INNER QUASIDIAGONALITY AND STRONG NF ALGEBRAS

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Continuing the study of generalized inductive limits of finite-dimensional C^*-algebras, we define a refined notion of quasidiagonality for C^*-algebras, called inner quasidiagonality, and show that a separable C^*-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal. Many natural classes of NF algebras are strong NF, including all simple NF algebras, all residually finite-dimensional nuclear C^*-algebras, and all approximately subhomogeneous C^*-algebras. Examples are given of NF algebras which are not strong NF.

1. Introduction.

This paper is a sequel to Blackadar & Kirchberg [**BKb**], to which we will frequently refer. In Blackadar & Kirchberg, we studied a generalized inductive limit construction for C^*-algebras and gave various characterizations of C^*-algebras which can be written as generalized inductive limits of finite-dimensional C^*-algebras. We recall the definitions for the convenience of the reader:

Definition 1.1. A separable C^*-algebra A is an *MF algebra* if it can be written as $\varinjlim(A_n, \phi_{m,n})$ for a generalized inductive system with the A_n finite-dimensional. If the connecting maps $\phi_{m,n}$ can be chosen to be completely positive contractions, then A is an *NF algebra*, and A is a *strong NF algebra* if the $\phi_{m,n}$ can be chosen to be complete order embeddings.

There is a close relation between these notions and quasidiagonality and nuclearity: a (separable) C^*-algebra A is an MF algebra if and only if it has an essential quasidiagonal extension by the compact operators $\mathbf{K}$ [**BKb**, 3.2.2], and A is an NF algebra if and only if it is nuclear and quasidiagonal [**BKb**, 5.2.2]. A number of other characterizations of MF, NF, and strong NF algebras are given in [**BKb**].

One major problem left unresolved in [**BKb**] is whether every NF algebra is a strong NF algebra. The purpose of this paper is to answer this question. We will characterize strong NF algebras in terms of a sharpened version of quasidiagonality we call *inner quasidiagonality*. The exact definition of inner quasidiagonality will be given in Section 2; roughly (and possibly not quite

correctly), a C^*-algebra is inner quasidiagonal if it has a separating family of quasidiagonal irreducible representations.

As a consequence, we show that "most", but not all, NF algebras are strong NF. In particular, we show that the following C^*-algebras are strong NF:

> All (separable) strongly quasidiagonal nuclear C^*-algebras.
> All (separable) residually finite-dimensional nuclear C^*-algebras.
> All (separable) approximately subhomogeneous C^*-algebras.
> All prime antiminimal NF algebras.
> All simple NF algebras.

We actually show that a prime strong NF algebra has a strong NF system $(A_n, \phi_{m,n})$ in which each A_n is a single matrix algebra.

On the other hand, there are NF algebras which are not strong NF: if A is a (separable) prime nuclear C^*-algebra containing an ideal isomorphic to $\mathbf{K}$, then A is strong NF if and only if its (unique) faithful irreducible representation is quasidiagonal. Thus the examples of [**Bn**] and [**BnD**] are not strong NF. We also show that if A is a separable nuclear C^*-algebra which is not strong NF, then CA and SA (which are NF by [**BKb**, 5.3.3]) are not strong NF, answering [**BKb**, 6.2.3(c)].

2. Inner quasidiagonality.

We begin by noting the following characterization of quasidiagonality from [**Vo2**, Theorem 1]:

Proposition 2.1. *A C^*-algebra A is quasidiagonal if and only if, for every $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a representation π of A on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \mathbf{B}(\mathbf{H})$ with $\|p\pi(x_j)p\| > \|x_j\| - \varepsilon$ and $\|[p, \pi(x_j)]\| < \varepsilon$ for all j.*

Definition 2.2. A C^*-algebra A is *inner quasidiagonal* if, for every $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a representation π of A on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \pi(A)'' \subseteq \mathbf{B}(\mathbf{H})$ with $\|p\pi(x_j)p\| > \|x_j\| - \varepsilon$ and $\|[p, \pi(x_j)]\| < \varepsilon$ for all j.

It obviously suffices in this definition to assume that the x_j have norm 1.

The term "inner quasidiagonal" should really be "weakly inner quasidiagonal," but we have rejected this terminology on pedantic grounds.

An inner quasidiagonal C^*-algebra is obviously quasidiagonal. The converse is false (2.7).

Proposition 2.3. *In the definition of inner quasidiagonality (2.2), the representation π may be taken to be a direct sum of a finite number of mutually inequivalent irreducible representations.*

Proof. Suppose A is inner quasidiagonal, and let $x_1, \dots, x_m \in A$ and $\varepsilon > 0$. Choose π and p as in 2.2. Let z be the central support projection for p in $\pi(A)''$; then $\pi(A)''z$ and hence also $\pi(A)'z$ are type I von Neumann algebras with finite-dimensional centers. If q is an abelian projection in $\pi(A)'$ with central support z, then $\pi|_{q\mathbf{H}}$ and pq are the desired representation and projection. □

This will be generalized later (3.7).

The following is an immediate consequence of the definition.

Proposition 2.4. *If A has a separating family of quasidiagonal irreducible representations, then A is inner quasidiagonal. In particular, every residually finite-dimensional C^*-algebra is inner quasidiagonal.*

We do not know whether the converse of 2.4 is true. (See note added in proof.) But an important special case of the converse is true, even in stronger form:

Proposition 2.5. *If A is separable and prime, then A is inner quasidiagonal if and only if some (hence every) faithful irreducible representation of A is quasidiagonal.*

Proof. A C^*-algebra with a quasidiagonal faithful irreducible representation is obviously inner quasidiagonal. For the converse, consider the cases A antiliminal (NGCR) and A not antiliminal separately. If A is antiliminal, separable, prime, and inner quasidiagonal, then A is quasidiagonal, so by Voiculescu's Weyl-von Neumann Theorem [**Vo1**] every faithful representation not hitting the compacts (in particular, every faithful irreducible representation) of A is quasidiagonal.

Now suppose A is separable, prime, inner quasidiagonal, and not antiliminal. Then A has an essential ideal isomorphic to $\mathbf{K}$, and has a unique faithful irreducible representation π_0 on a Hilbert space $\mathbf{H}_0$. Let $\{x_i\}$ be a dense sequence in A, and let $\{e_{ij}\}$ be a set of matrix units in $\mathbf{K} \subseteq A$. For each n let π_n and p_n be as in 2.3 for the set $\{x_1, \dots, x_n, e_{11}, \dots, e_{nn}\}$ and for $\varepsilon = 1/n$. Then π_0 must be one of the irreducible subrepresentations of π_n for each n since $\|\pi_n(e_{11})\| > \|e_{11}\| - 1/n \geq 0$. Let q_n be the component of p_n in π_0. Then, for any j, $\lim_{n\to\infty} \|q_n\pi_0(e_{jj})q_n\| = 1$ and $\lim_{n\to\infty}[q_n, \pi_0(x_j)] = 0$, so $q_n \to 1_{\mathbf{H}_0}$ strongly and π_0 is quasidiagonal. □

2.5 will be generalized in 3.18.

Corollary 2.6. *Every separable antiliminal quasidiagonal prime C^*-algebra is inner quasidiagonal. Every separable simple quasidiagonal C^*-algebra is inner quasidiagonal.*

Example 2.7. (a) The examples of [**Bn**] and [**BnD**] are quasidiagonal, but not inner quasidiagonal by 2.5.

We recall for the reader that the example of [**Bn**] is an essential extension of the continuous functions on the real projective plane $\mathbf{R}P^2$ by the compacts:

$$0 \to \mathbf{K} \to A \to C(\mathbf{R}P^2) \to 0.$$

The examples of [**BnD**] are slight variations of this, and have the additional feature that the extension has real rank zero.

(b) A similar example is the C^*-algebra generated by the direct sum of the unilateral shift s and its adjoint. The C^*-algebra is not prime; we have the extension

$$0 \to \mathbf{K} \oplus \mathbf{K} \to C^*(s \oplus s^*) \to C(\mathbf{T}) \to 0.$$

$C^*(s \oplus s)$ is quasidiagonal, but has only two irreducible representations nonzero on the $\mathbf{K} \oplus \mathbf{K}$, neither of which is quasidiagonal; hence it is easily seen not to be inner quasidiagonal.

