Pacific Journal of Mathematics

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Volume 198 No. 1 March 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 198, No. 1, 2001

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By using the Cauchy–Fantappiè machinery, the nonhomogeneous Cauchy-Riemann equation on convex domain D for $(0,q)$ form f with $\overline{\partial} f = 0$, $\overline{\partial} u = f$, has a solution which is a linear combination of integrals on bD of the following differential forms

$$\frac{1}{A^{j+1}\beta^{n-j-1}}\partial_\zeta r \wedge (\overline{\partial}_\zeta \partial_\zeta r)^j \wedge \partial_\zeta \beta \wedge \left(\sum_{i=1}^{n} d\overline{\zeta}_i \wedge d\zeta_i\right)^{n-q-3-j} \wedge \left(\sum_{i=1}^{n} d\zeta_i \wedge d\overline{z}_i\right)^{q-1} \wedge f,$$

$j = 1, \cdots, n-q-3$, where $A = \langle \partial_\zeta r(\zeta), \zeta - z\rangle$, $\beta = |z-\zeta|^2$ and r is the defining function of D. In the case of finite strict type, Bruna et al. estimated $\langle \partial r(\zeta), \zeta - z\rangle$ by the pseudometric constructed by McNeal. We can estimate the above differential forms and their derivatives. Then, by using a method of estimating integrals essentially due to McNeal and Stein, we prove the following almost sharp Hölder estimate

$$\|u\|_{C^{\frac{1}{m}-\kappa}_{0,q-1}(\overline{D})} \leq C\|f\|_{L^\infty_{0,q}(\overline{D})}, \qquad 1 \leq q \leq n-1$$

for arbitary $\kappa > 0$. The constant only depends on κ, D and q.

1. Introduction.

Let D be a convex domain in $\mathbb{C}^n$ with smooth boundary, $D = \{r < 0\}$, $bD = \{r = 0\}$ and $dr \neq 0$ on bD. Suppose the defining function r is convex near the boundary bD. For $\zeta \in \overline{D}$, $T^{\mathbb{C}}_\zeta$ denotes the complex-tangential space to $\{r = r(\zeta)\}$ at ζ. By the results in [**Mc3**], we say $p \in bD$ of *type* m if the contact order of complex lines $L \subset T^{\mathbb{C}}_\zeta$ with bD at ζ is not greater than m, for all $\zeta \in bD$.

We say D is of *strict type* if there exists $C = C(D)$ such that for all $\zeta \in bD$, all directions $v \in T^{\mathbb{C}}_\zeta(bD)$, $|v| = 1$, and small t

$$\frac{1}{C} r(\zeta + tv) \leq r(\zeta + t\sqrt{-1}v) \leq C r(\zeta + tv). \tag{1.1}$$

The condition implies that the order of contact with bD at ζ of $\{\zeta + tv\}$ and $\{\zeta + t\sqrt{-1}v\}$ is the same. If D is both of finite type and of strict type, we say that D has *finite strict type*.

Now consider the *nonhomogeneous Cauchy-Riemann equation* on D

$$\overline{\partial} u = f \tag{1.2}$$

with $f \in L^{\infty}_{0,q}(\overline{D})$, where $L^{\infty}_{0,q}(\overline{D})$ is the space of $(0,q)$ forms on D with coefficients in $L^{\infty}(\overline{D})$. When D is of type m, i.e., each point $p \in bD$ is of type less than or equal to m, it is natural to expect that the following Hölder estimate holds: For $f \in L^{\infty}_{0,q}(\overline{D})$ and $\overline{\partial} f = 0$, Equation (1.2) has a solution u satisfying

$$\|u\|_{C^{\frac{1}{m}}_{0,q-1}(\overline{D})} \leq C\|f\|_{L^{\infty}_{0,q}(\overline{D})}, \qquad 1 \leq q \leq n-1, \tag{1.3}$$

where C is a constant depending on D, q.

One method to study this problem is to establish an integral representation formula using the Cauchy-Fantappiè machinery (see [**R2**]). The key point of the Cauchy-Fantappiè machinery is to find a barrier form

$$w(\zeta, z) = \sum_{i=1}^{n} w_i(\zeta, z) d\zeta_i \tag{1.4}$$

satisfing

$$\sum_{i=1}^{n} w_i(\zeta, z)(\zeta_i - z_i) = 1 \tag{1.5}$$

for $\zeta \in bD, z \in D$, where $w_i(\zeta, z), i = 1, \cdots, n$, are holomorphic in z. For the strongly pseudoconvex domains, this barrier form, hence the integral representation formula, is constructed and the sharp estimate is obtained (see [**R2**], for example). For the weakly pseudoconvex domains, little is known, but some important results have been obtained. Range [**R1**] proved sharp Hölder estimate for some convex domains of finite type in $\mathbb{C}^2$ and generalized by Bruna and Castillo [**BC**]. Fornæss [**Fo**] constructed a barrier form for a kind of pseudoconvex domains of finite type in $\mathbb{C}^2$ and proved sup norm estimate. Diederich et al. [**DFW**] and Chen et al. [**CKM**] obtained the sharp estimate for ellipsoids. By using Skoda's estimates, Range [**R3**] constructed a barrier form for pseudoconvex domain of finite type in $\mathbb{C}^2$. His method was generalized to pseudoconvex domains in $\mathbb{C}^n$ by Michel [**M**]. We should also mention the work of Chaumat and Chollet for convex domain which needn't be pseudoconvex. Chaumat and Chollet's and Michel's results, which are far from sharp, are C^{∞} estimates. Bruna et al. [**BCD**] proved L^1 estimate for $(0,1)$ forms on convex domains of finite strict type. Hölder estimates can also be obtained by studying the associated $\overline{\partial}_b$ and singular integral operators on the boundary (see [**FK**], [**FKM**], [**Ch**]).

When the domain is convex, there is a natural barrier form

$$w(\zeta, z) = \frac{\partial r(\zeta)}{\langle \partial r(\zeta), \zeta - z\rangle} \tag{1.6}$$

where r is the defining function of the domain,

(1.7)

$$\partial r(\zeta) = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta) d\zeta_i \qquad \text{and} \qquad \langle \partial r(\zeta), \zeta - z\rangle = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta)(\zeta_i - z_i).$$

By using the barrier form, we can construct the integral representation formula for $(0, q)$ form by applying Cauchy-Fantappiè machinery. Therefore, to estimate the solution of $\overline{\partial}$, we need to estimate the quantity $\langle \partial r(\zeta), \zeta - z\rangle$ for $\zeta \in bD, z \in D$. Thanks to the work of McNeal [**Mc4**] and [**BCD**], there exists a a pseudometric on D, and we can estimate $\langle \partial r(\zeta), \zeta - z\rangle$ by this pseudometric in the case of finite strict type. Then the integral kernel and its derivatives are estimated. All these results allow us to prove the following theorem.

Theorem 1.1. *Let $D \subset\subset C^n$ be a bounded convex domain of strict finite type m, and let $f \in L^{\infty}_{0,q}(\overline{D})$ be $\overline{\partial}$-closed, $1 \le q \le n-1$. Then the nonhomogeneous Cauchy-Riemann Equation* (1.2) *has a solution u satisfying*

$$\|u\|_{C^{\frac{1}{m}-\kappa}_{0,q-1}(\overline{D})} \le C\|f\|_{L^{\infty}_{0,q}(\overline{D})} \tag{1.8}$$

for arbitary $\kappa > 0$. The constant only depends on κ, D and q.

Here is the plan of this paper. In Section 2, we state McNeal's pseudometric and some propositions for our purpose. In Section 3, we deduce the integral representation formula and some estimates associated to it. In Section 4, we estimate the derivatives of the integral kernel. Then we use a method essentially due to McNeal and Stein [**MS**] to prove the main Theorem.

In this paper, we will us the following notations. The expression $X \lesssim Y$ and $X \gtrsim Y$ means that there exists some constant $C > 0$, which is independent on the obvious parameters, so that $X \le CY$ and $Y \ge CY$, respectively. $X \approx Y$ means $X \lesssim Y$ and $Y \lesssim X$ simultaneously.

