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GEOMETRIC PROPERTIES OF JULIA SETS OF THE COMPOSITION OF POLYNOMIALS OF THE FORM $z^2 + c_n$

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For a sequence (c_n) of complex numbers we consider the quadratic polynomials $f_{c_n}(z) := z^2 + c_n$ and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$. The Fatou set $\mathcal{F}_{(c_n)}$ is by definition the set of all $z \in \widehat{\mathbb{C}}$ such that (F_n) is normal in some neighbourhood of z, while the complement of $\mathcal{F}_{(c_n)}$ is called the Julia set $\mathcal{J}_{(c_n)}$. The aim of this article is to study geometric properties, Lebesgue measure and Hausdorff dimension of the Julia set $\mathcal{J}_{(c_n)}$ provided that the sequence (c_n) is bounded.

1. Introduction.

For a sequence (c_n) of complex numbers we consider the quadratic polynomials $f_{c_n}(z) := z^2 + c_n$ and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$. (Note that F_n depends on $c_1, \ldots, c_n$ which we do not indicate explicitly in the notation.) If $c_n = c$ for all n, we write f_c^n instead of F_n. The *Fatou set* $\mathcal{F}_{(c_n)}$ is by definition the set of all $z \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that (F_n) is normal (in the sense of Montel) in some neighbourhood of z, while the complement of $\mathcal{F}_{(c_n)}$ (in $\widehat{\mathbb{C}}$) is called the *Julia set* $\mathcal{J}_{(c_n)}$. A component of the Fatou set is called a *stable domain*. For iteration theory of a fixed function we refer the reader to the books of Beardon [**Be**], Carleson and Gamelin [**CG**], Milnor [**M**] or Steinmetz [**St**]. We also mention the survey articles of Blanchard [**Bl**], Lyubich [**L2**] or Eremenko and Lyubich [**EL**].

We always assume that $|c_n| \leq \delta$ for some $\delta > 0$. Then from [**Bü2**] it is known that to some extent the sequence (F_n) behaves similar to the sequence (f_c^n). There exists a stable domain $\mathcal{A}_{(c_n)}(\infty)$ which contains the point ∞ and wherein $F_n \to \infty$ as $n \to \infty$ locally uniformly. This domain need not be *invariant* (i.e., $f_{c_k}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$ for all k) or *backward invariant* (i.e., $f_{c_k}^{-1}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$ for all k), but there exists an invariant domain $M = M_\delta \subset \mathcal{A}_{(c_n)}(\infty)$ which contains the point ∞ and which satisfies $\mathcal{A}_{(c_n)}(\infty) = \{\, z \in \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \,\}$. Therefore, the *filled Julia set* $\mathcal{K}_{(c_n)} := \widehat{\mathbb{C}} \setminus \mathcal{A}_{(c_n)}(\infty)$ and the Julia set $\mathcal{J}_{(c_n)}$ are compact in $\mathbb{C}$, and $\mathcal{K}_{(c_n)}$ is the set of all $z \in \mathbb{C}$ such that $(F_k(z))_{k=1}^{\infty}$ is bounded. Furthermore, we have $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty) = \partial \mathcal{K}_{(c_n)}$. Also $\mathcal{J}_{(c_n)}$ and $\mathcal{K}_{(c_n)}$ are perfect sets.

Finally, $\mathcal{J}_{(c_n)}$ and $\mathcal{F}_{(c_n)}$ are invariant in the sense that $F_k^{-1}(F_k(\mathcal{J}_{(c_n)})) = \mathcal{J}_{(c_n)}$ and $F_k^{-1}(F_k(\mathcal{F}_{(c_n)})) = \mathcal{F}_{(c_n)}$ for all $k \in \mathbb{N}$. For further results we also refer to [**Brü**], [**BBR**], [**Bü1**] and [**FS**].

The *Mandelbrot set* $\mathcal{M}$ is defined as the set of all $c \in \mathbb{C}$ such that $(f_c^n(0))_{n=1}^{\infty}$ is bounded, and $\mathcal{M}$ is compact in $\mathbb{C}$. It plays an important role in iteration of a fixed quadratic polynomial f_c. We recall that the largest disk with center 0 which is contained in $\mathcal{M}$ has radius $\frac{1}{4}$.

The plan of this article is as follows. After introducing some notations and known auxiliary results (Section 2) we show that the Julia set $\mathcal{J}_{(c_n)}$ is always uniformly perfect (Section 3).

Our main result (Section 4) states that the Julia set $\mathcal{J}_{(c_n)}$ is a quasicircle provided that $|c_n| \le \delta$ for some $\delta < \frac{1}{4}$. This is done by proving that $\mathcal{F}_{(c_n)}$ consists of two simply connected John domains $\mathcal{A}_{(c_n)}(0)$ and $\mathcal{A}_{(c_n)}(\infty)$ which have $\mathcal{J}_{(c_n)}$ as their common boundaries.

Concerning the two-dimensional Lebesgue measure $m_2(\mathcal{J}_{(c_n)})$ of Julia sets (Section 5) we show that it is almost surely zero provided that the c_n are randomly chosen in $\{ z \in \mathbb{C} : |z| \le \delta \}$ for some $\delta > \frac{1}{4}$. For $\delta < \frac{1}{4}$ we always have $m_2(\mathcal{J}_{(c_n)}) = 0$.