Proposition 2.8. *Let A be a separable C^*-algebra. The following are equivalent:*

   (i) *Every quotient of A is inner quasidiagonal.*
  (ii) *Every primitive quotient of A is inner quasidiagonal.*
 (iii) *Every irreducible representation of A is quasidiagonal.*
 (iv) *A is strongly quasidiagonal, i.e., every representation of A is quasidiagonal.*

Proof. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) is 2.5. To show (iii) $\Rightarrow$ (iv), let π be a representation, which we may assume is nondegenerate, of A on a Hilbert space $\mathbf{H}$, which we may assume is separable. Let $J = \pi^{-1}(\mathbf{K})$. Then $\pi = \pi_1 \oplus \pi_2$, where $\pi_1|_J$ is nondegenerate and $\pi_2(J) = 0$. Then π_1 is a direct sum of irreducible representations, hence quasidiagonal, and π_2 is quasidiagonal by Voiculescu's Weyl-von Neumann Theorem since $\pi_2(A)$ is a quasidiagonal C^*-algebra by (i). $\qquad\square$

Proposition 2.9. *Let A be a C^*-algebra, and J_1, J_2 ideals of A. Set $J = J_1 \cap J_2$. If A/J_1 and A/J_2 are inner quasidiagonal, then A/J is inner quasidiagonal.*

Proof. We may clearly assume $J = 0$. Let $x_1, \dots, x_m \in A$ and $\varepsilon > 0$. Let ρ_k $(k = 1, 2)$ be the quotient map from A to A/J_k. Then, for each j, $\|x_j\| = \max(\|\rho_1(x_j)\|, \|\rho_2(x_j)\|)$. By reordering we may assume $\|\rho_1(x_j)\| = \|x_j\|$ for $1 \le j \le r$ and $\|\rho_2(x_j)\| = \|x_j\| > \|\rho_1(x_j)\|$ for $r + 1 \le j \le m$. We may also assume ε is small enough that $\|\rho_1(x_j)\| < \|x_j\| - \varepsilon$ for $r + 1 \le j \le m$.

Let σ_k $(k = 1, 2)$ be representations of A/J_k as in 2.3 for $\{\rho_k(x_1), \dots, \rho_k(x_m)\}$ and the given ε, with projections p_k. Let σ_0 be the subrepresentation of σ_2 consisting of those irreducible subrepresentations σ such that $\|p\sigma(\rho_2(x_j))p\| > \|x_j\| - \varepsilon$ for at least one j, $r + 1 \le j \le m$ (where p denotes the σ-component of p_2), and p_0 the component of p_2 in σ_0. Then all of the

irreducible subrepresentations of σ_0, regarded as representations of A, are disjoint from $\sigma_1 \circ \rho_1$ since each such σ satisfies $\|\sigma(\rho_2(x_j))\| > \|\rho_1(x_j)\|$ for at least one j. Thus $\pi = \sigma_1 \circ \rho_1 \oplus \sigma_0 \circ \rho_2$ and $p = p_1 \oplus p_0$ have the desired properties from 2.2. $\qquad\square$

Corollary 2.10. *A C^*-algebra A is inner quasidiagonal if A contains a collection $\{J_i\}$ of ideals with A/J_i inner quasidiagonal for all i and $\cap J_i = 0$.*

Proof. A direct proof can be given along the lines of the proof of 2.9. Alternatively, note that the result is immediate from the definition of inner quasidiagonality if the J_i are directed by inclusion. In general, let $J_{i_1,\dots,i_n} = J_{i_1} \cap \dots \cap J_{i_n}$, and use 2.9 to conclude that $A/J_{i_1,\dots,i_n}$ is inner quasidiagonal. $\qquad\square$

Remark 2.11. It is obvious from the definition that if $\{A_i, \phi_{ij}\}$ is an (ordinary) inductive system (indexed by any directed set) of inner quasidiagonal C^*-algebras with *injective* connecting maps ϕ_{ij}, then the inductive limit is inner quasidiagonal. It is not true that the inductive limit of an inductive system with noninjective connecting maps is necessarily inner quasidiagonal, as the the following example shows. (It is an inductive system with surjective connecting maps.) The same question can be asked about quasidiagonality, where it appears to have a positive answer; the closely related classes of MF and NF algebras are closed under (ordinary) inductive limits with noninjective connecting maps, as well as certain generalized inductive limits [**BKb**, 3.4.4 and 5.3.5].

Example 2.12. Let B be a (separable) quasidiagonal C^*-algebra which is not inner quasidiagonal, e.g., the example of [**Bn**] (2.7). Let π be a faithful representation of B of infinite multiplicity on a separable Hilbert space $\mathbf{H}_0$; then π is quasidiagonal. Let $\langle \mathbf{H}_n \rangle_{n \geq 1}$ be a sequence of separable infinite-dimensional Hilbert spaces, with distinguished unit vectors ξ_n, and set $\mathbf{H}^{(n)} = \otimes_{k \geq n}(\mathbf{H}_k, \xi_k)$. If $\mathbf{H} = \mathbf{H}_0 \otimes \mathbf{H}^{(1)}$, define C^*-subalgebras of $\mathbf{B}(\mathbf{H})$ by $C_n = \mathbf{K}(\mathbf{H}_0) \otimes 1_{\mathbf{H}_1} \otimes \cdots \otimes 1_{\mathbf{H}_{n-1}} \otimes \mathbf{K}(\mathbf{H}^{(n)})$ $[C_1 = \mathbf{K}(\mathbf{H})]$, $J_n = C_1 + \cdots + C_n$, $A_n = J_n + \rho(B)$, $J = [\cup J_n]^-$, $A = J + \rho(B) = [\cup A_n]^-$, where $\rho = \pi \otimes 1_{\mathbf{H}^{(1)}}$. Then $\langle J_n \rangle$ is an increasing sequence of ideals of A. Each A_n is inner quasidiagonal by repeated applications of 2.5; thus A is inner quasidiagonal by 2.11. A/J_n is isomorphic to A for any n, hence inner quasidiagonal; but $A/J \cong B$ is not inner quasidiagonal.

3. Variations and technicalities.

A somewhat cleaner alternative definition of inner quasidiagonality can be given using the socle of the bidual. See [**BoD**] for the general theory of socles of Banach algebras.

Definition 3.1. If B is a C^*-algebra, then a projection $p \in B$ is in the *socle* of B if pBp is finite-dimensional.

Proposition 3.2. *A C^*-algebra A is inner quasidiagonal if and only if, for any $x_1, \dots, x_m \in A$ and $\varepsilon > 0$, there is a projection p in the socle of A^{**} with $\|px_j p\| > \|x_j\| - \varepsilon$ and $\|[p, x_j]\| < \varepsilon$ for all j.*

Proposition 3.3. *Let B be a C^*-algebra, $x \in B$, and p a projection in B. Then*

$$\|[x, p]\| = \max(\|(1 - p)xp\|, \|px(1 - p)\|)$$
$$= \max(\|px^*xp - px^*pxp\|^{1/2}, \|pxx^*p - pxpx^*p\|^{1/2}).$$

Proof. Set $a = (1 - p)xp$, $b = px(1 - p)$. Then calculation shows that $[x, p]^*[x, p] = a^*a + b^*b$, so $\|[x, p]\|^2 = \|a^*a + b^*b\| = \max(\|a^*a\|, \|b^*b\|)$ since a^*a and b^*b are orthogonal. Also, $\|[x, p]\|^2 = \max(\|a^*a\|, \|bb^*\|)$, and $a^*a = px^*xp - px^*pxp$ and $bb^* = pxx^*p - pxpx^*p$. $\qquad\square$

The following fact, which is a slight sharpening of the Kadison Transitivity Theorem, follows immediately from [**Pd**, 2.7.5 and 3.11.9].

Proposition 3.4. *Let A be a C^*-algebra, and p a projection in the socle of A^{**}. Set $N_p = \{x \in A : [p, x] = 0\} = \{p\}' \cap A$. Then*

(a) *$pN_p = pN_p p = pA^{**}p \ (= pAp)$.*
(b) *The weak closure of N_p in A^{**} is $pA^{**}p + (1 - p)A^{**}(1 - p)$.*

Corollary 3.5. *Let A, p, N_p be as in 3.4, and let $x \in A$. Then $d(x, N_p) = \|[x, p]\|$.*

Proof. $d(x, N_p) = d(x, N_p^{**})$ by the Hahn-Banach Theorem. We have $y = pxp + (1 - p)x(1 - p) \in N_p^{**}$ by 3.4, and

$$\|x - y\| = \|(1 - p)xp + px(1 - p)\|$$
$$= \max(\|(1 - p)xp\|, \|px(1 - p)\|) = \|[x, p]\|.$$

So $d(x, N_p) \leq \|[x, p]\|$. Conversely, if $y \in N_p$, then

$$\|[x, p]\| = \|[x - y, p]\|$$
$$= \max(\|(1 - p)(x - y)p\|, \|p(x - y)(1 - p)\|) \leq \|x - y\|.$$
$$\qquad\square$$

We now show that in many instances, the study of inner quasidiagonality can be reduced to the separable case.