2. McNeal's pseudometric on the convex domain of finite type.

Let $S^n = \{\zeta \in \mathbb{C}^n; |\zeta| = 1\}$. Each element of S^n, together with a point $q \in \mathbb{C}^n$, determine a complex line in $\mathbb{C}^n$. Without loss of generality, then, we may assume that our defining function r has the property that all the set $\{z; r(z) < \eta\}$ are convex for $-\eta_0 < \eta < \eta_0$, for some $\eta_0 > 0$. If $\gamma \in S^n$, $-\eta_0 < \eta < \eta_0$, we denote the distance from q to the level set $\{z; r(z) = \eta\}$ along the complex line γ by $\delta_\eta(q, \gamma)$.

Now recall the definition of ε-*extremal basis of McNeal*. Let $bD_{q,\varepsilon} = \{z \in U; r(z) = r(q)+\varepsilon\}$. Let $p \in bD$. If U is sufficiently small a neighbourhood of p , the distance from $q \in U$ to $bD_{q,\varepsilon}$ is well defined. Let n be the real normal line of $\{z, r(z) = r(q)\}$ at q and p_1 be the intersection of n with $bD_{q,\varepsilon}$. Set $\tau_1(q,\varepsilon) = |q-p_1|$. Choose a parametrization of the complex line from q to p_1, by z_1, with $z_1(0) = q$ and p_1 lying on the positive $\mathrm{Re} z_1$ axis. Now consider the orthogonal complement of the span of the coordinate z_1, OC_1, which is the complex subspace of the tangent space to $\{z; r(z) = r(q)\}$ at q. For any $\gamma \in OC_1 \cap S^n$, compute $\delta_{r(q)+\varepsilon}(q,\gamma)$. Let $\tau_2(q,\varepsilon)$ be the largest such distance and $p_2 \in bD_{q,\varepsilon}$ be any point achieving this distance. The coordinate z_2 is defined by parametrization the complex line from q to p_2 on such a way that $z_2(0) = q$ and p_2 lying on the positive $\mathrm{Re} z_2$ axis. Continuing this process, we obtain the n coordinate functions $z_1, \cdots z_n$, n quantities $\tau_1(q,\varepsilon), \cdots, \tau_n(q,\varepsilon)$ and n points $p_1, \cdots, p_n$. Let $z_j = x_j + \sqrt{-1}x_{n+j}$ for $1 \le j \le n$. Following [**BCD**], we call coordinates $\{z_1, \cdots, z_n\}$ $\varepsilon-$*extremal coordinates centered at* q.

If we define the polydisc

$$P_\varepsilon(q) = \{z \in U; |z_1| < \tau_1(q,\varepsilon), \dots, |z_n| < \tau_n(q,\varepsilon)\}, \tag{2.1*}$$

then there exist a constant $C > 0$, independent on $q \in U \cap D$, such that $CP_\varepsilon(q) \subset \{z \in U; r(z) < r(q) + \varepsilon\}$.

It is obvious that $\tau_1(q,\varepsilon) \approx \varepsilon$. Let

$$A^i_k(q) = |a^i_k(q)|, \qquad a^i_k(q) = \frac{\partial^k}{\partial x_i^k} r(z(0,\dots,0,x_i,0,\dots,0)), \qquad 2 \le i \le n \tag{2.1}$$

and set

$$\sigma_i(q,\varepsilon) = \min\left\{\left(\frac{\varepsilon}{A^i_k(q)}\right)^{\frac{1}{k}},\ 2 \le k \le m\right\}, \tag{2.2}$$

then

$$\sigma_i(q,\varepsilon) \approx \tau_i(q,\varepsilon) \tag{2.3}$$

[**Mc4**]. It follows directly from (2.1)-(2.3) that for each $2 \le i \le n$,

$$\left|\frac{\partial^k}{\partial x_i^k} r(q)\right| \lesssim \varepsilon \tau_i(q,\varepsilon)^{-k},\ k = 1,\dots,m. \tag{2.4}$$

We have the following three propositions. See [**Mc4**] for their proofs.

Proposition 2.1. *For $q \in U \cap D$, let $\gamma_1, \dots, \gamma_n$ be the orthogonal unit vector determined by ε-extremal coordinate centered at q. Suppose $\lambda \in S^n$*

can be written as $\lambda = \sum_{i=1}^{n} a_i\gamma_i$, $a_i \geq 0, \sum_{i=1}^{n} a_i = 1$. *For small* $\varepsilon > 0$, *set* $\eta = r(q) + \varepsilon$, *then*

$$\left(\sum_{i=1}^{n} \frac{a_i}{\tau_i(q,\varepsilon)}\right)^{-1} \approx \delta_\eta(q,\varepsilon). \tag{2.5}$$

Proposition 2.2. *There is a constant* C *independent of* $q, q^1, q^2 \in D \cap U, \varepsilon > 0$, *so that if* $P_\varepsilon(q^1) \cap P_\varepsilon(q^2) \neq \emptyset$, *then*

$$P_\varepsilon(q^1) \subset CP_\varepsilon(q^2) \text{ and } P_\varepsilon(q^2) \subset CP_\varepsilon(q^1) \tag{2.6}$$

and

$$P_{r\varepsilon}(q) \subset CP_\varepsilon(q), \qquad 0 \leq r \leq 2. \tag{2.7}$$

When $r = 2$, (2.7) is Proposition 2.5 in [**Mc4**]. His proof works in the case of $0 \leq r \leq 2$ (because $\tau_i(q, r\varepsilon) \approx \sigma_i(q, r\varepsilon) \approx \min\{(\frac{r\varepsilon}{A_k^i(q)})^{\frac{1}{k}}, 2 \leq k \leq m\} \lesssim \sigma_i(q,\varepsilon)$). Note $r\varepsilon$-extremal coordinates centered at q may be different from ε-coordinates centered at q ($r \leq 1$). $P_{r\varepsilon}(q) \subset P_\varepsilon(q)$ may not hold for $0 \leq r \leq 1$.

Suppose $q^1, q^2 \in U \cap D$, define

$$d(q^1, q^2) = \inf\{\varepsilon; q^2 \in P_\varepsilon(q^1)\}, \tag{2.8}$$

where $P_\varepsilon(q^1)$ defined by (2.1).

Proposition 2.3. $d(\cdot,\cdot)$ *defines a local pseudometric on* $U \cap D$, *i.e., for* $q^1, q^2, q^3 \in U \cap D$,

(1) $d(q^1, q^2) = 0$ *iff* $q^1 = q^2$;
(2) $d(q^1, q^2) \approx d(q^2, q^1)$;
(3) $d(q^1, q^3) \lesssim d(q^1, q^2) + d(q^2, q^3)$.

Corollary 2.4. *Let* $\varepsilon > 0, q, q' \in U \cap D$, $\varepsilon \leq d(q, q') \leq 2\varepsilon$, *then, in the* ε*-extremal coordinates centered in* q, $q' = (q'_1, \cdots, q'_n)$,

$$d(q, q') \approx |q'_1| + \sum_{i=2}^{n}\sum_{l=2}^{m} A_l^i(q)|q'_i|^l. \tag{2.9}$$

Proof. Since q' lies in the boundary of polydisc $P_{d(q,q')}(q)$, and $\frac{1}{C}P_\varepsilon(q) \subset P_{d(q,q')}(q) \subset CP_\varepsilon(q)$ for some constant $C > 0$, by Proposition 2.2, we find

$$|q'_i| \leq C\tau_i(q,\varepsilon), \qquad i = 1, \cdots, n, \tag{2.10}$$

and there exists i_0 such that $|q'_{i_0}| \geq \frac{1}{C}\tau_{i_0}(q,\varepsilon)$. Thus

$$A_l^{i_0}(q)|q'_{i_0}|^l \gtrsim \varepsilon \tag{2.11}$$

for some l by (2.1)-(2.3), the right side of (2.9) $\gtrsim \varepsilon$. The right side of (2.9) $\lesssim \varepsilon$ by (2.10) and (2.1)-(2.3).

