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Given a bounded, non-negative operator W and a projection P on a Hilbert space, we find necessary and sufficient conditions for the existence of a non-trivial, non-negative operator V such that P is bounded from $L^2(W)$ to $L^2(V)$. This leads to a vector-valued version of a theorem of Koosis and Treil' concerning the boundedness of the Riesz projection in spaces with weights.

1. Introduction.

Let $\partial\mathbf{D}$ be the unit circle in the complex plane, define the function χ on $\partial\mathbf{D}$ by $\chi(e^{i\theta}) = e^{i\theta}$, and set $\mathcal{P} = \{p : p = \sum_{k=-N}^{N} c_k\chi^k\}$. Let σ be normalized Lebesgue measure on $\partial\mathbf{D}$. The *Riesz projection* P_+ is defined on $\mathcal{P}$ by the formula $P_+(\sum_{k=-N}^{N} c_k\chi^k) = \sum_{k=0}^{N} c_k\chi^k$.

In [4], Paul Koosis proved:

Theorem 1 (Koosis). *Given a non-negative function $w \in L^1$, there exists a non-negative, non-trivial function $v \in L^1$ such that*

$$\int_{\partial\mathbf{D}} |P_+f|^2 v\, d\sigma \le \int_{\partial\mathbf{D}} |f|^2 w\, d\sigma \qquad \forall f \in \mathcal{P}$$

if and only if $\frac{1}{w} \in L^1$.

The $w^{-1} \in L^1$ requirement may look familiar to readers acquainted with the theorems of prediction theory, and indeed, in his proof of Theorem 1, Koosis observes that the necessity of the $w^{-1} \in L^1$ condition is a consequence of:

Theorem 2 (Kolmogorov's infimum). *Given $w \ge 0$ in L^1,*

$$\inf\left\{\int_{\partial\mathbf{D}} |1-p|^2\omega d\sigma : \; p \in \mathcal{P}, \int_{\partial\mathbf{D}} p d\sigma = 0\right\} = \left[\int_{\partial\mathbf{D}} \frac{1}{\omega}\, d\sigma\right]^{-1},$$

where the infimum is understood to be zero if $w^{-1} \notin L^1$.

Koosis' proof that $w^{-1} \in L^1$ is sufficient in Theorem 1 is short and elegant, but it uses techniques from analytic function theory that tie it to the scalar-valued setting. A version of Theorem 1 for vector-valued functions and operator-valued weights was proved in a very different way by S.R. Treil' in [**6**]. Treil' takes an interesting geometric approach, and it is this viewpoint that prompted us to study more deeply the nature of the relationship between the weights w, v, and the projection P_+.

Starting with an extremely general formulation of the Koosis result in Section 2, we prove a version of Theorem 1 for a projection P and a non-negative, bounded operator W on an arbitrary Hilbert space $\mathcal{L}$. The resulting theorem (Theorem 4) has some interesting implications when we specialize to L^2 of the unit circle. We cannot, however, use it to recover the Koosis result (for bounded weight functions) since the positive operator V that appears in the theorem need not be a multiplication operator. This issue is addressed in Section 3, where we introduce a bilateral shift U on $\mathcal{L}$ and require that our weights W and V commute with U. The main result of this section (Theorem 7) is a strengthening of Treil's vector-valued result referenced above.

This research owes a great debt to Treil' in that the proof of Theorem 7 uses the same line of attack discovered by him, albeit with two notable differences. One substantial simplification comes from the use of Theorem 4 below which is essentially a corollary to the main result in [**1**]. A second, more significant, improvement is achieved by replacing Treil's geometric construction with an algebraic argument that enables us to drop the hypothesis of invertibility assumed in Treil's work. (See Corollary 8.) The result is a stronger theorem with, what is in our opinion, a more elegant proof.