Proposition 3.6. *Let A be an inner quasidiagonal C^*-algebra, and B a separable C^*-subalgebra of A. Then there is a separable inner quasidiagonal C^*-subalgebra E of A containing B.*

Proof. We show that if $x_1, \dots, x_m \in B$ and $\varepsilon > 0$, then there is a separable C^*-subalgebra D of A containing B, and a projection q in the socle of D^{**} with $\|[q, x_j]\| < \varepsilon$ and $\|qx_j q\| > \|x_j\| - \varepsilon$ for $1 \leq j \leq m$. Choose a projection

p in the socle of A^{**} with $\|[p, x_j]\| < \varepsilon/2$ and $\|px_jp\| > \|x_j\| - \varepsilon/2$ for all j. Let C be a separable C^*-subalgebra of N_p with $d(x_j, C) = d(x_j, N_p)$ for all j and $pC = pN_p = pAp$, and let D be the C^*-subalgebra of A generated by B and C. Then D is separable. Regard the homomorphism $\pi : C \to pC = pAp$ as a representation of C on a finite-dimensional Hilbert space $\mathbf{H}$, and extend π to a representation p of D on a larger Hilbert space $\widetilde{\mathbf{H}}$ such that ρ is a direct sum of finitely many mutually inequivalent irreducible representations (by extending each irreducible subrepresentation of π to an irreducible representation of D and identifying together equivalent ones if necessary). If Q is the projection from $\widetilde{\mathbf{H}}$ onto $\mathbf{H}$, then Q may be regarded as a projection q in the socle of D^{**}, and $C \subseteq N_q$. We have by 3.5, for each j,

$$\|[q, x_j]\| = d(x_j, N_q) \leq d(x_j, C) = \|[p, x_j]\| < \varepsilon.$$

Also, there is an isometry ψ from qDq to pAp induced by ρ, and $\psi(qaq) = pap$ for $a \in C$. Since $d(x_j, C) < \varepsilon/2$, we have, for each j,

$$\|qx_jq\| > \|px_jp\| - \varepsilon/2 > \|x_j\| - \varepsilon.$$

As a consequence, if $B = B_1$ is a separable C^*-subalgebra of A, then there is a larger separable C^*-subalgebra B_2 of A such that, if $x_1, \ldots, x_m \in B_1$ and $\varepsilon > 0$, there is a projection q in the socle of B_2^{**} with $\|[q, x_j]\| < \varepsilon$ and $\|qx_jq\| > \|x_j\| - \varepsilon$ for $1 \leq j \leq m$. Iterating this construction, obtain an increasing sequence $\langle B_n \rangle$, and set $E = [\cup B_n]^-$. Then E is separable and inner quasidiagonal. $\qquad\square$

Separability is nice because of the following characterizations of inner quasidiagonality. By a slight extension of usual terminology, we will say an irreducible representation π of a C^*-algebra A is a *GCR representation* if $\pi(A)$ contains the compact operators.

Proposition 3.7. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *A is inner quasidiagonal.*

(ii) *Given $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is an irreducible representation π of A on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \mathbf{B}(\mathbf{H})$ such that $\|[p, \pi(x_j)]\| < \varepsilon$ for $1 \leq j \leq m$ and $\|p\pi(x_1)p\| > \|x_1\| - \varepsilon$.*

(iii) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ of A on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$.*

(iv) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ of A on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$, such that any representation occuring more than once (up to equivalence) in the sequence is quasidiagonal and GCR.*

Proof. (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) are obvious. To prove (iv) $\Rightarrow$ (i), let $x_1, \dots, x_m \in A$ and $\varepsilon > 0$, and let $\langle (\pi_n, p_n) \rangle$ be as in (iv). Choose $(\pi_{n_1}, p_{n_1}), \dots, (\pi_{n_m}, p_{n_m})$ with $\|p_{n_k} \pi_{n_k}(x_k) p_{n_k}\| > \|x_k\| - \varepsilon$ and $\|[\pi_{n_k}(x_j), p_{n_k}]\| < \varepsilon$ for all j and k. If there is a subset F of $\{1, \dots, m\}$ with $|F| > 1$ such that $\pi_{n_k} = \pi_0$ for $n \in F$, then π_0 is quasidiagonal, so there is a finite-rank projection q in $\pi_0(A)''$ with $\|q\pi_0(x_k)q\| > \|x_k\| - \varepsilon$ for all $k \in F$. Replace $\oplus_{k=1}^m (\pi_{n_k}, p_{n_k})$ by $(\pi_0, q) \oplus \oplus_{k \in F}(\pi_{n_k}, p_{n_k})$. Repeating the process finitely many times if necessary, we obtain a representation as in 2.3.

So we need only prove (ii) $\Rightarrow$ (iv). Let $\langle x_k \rangle$ be a dense sequence in A, and choose a doubly indexed sequence $\langle \rho_{ij} \rangle$ $(1 \leq k \leq j)$ of irreducible representations of A on $\mathbf{H}_{jk}$, and finite-rank projections $q_{jk} \in \mathbf{B}(\mathbf{H}_{jk})$, such that $\|[q_{jk}, \rho_{jk}(x_i)]\| < j^{-1}$ for $1 \leq i \leq j$ and $\|q_{jk}\rho_{jk}(x_i)q_{jk}\| > \|x_k\| - j^{-1}$. (Note that this sequence satisfies the conditions of (iii).) The rest of the proof will consist of two parts.

(1) We first show that if infinitely many ρ_{jk} are equivalent to a single ρ, then either ρ is quasidiagonal and GCR or the sequence can be modified to a new sequence in which no ρ_{jk} is equivalent to ρ. If $\rho(A) \cap \mathbf{K} = \{0\}$, then there are infinitely (in fact, uncountably) many mutually inequivalent irreducible representations of A with the same kernel as ρ (see Appendix), and if π is any such representation, on a Hilbert space $\widetilde{\mathbf{H}}$, and $\varepsilon > 0$, then for any j, k with $\rho_{jk} \approx \rho$, by [**Vo1**, Lemma 1] there is a unitary u from $\mathbf{H}_{jk}$ to $\widetilde{\mathbf{H}}$ and a finite-rank projection $p \in \mathbf{B}(\widetilde{\mathbf{H}})$ such that

$$\|uq_{jk}\rho_{jk}(x_i)q_{jk}u^* - p\pi(x_i)p\| < \varepsilon$$

for $1 \leq i \leq j$. For sufficiently small ε, (ρ_{jk}, q_{jk}) can be replaced by (π, p), and a different π can be used for each ρ_{jk} equivalent to ρ.

Now suppose that ρ is GCR. If $J = \ker \rho$, then there is an ideal K of A with K/J essential in A/J and isomorphic to $\mathbf{K}$. By identifying $\mathbf{H}_{jk}$ with $\mathbf{H}$ (the Hilbert space on which ρ acts) for each ρ_{jk} equivalent to ρ, the projections q_{jk} become a sequence $\langle r_n \rangle$ of finite-rank projections in $\mathbf{B}(\mathbf{H})$ which asymptotically commute in norm with $\rho(A)$, and in particular with $\mathbf{K}$; thus the only possible weak operator limit points of the sequence are 0 and 1 [any limit point must be a scalar by irreducibility, and if $\lambda 1$ is a limit point and p a finite-rank projection, then $\langle pr_n p \rangle$ has a subsequence converging in norm to λp; but $pr_n p$ is approximately a projection for large n, so any norm limit point must be a projection]. If 1 is a limit point, then there is a subsequence of $\langle r_n \rangle$ converging weakly, hence strongly, to 1, so ρ is quasidiagonal.

If 1 is not a limit point, then $r_n \to 0$ weakly, hence strongly, and so $r_n a r_n \to 0$ in norm for all $a \in \mathbf{K}$, i.e., $\|r_n \rho(x) r_n\| \to 0$ for all $x \in K$. Fix $x_0 \in K$ of norm 1. We can then find a subsequence $\langle \rho_n \rangle$ of $\{\rho_{jk}\}$, with associated Hilbert spaces $\mathbf{H}_n$ and projections q_n, such that $\|[q_n, \rho_n(x)]\| \to 0$ for all $x \in A$, $\|q_n \rho_n(x_0) q_n\| \to 1$, and such that ρ_n is not equivalent to ρ

for any n. Define $\phi : A \to \prod q_n \mathbf{B}(\mathbf{H}_n)q_n$ by $\phi(x) = \prod q_n \rho_n(x)q_n$. ϕ then drops to a *-homomorphism σ from A to $(\prod q_n \mathbf{B}(\mathbf{H}_n)q_n)/(\oplus q_n \mathbf{B}(\mathbf{H}_n)q_n)$. Set $I = \ker \sigma$. Then $I \cap K \subseteq J$ since $x_0 \notin I$, so $(I + J)/J$ is an ideal of A/J orthogonal to K/J. But K/J is essential in A/J, so $(I + J)/J = 0$, $I \subseteq J$, and so $\|\sigma(x)\| \geq \|\rho(x)\|$ for all $x \in A$. Thus the subsequence of $\{\rho_{jk}\}$ consisting of those which are not equivalent to ρ still satisfies the conditions of (iii), and a smaller subsequence will have the same specific properties as the full double sequence $\{\rho_{jk}\}$ if suitably reindexed.

(2) We now construct a doubly indexed sequence $\langle(\pi_{jk}, p_{jk})\rangle$ satisfying the conditions of (iv). Suppose $\{(\pi_{jk}, p_{jk})\}$ have been chosen from among the (ρ_{nr}, q_{nr}) for $1 \leq k \leq j < m$, satisfying the following properties:

(a) $\|[p_{jk}, \pi_{jk}(x_i)]\| < j^{-1}$ for $1 \leq i \leq j$.
(b) $\|p_{jk}\pi_{jk}(x_i)p_{jk}\| > \|x_k\| - j^{-1}$.
(c) No irreducible representation occurs more than once among the π_{jk} chosen so far unless it is quasidiagonal and GCR.

