction_{\Delta_s}] \in \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Delta_s)$ do not vanish either (see Case III in §3), therefore $[f \circ c] = (1,1) \in \mathrm{H}^2(\mathbf{Z}_2 \oplus \mathbf{Z}_2; \Delta)$. □

(4.1) *The Hantzsche-Wendt module.* Consider the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module Λ, of rank 3, given by the representation $\chi_1 \oplus \chi_2 \oplus \chi_3$. Notice that it is a faithful module and clearly $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$.

The Hantzsche-Wendt manifold (see Introduction) is built on this module (see §5). Thus we will call it the *Hantzsche-Wendt module.*

By Proposition 4.3 a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-module Λ admits a special cohomology class if and only if Λ contains a submodule equivalent to the Hantzsche-Went module.

We can state the main theorem which is now an immediate consequence of Proposition 4.3, Lemma 4.4 and (4.1).

Theorem 4.5. *The affine equivalence classes of compact Riemannian flat manifolds with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and first Betti number zero are in a bijective correspondence with the $\mathbf{Z}[\mathbf{Z}_2 \oplus \mathbf{Z}_2]$-modules Λ, such that:*

(1) *As abelian group Λ is free and of finite rank;*
(2) $\Lambda^{\mathbf{Z}_2 \oplus \mathbf{Z}_2} = 0$*;*
(3) Λ *contains a submodule equivalent to the Hantzsche-Wendt module.*

For completeness we should treat step (i) at the begining of §4. Actually, after Theorem 4.5, it would suffice to determine the semi-equivalence classes of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-modules given by

$$\begin{aligned}\rho &= m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu \\ &= (m_1, m_2, m_3, k_1, k_2, k_3, s, t)\end{aligned} \tag{4.2}$$

with $m_1, m_2, m_3 \geq 1$, which are already faithful.

For each $\sigma \in S_3 = \mathrm{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ denote by $\sigma\rho$ the representation $\rho \circ \sigma$. If ρ is given by matrices B_1, B_2 and B_3, then $\sigma\rho$ is given by $B'_1 = B_{\sigma(1)}$, $B'_2 = B_{\sigma(2)}$ and $B'_3 = B_{\sigma(3)}$.

It is clear that ρ and ρ' are semi-equivalent if and only if there exist $\sigma \in S_3$, such that $\rho' \sim \sigma\rho$.

Notice that if ρ is as in (4.2), then

$$\sigma\rho = m_1\sigma\chi_1 \oplus m_2\sigma\chi_2 \oplus m_3\sigma\chi_3 \oplus k_1\sigma\rho_1 \oplus k_2\sigma\rho_2 \oplus k_3\sigma\rho_3 \oplus s\sigma\mu \oplus t\sigma\nu.$$

Proposition 4.6. *Let ρ and ρ' be the representations given, respectively, by the 8-tuples $(m_1, m_2, m_3, k_1, k_2, k_3, s, t)$ and $(m'_1, m'_2, m'_3, k'_1, k'_2, k'_3, s', t')$. Then ρ and ρ' are semi-equivalent if and only if $s = s'$, $t = t'$ and there exist $\sigma \in S_3$, such that $m'_i = m_{\sigma(i)}$ and $k'_i = k_{\sigma(i)}$ for $1 \le i \le 3$.*

Proof. Observe that $\sigma\chi_i = \chi_{\sigma(i)}$ and, since $J \sim -J$, then also $\sigma\rho_i \sim \rho_{\sigma(i)}$. Finally, from Proposition 2.12 it follows that for any σ it verifies $\sigma\mu \sim \mu$ and $\sigma\nu \sim \nu$. Thus, the proposition is a consequence of Theorem 2.6. □

Remark. The total number of primitive $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifolds of dimension n (e_n) can be estimated by estimating the number of 8-tuples $(m_1, m_2, m_3, k_1, k_2, k_3, s, t)$ with $m_1, m_2, m_3 \ge 1$ modulo the equivalence imposed by Proposition 4.6.

It can be shown that $e_n \sim En^7$, as $n \to \infty$, with $E = \frac{1}{2^8 \cdot 3^5 \cdot 5 \cdot 7}$ (c.f. Theorem 3.5 in [**RT**]).

5. Realizations.

In this section we construct discrete subgroups Γ of isometries of $\mathbf{R}^n$ to exhibit each classified manifold as a quotient $\mathbf{R}^n/\Gamma$.