2. Koosis' Theorem for an Arbitrary Projection.

Let $\mathcal{L}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $\mathcal{B}(\mathcal{L})$ be the algebra of bounded linear operators on $\mathcal{L}$. Given a projection $P \in \mathcal{B}(\mathcal{L})$ onto a subspace $\mathcal{C} \subseteq \mathcal{L}$ and a non-negative operator $W \in \mathcal{B}(\mathcal{L})$, we ask when there exists a non-trivial, non-negative operator $V \in \mathcal{B}(\mathcal{C})$ satisfying

$$\langle VPf, Pf\rangle \leq \langle Wf, f\rangle \quad \forall f \in \mathcal{L}.$$

It may seem surprising that one could say anything interesting at all without the addition of some more hypotheses, but we get hope from the fact that Kolmogorov's infimum has a useful analogue in this very general setting. The result appears in [**1**], and is stated here as:

Theorem 3. *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative, and let $P \in \mathcal{B}(\mathcal{L})$ be the projection onto a subspace $\mathcal{C} \subseteq \mathcal{L}$. If $k \in \mathcal{C}$, then*

$$\inf\{\langle W(k+f), k+f\rangle : Pf = 0\} = \lim_{\epsilon\to 0^+}\langle [PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}k, k\rangle, \tag{1}$$

where $W_\epsilon = W + \epsilon I$, and I is the identity operator on $\mathcal{L}$.

The two inverses in Equation (1) refer to different spaces. For each $\epsilon > 0$, the operator W_ϵ is invertible in $\mathcal{B}(\mathcal{L})$. Letting $A_\epsilon = PW_\epsilon^{-1}|_{\mathcal{C}} \in \mathcal{B}(\mathcal{C})$, we have that A_ϵ is bounded below and thus is invertible in $\mathcal{B}(\mathcal{C})$. The limit in Equation (1) is monotone decreasing with decreasing ϵ, and a polarization argument ensures that $\lim_{\epsilon\to 0^+}\langle [PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}f, g\rangle$ exists for all $f, g \in \mathcal{C}$. Thus it makes sense to define $V \in \mathcal{B}(\mathcal{C})$ to be the weak limit of $[PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}$ as ϵ tends to zero from the right.

Combining these observations with Treil's geometric insight into Koosis' theorem gives us:

Theorem 4. *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative, and let $P \in \mathcal{B}(\mathcal{L})$ be a projection onto $\mathcal{C} \subseteq \mathcal{L}$. Then $V = wk\text{-}\lim\limits_{\epsilon\to 0^+}[PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}$ satisfies*

$$\langle VPf, Pf\rangle \leq \langle Wf, f\rangle \qquad \forall f \in \mathcal{L}, \tag{2}$$

and is maximal in the sense that $V \geq B$ for any B that also satisfies (2).

Proof. For $f \in \mathcal{L}$, write $f = k + g$ where $k \in \mathcal{C}$ and $g \in \mathcal{C}^\perp$. By Theorem 3,

$$\langle VPf, Pf\rangle = \langle Vk, k\rangle = \inf_{Pg'=0}\langle W(k+g'), k+g'\rangle \leq \langle Wf, f\rangle.$$

If B satisfies (2), then for any $g' \in \mathcal{C}^\perp$ it must be that $\langle Bk, k\rangle \leq \langle W(k + g'), k + g'\rangle$. Thus

$$\langle Bk, k\rangle \leq \inf_{Pg'=0}\langle W(k+g'), k+g'\rangle = \langle Vk, k\rangle.$$

$\square$

Corollary 5. *Given W and P in $\mathcal{B}(\mathcal{L})$ as in Theorem 4, there exists a non-negative, non-trivial $V \in \mathcal{B}(\mathcal{C})$ satisfying (2) if and only if $\lim\limits_{\epsilon\to 0^+}\langle [PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}k, k\rangle > 0$ for some $k \in \mathcal{C}$.*

Corollary 5 is just a slightly weaker reformulation of Theorem 4 that more accurately parallels the statement of Koosis' result (Theorem 1). The next proposition gives a condition sufficient for proving the existence of a non-trivial weight V. Although it is no longer necessary, this condition is somewhat easier to verify than the one given in Corollary 5.