If $m = 1$, choose (π_{11}, p_{11}) to satisfy (a) and (b). If $m > 1$, the π_{jk} already chosen come from the ρ_{nr} for $n \leq n_0$ for some n_0. The tail $\{\rho_{nr} : n > n_0\}$ can be modified as in (1), and further truncated by increasing n_0 if necessary, so that none of the $\{\pi_{jk} : 1 \leq k \leq j < m\}$ occurs in the tail unless it is quasidiagonal and GCR. Then a suitable element of the modified tail satisfies (a) and (b) and can be chosen as π_{m1}. After again modifying and truncating the tail to eliminate subsequent appearances of π_{m1} if necessary, π_{m2} can be chosen. The process can be continued inductively to get the desired representations and projections.

This completes the proof. □

Lemma 3.8. *Let A be a C^*-algebra. Then, for any k, A is inner quasidiagonal if and only if $M_k(A) = A \otimes \mathbf{M}_k$ is inner quasidiagonal. The projections for $A \otimes \mathbf{M}_k$ may be chosen of the form $p \otimes 1_k$, where p is in the socle of A^{**}.*

Proof. Suppose A is inner quasidiagonal. It follows from 3.6 that we may assume A is separable (or the following argument may be easily modified to apply to the nonseparable case). Let $\langle(\pi_n, p_n)\rangle$ be a sequence as in 3.7(iii). The map $\phi : A \to \prod p_n \mathbf{B}(\mathbf{H}_n)p_n$ given by $\phi(x) = \prod p_n \pi_n(x)p_n$ drops to an injective *-homomorphism σ from A to $(\prod p_n \mathbf{B}(\mathbf{H}_n)p_n)/\oplus p_n \mathbf{B}(\mathbf{H}_n)p_n$, and hence $\sigma \otimes 1_k$ is an injective *-homomorphism from $A \otimes M_k$ to

$$\left[\left(\prod p_n \mathbf{B}(\mathbf{H}_n)p_n\right) / \oplus p_n \mathbf{B}(\mathbf{H}_n)p_n\right] \otimes M_k$$
$$\cong \left(\prod (p_n \otimes 1_k)[\mathbf{B}(\mathbf{H}_n) \otimes M_k](p_n \otimes 1_k)\right) / \oplus (p_n \otimes 1_k)$$
$$\cdot [\mathbf{B}(\mathbf{H}_n) \otimes M_k](p_n \otimes 1_k).$$

The converse is trivial. □

Proposition 3.9. *Let A and B be C^*-algebras, with A inner quasidiagonal and B residually finite-dimensional (e.g., commutative). Then $A \otimes B = A \otimes_{\min} B$ is inner quasidiagonal.*

Proof. Let $\{J\}$ be a collection of ideals of B with intersection 0, with B/J_i finite-dimensional for all i. Then $\{A \otimes J_i\}$ is a collection of ideals of $A \otimes B$, with intersection 0, and $(A \otimes B)/(A \otimes J_i) = A \otimes (B/J_i)$ is a finite direct sum of matrix algebras over A, hence inner quasidiagonal. $\qquad\square$

We will prove a partial converse to 3.9 in 3.10 below. We believe that a tensor product of any two inner quasidiagonal C^*-algebras is inner quasidiagonal, at least if one of the factors is nuclear. This would follow if the converse to 2.4 is true. (See note added in proof.) Note that a tensor product of strong NF algebras is strong NF (5.17).

Proposition 3.10. *Let A and C be C^*-algebras, with C commutative. If $A \otimes C$ is inner quasidiagonal, then A is also inner quasidiagonal.*

Proof. We reduce to the case where A is separable. If $A \otimes C$ is inner quasidiagonal, and S is a countable subset of A, then by 3.6 and an obvious additional construction we may construct an increasing sequence $\langle B_n \rangle$ of separable C^*-subalgebras of $A \otimes C$ such that B_n is inner quasidiagonal for n odd and $B_n = D_n \otimes E_n$ for n even, for separable C^*-subalgebras D_n of A containing S and E_n of C. Then $B = [\cup B_n]^-$ is separable, inner quasidiagonal, and equal to $D \otimes E$ for $D = [\cup D_n]^- \subseteq A$, which contains S, and $E = [\cup E_n]^- \subseteq C$.

Now suppose A is separable. Write $C = C_0(X)$ for a locally compact Hausdorff space X; then $A \otimes C = C_0(X, A)$. Suppose $A \otimes C$ is inner quasidiagonal; let $x_1, \dots, x_m \in A$, all of norm 1, and $0 < \varepsilon < 1$. Choose $\delta > 0$ such that $\frac{\delta}{1-\delta} < \varepsilon$. Let U be an open set in X with compact closure, $g \in C_0(X)$ of norm 1 supported in U, and $f \in C_0(X)$ of norm 1 and identically 1 on U, with f and g taking values in $[0,1]$. Let $V = \{x : g(x) > 1 - \delta\} \subseteq U$. Let π be an irreducible representation of $A \otimes C = C_0(X, A)$ and p a finite-rank projection such that $\|p\pi(x_1 \otimes g)p\| > 1 - \delta$ and $[p, \pi(x_j \otimes g)] < \delta$ for all j. Then π is supported on a point x_0 of V, so π may be regarded as a representation ρ of A by $\rho(x) = \pi(x \otimes f)$. For each j, $\pi(x_j \otimes f)$ is a scalar multiple of $\pi(x_j \otimes g)$, with a scalar $\lambda = g(x_0)^{-1}$ satisfying $1 \leq \lambda \leq (1-\delta)^{-1}$; thus $\|[q, \pi(x_j \otimes f)]\| \leq (1-\delta)^{-1}\|[q, \pi(x_j \otimes g)]\| < \varepsilon$, and $\|p\pi(x_1 \otimes f)p\| \geq \|p\pi(x_j \otimes g)p\| > 1 - \delta > 1 - \varepsilon$. Thus ρ and p satisfy condition (ii) of 3.7, so A is inner quasidiagonal. $\qquad\square$

Corollary 3.11. *SA is inner quasidiagonal if and only if A is inner quasidiagonal, and similarly for CA.*

Proof. Combine 3.10 with 3.9. $\qquad\square$

Remark 3.12. 3.11 shows that unlike quasidiagonality [**Vo2**], inner quasidiagonality is not a homotopy invariant for C^*-algebras.

We have the following refinement of the notion of inner quasidiagonality:

Definition 3.13. Let A be a C^*-algebra, and z a central projection in A^{**}. Then A is *z-inner quasidiagonal* if, for any $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a projection p in the socle of zA^{**} with $\|px_jp\| > \|x_j\| - \varepsilon$ and $\|[p, x_j]\| < \varepsilon$ for all j. If Π is a subset of $\widehat{A}$, A is *Π-inner quasidiagonal* if A is z-inner quasidiagonal, where z is the support projection of Π in the center of A^{**}.

A is inner quasidiagonal if and only if A is $1_{A^{**}}$-inner quasidiagonal. A is z-inner quasidiagonal if and only if A is Π_z-inner quasidiagonal, where Π_z is the set of irreducible representations of A with central support $\leq z$.

Example 3.14. Let A be a C^*-algebra with a quasidiagonal faithful irreducible representation π, and let z be the support projection of π in A^{**}. Then A is z-inner quasidiagonal, and zA^{**} is a type I factor. More generally, if $\{\pi_j\}$ is a separating family of quasidiagonal irreducible representations of A, and z is the support projection of $\oplus \pi_j$, then A is z-inner quasidiagonal.

There are versions of 3.6-3.8 for z-inner or Π-inner quasidiagonality, although 3.7(iv) must be weakened (but see 3.18). If A is a C^*-algebra, B a C^*-algebra, and Π a subset of $\widehat{A}$, let $\Pi|_B$ be the subset of $\widehat{B}$ consisting of all irreducible representations (actually, not just weakly) contained in $\pi|_B$ for some $\pi \in \Pi$.