Consider the group $\Gamma = \langle \mathbf{Z}^n, B_1L_{b_1}, B_2L_{b_2}\rangle$, with B_1 and B_2 in $\mathrm{O}(n)$, b_1 and b_2 in $\mathbf{R}^n$ and $\langle B_1, B_2\rangle \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, such that Γ satisfies the exact sequence

$$0 \longrightarrow \mathbf{Z}^n \longrightarrow \Gamma \longrightarrow \langle B_1, B_2\rangle \longrightarrow 1,$$

where $\langle B_1, B_2\rangle$ acts on $\mathbf{Z}^n$ by evaluation, $B_i \cdot \lambda = B_i(\lambda)$.

Let us construct $\Gamma(m_1, m_2, m_3, k_1, k_2, k_3, s, t)$. Consider the following matrices, that we may assume orthogonal since the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is finite. Set

$$(5.1) \qquad B_1 = \begin{pmatrix} I_{m_1} &&&&&&& \\ & -I_{m_2} &&&&&& \\ && -I_{m_3} &&&&& \\ &&& -I_{2k_1} &&&& \\ &&&& K_{k_2} &&& \\ &&&&& K_{k_3} && \\ &&&&&& M^1_s & \\ &&&&&&& N^1_t \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -I_{m_1} & & & & & & & \\ & I_{m_2} & & & & & & \\ & & -I_{m_3} & & & & & \\ & & & K_{k_1} & & & & \\ & & & & -I_{2k_2} & & & \\ & & & & & -K_{k_3} & & \\ & & & & & & M_s^2 & \\ & & & & & & & N_t^2 \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix, K_m is the direct sum of m matrices $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, M_s^1 (resp. M_s^2) is the direct sum of s matrices $\left(\begin{smallmatrix} -1 & & \\ & 0 & 1 \\ & 1 & 0 \end{smallmatrix}\right)$ $\left(\text{resp. } \left(\begin{smallmatrix} -1 & 1 & -1 \\ & 0 & -1 \\ & -1 & 0 \end{smallmatrix}\right)\right)$ and N_t^1 (resp. N_t^2) is the direct sum of t matrices $\left(\begin{smallmatrix} -1 & 1 & \\ & 1 & \\ & & -1 \end{smallmatrix}\right)$ $\left(\text{resp. } \left(\begin{smallmatrix} -1 & 0 & 1 \\ & -1 & 0 \\ & & 1 \end{smallmatrix}\right)\right)$ (see 3.4).

We shall find translations L_{b_1} and L_{b_2} in order to have Γ torsionfree. We already know that every pair of translations producing a torsionfree Γ, will give isomorphic groups (see Theorem 4.5).

From Proposition 4.3 it follows that it suffices to consider the submatrices (of B_1 and B_2)

$$C_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Notice that $\mathbf{Z}^3$ with the action of $\langle C_1, C_2 \rangle$ is the Hantzsche-Wendt module (see (4.1)). Let $\Gamma_C = \langle \mathbf{Z}^3, C_1 L_{b_1}, C_2 L_{b_2} \rangle$ with $b_1 = (b_1^1, b_1^2, b_1^3)$ and $b_2 = (b_2^1, b_2^2, b_2^3)$.

The extension class $\alpha \in \mathrm{H}^2(\langle C_1, C_2 \rangle; \mathbf{Z}^3)$ of Γ_C as an extension of $\langle C_1, C_2 \rangle$ by $\mathbf{Z}^3$ is $\alpha = [f]$; where $f : \langle C_1, C_2 \rangle \times \langle C_1, C_2 \rangle \longrightarrow \mathbf{Z}^3$ is defined by $f(x, y) = s(x)s(y)s(xy)^{-1}$, for $x, y \in \langle C_1, C_2 \rangle$ and s is any section $\langle C_1, C_2 \rangle \longrightarrow \Gamma_C$.

Take the section s defined by $s(I) = I$, $s(C_i) = C_i L_{b_i}$ $(i = 1, 2)$ and $s(C_1 C_2) = s(C_1)s(C_2)$. We have explicitly,

f	C_1	C_2	C_3
C_1	$\begin{pmatrix} 2b_1^1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2b_1^1 \\ 0 \\ 0 \end{pmatrix}$
C_2	$\begin{pmatrix} -2b_1^1 \\ 2b_2^2 \\ 2(b_1^3 - b_2^3) \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2b_2^2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2b_1^1 \\ 0 \\ 2(b_1^3 - b_2^3) \end{pmatrix}$
C_3	$\begin{pmatrix} 0 \\ -2b_2^2 \\ 2(-b_1^3 + b_2^3) \end{pmatrix}$	$\begin{pmatrix} 0 \\ -2b_2^2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 2(-b_1^3 + b_2^3) \end{pmatrix}$

Since $[f] = [f]_1 + [f]_2 + [f]_3$ lies in

$$\mathrm{H}^2(\langle C_1, C_2 \rangle; \mathbf{Z}) \oplus \mathrm{H}^2(\langle C_1, C_2 \rangle; \mathbf{Z}) \oplus \mathrm{H}^2(\langle C_1, C_2 \rangle; \mathbf{Z}) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

we should choose b_1 and b_2 in such a way that $[f]_1 = 1$, $[f]_2 = 1$ and $[f]_3 = 1$. Recalling the computations in §3, it turns out that $b_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$ and $b_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$ fit our needs.