Corollary 6. *Given W and P in $\mathcal{B}(\mathcal{L})$ as in Theorem 4, there exists a non-trivial, non-negative operator $V \in \mathcal{B}(\mathcal{C})$ satisfying (2) provided $\lim\limits_{\epsilon\to 0^+}\langle W_\epsilon^{-1}k, k\rangle < \infty$ for some non-trivial $k \in \mathcal{C}$.*

Proof. Let P_k be the projection onto the one dimensional subspace spanned by the vector k. A straightforward calculation shows that the operator $P_kW_\epsilon^{-1}|_{P_k\mathcal{L}}$ is just multiplication by the constant $\langle W_\epsilon^{-1}\frac{k}{\|k\|}, \frac{k}{\|k\|}\rangle$. Since $k \in \operatorname{ran}P$,

$$\inf_{Pf=0}\langle W(k+f), k+f\rangle \geq \inf_{P_kf=0}\langle W(k+f), k+f\rangle.$$

Now using Theorem 3 we can write

$$\lim_{\epsilon\to 0^+}\langle [PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}k,k\rangle \geq \lim_{\epsilon\to 0^+}\langle [P_kW_\epsilon^{-1}|_{P_k\mathcal{L}}]^{-1}k,k\rangle = \lim_{\epsilon\to 0^+}\frac{\|k\|^4}{\langle W_\epsilon^{-1}k,k\rangle},$$

and the result follows from Corollary 5. □

3. Laurent Operators.

The generality of Theorem 4 has a strong appeal; however, the original Koosis result deals with multiplication operators, and this quality is ignored in Theorem 4. Consider this example on L^2 of the unit circle.

Let $w \geq 0$ be a bounded function satisfying (i) $\log w \in L^1$ and (ii) $\frac{1}{w} \notin L^1$, and define W to be multiplication by w on $\mathcal{L} = L^2$. The Hardy space $H^2 = \{f \in L^2 : \hat{f}(n) = 0, \forall n < 0\}$ is a closed subspace of L^2 and the orthogonal projection P_H onto H^2 agrees with the Riesz projection P_+ on polynomials. Now condition (i) implies that there exists an $h \in H^2$ such that $|h|^2 = w$ a.e. on the unit circle, which means that $\lim\limits_{\epsilon\to 0^+}\langle W_\epsilon^{-1}h,h\rangle = 1$. By Corollary 6, then, there exists a non-trivial, non-negative operator $V \in \mathcal{B}(H^2)$ satisfying $\langle VP_Hf, P_Hf\rangle \leq \displaystyle\int_{\partial\mathbf{D}} |f|^2 w\, d\sigma$ for all $f \in L^2$. However, using Theorem 1, we see that condition (ii) above implies that there is no way to extend V to be multiplication by some non-negative function v on L^2.

This example illustrates that to fully recover Koosis' theorem from the abstract setting, we must introduce a bilateral shift $U \in \mathcal{B}(\mathcal{L})$ and consider operators that commute with U.

Definition. A unitary operator $U \in \mathcal{B}(\mathcal{L})$ is a *bilateral shift* if there exists a projection $P_0 \in \mathcal{B}(\mathcal{L})$ satisfying

(i) $P_0U^jP_0 = \delta_{j,0}P_0, \quad \forall j \in \mathbf{Z}$; and,

(ii) as $n \to \infty$, $\displaystyle\sum_{j=-n}^{n} U^jP_0U^{*j}$ converges strongly to the identity on $\mathcal{L}$.

Letting $\mathcal{P}_0 = P_0\mathcal{L}$, we can write $\mathcal{L} = \displaystyle\sum_{j=-\infty}^{\infty} \oplus U^j\mathcal{P}_0$. Theorem 7 will deal specifically with the projection

$$P_{\mathcal{H}} = \sum_{j=0}^{\infty} U^jP_0U^{*j}$$

onto the half-space $\mathcal{H} = \displaystyle\sum_{j=0}^{\infty} \oplus U^j\mathcal{P}_0$.

Definition. An operator $A \in \mathcal{B}(\mathcal{L})$ is *Laurent* (*with respect to* U) if $AU = UA$.

In the case of the unit circle, if $U \in \mathcal{B}(L^2)$ is given by $Uf = \chi f$, then $A \in \mathcal{B}(L^2)$ is Laurent if and only if $Af = \phi f$ for some $\phi \in L^\infty$. An analogous fact holds in the vector-valued case ([**5**, p. 110]).

We are now ready to prove:

Theorem 7. *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative and Laurent. Then there exists a non-trivial, non-negative Laurent operator $V \in \mathcal{B}(\mathcal{L})$ satisfying*

$$\langle VP_{\mathcal{H}}f, P_{\mathcal{H}}f\rangle \le \langle Wf, f\rangle \quad \forall f \in \mathcal{L} \tag{3}$$

if and only if $V_0 = wk\text{-}\lim_{\epsilon\to 0}[P_0W_\epsilon^{-1}|_{\mathcal{P}_0}]^{-1}$ is non-trivial.