Section 6 deals with Hausdorff dimension $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ of Julia sets. We give a lower estimate for $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ depending only on δ which implies that $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ is always positive. For that purpose we prove that the Green function of $\mathcal{A}_{(c_n)}(\infty)$ (which is known to exist) is Hölder continuous. Furthermore, for $\delta < \frac{1}{4}$ it follows that $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} < 2$.

A point $\zeta \in \mathbb{C}$ is called a repelling fixpoint of the sequence of iterates (F_n) if $F_k(\zeta) = \zeta$ for some $k \in \mathbb{N}$ and $|F_k'(\zeta)| > 1$. The set of all those points is denoted by $\mathcal{R}_{(c_n)}$. In this general setting it is not necessarily true that $\mathcal{R}_{(c_n)} \subset \mathcal{J}_{(c_n)}$. But we prove (Section 7) that if $|c_n| \le \delta < \frac{1}{4}$, then the derived set of $\mathcal{R}_{(c_n)}$ coincides with $\mathcal{J}_{(c_n)}$. In the last section we investigate the asymptotic distribution of certain predecessors.

2. Notations and auxiliary results.

We introduce a few further notations and collect some known auxiliary results that are frequently used in the sequel. If $E \subset \mathbb{C}$, then E' denotes the derived set (that is the set of points $z \in \mathbb{C}$ such that every neighbourhood of z contains a point $w \in E \setminus \{z\}$), $\overline{E}$ the closure and E° the set of interior points of E. Furthermore, the diameter of E is defined by $\operatorname{diam} E := \sup \{ |z-w| : z, w \in E \}$, and the distance of a point $z \in \mathbb{C}$ from E by $\operatorname{dist}(z, E) := \inf \{ |z-w| : w \in E \}$. For $a \in \mathbb{C}$ and $r > 0$ we set $D_r(a) := \{ z \in \mathbb{C} : |z-a| < r \}$, $D_r := D_r(0)$, $\mathbb{D} := D_1$ and $K_r := \overline{D}_r$. Finally, for $R > 0$ let $\Delta_R := \{ z \in \widehat{\mathbb{C}} : |z| > R \}$.

If $(c_n) \in K_\delta^{\mathbb{N}}$, then the invariant domain $M \subset \mathcal{A}_{(c_n)}(\infty)$ may be chosen as $M = \Delta_R$ for any

$$R \geq R_\delta := \tfrac{1}{2}\left(1 + \sqrt{1+4\delta}\,\right).$$

More precisely, if $R > R_\delta$, then $f_c(\Delta_{R_\delta}) \subset \Delta_{R_\delta}$ and $f_c(\overline{\Delta}_R) \subset \Delta_R$ for all $c \in K_\delta$. This implies that $\mathcal{K}_{(c_n)} \subset K_{R_\delta}$. If $\delta \leq \frac{1}{4}$, we set

$$r_\delta := \tfrac{1}{2}\left(1 + \sqrt{1-4\delta}\,\right) \in \left[\tfrac{1}{2}, 1\right], \qquad s_\delta := \tfrac{1}{2}\left(1 - \sqrt{1-4\delta}\,\right) \in \left[0, \tfrac{1}{2}\right].$$

Then we have $f_c(D_{s_\delta}) \subset D_{s_\delta}$, $f_c(D_{r_\delta}) \subset D_{r_\delta}$ and $f_c(\overline{D}_r) \subset D_r$ for all $c \in K_\delta$ and all $r \in (s_\delta, r_\delta)$. This implies that there exists a stable domain $\mathcal{A}_{(c_n)}(0) \supset D_{r_\delta}$, and there holds $\mathcal{J}_{(c_n)} \subset K_{R_\delta} \cap \overline{\Delta}_{r_\delta}$.

From [**FS**, Theorem 2.1] it follows that $\mathcal{A}_{(c_n)}(\infty)$ is regular for logarithmic potential theory which means that the Green function of $\mathcal{A}_{(c_n)}(\infty)$ with pole at infinity exists. More precisely, the function $g_{(c_n)}$ defined by

$$g_{(c_n)}(z) := \lim_{k\to\infty} \frac{1}{2^k} \log^+ |F_k(z)| \tag{2.1}$$

is continuous in $\mathbb{C}$, $g_{(c_n)}(z) = 0$ for $z \in \mathcal{K}_{(c_n)}$, and it is the Green function of $\mathcal{A}_{(c_n)}(\infty)$ with pole at infinity.

Furthermore, we introduce the *critical set* (or set of critical points)

$$\mathcal{C}_{(c_n)} := \{\, z \in \mathbb{C} : F_j(z) = 0 \text{ for some } j \in \mathbb{N}_0 \,\}$$

of (F_n), where $F_0(z) := z$. This is motivated by the fact that

$$F_k'(z) = 2^k \prod_{j=0}^{k-1} F_j(z)$$

so that $F_k'(z) = 0$ if and only if $F_j(z) = 0$ for some $j \in \{0, 1, \ldots, k-1\}$. We call a point $w \in \mathbb{C}$ a *critical value* of (F_n), if $w = F_k(z)$ and $F_k'(z) = 0$ for some $k \in \mathbb{N}$ and some $z \in \mathbb{C}$. If $w \in \mathbb{C}$ is not a critical value of F_k, then in some sufficiently small disk $D_\varepsilon(w)$ there exist 2^k analytic branches of the inverse function of F_k.