Definition. We will denote by $M(m_1, m_2, m_3, k_1, k_2, k_3, s, t)$, with $m_i \geq 1$ $(i = 1, 2, 3)$, the manifold $\mathbf{R}^n/\Gamma$, where $\Gamma = \langle \mathbf{Z}^n, B_1 l_{b_1}, B_2 L_{b_2} \rangle$ being B_1 and B_2 as in (5.1) and

$$b_1 = \Big(\underbrace{\frac{1}{2}, 0, \ldots, 0,}_{m_1+m_2} \frac{1}{2}, 0, \ldots, 0\Big) \quad \text{and} \quad b_2 = \Big(\underbrace{0, \ldots, 0,}_{m_1} \frac{1}{2}, 0, \ldots, 0\Big). \tag{5.2}$$

It follows from Theorem 4.5 that any primitive $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-manifold is affinely equivalent to one and only one of these manifolds.

Cobb's manifolds and other known families. We first observe that $M(1, 1, 1)$ is (must be) the Hantzsche-Wendt manifold.

To identify Cobb's manifolds let X, Y, r_i, s_i, t_i, u_i be as in [**Co**], for $0 \leq i \leq m - 1$. We have that $\mathcal{V} = \{u_i\}$ is a lattice Λ of rank m. Notice that the vectors u_i are orthogonal and are of the same length. One can check that, in the basis $\mathcal{V}$, X and Y act by

$$\begin{aligned} X(r_i) &= r_i, & X(s_i) &= -s_i, & X(t_i) &= -t_i; \\ Y(r_i) &= -r_i, & Y(s_i) &= s_i, & Y(t_i) &= -t_i. \end{aligned}$$

After reordering, if necessary, we may assume

$$\mathcal{V} = \{r_0, \ldots, r_{m_2-1}, s_0, \ldots, s_{m-m_2-m_1-1}, t_0, \ldots, t_{m_1-1}\}$$

hence,

$$X = \begin{pmatrix} I_{m_1} & & \\ & -I_{m_2} & \\ & & -I_{m_3} \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} -I_{m_1} & & \\ & I_{m_2} & \\ & & -I_{m_3} \end{pmatrix}.$$

Cobb considered the elements $x = B_1 L_{\frac{r_0+t_0}{2}}$ and $y = B_2 L_{\frac{r_0+s_0}{2}}$ and proved that the group $\Gamma = \langle x, y, \Lambda \rangle$ is a Bieberbach group.

Hence, it follows from Theorem 4.5 and Definition 5.1 that Cobb's family is exactly $\mathcal{C} = \{M(m_1, m_2, m_3)\}$.

It is even more clear, from Definition 5.1 and §3 of [**RT**], that the family considered in [**RT**] is exactly $\mathcal{D} = \{M(m_1, m_2, m_3, k_1, k_2, k_3)\}$.

Then, it turns out that Cobb's manifolds can be characterized as all the primitive manifolds, with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, where the holonomy representation (§2) decomposes as a sum of 1-dimensional representations and that the family considered in [**RT**] consist of all primitive manifolds, with holonomy group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, such that the holonomy representation decomposes as a sum of 1-dimensional and 2-dimensional indecomposable representations, being those in which the holonomy representation has indecomposable summands of dimension 3 all new.

Integral Homology. Since we have explicit realizations for all the manifolds classified it is not difficult to compute their first integral homology group by the formula $\mathrm{H}_1(M;\mathbf{Z}) = \Gamma/[\Gamma,\Gamma]$.

In all cases $\Gamma = \langle \omega_1, \omega_2, \Lambda\rangle$, where $\Lambda = \langle L_{e_1}, \dots, L_{e_n}\rangle$, $\omega_1 = B_1 L_{b_1}$ and $\omega_2 = B_2 L_{b_2}$ (see (5.1) and (5.2)). Hence, $[\Gamma,\Gamma] = \langle [\omega_1, L_{e_i}]; [\omega_2, L_{e_i}]; [\omega_1,\omega_2]\rangle$.