Moreover, if V_0 is non-trivial, then V can be constructed to satisfy

$$\langle Vc, c\rangle \ge \frac{1}{4}\langle V_0c, c\rangle \quad \forall c \in \mathcal{P}_0. \tag{4}$$

Proof. Assume V exists. Then for any $f \in \mathcal{L}$,

$$\begin{aligned}
\langle VP_0f, P_0f\rangle^{\frac{1}{2}} &= \|V^{\frac{1}{2}}P_0f\| = \|V^{\frac{1}{2}}(P_{\mathcal{H}} - UP_{\mathcal{H}}U^*)f\| \\
&\le \|V^{\frac{1}{2}}P_{\mathcal{H}}f\| + \|V^{\frac{1}{2}}UP_{\mathcal{H}}U^*f\| \\
&\le \|V^{\frac{1}{2}}P_{\mathcal{H}}f\| + \|V^{\frac{1}{2}}P_{\mathcal{H}}U^*f\| \\
&\le \langle Wf, f\rangle^{\frac{1}{2}} + \langle WU^*f, U^*f\rangle^{\frac{1}{2}} \\
&= 2\langle Wf, f\rangle^{\frac{1}{2}}.
\end{aligned}$$

Thus, $\frac{1}{4}\langle VP_0f, P_0f\rangle \le \langle Wf, f\rangle$ for all $f \in \mathcal{L}$, and so by Theorem 4, $\frac{1}{4}P_0V|_{\mathcal{P}_0} \le V_0$. Since V is non-trivial and Laurent, its kernel cannot contain $\mathcal{P}_0$ and it follows that V_0 is non-trivial as well.

Conversely, assume $V_0 = \text{wk-}\lim_{\epsilon\to 0^+}[P_0W_\epsilon^{-1}|_{\mathcal{P}_0}]^{-1}$ is non-trivial. For $n \ge 1$, define $P_n = \sum_{j=0}^{n} U^jP_0U^{*j}$ to be the projection onto the subspace $\mathcal{P}_n = P_n\mathcal{L}$, and let $V_n = \text{wk-}\lim_{\epsilon\to 0^+}[P_nW_\epsilon^{-1}|_{\mathcal{P}_n}]^{-1}$. By Theorem 4,

$$\langle V_nP_nf, P_nf\rangle \le \langle Wf, f\rangle \quad \forall f \in \mathcal{L},$$

and the sequence V_n is monotone in the sense that if $0 \le m < n$ and $p_m \in \mathcal{P}_m$ then

$$\begin{aligned}
\langle V_mp_m, p_m\rangle &= \inf_{P_mf=0}\langle W(p_m + f), p_m + f\rangle \\
&\le \inf_{P_nf=0}\langle W(p_m + f), p_m + f\rangle = \langle V_np_m, p_m\rangle.
\end{aligned}$$

Roughly speaking, we intend to define V via the limit of the monotone sequence V_n. The dilemma is that the argument will require each successive operator to be a dilation of the previous one (i.e., $P_nV_{n+1}|_{\mathcal{P}_n} = V_n$) which is not true of the sequence V_n. Thus, we first need to move to a new sequence A_n satisfying $0 \le A_n \le V_n$ which does have this property.

To this end set $A_0 = V_0$, and define A_{n+1} inductively as follows. Write $\mathcal{P}_{n+1} = \mathcal{P}_n \oplus U^{n+1}\mathcal{P}_0$, and denote $V_{n+1} \in \mathcal{B}(\mathcal{P}_{n+1})$ by the 2×2 matrix

$$V_{n+1} = \begin{pmatrix} B & D \\ D^* & C \end{pmatrix}$$

where B, C, and D are acting on the appropriate spaces. Now $V_{n+1} \geq 0$ is equivalent to $B \geq 0$, $C \geq 0$ and the existence of a contraction $W : \overline{\operatorname{ran} C} \to \overline{\operatorname{ran} B}$ satisfying $D = B^{\frac{1}{2}} W C^{\frac{1}{2}}$ [**3**, p. 547]. Letting $X = WC^{\frac{1}{2}}$ leads to the LU-factorization