Finally, we recall a result of Büger [**Bü1**] that the Julia set $\mathcal{J}_{(c_n)}$ is self-similar. This means that for any open set D meeting $\mathcal{J}_{(c_n)}$ there exists $k_0 \in \mathbb{N}$ such that $F_k(\mathcal{J}_{(c_n)} \cap D) = F_k(\mathcal{J}_{(c_n)})$ for all $k \geq k_0$.

3. Uniform perfectness of Julia sets.

An open set $A \subset \widehat{\mathbb{C}}$ is called a *conformal annulus*, if it can be mapped conformally onto an annulus $\{\, z \in \mathbb{C} : 1 < |z| < \varrho \,\}$ for some $\varrho > 1$. Then the number ϱ is uniquely determined and $\operatorname{mod} A := \frac{1}{2\pi} \log \varrho$ is called the *modulus* of A. Now, let $E \subset \widehat{\mathbb{C}}$ be a compact set. A conformal annulus A *seperates* E, if both components of $\widehat{\mathbb{C}} \setminus A$ meet E. The set E is called *uniformly perfect*, if it is not a single point and if there is a constant $\alpha > 0$ such that for any conformal annulus A which seperates E there holds $\bmod A \leq \alpha$. Obviously, a uniformly perfect set is also perfect (that is $E' = E$), and every connected compact set with at least two points is uniformly perfect. Uniformly perfect sets were introduced by Beardon and Pommerenke [**BeP**] (see also [**P1**]). It is known that the Julia set of a fixed rational function is always uniformly perfect [**MR**] (see also [**CG**, p. 64]). We show that this result extends to our situation.

Theorem 3.1. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then the Julia set $\mathcal{J}_{(c_n)}$ is uniformly perfect.*

Proof. We assume that $\mathcal{J}_{(c_n)}$ is not uniformly perfect. Then there exists a sequence of conformal annuli $A_k \subset \mathcal{F}_{(c_n)}$ which seperate $\mathcal{J}_{(c_n)}$ and $\bmod A_k \to \infty$ as $k \to \infty$. Let E_k be the component of $\widehat{\mathbb{C}} \setminus A_k$ with the smaller chordal diameter (which we denote by $\operatorname{diam}_\chi E_k$). Then we have $\operatorname{diam}_\chi E_k \to 0$ as $k \to \infty$. If $\lambda_k \colon \mathbb{D} \to A_k \cup E_k$ is a conformal map of $\mathbb{D}$ onto $A_k \cup E_k$ with $\lambda_k(0) \in E_k$, and if $M_k := \lambda_k^{-1}(E_k) \subset \mathbb{D}$, then M_k is compact and connected, $0 \in M_k$ and $\operatorname{diam}_\chi M_k \to 0$ as $k \to \infty$.

It is elementary to see that (f_{c_n}) satisfies a uniform Lipschitz condition with respect to the chordal metric χ, that means that there exists a constant $L > 0$ (which depends only on δ but not on n) such that $\chi(f_{c_n}(z), f_{c_n}(w)) \leq L\chi(z, w)$ for all z, $w \in \widehat{\mathbb{C}}$ and all $n \in \mathbb{N}$. From Lemma 4.1 in [**BBR**] we know that $\operatorname{diam} F_k(\mathcal{J}_{(c_n)}) \geq 1$ for all $k \in \mathbb{N}_0$ so that $\operatorname{diam}_\chi F_k(\mathcal{J}_{(c_n)}) \geq C := 2(1 + R_\delta^2)^{-1}$.

We choose $\varepsilon > 0$ with $\varepsilon < C$ and

$$\frac{C}{3} > L\varepsilon. \tag{3.1}$$

Let $k_0 \in \mathbb{N}$ such that $\operatorname{diam}_\chi E_k < \varepsilon$ for all $k \geq k_0$. Since $(A_k \cup E_k) \cap \mathcal{J}_{(c_n)} \neq \varnothing$ and since $\mathcal{J}_{(c_n)}$ is self-similar (cf. [**Bü1**]), for every $k \geq k_0$ there exists a smallest index $m(k) \in \mathbb{N}$ such that $\operatorname{diam}_\chi F_{m(k)}(E_k) > \varepsilon$. Setting $G_k := F_{m(k)} \circ \lambda_k$ we obtain

$$\operatorname{diam}_\chi G_k(M_k) > \varepsilon \tag{3.2}$$

for all $k \geq k_0$. By the choice of $m(k)$ we have $\operatorname{diam}_\chi F_{m(k)-1}(E_k) \leq \varepsilon$ and thus $\operatorname{diam} F_{m(k)}(E_k) = \operatorname{diam}_\chi f_{c_{m(k)}}(F_{m(k)-1}(E_k)) \leq L\varepsilon$ for all $k \geq k_0$.