Proposition 3.15. *Let A be a C^*-algebra and Π a subset of $\widehat{A}$. If A is Π-inner quasidiagonal and B is separable C^*-subalgebra of A, then there is a separable C^*-subalgebra E of A, containing B, which is $\Pi|_E$-inner quasidiagonal.*

Proposition 3.16. *Let A be a separable C^*-algebra, and Π a subset of $\widehat{A}$. The following are equivalent:*

  (i) *A is Π-inner quasidiagonal.*
  (ii) *Given $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is an irreducible representation $\pi \in \Pi$ on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \mathbf{B}(\mathbf{H})$ such that $\|[p, \pi(x_j)]\| < \varepsilon$ for $1 \leq j \leq m$ and $\|[p\pi(x_j)p]\| > \|x_1\| - \varepsilon$.*
  (iii) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ in Π on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$.*
  (iv) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ in Π on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$, such that any GCR representation occuring more than once (up to equivalence) in the sequence is quasidiagonal.*

Proposition 3.17. *Let A be a C^*-algebra, z a central projection in A^{**}. Then, for any k, A is z-inner quasidiagonal if and only if $M_k(A) = A \otimes \mathbf{M}_k$ is $(z \otimes 1)$-inner quasidiagonal. The projections for $A \otimes \mathbf{M}_k$ may be chosen of the form $p \otimes 1_k$, where p is in the socle of zA^{**}.*

Actually, z-inner or Π-inner quasidiagonality is not really a stronger condition than inner quasidiagonality for separable C^*-algebras, as the next result shows. This is a generalization of 2.5, and is closely related to 3.7.

Theorem 3.18. *Let A be a separable inner quasidiagonal C^*-algebra, and Π a separating set of mutually inequivalent irreducible representations of A. Then A is Π-inner quasidiagonal.*

Lemma 3.19. *Let A be a C^*-algebra, Π a faithful family of irreducible representations of A, $\pi_1, \ldots, \pi_n \in \Pi$, p_k a finite-rank projection in $\pi_k(A)''$ for $1 \le k \le n$, X a finite subset of A, and $\eta > 0$. If ρ is an representation of A not equivalent to any $\pi \in \Pi$, and q a finite-rank projection on $\mathbf{H}_\rho$, then there is a $\pi \in \Pi$ and finite-rank projection p on $\mathbf{H}_\pi$, with $p \perp p_k$ if $\pi \sim \pi_k$, and an isometry z from $q\mathbf{H}_\rho$ onto $p\mathbf{H}_\pi$, such that $\|q\rho(x)q - z^*p\pi(x)pz\| < \eta$ for all $x \in X$.*

Proof. First suppose $\pi_0 \in \Pi$ is GCR, with kernel J, and let $K = \pi_0^{-1}(\mathbf{K})$. If $I = \cap\{\ker \pi : \pi \in \Pi, \ \pi \not\sim \pi_0\}$, we have $I \cap J = 0$, so if $L \in \mathrm{Prim}(A)$, $L \ne J$, then either $I \subseteq L$ or $K \subseteq L$. Thus if p is any finite-rank projection on $\mathbf{H}_{\pi_0}$, and ρ is an irreducible representation of A not equivalent to π_0, then for any $x \in A$ we have

$$\|\rho(x)\| \le \max\{\|(1-p)\pi_0(x)(1-p)\|, \ \max\{\|\pi(x)\| : \pi \in \Pi, \ \pi \not\sim \pi_0\}\}.$$

This formula also holds if π_0 is not GCR, for any ρ and p, since then the right-hand side is equal to $\|x\|$.

Now let ρ and q be as in the statement of the lemma, with $r = \mathrm{rank}\, q$. By replacing A by $M_r(A)$ and using the standard identifications of [**Pa**, §5] (cf. [**BKb**, 4.3]), we may assume $r = 1$. If $p_1, \ldots, p_n$ are given, let S be the set of vector states coming from representations in Π, where only vector states from π_k coming from vectors orthogonal to p_k are included. Then by the first part of the proof, for any $x = x^*$ in A we have $\|\rho(x)\| \le \sup_{f \in S} |f(x)|$. So by the bipolar theorem, the weak-* closure of S contains all pure states of $\rho(A)$, proving the lemma. $\square$

Proof of Theorem 3.18. Let $x_1, \ldots, x_m \in A$, of norm 1, and $0 < \varepsilon < 1$. Choose mutually inequivalent irreducible representations $\rho_1, \ldots, \rho_n$ of A and finite-rank projections $q_1, \ldots, q_n$ such that $\|[q_k, \rho_k(x_j)]\| < \varepsilon/2m$ for all j and k and such that, for each j, there is at least one k with $\|q_k\rho_k(x_j)q_k\| > \|x_j\| - \varepsilon/2$ (2.3). Define new pairs (π_k, p_k) for $1 \le k \le n$ inductively as follows, with each π_k in Π. Suppose $(\pi_1, p_1), \ldots, (\pi_{k-1}, p_{k-1})$ have been defined. If ρ_k is in Π, set $(\pi_k, p_k) = (\rho_k, q_k)$. Otherwise, choose π, p, z as

in 3.19 for $\eta = (\varepsilon/2m)^2$, $X = \{x_j, x_j^* x_j, x_j x_l^* : 1 \le j \le m\}$, and $\rho = \rho_k$, $q = q_k$, such that $p \perp p_i$ for all $i < k$ for which $\pi = \pi_i$. Then $\|p\pi(x_j)p\| > \|q_k \rho_k(x_j) q_k\| - \varepsilon/2m$ for all j. Also, for each j,

$$\|p\pi(x_j)^* \pi(x_j)p - p\pi(x_j)^* p\pi(x_j)p\|$$
$$\le \|z^* p\pi(x_j^* x_j)pz - q\rho(x_j^* x_j)q\|$$
$$+ \|q\rho(x_j^* x_j)q - q\rho(x_j)^* q\rho(x_j)q\|$$
$$+ \|q\rho(x_j)^* q\rho(x_j)q - z^* p\pi(x_j)^* pzq\rho(x_j)q\|$$
$$+ \|z^* p\pi(x_j)^* pzq\rho(x_j)q - z^* p\pi(x_j)^* p\pi(x_j)pz\|$$
$$< \left(\frac{\varepsilon}{2m}\right)^2 + \left(\frac{\varepsilon}{2m}\right)^2 + \left(\frac{\varepsilon}{2m}\right)^2 + \left(\frac{\varepsilon}{2m}\right)^2 = \left(\frac{\varepsilon}{m}\right)^2$$

(see 3.3 for the second term). Similarly, $\|p\pi(x_j x_j^*)p - p\pi(x_j)p\pi(x_j)^* p\| < (\varepsilon/m)^2$, so $\|[p, \pi(x_j)]\| < \varepsilon/m$ by 3.3. Set $(\pi_k, p_k) = (\pi, p)$.

We have now obtained a set $\{(\pi_k, p_k) : 1 \le k \le n\}$ of representations in Π and finite-rank projections such that $\|[p_k, \pi_k(x_j)]\| < \varepsilon/m$ for all j and k and such that, for each j, there is at least one k for which $\|[p_k, \pi_k(x_j)p_k]\| > \|x_j\| - \varepsilon$. The π_k are, however, not necessarily distinct. Suppose, for some set F, each π_k for $k \in F$ is equal to a representation π_0 in Π. Then $\{p_k : k \in F\}$ are mutually orthogonal, and if $p_0 = \sum_{k \in F} p_k$, then

$$\|[p_0, \pi_0(x_j)]\| \le \sum_{k \in F} \|[p_k, \pi_k(x_j)]\| < \frac{\varepsilon|F|}{m} \le \varepsilon$$
$$\|p_0 \pi_0(x_j)p_0\| \ge \max_{k \in F} \|p_k \pi_k(x_j)p_k\|$$

for all j, so $\oplus(\pi_k, p_k)$ may be replaced by $(\pi_0, p_0) \oplus \oplus_{k \in F}(\pi_k, p_k)$. After finitely many such procedures, a direct sum of mutually inequivalent irreducible representations in Π is obtained, satisfying the definition of Π-inner quasidiagonality for $x_1, \ldots, x_m, \varepsilon$. $\qquad\square$

4. The Main Theorem.

We first show that strong NF algebras are inner quasidiagonal. We begin by recalling one of the characterizations of strong NF algebras from [**BKb**]; we state a slightly refined form for later use. The proof is essentially identical to the proof of [**BKb**, 6.1.1] (note that that proof works throughout if the finite-dimensional algebras are restricted to be in a given class **B**).

Proposition 4.1. *Let* **B** *be a set of finite-dimensional C^*-algebras, and A a separable C^*-algebra. Then the following are equivalent:*

(i) *A can be written as $\varinjlim(A_n, \phi_{m,n})$ for a generalized inductive system $(A_n, \phi_{m,n})$ with each A_n isomorphic to an algebra in* **B** *and each $\phi_{m,n}$ a complete order embedding (completely positive complete isometry).*

(ii) *For every $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a $B \in \mathbf{B}$, elements $b_1, \ldots, b_m \in B$, and a complete order embedding $\phi : B \to A$ with $\|x_j - \phi(b_j)\| < \varepsilon$ for all j.*

The strong NF algebras are exactly the A for which these conditions hold for $\mathbf{B}$ the set of all finite-dimensional C^*-algebras. We may take condition (ii) (with $\mathbf{B}$ the set of all finite-dimensional C^*-algebras) to be the definition of a strong NF algebra even in the nonseparable case.