$$V_{n+1} = \begin{pmatrix} B & D \\ D^* & C \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ X^* & Y^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & X \\ 0 & Y^{\frac{1}{2}} \end{pmatrix} \tag{5}$$

where $Y = C - X^*X = C^{\frac{1}{2}}(I - W^*W)C^{\frac{1}{2}} \geq 0$. By the induction hypothesis, $0 \leq A_n \leq B$ which implies that $A_n = B^{\frac{1}{2}} Z B^{\frac{1}{2}}$ for a positive contraction Z [**2**, p. 413]. Now it is straightforward to verify that

$$A_{n+1} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ X^* & Y^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & X \\ 0 & Y^{\frac{1}{2}} \end{pmatrix}$$

satisfies $0 \leq A_{n+1} \leq V_{n+1}$, and $P_n A_{n+1}|_{\mathcal{P}_n} = A_n$.

By construction, $\langle A_n P_n f, P_n f\rangle \leq \langle V_n P_n f, P_n f\rangle \leq \langle Wf, f\rangle$ for all $n \geq 0$ and $f \in \mathcal{L}$, and the sesquilinear form $a(p,q) = \lim_{n\to\infty} \langle A_n p, q\rangle$ is well defined for $p, q \in \mathcal{P}_+ = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. The operators A_n are uniformly bounded on the diagonal by $\|W\|$, so a is as well, and hence there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\langle Ap, q\rangle = a(p,q)$ for all $p, q \in \mathcal{P}_+$. The operator A satisfies $\langle AP_n f, P_n f\rangle \leq \langle Wf, f\rangle$ for all $f \in \mathcal{L}$ from which we can conclude that $\langle AP_{\mathcal{H}} f, P_{\mathcal{H}} f\rangle \leq \langle Wf, f\rangle$.

We now use A to construct a Laurent operator $V \in \mathcal{B}(\mathcal{L})$ with the required properties. For $k \geq 0$, let F_k be the operator on $\mathcal{L}$ defined by $F_k = \frac{1}{k+1}\sum_{n=0}^{k} U^{*n} A P_{\mathcal{H}} U^n$. For $n \geq 1$ and $f \in \mathcal{L}$,

$$\begin{aligned} \langle AU^n P_{\mathcal{H}} f, U^n P_{\mathcal{H}} f\rangle^{\frac{1}{2}} &= \|A^{\frac{1}{2}} U^n P_{\mathcal{H}} f\| = \|A^{\frac{1}{2}}(P_{\mathcal{H}} - P_{n-1})U^n f\| \\ &\leq \|A^{\frac{1}{2}} P_{\mathcal{H}} U^n f\| + \|A_{n-1}^{\frac{1}{2}} P_{n-1} U^n f\| \\ &\leq 2\langle Wf, f\rangle^{\frac{1}{2}}. \end{aligned}$$

This implies $\langle F_k P_{\mathcal{H}} f, P_{\mathcal{H}} f\rangle \leq \langle 4Wf, f\rangle$ for all $k \geq 0$. Letting V be a weak limit point of the set $\{\frac{1}{4}F_k : k \geq 0\}$, it follows that V satisfies (3) and is Laurent as desired.

It remains to show that V satisfies (4), which will follow if we can demonstrate that $\langle AU^n c, U^n c\rangle \geq \langle V_0 c, c\rangle$ for all $n \geq 0$ and $c \in \mathcal{P}_0$. If $n = 0$,

$A_0 = V_0$ and the result is clear. For a fixed $n \geq 0$, the inductive construction of A_{n+1} yields

$$\begin{aligned}\langle AU^{n+1}c, U^{n+1}c\rangle &= \langle A_{n+1}U^{n+1}c, U^{n+1}c\rangle \\ &= \langle X^*ZXU^{n+1}c, U^{n+1}c\rangle + \langle YU^{n+1}c, U^{n+1}c\rangle \\ &\geq \langle YU^{n+1}c, U^{n+1}c\rangle.\end{aligned}$$

Thus it is sufficient to prove $\langle YU^{n+1}c, U^{n+1}c\rangle \geq \langle V_0c, c\rangle$ for all $c \in \mathcal{P}_0$.