Note (2.9) may not hold for each $q' \in U \cap D$ because ε-extremal coordinates centered at q may change abruptly as ε. We can also prove:

Lemma 2.5. *If $d(z,\zeta) \leq \varepsilon$, then, in the ε-extremal coordinates $\{w_1, \dots, w_n\}$ centered at z, $\zeta = \{\zeta_1, \cdots, \zeta_n\}$*

$$|D^\beta r(\zeta)| \lesssim \frac{\varepsilon}{\tau(z,\varepsilon)^\beta} \tag{2.12}$$

for all multiindices $\beta = (\beta_1, \overline{\beta}_1, \dots, \overline{\beta}_n)$, where

$$D^\beta r = \frac{\partial^{\beta_1 + \overline{\beta}_1 + \cdots + \overline{\beta}_n} r}{\partial \zeta_1^{\beta_1} \partial \overline{\zeta}_1^{\overline{\beta}_1} \cdots \partial \overline{\zeta}_n^{\overline{\beta}_n}}, \qquad \tau^\beta(z,\varepsilon) = \tau_1^{\beta_1 + \overline{\beta}_1}(z,\varepsilon) \cdots \tau_n^{\beta_n + \overline{\beta}_n}(z,\varepsilon).$$

Proof. Note (2.12) is obvious by $|\tau(z,\varepsilon)^\beta| \lesssim \varepsilon$ if $|\beta| = \sum_{i=1}^n \beta_i + \overline{\beta}_i \geq m$. When $\zeta = z$,

$$|D^\beta r(z)| \lesssim \frac{\varepsilon}{\tau_i(z,\varepsilon)^\beta} \tag{2.13}$$

is proved for β with $\beta_i + \overline{\beta}_i \neq 0$ only for two i in [**BCD**, p. 398-399], their proof works in the general case. Since $P_{d(z,\zeta)}(z) \subset CP_\varepsilon(z)$ for constant C by Proposition 2.2, $|\zeta_i| \lesssim \tau_i(z,\varepsilon)$, $i = 1, \dots, n$. Then

$$\left| \frac{\partial r}{\partial w_i}(\zeta) - \frac{\partial r}{\partial w_i}(z) \right| \lesssim \sum_{|\beta|=1}^{m} D^\beta \frac{\partial r}{\partial w_i}(z) \tau^\beta(z,\varepsilon) + \circ(|\tau(z,\varepsilon)|^m), \tag{2.14}$$

where $|\tau(z,\varepsilon)| = \max_{1 \leq k \leq m} \tau_k(z,\varepsilon)$. It

$$\lesssim \sum_{|\beta|=1}^{m} \varepsilon \tau^{-\beta}(z,\varepsilon) \tau_i^{-1}(z,\varepsilon) \tau^\beta(z,\varepsilon) + \circ(|\tau(z,\varepsilon)|^m) \lesssim \frac{\varepsilon}{\tau_i(z,\varepsilon)}$$

by (2.13) and $|\tau(z,\varepsilon)| \lesssim \varepsilon^{\frac{1}{m}}$. Thus

$$\left| \frac{\partial r}{\partial w_i}(\zeta) \right| \lesssim \frac{\varepsilon}{\tau_i(z,\varepsilon)}.$$

For general β, (2.12) can be proved similarly. This completes the proof of Lemma 2.5.

3. The integral representation formula and some estimates.

It is well known that for convex domains with smooth boundaries, we have the following explicit integral representation for $\overline{\partial}$ problem.

Proposition 3.1 ([**R2**, p. 176])**.** *Let $D \subset\subset C^n$ be convex with smooth boundary and let $r \in C^2$ be a defining function for D. Let*

$$C^{(r)}(\zeta, z) = \frac{\partial r(\zeta)}{\langle \partial r(\zeta), \zeta - z \rangle}, \tag{3.1}$$

where $\langle \partial r(\zeta), \zeta - z\rangle = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta)(\zeta_i - z_i)$ *and*

$$\hat{C}^{(r)}(\zeta, z) = \lambda C^{(r)}(\zeta, z) + (1-\lambda)B(\zeta, z), \tag{3.2}$$

where $0 < \lambda < 1$*,*

$$B(\zeta, z) = \frac{\partial \beta}{\beta}, \quad \beta = |z-\zeta|^2. \tag{3.3}$$

Define the Cauchy-Fantappie kernel associated to $\hat{C}^{(r)}$ *by*

$$\Omega_q(\hat{C}^{(r)}) = \frac{(-1)^{\frac{q(q-1)}{2}}}{(2\pi i)^n}\binom{n-1}{q}\hat{C}^{(r)} \wedge (\overline{\partial}_{\zeta,\lambda}\hat{C}^{(r)})^{n-q-1} \wedge (\overline{\partial}_z \hat{C}^{(r)})^q \tag{3.4}$$

for $0 \leq q \leq n-1$*,* $\overline{\partial}_{\zeta,\lambda} = \overline{\partial}_\zeta + d_\lambda$*, and the Bochner-Martinelli-Koppelman kernel*

$$K_q = \frac{(-1)^{\frac{q(q-1)}{2}}}{(2\pi i)^n}\binom{n-1}{q} B \wedge (\overline{\partial}_\zeta B)^{n-q-1} \wedge (\overline{\partial}_z B)^q. \tag{3.5}$$

Then the operator $T_q^{(r)} : C_{0,q}(D) \longrightarrow C_{0,q-1}(D)$ *defined by*

$$T_q^{(r)} f = \int_{bD\times I} f \wedge \Omega_{q-1}(\hat{C}^{(r)}) - \int_D f \wedge K_{q-1}, \qquad I = [0,1] \tag{3.6}$$

satisfies

$$\overline{\partial} T_q^{(r)} f = f$$

on D *if* $f \in C_{0,q}(D)$ *and* $\overline{\partial} f = 0$*.*

We decompose

$$\Omega_q(\hat{C}^{(r)}) = \Omega_q^{(1)} \wedge d\lambda + \Omega_q^{(0)}, \tag{3.7}$$

where $\Omega_q^{(0)}, \Omega_q^{(1)}$ is of degree 0 in λ. By simple calculation, we can prove the following:

Lemma 3.2 ([**R2**, p. 206])**.** *For* $0 \leq q \leq n-2$ *and* $f \in C_{0,q+1}(bD)$*, one has*

$$\int_{bD\times I} f \wedge \Omega_q(\hat{C}^{(r)}) = \int_{bD} f \wedge A_q(C^{(r)}; B) \tag{3.8}$$

where

$$A_q(C^{(r)}; B) = \sum_{j=0}^{n-q-2}\sum_{k=0}^{q} a_q^{j,k} A_q^{j,k}(C^{(r)}; B) \tag{3.9}$$

with universal constants $a_q^{j,k}$ *and*

$$\begin{aligned} A_q^{j,k}&(C^{(r)}; B) \\ &= C^{(r)} \wedge B \wedge (\overline{\partial}_\zeta C^{(r)})^j \wedge (\overline{\partial}_\zeta B)^{n-q-2-j} \wedge (\overline{\partial}_z C^{(r)})^k \wedge (\overline{\partial}_z B)^{q-k}. \end{aligned} \tag{3.10}$$

A straightforward computation gives

(3.11)

$$A_q^{j,k}(C^{(r)};B) = \frac{\partial_\zeta r(\zeta)\wedge\partial_\zeta\beta\wedge(\overline{\partial}_\zeta\partial_\zeta r)^j\wedge(\overline{\partial}_\zeta\partial_\zeta\beta)^{n-q-2-j}\wedge(\overline{\partial}_z\partial_\zeta r)^k\wedge(\overline{\partial}_z\partial_\zeta\beta)^{q-k}}{\langle\partial r(\zeta),\zeta-z\rangle^{j+k+1}\beta^{n-(j+k+1)}}$$

(see [**R2**, p. 206] for a general formula) and

$$A_q^{j,k}=0, \quad \text{if } \ k\geq 1$$

by $\overline{\partial}_z\partial_\zeta r(\zeta)=0$.

Because the Bochner-Martinelli-Koppelman kernel K_{q-1} is a kind of Caldéron-Zygmund kernel, we have the following regularity result. See [**R2**, p. 156], for example.

Proposition 3.3. *For $0<\alpha<1$, we have*

$$\left\|\int_D f\wedge K_{q-1}\right\|_{C^\alpha_{0,q-1}(D)} \lesssim \|f\|_{L^\infty_{0,q}(D)}. \tag{3.12}$$

In the kernels in (3.11), there is a factor $\langle\partial r(\zeta),\zeta-z\rangle(\zeta\in bD)$ in the dominators. We should estimate this quantity.