Proposition 4.2. *Every strong NF algebra is inner quasidiagonal.*

Proof. Let A be a strong NF algebra, $x_1, \ldots, x_m \in A$, and $\varepsilon > 0$. Choose a finite-dimensional C^*-algebra B, elements $b_1, \ldots, b_m \in B$, and a complete order embedding $\phi : B \to A$ such that if $y_j = \phi(b_j)$, then $\|x_j - y_j\| < \varepsilon/2$ for all j. Let D be the C^*-subalgebra of A generated by $\phi(B)$. By [**CE1**, 4.1] (cf. [**BKb**, 4.2.2]), there is a *-homomorphism π from D onto B with $\pi(y_j) = b_j$ for all j. If $B = B_1 \oplus \cdots \oplus B_n$ with each B_i a full matrix algebra, and $\pi = \pi_1 \oplus \cdots \oplus \pi_n$, then π_i can be regarded as an irreducible representation of D on a finite-dimensional Hilbert space $\mathbf{H}_i$. Extend π_i to an irreducible representation $\widetilde{\pi}_i$ of A on a larger Hilbert space $\widetilde{\mathbf{H}}_i$, and let p_i be the projection of $\widetilde{\mathbf{H}}_i$ onto $\mathbf{H}_i$. The $\widetilde{\pi}_i$ are not in general inequivalent; we may assume that $\widetilde{\pi}_1, \ldots, \widetilde{\pi}_r$ are a set of representatives for the equivalence classes. Set $\widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}_1 \oplus \cdots \oplus \widetilde{\mathbf{H}}_r$ and $\widetilde{\pi} = \widetilde{\pi}_1 \oplus \cdots \oplus \widetilde{\pi}_r$. For $i > r$, choose $k \leq r$ with $\widetilde{\pi}_i \approx \widetilde{\pi}_k$ and identify $\widetilde{\mathbf{H}}_i$ with $\widetilde{\mathbf{H}}_k$, and p_i with the corresponding projection on $\widetilde{\mathbf{H}}_k$. Let $p \in \mathbf{B}(\widetilde{\mathbf{H}})$ be the sum of the p_i (note that for a fixed k the p_i on $\widetilde{\mathbf{H}}_k$ are orthogonal since the π_i are disjoint). Then $p \in \widetilde{\pi}(A)''$; and for each j, $[p, \widetilde{\pi}(y_j)] = 0$. so $\|[p, \widetilde{\pi}(x_j)]\| \leq 2\|x_j - y_j\| < \varepsilon$. For each j we have $\|p\widetilde{\pi}(y_j)p\| = \|y_j\|$ (since $\|\pi(y_j)\| = \|y_j\|$); so

$$\|p\widetilde{\pi}(x_j)p\| \geq \|y_j\| - \|x_j - y_j\| > \|y_j\| - \varepsilon/2 > \|x_j\| - \varepsilon.$$

$\square$

The next proposition gives an important technical characterization of inner quasidiagonality.

Proposition 4.3. *Let A be a C^*-algebra, and z a central projection in A^{**}. Then A is z-inner quasidiagonal if and only if, for any $x_1, \ldots, x_m \in A$, completely positive contraction $\phi : A \to \mathbf{M}_n$, and $\varepsilon > 0$, there is a projection p in the socle of zA^{**} with $\|[p, x_j]\| < \varepsilon$ for all j, and a completely positive contraction $\psi : pA^{**}p \to \mathbf{M}_n$ with $\|\phi(x_j) - \psi(px_jp)\| < \varepsilon$ for all j.*

Proof. The "if" part is obvious (consider the case $n = 1$). Conversely, suppose A is z-inner quasidiagonal; we may assume A is unital. Fix $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$. By 3.15 we may assume A is separable. Then we may assume there is a set Π of irreducible representations as in 3.16(iv) such that z is the support projection of Π. Because of 3.17 and the identifications described

in [**Pa**, §5] (cf. [**BKb**, 4.3]), we may assume $n = 1$. For each $\delta > 0$ let S_δ be the weak-* closure of the set of all states ω of A of the form $\omega(x) = \psi(pxp)$, where p is in the socle of zA^{**}, $\|[p, x_j]\| < \delta$ for all j, and ψ is a state on $pA^{**}p$. We want to show that S_ε is the entire state space of A. For any δ, S_δ is norming for A, i.e., for $x = x^* \in A$, $\sup\{|\phi(x)| : \phi \in S_\delta\} = \|x\|$ by Π-inner quasidiagonality. Therefore, if $\|x\| \leq 1$ and $\phi(x) \geq 0$ for all $\phi \in S_\delta$, then $\phi(1 - x) \leq 1$ for all $\phi \in S_\delta$, so $\|1 - x\| \leq 1$, $x \geq 0$. Thus S_δ contains all pure states of A by [**Dx**, 3.4.1], so it suffices to show that a convex combination of two elements of $S_{\delta/2}$ is in S_δ. So let p_1, p_2 be projections in the socle of zA^{**}, $\|[p_i, x_j]\| < \delta/2$ for $i = 1, 2$, $1 \leq j \leq n$, and ω_i states on A of the form $\omega_i(x) = \psi_i(p_i x p_i)$ for states ψ_i on $p_i A^{**} p_i$. As in 2.3, there are representations π_i of A, each of which is a direct sum of mutually inequivalent irreducible representations in Π, such that $\pi_i(p_i)\pi_i(A)\pi_i(p_i) = p_i A p_i$, and ω_i is a linear combination of vector states from vectors in the range of $\pi(p_i)$. If $0 < \lambda < 1$ is fixed, we must show that $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ is approximately of the same form.

The difficulty comes when one or more of the irreducible subrepresentations of π_1 is equivalent to a subrepresentation of π_2. By the choice of Π, any such representation ρ is either quasidiagonal or not GCR. We will separately work within each such ρ, so fix ρ, on a Hilbert space $\mathbf{H}$.

If ρ is quasidiagonal, identify the components of $\pi(p_i)$ $(i = 1, 2)$ in ρ with $q_i \in \mathbf{B}(\mathbf{H})$. Then, for any $\eta > 0$, there is a finite-rank projection r such that $\|q_i - rq_i\| < \eta$ $(i = 1, 2)$ and $\|[r, \rho(x_j)]\| < \varepsilon$ for $1 \leq j \leq m$; the component of ω corresponding to ρ can thus be approximated within η in norm by a convex combination of vector states in the range of r, and such states are in S_ε.

Now suppose ρ is not GCR, i.e., $\rho(A) \cap \mathbf{K} = \{0\}$. Identify the subrepresentation ρ_1 of π_1 equivalent to ρ with ρ, giving a projection q_1; then the component of ω_1 corresponding to ρ is a convex combination of vector states in the range of q_1. Let ρ_2 on $\mathbf{H}_2$ be the subrepresentation of π_2 equivalent to ρ, and r_2 the corresponding projection. By [**Vo1**, Lemma 1], for any $y_1, \ldots, y_r \in A$ and $\eta > 0$, there is a unitary u from $\mathbf{H}_2$ to $\mathbf{H}$ and a finite-rank projection $q_2 \in \mathbf{B}(\mathbf{H})$ orthogonal to q_1 with $q_2 = ur_2u^*$ and $\|q_2\rho(x_j)q_2 - ur_2\rho_2(x_j)r_2u^*\| < \eta$, $\|q_2\rho(y_k)q_2 - ur_2\rho_2(y_k)r_2u^*\| < \eta$ for all j, k. Thus every weak-* neighborhood of the component of ω_2 in ρ_2 contains a state $\widetilde{\omega}_2$ which is a convex combination of vector states in the range of some such q_2, and $\widetilde{\omega}_2$ is in $S_{\delta/2}$ for sufficiently small η. Then the component of ω corresponding to ρ is approximated by a convex combination of vector states in the range of $q_1 + q_2$, and $\|[q_1 + q_2, x_j]\| \leq \|[q_1, x_j]\| + \|[q_2, x_j]\| < \delta$ for all j, so $\omega \in S_\delta$. $\square$

Theorem 4.4. *Let A be a nuclear C^*-algebra, and z a central projection in A^{**}. If A is z-inner quasidiagonal, then A satisfies condition* (ii) *of* 4.1 *with*

$\mathbf{B} = \{pAp : p$ is in the socle of $zA^{**}\}$. So if A is separable, then A can be written as $\varinjlim(A_n, \phi_{m,n})$ for a generalized inductive system $(A_n, \phi_{m,n})$ where each A_n is isomorphic to $p_n A^{**} p_n$ for some p_n in the socle of zA^{**}, and each $\phi_{m,n}$ is a complete order embedding.

Putting together 4.4 with 4.2 we obtain:

Theorem 4.5. *Let A be a separable C^*-algebra. Then A is a strong* NF *algebra if and only if A is nuclear and inner quasidiagonal.*

Proof of Theorem 4.4. Suppose A is nuclear and z-inner quasidiagonal. Let $x_1, \dots, x_m \in A$ and $\varepsilon > 0$. Choose a matrix algebra $\mathbf{M}_n$ and completely positive contractive maps $\alpha : A \to \mathbf{M}_n$ and $\beta : \mathbf{M}_n \to A$ such that $\|\beta \circ \alpha(x_j) - x_j\| < \varepsilon/7$ for $1 \le j \le m$. Then by 4.3 choose p in the socle of zA^{**} with $\|[p, x_j]\| < \varepsilon/7$ and $\sigma : pAp \to \mathbf{M}_n$ a completely positive contraction such that $\|\sigma(px_jp) - \alpha(x_j)\| < \varepsilon/7$ for $1 \le j \le m$. Set $B = pAp$ and $\omega = \beta \circ \sigma : B \to A$. Then $\|\omega(px_jp) - x_j\| < \varepsilon/7$ for all j.