Because of (3.1) there exist at least three different points $a_{1,k}$, $a_{2,k}$, $a_{3,k} \in F_{m(k)}(\mathcal{J}_{(c_n)})$ whose chordal distance is greater than $L\varepsilon$. We have $G_k(\mathbb{D} \setminus M_k) = F_{m(k)}(\lambda_k(\mathbb{D} \setminus M_k)) = F_{m(k)}(A_k) \subset F_{m(k)}(\mathcal{F}_{(c_n)})$ and $\operatorname{diam}_\chi G_k(M_k) = \operatorname{diam}_\chi F_{m(k)}(E_k) \leq L\varepsilon$ for all $k \geq k_0$. This implies that G_k omits at least two of the values $a_{1,k}$, $a_{2,k}$, $a_{3,k}$ in $\mathbb{D}$ and hence (G_k) is normal in $\mathbb{D}$ by a generalized version of Montel's theorem (cf. [**Be**, p. 57]). Since $\operatorname{diam}_\chi M_k \to 0$ as $k \to \infty$ and $0 \in M_k$ we get $\operatorname{diam}_\chi G_k(M_k) \to 0$ as $k \to \infty$ which contradicts (3.2). □

4. Julia sets and quasicircles.

From iteration theory of a fixed function it is known that $\mathcal{J}(f_c)$ is a quasicircle if c is in the interior of the main cardioid of the Mandelbrot set (cf. Yakobson [**Y**], see also [**CG**, p. 103]). The goal of this section is to show that this result remains valid in our general situation provided that $\delta < \frac{1}{4}$. We do this in several steps, and we first recall some facts on quasicircles and John domains.

A *quasicircle* $\Gamma \subset \mathbb{C}$ is the image of the unit circle $\partial\mathbb{D}$ under a quasiconformal homeomorphism of $\mathbb{C}$ onto itself. An equivalent geometric definition is the three-point property, i.e., there exists a constant $a > 0$ such that if z_1, z_2, $z_3 \in \Gamma$ and z_2 is on the arc between z_1 and z_3 with the smaller diameter, then $|z_1 - z_2| + |z_2 - z_3| \le a|z_1 - z_3|$. A quasicircle may be non-rectifiable but it has no cusps. For details we refer, for example, to the books of Ahlfors [**A**] or Lehto and Virtanen [**LV**].

A domain $G \subset \widehat{\mathbb{C}}$ with $\partial G \subset \mathbb{C}$ is called a *John domain*, if there exists a constant $b > 0$ and a point $w_0 \in G$ such that for any $z_0 \in G$, there is an arc $\gamma = \gamma(z_0) \subset G$ joining z_0 and w_0 and satisfying $\operatorname{dist}(z, \partial G) \ge b|z - z_0|$ for all $z \in \gamma$. A simply connected John domain G has locally connected boundary ∂G so that by Carathéodory's theorem (cf. [**P2**, p. 20]) the Riemann map from $\mathbb{D}$ onto G extends continuously to $\overline{\mathbb{D}}$. The image of a John domain under a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto itself is again a John domain. Thus, the two complementary domains of a quasicircle are John domains. Conversely, if the two complementary components of a Jordan curve (a homeomorphic image of the unit circle) Γ are John domains, then Γ is a quasicircle. For this and further background material we refer to [**NV**].

For $\delta \le \frac{1}{4}$ we know that $\mathcal{J}_{(c_n)}$ is connected [**BBR**], and since $\mathcal{J}_{(c_n)} = \partial\mathcal{A}_{(c_n)}(\infty)$, the stable domain $\mathcal{A}_{(c_n)}(\infty)$ is simply connected. Furthermore, there exists a stable domain $\mathcal{A}_{(c_n)}(0)$ containing D_{r_δ}. We now show:

Theorem 4.1. *Let* $\delta \le \frac{1}{4}$, $(c_n) \in K_\delta^{\mathbb{N}}$ *and* $s_\delta \le r \le r_\delta$. *Then there holds* $\mathcal{A}_{(c_n)}(0) = \bigcup_{k=0}^{\infty} F_k^{-1}(D_r)$ *and* $\partial\mathcal{A}_{(c_n)}(0) = \mathcal{J}_{(c_n)}$. *In particular,* $\mathcal{A}_{(c_n)}(0)$ *is simply connected and* $\mathcal{F}_{(c_n)} = \mathcal{A}_{(c_n)}(0) \cup \mathcal{A}_{(c_n)}(\infty)$.

Proof. We set $A := \bigcup_{k=0}^{\infty} U_k$ with $U_k := F_k^{-1}(D_r)$. It is elementary to see that each U_k is a domain containing D_r, and since D_r is invariant, we get $U_k \subset \mathcal{F}_{(c_n)}$. Thus, A is a domain with $D_r \subset A \subset \mathcal{F}_{(c_n)}$ which gives $A \subset \mathcal{A}_{(c_n)}(0)$.