Proposition 3.4 ([**BCD**, Lemma 4.2])**.** *If D is of finite strict type, then*

$$d(\zeta,z)\approx|\langle\partial r(\zeta),\zeta-z\rangle| \tag{3.13}$$

for $\zeta\in bD, z\in\overline{D}$.

In order to prove the Hölder estimate in the main Theorem 1.1, we will use the following elementary real variable fact.

Lemma 3.5 ([**R2**, p. 204])**.** *Let $D\subset\subset R^n$ be a bounded domain with C^1 boundary. Suppose g differentiable on D and that for some $0<\alpha<1$, there is a constant C such that*

$$|dg(x)|\lesssim C\delta_D(x)^{\alpha-1}, \quad x\in D, \tag{3.14}$$

where $\delta_D(x)$ is the distance from x to the boundary bD. Then $g\in C^\alpha(\overline{D})$ and there exists a compact subset K of D such that

$$\|g\|_{C^\alpha(\overline{D})}\lesssim C+\|g\|_{L^\infty(K)}. \tag{3.15}$$

Therefore, if we can prove the following proposition, the Hölder estimate of $\overline{\partial}$-problem is proved by Lemma 3.5.

Proposition 3.6. *Using the above notation, we have*

$$\left|\int_{bD} d_z A_q^{j,0}(z,\zeta)\wedge f\right|\lesssim\delta_D(z)^{-1+\frac{1}{m}-\kappa}\|f\|_{L^\infty_{0,q}} \tag{3.16}$$

for $0<\kappa<1-\frac{1}{m}$, where the constant depends only on D,j,q,κ.

Theorem 1.1 for $\kappa \geq 1 - \frac{1}{m}$ obviously follows from $\kappa < 1 - \frac{1}{m}$. We will prove (3.16) in Section 4.

We will need the following estimates.

Lemma 3.7. *If $a, b, a', \alpha, \alpha' > 0, k \geq 1$, then*

$$\int_{\mathbb{C}} \frac{b^{\frac{1}{k}}}{(a+b|\zeta|^k)^{\alpha+\frac{1}{k}}} \cdot \frac{1}{(a'+|\zeta|)^{\alpha'+1}} dV(\zeta) \lesssim \frac{1}{a^\alpha}\frac{1}{a'^{\alpha'}}, \tag{3.17}$$

where $dV(\zeta)$ is the volume element of $\mathbb{C}$.

Proof. Denote $\zeta = x + iy$, then the integral

$$\begin{aligned} &\leq \int_{\mathbb{R}} \frac{b^{\frac{1}{k}}}{(a+b|x|^k)^{\alpha+\frac{1}{k}}} dx \cdot \int_{\mathbb{R}} \frac{dy}{(a'+|y|)^{\alpha'+1}} \\ &= \int_{\mathbb{R}} \frac{dx}{(a+|x|^k)^{\alpha+\frac{1}{k}}} \cdot \int_{\mathbb{R}} \frac{dy}{(a'+|y|)^{\alpha'+1}} \\ &\lesssim \frac{1}{a^\alpha a'^{\alpha'}}. \end{aligned}$$

Lemma 3.8. *If $a, b, \alpha > 0, k \geq 1$, then*

$$\int_{\mathbb{C}} \frac{b^{\frac{2}{k}}}{(a+b|\zeta|^k)^{\alpha+\frac{2}{k}}} dV(\zeta) \lesssim \frac{1}{a^\alpha}, \tag{3.18}$$

where $dV(\zeta)$ is the volume element of $\mathbb{C}$.

Proof. Define

$$D_1 = \{\zeta; b|\zeta|^k \geq a\} \tag{3.19}$$

and

$$D_2 = \{\zeta; b|\zeta|^k < a\}. \tag{3.20}$$

It follows that on the region D_1, we have

$$\begin{aligned} \int_{D_1} \frac{b^{\frac{2}{k}}}{(a+b|\zeta|^k)^{\alpha+\frac{2}{k}}} dV(\zeta) &\lesssim \int_{D_1} \frac{b^{\frac{2}{k}}}{(b|\zeta|^k)^{\alpha+\frac{2}{k}}} dV(\zeta) \\ &\lesssim \int_L^\infty b^{-\alpha}(\rho^k)^{-\frac{2}{k}-\alpha}\rho d\rho \lesssim a^{-\alpha}, \end{aligned}$$

where $L = (\frac{a}{b})^{\frac{1}{k}}$. On the region D_2, we obtain the same upper bound

$$\int_{D_2} \frac{b^{\frac{2}{k}}}{(a+b|\zeta|^k)^{\alpha+\frac{2}{k}}} dV(\zeta) \lesssim \frac{b^{\frac{2}{k}}}{a^{\frac{2}{k}+\alpha}} \mathrm{Vol}(D_2) \lesssim a^{-\alpha}.$$

This completes the proof of Lemma 3.8.

Lemmas 3.7 and 3.8 were used in [**MS**] implicitly to estimate the Bergman projection operator in the convex domain of finite type.

4. The estimate of the integral.

The purpose of this section is to prove Proposition 3.6.

Let β be a multiindex, $\beta = (\beta_1, \overline{\beta}_1, \dots, \overline{\beta}_n)$. Define

$$D^\beta = \frac{\partial^{\beta_1}}{\partial z_1^{\beta_1}} \frac{\partial^{\overline{\beta}_1}}{\partial \overline{z}_1^{\overline{\beta}_1}} \cdots \frac{\partial^{\beta_n}}{\partial z_n^{\beta_n}} \frac{\partial^{\overline{\beta}_n}}{\partial \overline{z}_n^{\overline{\beta}_n}}, \qquad |\beta| = \sum_{i=1}^n \beta_i + \overline{\beta}_i \tag{4.1}$$

and

$$\tau^\beta(w,\varepsilon) = \tau^{\beta_1+\overline{\beta}_1}(w,\varepsilon) \cdots \tau_n^{\beta_n+\overline{\beta}_n}(w,\varepsilon) \tag{4.2}$$

for $w \in D \cap U, \varepsilon > 0$. If $0 \le \beta_i, \overline{\beta}_i \le 1$, define

$$d\zeta^\beta = d\zeta_1^{\beta_1} \wedge d\overline{\zeta}_1^{\overline{\beta}_1} \wedge \cdots \wedge d\overline{\zeta}_n^{\overline{\beta}_n}, \tag{4.3}$$

where $d\zeta_i$ or $d\overline{\zeta}_i$ absents if $\beta_i = 0$ or $\overline{\beta}_i = 0$, respectively.

Suppose bD is covered by $U_1, \dots, U_N$, where U_i are balls centered at $z_i \in bD$ with radius r_i. Furthermore, all results of Proposition 2.1-2.5 hold for $U = U_i', i = 1, \dots, N$, where $U_i' = B(z_i, 2r_i)$. Let $V = \cup_{i=1}^N U_i'$ and set

$$A_{q-1}^{j,0} = \sum_J A_J^j d\overline{z}^J \tag{4.4}$$

where A_J^j are $(n, n-q-1)$ forms in ζ and $\overline{\zeta}$, and J takes over all multiindices with $|J| = q-1, 1 \le J_i, \overline{J}_i \le 1$. The estimate for $d_z A_J^j$ is as follows. We will use the following notation. For (p,p') differential form $A(\zeta)$, $\|A(\zeta)\|_{bD}$ denote the norm of $A(\zeta)$ acting on $(\bigotimes T_\zeta(bD))^{p+p'}$.

Proposition 4.1.