We have $d(x_j, N_p) = \|[p, x_j]\|$ for each j by 3.5; let $y_j \in N_p$ with $\|x_j - y_j\| < \varepsilon/7$. Then $\|\omega(py_j) - y_j\| < 4\varepsilon/7$ for all j. The map $x \to px = pxp$ is a *-homomorphism from N_p onto B; let J be the kernel, $\psi : B \to N_p$ a completely positive contractive cross section for the quotient map, and $\{e_i\}$ a quasicentral approximate identity for J in N_p. For each i, define $\phi_i : B \to A$ by

$$\phi_i(b) = (1 - e_i)^{1/2} \psi(b) (1 - e_i)^{1/2} + e_i^{1/2} \omega(b) e_i^{1/2}.$$

For each i, ϕ_i is a complete order embedding since $p\phi_i(b)p = b$ for all $b \in B$. For i sufficiently large, $\|y_j - ((1 - e_i)^{1/2} y_j (1 - e_i)^{1/2} + e_i^{1/2} y_j e_i^{1/2})\| < \varepsilon/7$ for all j since $\{e_i\}$ is quasicentral. We also have
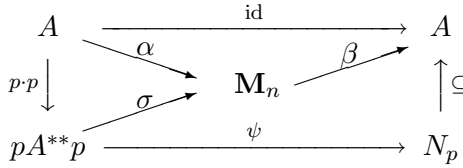
$$\|(1 - e_i)^{1/2} (y_j - \psi(py_j))(1 - e_i)^{1/2}\| < \varepsilon/7$$

for each j, for i large, since $y_j - \psi(py_j) \in J$. Thus, for i sufficiently large, we have, for all $1 \le j \le m$,

$$\|x_j - \phi_i(py_j)\|$$
$$\le \|x_j - y_j\| + \left\| y_j - \left((1 - e_i)^{1/2} y_j (1 - e_i)^{1/2} + e_i^{1/2} y_j e_i^{1/2} \right) \right\|$$
$$+ \left\| (1 - e_i)^{1/2} (y_j - \psi(py_j))(1 - e_i)^{1/2} \right\| + \left\| e_i^{1/2} (y_j - \omega(py_j)) e_i^{1/2} \right\|$$
$$< \frac{\varepsilon}{7} + \frac{\varepsilon}{7} + \frac{\varepsilon}{7} + \frac{4\varepsilon}{7} = \varepsilon,$$

so A satisfies condition (ii) of 4.1. If A is separable, the last statement of 4.4 follows from 4.1.

The following diagram summarizes the maps used in the proof.

$$
\begin{array}{ccc}
A & \xrightarrow{\mathrm{id}} & A \\
{\scriptstyle p\cdot p}\downarrow \;\; {\scriptstyle \alpha} \;\; {\scriptstyle \beta} & \mathbf{M}_n & \;\uparrow {\scriptstyle \subseteq} \\
pA^{**}p & \xrightarrow{\psi} & N_p
\end{array}
\qquad \square
$$

5. Corollaries.

Corollary 5.1. *A separable nuclear C^*-algebra with a separating family of quasidiagonal irreducible representations is a strong NF algebra. In particular, every separable nuclear residually finite-dimensional C^*-algebra is a strong NF algebra, and every separable nuclear strongly quasidiagonal C^*-algebra is a strong NF algebra.*

Corollary 5.2. *Let A be a separable subhomogeneous C^*-algebra, and suppose each irreducible representation of A is of dimension $\leq k$. Then A is a strong NF algebra, and has a strong NF system $(A_n, \phi_{m,n})$ where each A_n is a (finite) direct sum of matrix algebras of size not more than $k \times k$.*

Since the class of strong NF algebras is closed under inductive limits [**BKb**, 6.1.3], we obtain:

Corollary 5.3. *Every (separable) approximately subhomogeneous C^*-algebra is a strong NF algebra.*

Corollary 5.4. *If A is separable, nuclear, and prime, then A is a strong NF algebra if and only if some (hence every) faithful irreducible representation of A is quasidiagonal.*

Corollary 5.5. *Every antiliminal prime NF algebra is a strong NF algebra. Every simple NF algebra is a strong NF algebra.*

Example 5.6. The examples of 2.7 are NF but not strong NF.

Corollary 5.7. *Let A be a separable nuclear C^*-algebra. The following are equivalent:*
  (i) *Every quotient of A is a strong NF algebra.*
 (ii) *Every primitive quotient of A is a strong NF algebra.*
(iii) *Every irreducible representation of A is quasidiagonal.*
(iv) *A is strongly quasidiagonal.*

Corollary 5.8. *Let A be any NF algebra, and let B be a split essential extension of A by $\mathbf{K}$. Then B is a strong NF algebra. So A can be embedded as a C^*-subalgebra of a strong NF algebra B with a retraction (homomorphic conditional expectation) from B onto A. In particular, A is a quotient of a strong NF algebra. So every separable nuclear C^*-algebra is a quotient of a strong NF algebra [**BKb**, 6.1.8].*

We can obtain a refinement of 5.4-5.5.

Definition 5.9. A strong NF algebra is *of monomial type* if it can be written as $\varinjlim(A_n, \phi_{m,n})$, with each A_n a single matrix algebra and each $\phi_{m,n}$ a complete order embedding.

We have used the terminology "monomial type" instead of "UHF type" or "matroid type" since the class of AF algebras which are strong NF of monomial type is considerably larger than the class of UHF or matroid C^*-algebras. (In fact, an AF algebra is strong NF of monomial type if and only if it is prime.)

Proposition 5.10 (cf. [**Dx**, 1.9.13]). *A C^*-algebra B is prime if and only if, for every nonzero $x, y \in B$, there is an irreducible representation π of B with $\pi(x)$ and $\pi(y)$ both nonzero.*

Proof. If I and J are nonzero ideals of B with $I \cap J = 0$, then every irreducible representation of A must annihilate either I or J, so if $x \in I$ and $y \in J$ are nonzero, then no irreducible representation of B can be nonzero on both x and y. Conversely, if B is prime, and x, y are nonzero elements of B, then there is a $z \in B$ with $xzy \neq 0$ (otherwise the two-sided ideals generated by x and y annihilate each other); if π is an irreducible representation of B with $\pi(xzy) \neq 0$, then $\pi(x)$ and $\pi(y)$ are both nonzero. □

Remark 5.11. The second half of the proof can be simplified if B is primitive (e.g., if B is separable). It is still an open question whether every prime C^*-algebra is primitive.

Proposition 5.12. *A strong NF algebra of monomial type is prime.*

Proof. If A is strong NF of monomial type and x_1, x_2 are nonzero elements of A, then by 4.1 there is a complete order embedding ϕ of a full matrix algebra B into A such that $\|x_j - \phi(b_j)\| < \|x_j\|/2$ for $j = 1, 2$, for some $b_j \in B$. ϕ^{-1} extends to an irreducible representation $\widetilde{\pi}$ of A as in proof of 4.2, and $\widetilde{\pi}(x_j) \neq 0$ for $j = 1, 2$. □

Theorem 5.13. *Let A be a prime separable nuclear C^*-algebra. Then the following are equivalent:*

(i) *A is a strong NF algebra.*
(ii) *A is a strong NF algebra of monomial type.*
(iii) *For some faithful irreducible representation π, $\pi(A)$ is a quasidiagonal C^*-algebra of operators.*
(iv) *For every faithful irreducible representation π, $\pi(A)$ is a quasidiagonal C^*-algebra of operators.*

In particular, every antiliminal prime NF algebra and every simple NF algebra is a strong NF algebra of monomial type.

Proof. (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are trivial; (i) $\Rightarrow$ (iv) by 2.5 and 4.2, and (iii) $\Rightarrow$ (ii) by 4.4, letting z be the support of π in A^{**} (cf. 3.14). $\square$

We have the following versions of 2.9-2.10:

Proposition 5.14. *Let A be a (separable) C^*-algebra, and J_1, J_2 ideals of A. Set $J = J_1 \cap J_2$. If A/J_1 and A/J_2 are strong NF algebras, then A/J is strong NF.*

Proof. In light of 2.9 it suffices to note that if A/J_1 and A/J_2 are nuclear, then A/J is nuclear. This can be seen in several ways. Perhaps the easiest is to use the fact that a separable C^*-algebra B is nuclear if and only if every factor representation of B generates an injective factor, and note that every factor representation of A/J factors though either A/J_1 or A/J_2. Alternatively, A/J is an extension of A/J_1 by J_1/J, and $J_1/J \cong (J_1+J_2)/J_2$, which is nuclear since it is an ideal in A/J_2. $\square$

Corollary 5.15. *A separable nuclear C^*-algebra A is a strong NF algebra if (and only if) A contains a sequence $\langle J_n \rangle$ of ideals with A/J_n strong NF for all n and $\cap J_n = 0$.*

Note that neither of the assumptions that A be separable and nuclear follow from the other hypotheses of 5.15 (e.g., $A = \prod_n \mathbf{M}_n$, J_n the sequences vanishing in the n'th coordinate).