We show that $\mathcal{J}_{(c_n)} \subset \partial A$. For that purpose, let $z_0 \in \mathcal{J}_{(c_n)}$ and $D := D_\varepsilon(z_0)$ for $\varepsilon > 0$. By Montel's theorem the set $\widehat{\mathbb{C}} \setminus \bigcup_{k=0}^{\infty} F_k(D)$ contains at most two points so that there exists $w \in D_r$ such that $w \in F_m(D)$ for some $m \in \mathbb{N}_0$. Therefore, $D_r \cap F_m(D)$ is a non-empty open set, and this implies that there exists $\zeta \in D \setminus \mathcal{J}_{(c_n)}$ with $F_m(\zeta) \in D_r$. That means $\zeta \in A$, and since $\varepsilon > 0$ was arbitrary we arrive at $z_0 \in \partial A$. Summarizing, we have $A \subset \mathcal{A}_{(c_n)}(0)$ and $\partial \mathcal{A}_{(c_n)}(0) \subset \mathcal{J}_{(c_n)} \subset \partial A$ which gives the assertion. $\square$

For $\delta < \frac{1}{4}$ and $\frac{1}{2} < r < r_\delta$ we set $V := \Delta_r \supset \mathcal{J}_{(c_n)}$. Then V is backward invariant, and V does not contain any critical value of (F_n) so that in every disk $D \subset V$ there exist 2^n analytic branches F_n^{-1} of the inverse function of F_n. We prove:

Lemma 4.2. *Let $\delta < \frac{1}{4}$, $(c_n) \in K_\delta^{\mathbb{N}}$ and $\frac{1}{2} < r < r_\delta$. Furthermore, let $\gamma\colon [0,1] \to V$ be a rectifiable curve in $V := \Delta_r$, $z := \gamma(0)$, $w := \gamma(1)$ and let F_n^{-1} be an analytic branch of the inverse function of F_n in some disk $D \subset V$ with center z. Finally, we denote the analytic continuation of F_n^{-1} along γ also by F_n^{-1}. Then there holds*

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \le 1 + \alpha \ell(\gamma) e^{\alpha \ell(\gamma)},$$

where $\alpha := 4r(2r-1)^{-1}$ and $\ell(\gamma)$ denotes the length of γ. In particular, for any disk $D \subset V$ and any analytic branch F_n^{-1} in D there holds

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \le 1 + \alpha e^{\alpha d} |z - w|$$

for all z, $w \in D$ and $n \in \mathbb{N}$, where $d := \operatorname{diam} D$.

Proof. For $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n-1$ we set $F_{n,k} := f_{c_n} \circ \cdots \circ f_{c_{k+1}}$. Since

$$(F_n^{-1})'(z) = \frac{1}{F_n'(F_n^{-1}(z))} = \frac{1}{2^n \prod_{j=0}^{n-1} F_j(F_n^{-1}(z))} = \frac{1}{2^n \prod_{j=0}^{n-1} F_{n,j}^{-1}(z)}$$

and V is backward invariant we have

$$|(F_n^{-1})'(z)| \le q^n \qquad (z \in V),$$

or

$$|(F_{n,k}^{-1})'(z)| \le q^{n-k} \qquad (z \in V),$$

where $q := \frac{1}{2r} < 1$. This implies

$$|F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)| \le \left| \int_z^w |(F_{n,k}^{-1})'(\zeta)| |d\zeta| \right| \le q^{n-k} \ell(\gamma), \tag{4.1}$$

where we integrate over the curve γ. Furthermore, we have

$$\frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} = \prod_{k=0}^{n-1} \frac{F_k(F_n^{-1}(w))}{F_k(F_n^{-1}(z))} = \prod_{k=0}^{n-1} \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)}.$$

Writing

$$\frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)} = 1 + \frac{F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)}{F_{n,k}^{-1}(z)},$$

we obtain from (4.1)

$$\left| \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)} \right| \leq 1 + 2q^{n-k+1}\ell(\gamma).$$

This implies

$$\begin{aligned}\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| &\leq \prod_{k=0}^{n-1} (1 + 2q^{n-k+1}\ell(\gamma)) = \prod_{k=2}^{n+1} (1 + 2q^k \ell(\gamma)) \\ &\leq \prod_{k=0}^{\infty} (1 + 2q^k \ell(\gamma)) = \exp\left(\sum_{k=0}^{\infty} \log\left(1 + 2q^k \ell(\gamma)\right) \right) \\ &\leq \exp\left(\sum_{k=0}^{\infty} 2q^k \ell(\gamma) \right) = e^{\alpha \ell(\gamma)},\end{aligned}$$

where $\alpha := 2(1-q)^{-1}$. Finally, this gives the assertion since $e^x \leq 1 + xe^x$ for $x \geq 0$. □

Theorem 4.3. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\mathcal{A}_{(c_n)}(\infty)$ is a John domain.*

Proof. We first introduce a few notations. For z_1, $z_2 \in \mathbb{C}$ let $[z_1, z_2]$ denote the line segment joining z_1 and z_2. If $\zeta \in \mathbb{C}$, $\zeta \neq 0$, and if Γ is the ray from 0 to ∞ passing through ζ, then let Γ_ζ denote that part of Γ from ζ to ∞.