(1) *If $z \in U_i$ for some i, then*

$$\|d_z A_J^j(z,\zeta)\|_{bD} \lesssim \sum_\beta \frac{1}{\tau^\beta(z,\varepsilon)|z-\zeta|^{2n-2j-3}} \tag{4.5}$$

for $\zeta \in bD \cap U_i'$, where $\varepsilon = d(z,\zeta)$, and β takes over all multiindices satisfying the following condition C:

(C1) $\sum_{i=1}^n \beta_i + \overline{\beta}_i = 2j+2$;

(C2) *There exists at most one $i_0 > 1$ such that $\beta_{i_0} + \overline{\beta}_{i_0} = 3$ and $\beta_l + \overline{\beta}_l \le 2$ for all $l \ne i_0$. If such i_0 exists, we must have $\beta_1 + \overline{\beta}_1 = 1$.*

(2) *If $\zeta \notin U_i'$, then*

$$\|d_z A_J^j\|_{bD} \lesssim 1. \tag{4.6}$$

Note

$$\left|\int_{bD\cap U_i'} d_zA_J^j(z,\zeta)\wedge f\right| \lesssim \int_{bD\cap U_i'} \|d_zA_J^j(z,\zeta)\wedge f\|_{bD}dV(\zeta) \tag{4.7}$$
$$\lesssim \|f\|_{L^\infty_{0,q}}\int_{bD\cap U_i'}\|d_zA_J^j(z,\zeta)\|_{bD}dV(\zeta),$$

where $dV(\zeta)$ is the volume element of bD, and

$$\int_{bD\setminus U_i'}\|d_zA_J^j\|_{bD}\lesssim 1 \tag{4.8}$$

by (4.6) for $z\in U_i$, and $|r(z)|\approx\delta_D(z)$, the proof of Proposition 3.6 is reduced to the following estimate.

Lemma 4.2. *For β satisfying condition C and $z\in U_i$ for some i, we have*

$$I_\beta=\int_{bD\cap U_i'}\frac{1}{\tau^\beta(z,\varepsilon)|z-\zeta|^{2n-2j-3}}dV(\zeta)\lesssim|r(z)|^{-1-\kappa+\frac{1}{m}},\qquad \varepsilon=d(z,\zeta) \tag{4.9}$$

for $0<\kappa<1-\frac{1}{m}$, where $dV(\zeta)$ is the volume element of bD.

Before we begin to prove Proposition 4.1, we give a lemma. Since

$$\overline{\partial}_\zeta\partial_\zeta\beta=\sum_{i=1}^n d\overline{\zeta}_i\wedge d\zeta_i, \tag{4.10}$$
$$\overline{\partial}_z\partial_\zeta\beta=-\sum_{i=1}^n d\overline{z}_i\wedge d\zeta_i,$$
$$\overline{\partial}_z\beta=\sum_{i=1}^n(z_i-\zeta_i)d\overline{z}_i,$$

we get

$$A_{q-1}^{j,0}=\frac{1}{A^{j+1}\beta^{n-j-1}}\partial_\zeta r\wedge(\overline{\partial}_\zeta\partial_\zeta r)^j\wedge\partial_\zeta\beta \tag{4.11}$$
$$\wedge\left(\sum_{i=1}^n d\overline{\zeta}_i\wedge d\zeta_i\right)^{n-q-3-j}\wedge\left(\sum_{i=1}^n d\zeta_i\wedge d\overline{z}_i\right)^{q-1},$$

where $A=\langle\partial_\zeta r(\zeta),\zeta-z\rangle$. Set

$$C=\partial_\zeta r\wedge(\overline{\partial}_\zeta\partial_\zeta r)^j. \tag{4.12}$$

Lemma 4.3. *For $z\in U_i,\zeta\in bD\cap U_i'$ for some i, we have*

$$\|C\|_{bD}\lesssim\sum_L\frac{\varepsilon^{j+1}}{\tau^L(z,\varepsilon)},\qquad d_zC=0,\qquad \varepsilon=d(z,\zeta), \tag{4.13}$$

where the sum takes over all multiindices satisfiying

$$0 \le L_i, \overline{L}_i \le 1, \qquad \sum_{i=1}^{n} L_i + \overline{L}_i = 2j+1, \qquad L_1 = 0.$$

Proof of Lemma 4.3. $d_z C = 0$ is obvious since C does not depend on z. Now fix $z \in U_i$. Note formula (3.11) for $A_q^{j,0}$ is stated in the standard coordinates $\zeta_1, \cdots, \zeta_n$ in $\mathbb{C}^n$. Denote the $d(z,\zeta)$-extremal coordinates centered at z by $w_1, \cdots, w_n$. Then there exists an unitary matrix U_z, which is only depending on z, and the translation T_z from the origin to z, such that $U_z \circ T_z$ transforms coordinates $\zeta_1, \cdots, \zeta_n$ to coordinates $w_1, \cdots, w_n$. It follows from the invariance of differential forms under a linear transform that we can write $\partial_\zeta r, \overline{\partial}_\zeta \partial_\zeta r$ in coordinates $w_1, \cdots, w_n$ as

$$\partial_\zeta r = \partial_w r, \qquad \overline{\partial}_\zeta \partial_\zeta r = \overline{\partial}_w \partial_w r. \tag{4.14}$$

Thus

$$\begin{aligned} C &= \partial_w r \wedge (\overline{\partial}_w \partial_w r)^j \\ &= \sum_{\substack{l_1,\dots,l_j \\ t,k_1,\dots,k_j}} \frac{\partial r}{\partial w_t} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial \overline{w}_{l_i} \partial w_{k_i}} \cdot dw_t \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_j} \wedge dw_{k_j}, \end{aligned} \tag{4.15}$$

where $l_1 \cdots, l_j$ are different, and $t, k_1, \cdots, k_j$ are different. Notice $dr = 0$ when restricted to the space tangential to bD, we find that

$$\begin{aligned} \frac{\partial r}{\partial w_1} dw_1 = -\frac{\partial r}{\partial w_2} dw_2 - \cdots - \frac{\partial r}{\partial w_n} dw_n \\ - \frac{\partial r}{\partial \overline{w}_1} d\overline{w}_1 - \frac{\partial r}{\partial \overline{w}_2} d\overline{w}_2 - \cdots - \frac{\partial r}{\partial \overline{w}_n} d\overline{w}_n \end{aligned} \tag{4.16}$$

holds on tangential space $T(bD)$, dw_1 disappeared in the differential forms in the right side of (4.15) if we substitute (4.16) into (4.15) (see [**CKM**,

p. 133] for the same fact). Note if $k_s = 1$, and substitute (4.16) into (4.15),

(4.17)

$$\frac{\partial r}{\partial w_t} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial \overline{w}_{l_i} \partial w_{k_i}} \cdot dw_t \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_j} \wedge dw_{k_j}$$

$$= - \sum_{v \neq 1, t, k_1, \ldots, k_j} \frac{\partial r}{\partial w_t} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial \overline{w}_{l_i} \partial w_{k_i}} \cdot \frac{\frac{\partial r}{\partial w_v}}{\frac{\partial r}{\partial w_1}} dw_t \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_s}$$

$$\wedge dw_v \wedge \cdots \wedge dw_{k_j} - \sum_{v \neq l_1, \ldots, l_j} \frac{\partial r}{\partial w_t} \cdot \prod_{i=1}^{j} \frac{\partial^2 r}{\partial \overline{w}_{l_i} \partial w_{k_i}} \cdot \frac{\frac{\partial r}{\partial \overline{w}_v}}{\frac{\partial r}{\partial w_1}}$$

$$dw_t \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_s} \wedge d\overline{w}_v \wedge \cdots \wedge dw_{k_j}.$$