The situation with an increasing sequence of ideals, and hence with inductive limits with noninjective connecting maps, is quite different. Recall that an (ordinary) inductive limit, with injective connecting maps, of strong NF algebras is strong NF ([**BKb**, 6.1.3]; this is an immediate corollary of 4.1, or of 2.1 and 4.5).

Proposition 5.16. *An (ordinary) inductive limit of an inductive system of strong NF algebras with noninjective connecting maps is not necessarily strong NF.*

Proof. Example 2.12 is a counterexample. $\square$

For completeness, we note the following fact, which should have been included in [**BKb**]:

Proposition 5.17. *The class of strong NF algebras is closed under tensor products.*

Proof. By 4.1 it suffices to show that, if A_1, A_2, B_1, B_2 are C^*-algebras, with B_1, B_2 finite-dimensional, and $\phi_i : B_i \to A_i$ are complete order embeddings, then the finite-dimensional subspace $\phi_1(B_1) \otimes \phi_2(B_2)$ of $A_1 \otimes A_2$ is completely order isomorphic to a C^*-algebra. This follows immediately from [**CE2**, 3.1] (cf. [**BKb**, 4.2.1]), since if ω_i is an idempotent completely positive contraction from A_i onto $\phi_i(B_i)$, then $\omega_1 \otimes \omega_2$ is an idempotent completely positive contraction from $A_1 \otimes A_2$ onto $\phi_1(B_1) \otimes \phi_2(B_2)$. $\square$

Finally, the next proposition is an immediate consequence of 3.11 and
[**BKb**, 5.3.3].

Proposition 5.18. *Let A be a separable C^*-algebra. Then A is a strong
NF algebra if and only if SA is strong NF, and similarly for CA. Thus, if
A is a separable nuclear C^*-algebra which is not strong NF (e.g., if A is not
NF), then SA and CA are NF but not strong NF.*

Using 5.18, we get examples of NF algebras which are not strong NF
which are very different from those of 5.6:

Example 5.19. SO_2 is an antiliminal NF algebra which is not strong NF.

Appendix A.

This appendix contains a "folklore" result that we have been unable to find
in the literature. The arguments are slight variations of those of Glimm
[**Gl**], as presented in [**Dx**]. The word "ideal" will mean "closed two-sided
ideal".

If J is a primitive ideal in a C^*-algebra A, we will call J a *GCR ideal* if J
is the kernel of a GCR irreducible representation (one whose image contains
the compact operators). The next proposition is well known and easy to
prove (cf. [**Dx**, 4.1.10]).

Proposition A.1. *Let J be a primitive ideal in a separable C^*-algebra A.
Then the following are equivalent:*

   (i) *J is a GCR ideal.*
  (ii) *There is an ideal K of A, containing J, such that K/J is an essential
       ideal of A/J isomorphic to $\mathbf{K}$.*
 (iii) *A/J is not antiliminal.*

Theorem A.2. *Let J be a primitive ideal in a separable C^*-algebra A.
Then the following are equivalent:*

   (i) *J is not GCR ideal.*
  (ii) *A/J is antiliminal.*
 (iii) *J is the kernel of a non-type I factor representation of A.*
  (iv) *There are two inequivalent irreducible representations of A with kernel
       J.*
   (v) *There are uncountably many mutually inequivalent irreducible repre-
       sentations of A with kernel J.*

Proof. (i) $\Leftrightarrow$ (ii) is A.1, (v) $\Rightarrow$ (iv) is trivial, and (iv) $\Rightarrow$ (i) follows from
[**Dx**, 4.1.10].

By replacing A by A/J, we may and will assume that $J = 0$ in the rest
of the proof, to simplify notation.

(iii) $\Rightarrow$ (v) by a slight modification of the argument in [**Dx**, 9.1]: if π
is a faithful non-type-I factor representation of A on a separable Hilbert

space and C is a masa in $\pi(A)'$, the direct integral decomposition of π as $\int^{\oplus} \pi_x d\mu(x)$ with respect to C has almost all π_x faithful and irreducible by the argument of [**Dx**, 9.1]. If, for a set E of nonzero measure, each π_x for $x \in E$ is equivalent to a fixed representation π_0, then by [**Dx**, 8.1.7] $\int_E^{\oplus} \pi_x d\mu(x)$ is a subrepresentation of π equivalent to a multiple of π_0, a contradiction. Thus, for each x, the set $E_x = \{y : \pi_y \sim \pi_x\}$ has measure 0, so there must be uncountably many such sets.

It remains to prove (ii) $\Rightarrow$ (iii). This follows from the results of Glimm if A has a minimal nonzero ideal, but not directly otherwise. However, using the next three lemmas, Glimm's argument essentially works in our case.

Lemma A.3. *Let A be a separable primitive C^*-algebra. Then A contains a decreasing sequence $\langle J_n \rangle$ of nonzero (not necessarily proper) ideals, such that every nonzero ideal of A contains J_n for some n.*

Proof. This is an immediate consequence of the fact that $\mathrm{Prim}\,(A)$ is a second countable T_0 space and 0 is a dense point.   □

Lemma A.4 (cf. [**Dx**, 9.3.5]). *Let B be an antiliminal C^*-algebra and I an essential ideal in B. If $d \in B_+$ of norm 1 and $0 < \tau \leq 1$, then there exist w, w', d' in I satisfying the conclusions of [**Dx**, 9.3.5].*

Proof. The proof is identical to the proof of [**Dx**, 9.3.5] except that π is chosen to be an irreducible representation of B which is nonzero on I (this is possible since I is essential and is itself an antiliminal C^*-algebra), so $\pi|_I$ is irreducible, and c is chosen in I. Then d_0 and hence ν are in I, so d', w, and w' are also in I.   □

Lemma A.5 (cf. [**Dx**, 9.3.7]). *Let B be a unital antiliminal C^*-algebra, $\langle J_n \rangle$ a decreasing sequence of essential ideals of B, and let $(s_0, s_1, \dots)$ be a sequence of self-adjoint elements of B. Then there exist elements $\nu(a_1, \dots, a_k)$ $(k = 1, 2, \dots)$ of B satisfying all the conditions of [**Dx**, 9.3.7], and in addition $\nu(a_1, \dots, a_k) \in J_k$ for all k and all $(a_1, \dots, a_k)$.*

Proof. The proof is identical to the proof of [**Dx**, 9.3.7], with A.4 used (with $I = J_{n+1}$) in place of [**Dx**, 9.3.5].   □

Proof of A.2 (ii) $\Rightarrow$ (iii). Let $\langle J_n \rangle$ be a sequence of ideals of A as in A.3. Choose elements $\nu(a_1, \dots, a_k)$ as in A.5 (if A is nonunital, work in $\widetilde{A}$). Choose the states f and g as in [**Dx**, 9.4]; then the representation π_g is a type II factor representation of A. If $I = \ker \pi_g$ is nonzero, then π_g is zero on J_n for some n; but this is impossible since $\nu(a_1, \dots, a_k) \in J_n$ and $\pi_g(\nu(a_1, \dots, a_k)) \neq 0$. Thus $I = 0$ and π_g is faithful.   □

The same technique can be used to give the following version of Maréchal's refinement [**Ma**, §2] of Glimm's result:

Lemma A.6. *Let A be a separable unital primitive antiliminal C^*-algebra. Then there is a unital sub-C^*-algebra B of A and ideal J of B, such that:*

(a) *B/J is isomorphic to the CAR algebra D (write $\phi : B \to D$ for the quotient map).*

(b) *For any cyclic representation π of D, there is a faithful cyclic representation ρ of A, and a projection $F \in \rho(B)'' \cap \rho(B)'$, of central support 1 in $\rho(A)''$, such that the subrepresentation ρ_1 of $\rho|_B$ defined by F is equivalent to $\pi \circ \phi$ and $\rho_1(B)'' = F\rho(A)''F$.*

Corollary A.7. *Let A be a separable C^*-algebra and J a non-GCR primitive ideal of A. If M is any properly infinite injective von Neumann algebra (in particular, any infinite injective factor) with separable predual, then there is a representation π of A with kernel J, such that $\pi(A)'' \cong M$.*

Note added in proof. The authors have recently shown that the converse of 2.4 is true: an inner quasidiagonal C^*-algebra has a separating family of quasidiagonal irreducible representations. As a consequence, if A and B are inner quasidiagonal and one of them is nuclear, then $A \otimes B$ is inner quasidiagonal (see the comment after 3.9). Some of the other arguments in this paper can be simplified using this result.

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