Let $R > R_\delta$ such that $R^2 + \delta - R \leq \frac{1}{2}$, $\varepsilon := R - R_\delta \leq 1$ and $U_k := F_k^{-1}(\Delta_R)$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(\infty)$ and $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{k=1}^{\infty} U_k$. Furthermore, U_k is a simply connected domain (in $\widehat{\mathbb{C}}$) bounded by an analytic Jordan curve. For $z \in \mathcal{A}_{(c_n)}(\infty)$ let $d(z) := \operatorname{dist}(z, \mathcal{J}_{(c_n)})$. We prove a lower estimate for $d(z)$, if $z \in U_k$ for some $k \in \mathbb{N}$. We set $w := F_k(z)$. If U denotes the component of $F_k^{-1}(D_\varepsilon(w))$ containing z, there holds $U \subset \mathcal{A}_{(c_n)}(\infty)$. Let $\varrho > 0$ such that $D_\varrho(z) \subset U$. If $z' \in D_\varrho(z)$ and $w' := F_k(z')$, then

$$\begin{aligned}w' - w = F_k(z') - F_k(z) &= \int_z^{z'} F_k'(\zeta)\, d\zeta = F_k'(F_k^{-1}(w)) \int_z^{z'} \frac{F_k'(\zeta)}{F_k'(F_k^{-1}(w))}\, d\zeta \\ &= F_k'(z) \int_z^{z'} \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))}\, d\zeta,\end{aligned}$$

where we integrate over the line segment $[z, z']$. By Lemma 4.2 we obtain

$$\left| \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))} \right| \le 1 + \alpha e^{\alpha\varepsilon} |w - F_k(\zeta)| \le 1 + \alpha e^{\alpha\varepsilon} \varepsilon \le 1 + \alpha e^{\alpha}$$

and thus

$$|w' - w| \le |F_k'(z)||z' - z|(1 + \alpha e^{\alpha}) \le |F_k'(z)|\varrho(1 + \alpha e^{\alpha}).$$

Setting

$$\varrho := \frac{\varepsilon}{|F_k'(z)|(1 + \alpha e^{\alpha})}$$

we obtain $D_\varrho(z) \subset U$ and thus

$$d(z) \ge \frac{\varepsilon}{|F_k'(z)|(1 + \alpha e^{\alpha})} = \frac{\alpha_1}{|F_k'(z)|} \qquad (z \in U_k). \tag{4.2}$$

In order to prove the John property, let $w_0 := \infty$ and $z_0 \in \mathcal{A}_{(c_n)}(\infty)$. We may assume that $z_0 \in U_k \backslash U_{k-1}$ for some $k \in \mathbb{N}$. Then $R < |F_k(z_0)| \le R^2 + \delta$. We construct an arc in U_k joining z_0 and w_0 as follows. First, we join z_0 with ∂U_{k-1} by an arc $\gamma_k \subset U_k \setminus U_{k-1}$ such that $F_k(\gamma_k) \subset \Gamma_{F_k(z_0)}$, and we denote the endpoint of γ_k on ∂U_{k-1} by ζ_{k-1}. Then we join ζ_{k-1} with ∂U_{k-2} by an arc $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$ such that $F_{k-1}(\gamma_{k-1}) \subset \Gamma_{F_{k-1}(\zeta_{k-1})}$, and we denote the endpoint of γ_{k-1} on ∂U_{k-2} by ζ_{k-2}. Proceeding in this way we get an arc in $U_k \cap \overline{D}_R$ with endpoint ζ_0 on ∂D_R. Finally, we set $\gamma = \gamma(z_0) := \gamma_k \cup \cdots \cup \gamma_1 \cup \Gamma_{\zeta_0}$. We note that the line segments $F_j(\gamma_j)$ $(j = 1, \dots, k)$ all lie in $\overline{\Delta}_R \cap \overline{D}_{R^2+\delta}$ and thus have lengths at most $\frac{1}{2}$.

We now show that the arc γ has the John property. For that purpose, let $z \in \gamma$. We may assume that $z \in D_R$. First, let $z \in U_k \setminus U_{k-1}$. We deduce an upper estimate for $|z - z_0|$. There holds

$$\begin{aligned} z - z_0 = F_k^{-1}(F_k(z)) - F_k^{-1}(F_k(z_0)) &= \int_{F_k(z_0)}^{F_k(z)} (F_k^{-1})'(\zeta)\, d\zeta \\ &= (F_k^{-1})'(F_k(z)) \int_{F_k(z_0)}^{F_k(z)} \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))}\, d\zeta, \end{aligned}$$

where we integrate over the line segment $[F_k(z_0), F_k(z)]$. By Lemma 4.2 we obtain

$$\left| \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))} \right| \le 1 + \alpha e^{\alpha} |F_k(z) - \zeta| \le 1 + \alpha e^{\alpha} |F_k(z) - F_k(z_0)| \le 1 + \alpha e^{\alpha}$$

and thus

$$|z - z_0| \le |(F_k^{-1})'(F_k(z))|(1 + \alpha e^{\alpha})|F_k(z) - F_k(z_0)| \tag{4.3}$$

$$\leq \frac{1+\alpha e^{\alpha}}{|F_k'(z)|} = \frac{\alpha_2}{|F_k'(z)|} \qquad (z \in \gamma \setminus U_{k-1}).$$