Without loss of generality, we can assume $|\frac{\partial r}{\partial w_1}(w)| \approx 1$ for $w \in U_i'$. Since

$$\left| \frac{\partial^2 r}{\partial \overline{w}_{l_i} \partial \overline{w}_{k_i}} \right| \lesssim \frac{\varepsilon}{\tau_{l_i}(z,\varepsilon) \tau_{k_i}(z,\varepsilon)},$$

$$\frac{|\frac{\partial^2 r}{\partial \overline{w}_{l_s} \partial w_1}| \, |\frac{\partial r}{\partial w_v}|}{|\frac{\partial r}{\partial w_1}(\zeta)|} \lesssim \frac{\varepsilon}{\tau_1(z,\varepsilon) \tau_{l_s}(z,\varepsilon)} \cdot \frac{\varepsilon}{\tau_v(z,\varepsilon)} \approx \frac{\varepsilon}{\tau_v(z,\varepsilon) \tau_{l_s}(z,\varepsilon)}$$

by Lemma 2.5, where $\varepsilon = d(z, \zeta)$, we see that the absolute value of the coefficient of $dw_t \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_s} \wedge dw_v \wedge \cdots \wedge dw_{k_j}$ in the right side of (4.17)

$$\lesssim \frac{\varepsilon}{\tau_t(z,\varepsilon)} \cdot \prod_{i \neq s} \frac{\varepsilon}{\tau_{k_i}(z,\varepsilon) \tau_{l_i}(z,\varepsilon)} \cdot \frac{\varepsilon}{\tau_v(z,\varepsilon) \tau_{l_s}(z,\varepsilon)} \tag{4.18}$$

the coefficient of $dw_t \wedge d\overline{w}_{l_1} \wedge dw_{k_1} \wedge \cdots \wedge d\overline{w}_{l_s} \wedge d\overline{w}_v \wedge \cdots \wedge dw_{k_j}$ in the right side of (4.17) has the same bound (4.18). If $t = 1$ in the right side of (4.15), after substituting (4.16) into (4.15), we have the similar results. Now, we find that, as differential form acting on $(\bigotimes T(bD))^{2j+1}$,

$$C = \sum_L a_L dw^L, \qquad |a_L| \lesssim \frac{\varepsilon^{j+1}}{\tau^L(z,\varepsilon)} \tag{4.19}$$

where multiindices L satisfy $0 \leq L_i, \overline{L}_i \leq 1, L_1 = 0$ and $\sum_{i=1}^n L_i + \overline{L}_i = 2j+1$.

By using the inverse of transformation $U_z \circ T_z$, we can write dw^L as a linear combination of differential forms $d\zeta^I$,

$$dw^L = \sum_I a_L^I d\zeta^I, \qquad |a_L^I| \lesssim 1$$

by each entry of the matrix U_z^{-1} has absolute value ≤ 1, where multiindices I satisfy $0 \leq I, \overline{I} \leq 1, |I| = 2j+1$. Thus, $\|C\|_{bD} \lesssim \sum_L |a_L|$, where the summation takes over all multiindices L satisfying $0 \le L_i, \overline{L}_i \le 1, L_1 = 0$ and $\sum_{i=1}^n L_i + \overline{L}_i = 2j+1$. This completes the proof of Lemma 4.3.

Now we can prove Proposition 4.1.

Proof of Proposition 4.1. (1) Note $\frac{\partial C}{\partial z_i} = 0$, $i = 1, \dots, n$,

$$\frac{\partial}{\partial z_i}\left(\frac{Cd\zeta^J}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^n (\overline{z}_i - \overline{\zeta}_i) d\zeta_i\right) = \frac{(j+1)Cd\zeta^J \cdot \frac{\partial r}{\partial \zeta_i}}{\langle \partial_\zeta r(\zeta), \zeta - z\rangle^{j+2} |z-\zeta|^{2n-2j-2}} \wedge \sum_{i=1}^n (\overline{z}_i - \overline{\zeta}_i) d\zeta_i + \frac{(n-j-1)Cd\zeta^J(\overline{\zeta}_i - \overline{z}_i)}{\langle \partial_\zeta r(\zeta), \zeta - z\rangle^{j+1} |z-\zeta|^{2n-2j}} \wedge \sum_{i=1}^n (\overline{z}_i - \overline{\zeta}_i) d\zeta_i \tag{4.20}$$

by $\frac{\partial}{\partial z_i}\langle \partial_\zeta r(\zeta), \zeta - z\rangle = -\frac{\partial r}{\partial \zeta_i}(\zeta)$. Note $|\frac{\partial r}{\partial \zeta_i}(\zeta)| \lesssim \sum_{j=1}^n |\frac{\partial r}{\partial w_j}| \lesssim \sum_{j=1}^n \frac{\varepsilon}{\tau_j(z,\varepsilon)}$ by each entry of matrix U_z having absolute value not bigger than 1 and $d(z,\zeta) \approx d(\zeta, z) \approx |\langle \partial r(\zeta), \zeta - z\rangle|$ for $\zeta \in bD \cap U_i'$ by Proposition 2.3 and Proposition 3.4. It follows that

$$\left\|\frac{\partial}{\partial z_i}\left(\frac{Cd\zeta^J}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^n (z_i - \zeta_i) d\zeta_i\right)\right\|_{bD} \lesssim \sum_I \frac{1}{d(z,\zeta)^{j+2}|z-\zeta|^{2n-2j-3}} \frac{\varepsilon^{j+1}}{\tau^I(z,\varepsilon)} \cdot \sum_{i=1}^n \frac{\varepsilon}{\tau_i(z,\varepsilon)} + \sum_I \frac{1}{d(z,\zeta)^{j+1}|z-\zeta|^{2n-2j-2}} \frac{\varepsilon^{j+1}}{\tau^I(z,\varepsilon)} \tag{4.21}$$

by Lemma 4.3 and $\varepsilon = d(z,\zeta)$, where I takes over all multiindices satisfying $0 \le I_i, \overline{I}_i \le 1, I_1 = 0$ and $\sum_{i=1}^n I_i + \overline{I}_i = 2j+1$. For such $I, 1 \le i \le n$, $\tau^I(z,\varepsilon)\tau_i(z,\varepsilon) = \tau^\beta(z,\varepsilon)$ for some multiindex β satisfying condition C, i.e., C1) $\sum_{i=1}^n \beta_i + \overline{\beta}_i = 2j+2$; C2) There exists at most one $i_0 > 1$ such that $\beta_{i_0} + \overline{\beta}_{i_0} = 3$ and $\beta_l + \overline{\beta}_l \le 2$ for all $l \ne i_0$. If such i_0 exists, we must have $\beta_1 + \overline{\beta}_1 = 1$. Notice

$$\tau_1(z,\zeta) \approx \varepsilon = d(z,\zeta) \lesssim |z-\zeta| \tag{4.22}$$

we get

$$\left\|\frac{\partial}{\partial z_i}\left(\frac{Cd\zeta^J}{A^{j+1}B^{n-j-1}} \wedge \sum_{i=1}^n (z_i - \zeta_i) d\zeta_i\right)\right\|_{bD} \lesssim \sum_\beta \frac{1}{\tau^\beta(z,\varepsilon)|z-\zeta|^{2n-2j-3}}, \tag{4.23}$$

similarly, we can prove

$$\left\| \frac{\partial}{\partial \overline{z}_i} \left(\frac{C d\zeta^J}{A^{j+1} B^{n-j-1}} \wedge \sum_{i=1}^{n} (z_i - \zeta_i) d\zeta_i \right) \right\|_{bD} \lesssim \sum_{\beta} \frac{1}{\tau^{\beta}(z,\varepsilon)|z-\zeta|^{2n-2j-3}},$$

where β takes over all multiindices satisfying condition C. This completes the proof of (1).

2) For $z \in \overline{U}_i$, $\zeta \in bD$ and $\zeta \notin U_i'$, $\langle \partial_\zeta r(\zeta), \zeta - z\rangle \neq 0$ and $|z-\zeta| \neq 0$. Note $U_1, \cdots, U_N$ covering bD. It follows

$$\left| \langle \frac{\partial r}{\partial \zeta}(\zeta), \zeta - z \rangle \right| \gtrsim 1, \qquad |z-\zeta| \gtrsim 1 \tag{4.24}$$

by compactness. It follows that the coefficients of differential forms $d_z A_q^{j,0}$ are bounded. The Proposition 4.1 is proved.

Now we are ready to prove Lemma 4.2.

Proof of Lemma 4.2. Define

$$\begin{aligned} S_0 &= \{i; \beta_i + \overline{\beta}_i = 0\} \\ S_1 &= \{i; \beta_i + \overline{\beta}_i = 1\} \\ S_2 &= \{i; \beta_i + \overline{\beta}_i = 2\} \\ S_3 &= \{i; \beta_i + \overline{\beta}_i = 3\} \end{aligned} \tag{4.25}$$

for the multiindex β satisfying condition C. Denote the cardinal of S_i by $n_i, i = 0, 1, 2, 3$. We know that $n_3 \leq 1$ from condition C. We consider three cases: Case A, $n_3 = 0$ and $1 \in S_2$; Case B, $n_3 = 0$ and $1 \notin S_2$; Case C, $n_3 = 1$.