Let $z \in \mathcal{P}_n$, so that $z + U^{n+1}c \in \mathcal{P}_n \oplus U^{n+1}\mathcal{P}_0 = \mathcal{P}_{n+1}$. Using the LU-factorization for V_{n+1} given in (5), we have

$$\begin{aligned}&\langle V_{n+1}(z + U^{n+1}c), z + U^{n+1}c\rangle \\ &= \langle Bz, z\rangle + 2Re\langle B^{\frac{1}{2}}z, XU^{n+1}c\rangle + \langle CU^{n+1}c, U^{n+1}c\rangle.\end{aligned}$$

Recall that the operator V_{n+1} was generated via Theorem 4. This, together with the assumption that W is Laurent allows us to write

$$\begin{aligned}&\langle V_{n+1}(z + U^{n+1}c), z + U^{n+1}c\rangle \\ &= \inf\left\{\langle W(z + U^{n+1}c + f), z + U^{n+1}c + f\rangle : P_{n+1}f = 0\right\} \\ &\geq \inf\left\{\langle W(U^{n+1}c + f), U^{n+1}c + f\rangle : U^{n+1}P_0U^{*(n+1)}f = 0\right\} \\ &= \inf\left\{\langle W(c + f), c + f\rangle : P_0f = 0\right\} \\ &= \langle V_0c, c, \rangle.\end{aligned}$$

Combining these observations we have

$$\langle Bz, z\rangle + 2Re\langle B^{\frac{1}{2}}z, XU^{n+1}c\rangle + \langle CU^{n+1}c, U^{n+1}c\rangle - \langle V_0c, c\rangle \geq 0 \tag{6}$$

for all $z \in \mathcal{P}_n$ and $c \in \mathcal{P}_0$. Let r be an arbitrary real number. Since $\mathrm{ran}X \subseteq \overline{\mathrm{ran}B} = \overline{\mathrm{ran}B^{\frac{1}{2}}}$, there exists a sequence z_m in $\mathcal{P}_n$ such that $B^{\frac{1}{2}}z_m \to rXU^{n+1}c$. Substituting into (6) and taking limits we get

$$r^2\|XU^{n+1}c\|^2 + 2r\|XU^{n+1}c\|^2 + \big(\langle CU^{n+1}c, U^{n+1}c\rangle - \langle V_0c, c\rangle\big) \geq 0.$$

Evidently, this quadratic equation in r has at most one real root which means that its discriminant is not positive. Translating this into a statement about the coefficients yields $\|XU^{n+1}c\|^2 \leq \langle CU^{n+1}c, U^{n+1}c\rangle - \langle V_0c, c\rangle$, which is equivalent to

$$\langle V_0c, c\rangle \leq \langle CU^{n+1}c, U^{n+1}c\rangle - \langle X^*XU^{n+1}c, U^{n+1}c\rangle = \langle YU^{n+1}c, U^{n+1}c\rangle.$$

Altogether then, if $c \in \mathcal{P}_0$, we have

$$\langle V_0c, c\rangle \leq \langle YU^{n+1}c, U^{n+1}c\rangle \leq \langle AU^{n+1}c, U^{n+1}c\rangle,$$

and the lower estimate in (4) follows. □

In the algebraic language of this paper, Treil's result in [**6**] essentially takes the form of:

Corollary 8 (Treil'). *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative and Laurent. Then there exists a non-trivial, non-negative Laurent operator $V \in \mathcal{B}(\mathcal{L})$ with $P_0 V|_{\mathcal{P}_0}$ invertible in $\mathcal{B}(\mathcal{P}_0)$ and satisfying*

$$\langle V P_{\mathcal{H}} f, P_{\mathcal{H}} f\rangle \leq \langle W f, f\rangle \quad \forall f \in \mathcal{L}$$

if and only if $V_0 = wk\text{-}\lim_{\epsilon\to 0}[P_0 W_\epsilon^{-1}|_{\mathcal{P}_0}]^{-1}$ is invertible.

Proof. If V exists, then as before, we can show that $V_0 \geq \frac{1}{4} P_0 V|_{\mathcal{P}_0}$. It follows that V_0 is bounded below and consequently invertible. Conversely, the construction in Theorem 7 yields an operator V satisfying (4). Thus, if V_0 is invertible then $P_0 V|_{\mathcal{P}_0}$ is invertible as well. □

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