Putting (4.2) and (4.3) together we arrive at

$$d(z) \geq \frac{\alpha_1}{\alpha_2}|z-z_0| = \alpha_3|z-z_0| \qquad (z \in \gamma \setminus U_{k-1}).$$

Now, let $z \in U_{k-m} \setminus U_{k-m-1}$ for some $m \in \{1, \ldots, k-1\}$. By (4.2) we have

$$d(z) \geq \frac{\alpha_1}{|F_{k-m}'(z)|}\,.$$

From the construction of γ and (4.3) we obtain

$$|z-z_0| \leq |z_0-\zeta_{k-1}| + |\zeta_{k-1}-\zeta_{k-2}| + \cdots + |\zeta_{k-m+1}-\zeta_{k-m}| + |\zeta_{k-m}-z|$$

$$\leq \alpha_2 \left(\frac{1}{|F_k'(\zeta_{k-1})|} + \frac{1}{|F_{k-1}'(\zeta_{k-2})|} + \cdots \right.$$

$$\left. + \frac{1}{|F_{k-m+1}'(\zeta_{k-m})|} + \frac{1}{|F_{k-m}'(z)|} \right)$$

and thus

$$\frac{d(z)}{|z-z_0|} \geq \frac{\alpha_3}{1+\displaystyle\sum_{j=1}^{m} \left| \frac{F_{k-m}'(z)}{F_{k-m+j}'(\zeta_{k-m+j-1})} \right|}. \tag{4.4}$$

In order to estimate the denominator of the right hand side we consider a single term

$$\frac{F_{k-m}'(z)}{F_{k-m+j}'(\zeta_{k-m+j-1})} = \frac{1}{2^j F_{k-m+j-1}(\zeta_{k-m+j-1}) \cdots F_{k-m}(\zeta_{k-m+j-1})}$$

$$\times \frac{F_{k-m}'(z)}{F_{k-m}'(\zeta_{k-m+j-1})}.$$

Because of $|F_{k-m+j-1}(\zeta_{k-m+j-1})| = R$ and the invariance of D_r we obtain

$$\left| \frac{F_{k-m}'(z)}{F_{k-m+j}'(\zeta_{k-m+j-1})} \right| \leq q^j \left| \frac{F_{k-m}'(z)}{F_{k-m}'(\zeta_{k-m+j-1})} \right|, \tag{4.5}$$

where $q := \frac{1}{2r} < 1$.

Now, we deduce an estimate of the right hand side of (4.5). For abbreviation we set $p := k-m$ and write

$$\frac{F_p'(z)}{F_p'(\zeta_{p+j-1})} = \frac{(F_p^{-1})'(F_p(\zeta_{p+j-1}))}{(F_p^{-1})'(F_p(z))}.$$

From Lemma 4.2 we get

$$\left|\frac{F_p'(z)}{F_p'(\zeta_{p+j-1})}\right| \le 1 + \alpha\ell(\sigma)e^{\alpha\ell(\sigma)}, \tag{4.6}$$

where $\sigma = \sigma_{p,j}$ is the curve $F_p(\gamma_p' \cup \gamma_{p+1} \cup \dots \cup \gamma_{p+j-1})$, and where γ_p' is that part of γ_p joining ζ_p with z. Hence, there holds $\ell(\sigma) \le \ell(F_p(\gamma_p)) + \dots + \ell(F_p(\gamma_{p+j-1}))$. We have $\ell(F_p(\gamma_p)) \le \frac{1}{2}$ and $F_p(\gamma_{p+\nu}) = F^{-1}_{p+\nu,p}(s_{p,\nu})$, where $s_{p,\nu} := F_{p+\nu}(\gamma_{p+\nu})$ is a line segment on $\Gamma_{F_{p+\nu}(\zeta_{p+\nu})}$ of length at most $\frac{1}{2}$ for $\nu \ge 1$. Furthermore, we know that $F_p(\gamma_{p+\nu}) \subset \Delta_r$. Therefore, we obtain

$$\ell(F_p(\gamma_{p+1})) = \int_{s_{p,1}} \frac{|dw|}{2\sqrt{|w - c_{p+1}|}} \le \frac{\ell(s_{p,1})}{2r} \le \frac{1}{4r}.$$

By induction we get $\ell(F_p(\gamma_{p+\nu})) \le \frac{1}{2(2r)^\nu} = \frac{1}{2}q^\nu$ and thus

$$\ell(\sigma) \le \frac{1}{2}(1 + q + \dots + q^{j-1}) \le \frac{1}{2(1-q)} = \alpha_4.$$

Setting $\alpha_5 := 1 + \alpha\alpha_4 e^{\alpha\alpha_4}$ we obtain together with (4.4), (4.5) and (4.6)

$$\frac{d(z)}{|z - z_0|} \ge \frac{\alpha_3}{1 + \alpha_5 \sum_{j=1}^{m} q^j} \ge \frac{\alpha_3(1-q)}{\alpha_5}$$

which finally shows that γ has the John property. □

Theorem 4.4. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\mathcal{A}_{(c_n)}(0)$ is a John domain.*

Proof. The proof is very similar to the proof of Theorem 4.3. The only difficulty that arises is that $\mathcal{A}_{(c_n)}(0)$ contains critical values which all lie in D_{s_δ}. Therefore, we only give a sketch and omit the details.