Note

$$\frac{1}{\tau_l(z,\varepsilon)} \lesssim \frac{1}{\varepsilon} \approx \frac{1}{\tau_1(z,\varepsilon)}. \tag{4.26}$$

If we replace τ_l by τ_1 in (4.9) for some $l \in S_2$, $\beta_1 + \overline{\beta}_1$ will increase 1. Case B is reduced to Case A. In Case C, $\beta_1 + \overline{\beta}_1 = 1$. If we replace τ_l by τ_1 for $l \in S_3$, $\beta_1 + \overline{\beta}_1$ will increase to 2. Case C is reduced to Case A. Thus we only need to consider Case A.

For such β: $\beta_i + \overline{\beta}_i \le 2, i = 1, \dots, n$ and $\beta_1 + \overline{\beta}_1 = 2$, we will calculate I_β as in [**MS**]. Recall the definition of σ_i and $\tau_i \approx \sigma_i$, we get

$$\begin{aligned} I_\beta &\lesssim \int_{bD\cap U_i'} \varepsilon^{-\beta_1-\overline{\beta}_1} \left(\sum_{i_2} [A^2_{i_2}(z)]^{\frac{\beta_2+\overline{\beta}_2}{i_2}} \varepsilon^{-\frac{\beta_2+\overline{\beta}_2}{i_2}} \right) \cdots \\ &\quad \cdot \left(\sum_{i_n} [A^n_{i_n}(z)]^{\frac{\beta_n+\overline{\beta}_n}{i_n}} \varepsilon^{-\frac{\beta_n+\overline{\beta}_n}{i_n}} \right) \cdot \frac{dV(\zeta)}{|z-\zeta|^{2n-2j-3}} \\ &\lesssim \sum_{(i_2,\cdots,i_n)} [A^2_{i_2}(z)]^{\frac{\beta_2+\overline{\beta}_2}{i_2}} \cdots [A^n_{i_n}(z)]^{\frac{\beta_n+\overline{\beta}_n}{i_n}} \\ &\quad \cdot \int_{bD\cap U_i'} \varepsilon^{-\beta_1-\overline{\beta}_1-\cdots-\frac{\beta_n+\overline{\beta}_n}{i_n}-\kappa} \frac{dV(\zeta)}{|z-\zeta|^{2n-2j-3+\kappa}} \end{aligned} \tag{4.27}$$

by (4.5) and $\varepsilon^{-\kappa} \gtrsim 1, \quad |z-\zeta|^{-\kappa} \gtrsim 1$, where the summation takes over all $(i_2, \cdots, i_n)$ with $2 \le i_2, \cdots, i_n \le m$, $\varepsilon = d(z,\zeta)$. Let

$$\begin{aligned} D_0 &= \{\zeta \in bD | d(z,\zeta) \le |r(z)|\}, \\ D_q &= \{\zeta \in bD | 2^{q-1}|r(z)| \le d(z,\zeta) \le 2^q|r(z)|\}, \qquad q = 1, 2, \cdots. \end{aligned}$$

Note for $\zeta \in D_q$, on the $2^{q-1}|r(z)|$-extremal coordinates centered at z, $\zeta = (\zeta_1, \dots, \zeta_n)$

$$d(z,\zeta) \approx 2^q|r(z)| + |\zeta_1| + \sum_{i=2}^{n}\sum_{l=2}^{m} A^i_l(z)|\zeta_i|^l \tag{4.28}$$

by Corollary 2.4. Note $|z-\zeta| \gtrsim d(z,\zeta) \ge 2^q|r(z)|$. (4.27) is less than

(4.29)

$$\begin{aligned} &\lesssim \sum_{q=0}^{\infty} \sum_{(i_2\cdots i_n)} [A^2_{i_2}(z)]^{\frac{\beta_2+\overline{\beta}_2}{i_2}} \cdots [A^n_{i_n}(z)]^{\frac{\beta_n+\overline{\beta}_n}{i_n}} \int_{bD\cap U_i'\cap D_q} \Bigg(2^q|r(z)| + |\zeta_1| \\ &\quad + \sum_{k=1}^{n}\sum_{l=2}^{m} A^k_l(z)|\zeta_k|^l \Bigg)^{-\beta_1-\overline{\beta}_1-\cdots-\frac{\beta_n+\overline{\beta}_n}{i_n}-\kappa} \frac{dV(\zeta)}{(2^q|r(z)|+|\zeta|)^{2n-2j-3+\kappa}} \\ &= \sum_{q=0}^{\infty} \sum_{(i_1\cdots i_n)} I^q_{\beta,i_2\cdots i_n}. \end{aligned}$$

We will prove

$$I^q_{\beta,i_2\cdots i_n} \lesssim |r(z)|^{-1+\frac{1}{m}-\kappa} 2^{q(-1+\frac{1}{m}-\kappa)} \tag{4.30}$$

for $0 < \kappa < 1 - \frac{1}{m}, 2 \le i_2, i_3, \cdots, i_n \le m, \beta$ satisfying condition C. Hence, $\sum_{q=0}^{\infty} I^q_{\beta,i_2\cdots i_n} < \infty$. This gives Lemma 4.2.

For $l \in S_2$, in the $2^{q-1}|r(z)|$-extremal coordinates centered at z,

(4.31)

$$I^q_{\beta,i_2\cdots i_n} \lesssim \Pi_{j\neq l}[A^j_{i_j}(z)]^{\frac{\beta_j+\overline{\beta}_j}{i_j}} \int_{\mathbb{R}^{2n-3}} \int_{\mathbb{C}} [A^l_{i_l}(z)]^{\frac{2}{i_l}} \left(2^q|r(z)| + \sum_{k\neq l} |\zeta_k|\right)^{-(2n-2j-3+\kappa)} \cdot \left(2^q|r(z)| + |\zeta_1| + \sum_{k\neq l}\sum_{t=2}^{m} A^k_t(z)|\zeta_k|^t + A^l_{i_l}(z)|\zeta_l|^{i_l}\right)^{-2-\cdots-\frac{2}{i_l}-\cdots\frac{\beta_n+\overline{\beta}_n}{i_n}-\kappa} dx_2\cdots dx_{2n}$$

by the volume element $dV(\zeta)$ on $bD \approx dx_2\cdots dx_n$. Now apply Lemma 3.8 to (4.31) with

$$b = A^l_{i_l}(z), \qquad k = i_l, \qquad \alpha = 2 + \sum_{j\neq l, j\geq 2} \frac{\beta_j + \overline{\beta}_j}{i_j} + \kappa$$

$$a = 2^q|r(z)| + |\zeta_1| + \sum_{k\neq l}\sum_{t=2}^{m} A^k_t(z)|\zeta_k|^t$$

to get

(4.32)

$$I^q_{\beta,i_2\cdots i_n} \lesssim \Pi_{j\neq l}[A^j_{i_j}(z)]^{\frac{\beta_j+\overline{\beta}_j}{i_j}} \int_{\mathbb{R}^{2n-3}} \left(2^q|r(z)| + \sum_{k\neq l} |\zeta_k|\right)^{-(2n-2j-3+\kappa)} \cdot \left(2^q|r(z)| + |\zeta_1| + \sum_{k\neq l}\sum_{t=2}^{m} A^k_t(z)|\zeta_k|^t\right)^{-2-\sum_{j\neq l,j\geq 2}\frac{\beta_j+\overline{\beta}_j}{i_j}-\kappa} dV(\zeta)$$

where $dV(\zeta)$ denote the volume element of R^{2n-3}. Repeating this procedure, we can integrate all variables ζ_i with $i \in S_2 \setminus \{1\}$. Then

(4.33)

$$I^q_{\beta,i_2\cdots i_n} \lesssim \Pi_{j\notin S_2\setminus\{1\}}[A^j_{i_j}(z)]^{\frac{\beta_j+\overline{\beta}_j}{i_j}} \cdot \int_{\mathbb{R}^{2n-2n_2+1}} \left(2^q|r(z)|+|\zeta_1|+\sum_{k\notin S_2}|\zeta_k|\right)^{-(2n-2j-3+\kappa)} \cdot \left(2^q|r(z)|+|\zeta_1|+\sum_{k\notin S_2}\sum_{t=2}^{m}A^k_t(z)|\zeta_k|^t\right)^{-(2+\sum_{j\notin S_2}\frac{\beta_j+\overline{\beta}_j}{i_j}+\kappa)} dV(\zeta).$$