We have,

$$[\omega_1, L_{e_i}] = B_1 e_i - e_i = (B_1 - I)e_i,$$
$$[\omega_2, L_{e_i}] = B_2 e_i - e_i = (B_2 - I)e_i.$$

Since B_1 and B_2 are block diagonal, with blocks of rank 1, 2 and 3, we proceed block by block.

Rank 1. Let $\Lambda = \langle e\rangle$ and suppose $B_1 = (1)$ and $B_2 = (-1)$. We then have

$$(B_1 - I)e = 0; \qquad (B_2 - I)e = -2e.$$

Rank 2. Let $\Lambda = \langle e, f\rangle$ and suppose $B_1 = J$ and $B_2 = -J$. We get immediately

$$(B_1 - I)e = -e + f, \qquad (B_1 - I)f = e - f;$$
$$(B_2 - I)e = -e - f, \qquad (B_2 - I)f = -e - f.$$

Rank 3. Let $\Lambda = \langle e, f, g\rangle$.

Let B_1 and B_2 be the first two matrices describing μ as in (3.4). Then,

$$\begin{aligned} (B_1 - I)e &= -2e, & (B_2 - I)e &= -2e,\\ (B_1 - I)f &= -f + g, & (B_2 - I)f &= e - f - g,\\ (B_1 - I)g &= f - g, & (B_2 - I)g &= -e - f - g. \end{aligned}$$

In this case writing

$$\frac{\Lambda}{\Lambda'} = \frac{\langle e, f, g\rangle}{\langle \mathrm{Im}\,(B_1 - I), \mathrm{Im}\,(B_2 - I)\rangle} = \langle \overline{e}, \overline{f}, \overline{g}\rangle,$$

it turns out that $|\overline{e}| = 2$, $4\overline{f} = 0$, $\overline{f} = \overline{g}$ and $\overline{e} \neq \overline{f}$, from which we conclude that

$$\frac{\Lambda}{\Lambda'} \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_4.$$

Now, let B_1 and B_2 be the first two matrices describing ν as in (3.4). Then,

$$\begin{aligned} (B_1 - I)e &= -2e, & (B_2 - I)e &= -2e,\\ (B_1 - I)f &= e, & (B_2 - I)f &= -2f,\\ (B_1 - I)g &= -2g, & (B_2 - I)g &= -e. \end{aligned}$$

Writing

$$\frac{\Lambda}{\Lambda'} = \frac{\langle e, f, g\rangle}{\langle \mathrm{Im}\,(B_1 - I), \mathrm{Im}\,(B_2 - I)\rangle} = \langle \overline{e}, \overline{f}, \overline{g}\rangle,$$

it turns out that $\overline{e} = 0$, $2\overline{f} = 2\overline{g} = 0$ and $\overline{f} \neq \overline{g}$, from which we conclude that

$$\frac{\Lambda}{\Lambda'} \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

It only remains to compute

$$\begin{aligned}[\omega_1, \omega_2] &= B_1 L_{b_1} B_2 L_{b_2} L_{-b_1} B_1 L_{-b_2} B_2 \\ &= B_1 B_2 L_{B_2 b_1 + b_2 - b_1} B_1 B_2 L_{-B_2 b_2} \\ &= L_{B_1 b_1 + B_1 B_2 b_2 - B_1 B_2 b_1 - B_2 b_2}.\end{aligned}$$

Since b_1 and b_2 have only three non-zero coordinates and B_1 and B_2 preserve this subspace, we restrict our attention to this subspace. Suppose $\Lambda = \langle e_1, e_2, e_3\rangle$ and let

$$B_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix},\ B_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix},\ b_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

We have that

$$B_1 b_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix},\ B_1 B_2 b_2 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix},\ B_1 B_2 b_1 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \text{ and } B_2 b_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Hence, $[\omega_1, \omega_2] = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. By observing that $\omega_1^2 = e_1$ and $\omega_2^2 = e_2$ and by the previous computations we can state the final result.

Proposition 5.1. *If $M = M(m_1, m_2, m_3, k_1, k_2, k_3, s, t)$ then, its first homology group is $\mathrm{H}_1(M; \mathbf{Z}) = \mathbf{Z}_2^{m-3+k+s+2t} \oplus \mathbf{Z}_4^{2+s}$.*

Cohomology. The cohomology of a compact Riemannian flat manifold coincides with the cohomology of its fundamental group (see [**Ch2**, p. 98]).

Lemma 5.2 ([**Hi**])**.** *If Γ is a Bieberbach group and $\Phi \longrightarrow \mathrm{Aut}\,(\Lambda)$ is its holonomy representation then,*

$$\mathrm{H}^q(\Gamma; \mathbf{Q}) \simeq (\mathit{\Lambda}^q \Lambda_{\mathbf{Q}}^*)^{\Phi},$$

where $\mathit{\Lambda}^q$ denotes the q-th exterior power and $\Lambda_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} \Lambda$.

It is straightforward to prove the following lemma.

Lemma 5.3. *The following $\mathbf{Q}$-equivalences hold,*

$$\begin{aligned} \rho_1 &\sim \chi_2 \oplus \chi_3, \\ \rho_2 &\sim \chi_1 \oplus \chi_3, \\ \rho_3 &\sim \chi_1 \oplus \chi_2, \end{aligned} \qquad \begin{aligned} \mu &\sim \chi_1 \oplus \chi_2 \oplus \chi_3, \\ \nu &\sim \chi_1 \oplus \chi_2 \oplus \chi_3. \end{aligned}$$

Hence, if $\rho = m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus k_1\rho_1 \oplus k_2\rho_2 \oplus k_3\rho_3 \oplus s\mu \oplus t\nu$ then,

$$\rho \sim_{\mathbf{Q}} n_1\chi_1 \oplus n_2\chi_2 \oplus n_3\chi_3,$$

where

$$\begin{aligned} n_1 &= m_1 + k_2 + k_3 + s + t, \\ n_2 &= m_2 + k_1 + k_3 + s + t, \\ n_3 &= m_3 + k_1 + k_2 + s + t. \end{aligned}$$

Let $M = \Lambda_{\mathbf{Q}}$. Thus, $M = M_1 \oplus M_2 \oplus M_3$, where M_i is of rank n_i and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ acts by the character χ_i. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ be a basis for M, so that $\mathcal{B}_i$ is a basis for M_i; denote by $\mathcal{B}_1 = \{f_i\}_{i=1}^{n_1}$, $\mathcal{B}_2 = \{g_i\}_{i=1}^{n_2}$ and $\mathcal{B}_3 = \{h_i\}_{i=1}^{n_3}$. A generic element of the canonical basis of $\Lambda^q M$ is

$$e = f_{i_1} \wedge \cdots \wedge f_{i_{a_1}} \wedge g_{j_1} \wedge \cdots \wedge g_{j_{a_2}} \wedge \cdots \wedge h_{l_1} \wedge \cdots \wedge h_{l_{a_3}},$$

with $a_1 + a_2 + a_3 = q$.

Since $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ acts by characters, then

$$\dim \mathrm{H}^q(\Gamma; \mathbf{Q}) = \big|\{e \in \mathcal{B} : \ B_i e = e, \ 1 \le i \le 3\}\big|.$$

Proposition 5.4. *For* $\Gamma(m_1, m_2, m_3, k_1, k_2, k_3, s, t)$ *one has:*

$$\dim \mathrm{H}^{2p}(\Gamma; \mathbf{Q}) = \sum_{q_1+q_2+q_3=p} \binom{n_1}{2q_1}\binom{n_2}{2q_2}\binom{n_3}{2q_3},$$

$$\dim \mathrm{H}^{2r+1}(\Gamma; \mathbf{Q}) = \sum_{q_1+q_2+q_3=r-1} \binom{n_1}{2q_1+1}\binom{n_2}{2q_2+1}\binom{n_3}{2q_3+1}.$$

Proof. Let $e \in \mathcal{B}$. If $e \in (\Lambda^q M)^{\mathbf{Z}_2 \oplus \mathbf{Z}_2}$, then

$$a_1 + a_2 \equiv a_1 + a_3 \equiv a_2 + a_3 \equiv 0 \mod 2.$$

(i) If q is even, then $a_1 + a_2 + a_3 \equiv 0 \mod 2$ and $a_1 \equiv a_2 \equiv a_3 \equiv 0 \mod 2$. Conversely, if $a_i \equiv 0 \mod 2$, for $1 \le i \le 3$, then every $e \in \mathcal{B}$ of the form

$$e = f_{i_1} \wedge \cdots \wedge f_{i_{a_1}} \wedge g_{j_1} \wedge \cdots \wedge g_{j_{a_2}} \wedge \cdots \wedge h_{l_1} \wedge \cdots \wedge h_{l_{a_3}},$$

is in $(\Lambda^q M)^{\mathbf{Z}_2 \oplus \mathbf{Z}_2}$. Therefore we have the first formula.

(ii) If q is odd, then $a_1 + a_2 + a_3 \equiv 1 \mod 2$. As in the previous case we get the second formula. □

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