Now integrate all variables ζ_i with $i \in S_0$ by

$$\int_{\mathbb{C}} \frac{d\zeta d\overline{\zeta}}{(|\zeta|+C)^k} \lesssim \frac{1}{C^{k-2}} \tag{4.34}$$

for $k > 2$, we get

(4.35)

$$I^q_{\beta,i_2\cdots i_n} \lesssim \Pi_{j\in S_1}[A^j_{i_j}(z)]^{\frac{\beta_j+\overline{\beta}_j}{i_j}} \cdot \int_{\mathbb{R}^{2n-2n_0-2n_2+1}} \left(2^q|r(z)|+\sum_{k\in S_1}|\zeta_k|+|\zeta_1|\right)^{-(2n-2j-2n_0-3+\kappa)} \cdot \left(2^q|r(z)|+|\zeta_1|+\sum_{k\in S_1}\sum_{t=2}^{m}A^k_t(z)|\zeta_k|^t\right)^{-\left(2+\sum_{j\in S_1}\frac{\beta_j+\overline{\beta}_j}{i_j}+\kappa\right)} dV(\zeta).$$

By condition C, $S_3 = \emptyset$ and $1 \in S_2$, we see that

$$\begin{aligned} 2n_0+2n_1+2n_2 &= 2n, \\ 2n_2+n_1 &= 2j+2. \end{aligned} \tag{4.36}$$

Therefore

$$\begin{aligned} 2n-2j-3-2n_0 &= n_1-1, \\ 2n-2n_2-2n_0+1 &= 2n_1+1. \end{aligned} \tag{4.37}$$

Now if $n_1 \geq 2$, $l \in S_1$, then

(4.38)

$$I^q_{\beta,i_2\cdots i_n} \lesssim \Pi_{j\in S_1\setminus\{l\}}[A^j_{i_j}(z)]^{\frac{1}{i_j}} \cdot \int_{\mathbb{R}^{2n_1+1}} [A^l_{i_l}(z)]^{\frac{1}{i_l}} \left(2^q|r(z)| + \sum_{k\in S_1\setminus\{l\}} |\zeta_k| + |\zeta_1| + |\zeta_l|\right)^{-n_1+1-\kappa} \cdot \left(2^q|r(z)| + |\zeta_1| + \sum_{k\in S_1\setminus\{l\}} \sum_{t=2}^m A^k_t(z)|\zeta_k|^t + A^l_{i_l}|\zeta_l|^{i_l}\right)^{-2-\sum_{j\in S_1, j\neq l}\frac{1}{i_j} - \frac{1}{i_l} - \kappa} dV(\zeta).$$

Now apply Lemma 3.7 to (4.38) with

(4.39)
$$a = 2^q|r(z)| + |\zeta_1| + \sum_{k\in S_1\setminus\{l\}} \sum_{t=2}^m A^k_t(z)|\zeta_k|^t$$
$$a' = 2^q|r(z)| + \sum_{k\in S_1, k\neq l} |\zeta_k| + |\zeta_1|$$
$$k = i_l, \alpha = 2 + \sum_{j\in S_1, j\neq l} \frac{1}{i_j} + \kappa, \alpha' = n_1 - 2 + \kappa > 0$$

to get

(4.40)

$$I^q_{\beta,i_2\cdots i_n} \lesssim \Pi_{j\in S_1\setminus\{l\}}[A^j_{i_j}(z)]^{\frac{1}{i_j}} \cdot \int_{\mathbb{R}^{2n_1-1}} \left(2^q|r(z)| + \sum_{k\in S_1\setminus\{l\}} |\zeta_k| + |\zeta_1|\right)^{-n_1+2-\kappa} \cdot \left(2^q|r(z)| + |\zeta_1| + \sum_{k\in S_1\setminus\{l\}} \sum_{t=2}^m A^k_t(z)|\zeta_k|^t\right)^{-2-\sum_{j\in S_1, j\neq l}\frac{1}{i_j} - \kappa} dV(\zeta).$$

Repeating this procedure, we can integrate out $(n_1 - 1)$ variables ζ_i with $i \in S_1$. Let ζ_s be the remaining variable. We get

$$(4.41)\qquad I^q_{\beta,i_2\cdots i_n} \lesssim \int_{\mathbb{R}^3} [A^s_{i_s}(z)]^{\frac{1}{i_s}} (2^q|r(z)|)^{-\kappa} \cdot \left(2^q|r(z)| + |\zeta_1| + \sum_{t=2}^{m} A^s_t(z)|\zeta_s|^t\right)^{-2-\frac{1}{i_s}-\kappa} dV(\zeta),$$

where $dV(\zeta) = dx_2 dx_s dx_{n+s}$. Now integrate out variable x_2 to get

$$\begin{aligned}(4.42)\qquad I^q_{\beta,i_2\cdots i_n} &\lesssim \int_{\mathbb{R}^2} [A^s_{i_s}(z)]^{\frac{1}{i_s}} (2^q|r(z)|)^{-\kappa} \cdot \left(2^q|r(z)| + \sum_{t=2}^{m} A^s_t(z)|\zeta_s|^t\right)^{-1-\frac{1}{i_s}-\kappa} dV(\zeta_s) \\ &\lesssim \int_{\mathbb{R}^2} [A^s_{i_s}(z)]^{\frac{1}{i_s}} (2^q|r(z)|)^{-\kappa} (2^q|r(z)| + A^s_{i_s}(z)|\zeta_s|^{i_s})^{-\frac{1}{i_s}-\frac{\kappa}{2}} \\ &\quad \cdot (2^q|r(z)| + A^s_{k_0}(z)|\zeta_s|^{k_0})^{-1-\frac{\kappa}{2}} dx_s dx_{n+s}\end{aligned}$$

where k_0 satisfies

$$(4.43)\qquad \sigma_s(z, 2^q|r(z)|) = \left(\frac{2^q|r(z)|}{A^s_{k_0}}\right)^{\frac{1}{k_0}}.$$

Then

$$\begin{aligned} I^q_{\beta,i_2\cdots i_n} &\lesssim \int_{\mathbb{R}} (2^q|r(z)|)^{-\kappa} [A^s_{i_s}(z)]^{\frac{1}{i_s}} (2^q|r(z)| + A^s_{i_s}(z)|x_s|^{i_s})^{-\frac{1}{i_s}-\frac{\kappa}{2}} dx_s \\ &\quad \cdot \int_{\mathbb{R}} (2^q|r(z)| + A^s_{k_0}(z)|x_{n+s}|^{k_0})^{-1-\frac{\kappa}{2}} dx_{n+s} \\ &\lesssim \frac{1}{(2^q|r(z)|)^{\kappa+\frac{\kappa}{2}}} \cdot \frac{1}{(A^s_{k_0})^{\frac{1}{k_0}}} \cdot \frac{(2^q|r(z)|)^{\frac{1}{k_0}}}{(2^q|r(z)|)^{1+\frac{\kappa}{2}}} \\ &= \frac{\sigma_s(z, 2^q|r(z)|)}{(2^q|r(z)|)^{1+2\kappa}} \lesssim |r(z)|^{\frac{1}{m}-1-2\kappa} 2^{(\frac{1}{m}-1-2\kappa)q}, \end{aligned}$$

by $\sigma_s(z, 2^q|r(z)|) \lesssim (2^q|r(z)|)^{\frac{1}{m}}$. This completes the proof of (4.30) (take κ to be $\frac{\kappa}{2}$), therefore Lemma 4.2.

Note added: This paper is the revised form of a paper titled *Hölder estimate for $\overline{\partial}$ on the convex domains of finite type* written in 1995, where Lemma 4.2 in [**BCD**] was incorrectly stated for all convex domains of finite type. The referee informed the author that Diederich and Fornæss announced similar results at the Hayama symposium in December, 1998.

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Received February 16, 1999 and revised May 24, 1999. This work was partially supported by National Nature Science Foundation (No. 10071070) and Nature Science Foundation of Zhejiang Province in China.

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