Let $\frac{1}{2} < r < r' < r_\delta$, $\varepsilon := r' - r \le 1$ and $U_k := F_k^{-1}(D_{r'})$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(0)$ and $\mathcal{A}_{(c_n)}(0) = \bigcup_{k=1}^\infty U_k$. For $z \in \mathcal{A}_{(c_n)}(0)$ let $d(z) := \operatorname{dist}(z, \mathcal{J}_{(c_n)})$. If $z \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$, $k \ge 2$ and $w := F_{k-1}(z)$, then $|w| \ge r'$ and thus $D_\varepsilon(w) \cap D_r = \varnothing$. Therefore, we obtain

$$d(z) \ge \frac{\alpha_1}{|F_{k-1}'(z)|} \qquad (z \in U_k \setminus U_{k-1}). \tag{4.2a}$$

In order to prove the John property, let $w_0 := 0$ and $z_0 \in \mathcal{A}_{(c_n)}(0)$. We may assume that $z_0 \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$, $k \ge 2$. Then $|F_{k-1}(z_0)| \ge r'$. We construct an arc in U_k joining z_0 and w_0 as follows. First, we join z_0 with ∂U_{k-1} by an arc $\gamma_k \subset U_k \setminus U_{k-1}$ such that $F_{k-1}(\gamma_k) \subset [0, F_{k-1}(z_0)]$, and we denote the endpoint of γ_k on ∂U_{k-1} by ζ_{k-1}. Then we join ζ_{k-1} with ∂U_{k-2} by an arc $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$ such that $F_{k-2}(\gamma_{k-1}) \subset [0, F_{k-2}(\zeta_{k-1})]$, and we denote the endpoint of γ_{k-1} on ∂U_{k-2} by ζ_{k-2}. Proceeding in this way we get an arc in $U_k \cap (\mathbb{C} \setminus U_1)$ with endpoint ζ_1 on ∂U_1. Finally, we set $\gamma = \gamma(z_0) := \gamma_k \cup \dots \cup \gamma_2 \cup [0, \zeta_1]$. We mention that $[0, \zeta_1] \subset \overline{U}_1$, since U_1

is a starlike domain with respect to 0 bounded by an analytic Jordan curve. Furthermore, we note that the line segments $F_{j-1}(\gamma_j)$ $(j = 2, \ldots, k)$ all lie in $\overline{\Delta}_{r'} \cap \overline{D}_{R_\delta}$ and thus have lengths at most one.

We now show that the arc γ has the John property. For that purpose, let $z \in \gamma$. We may assume that $z \notin U_1$. First, let $z \in U_k \setminus U_{k-1}$. Then we obtain the upper estimate for $|z - z_0|$

$$|z - z_0| \leq \frac{\alpha_2}{|F'_{k-1}(z)|} \qquad (z \in \gamma \setminus U_{k-1}). \tag{4.3a}$$

Putting (4.2a) and (4.3a) together we arrive at

$$d(z) \geq \alpha_3 |z - z_0| \qquad (z \in \gamma \setminus U_{k-1}).$$

Finally, the case that $z \in U_{k-m} \setminus U_{k-m-1}$ for some $m \in \{1, \ldots, k-2\}$ is handled as in the proof of Theorem 4.3. □

Corollary 4.5. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\mathcal{J}_{(c_n)}$ is a quasicircle.*

Proof. From Theorem 4.1 we know that $\mathcal{F}_{(c_n)} = \mathcal{A}_{(c_n)}(\infty) \cup \mathcal{A}_{(c_n)}(0)$. Then the assertion follows from Theorems 4.3 and 4.4 and the known results mentioned at the beginning of this section. □

If $(c_n) \in K_{1/4}^{\mathbb{N}}$, then $\mathcal{J}_{(c_n)}$ need not be a quasicircle. For example, if $c_n = \frac{1}{4}$ for all n, then it is known that $\mathcal{J}(f_{1/4})$ is still a Jordan curve (see for example [**CG**, p. 97] or [**St**, p. 124]) but it has cusps. Furthermore, Corollary 4.5 does not hold true in general when all c_n are contained in the interior of the main cardioid of the Mandelbrot set. This can be seen by the simple example $c_1 = -\frac{1}{2} - \eta$ and $c_n = \frac{1}{4} - \varepsilon$ for $n \geq 2$ with $0 < \eta < \frac{1}{4}$ and $0 < \varepsilon < \eta^2$. In this case we have $F_n(0) \to \infty$ as $n \to \infty$ so that by Theorem 1.1 in [**BBR**] the Julia set $\mathcal{J}_{(c_n)}$ is even disconnected. It would be of interest whether $\mathcal{J}_{(c_n)}$ is also a Jordan curve in our more general setting provided that $(c_n) \in K_{1/4}^{\mathbb{N}}$ or what holds when $(c_n) \in D_{1/4}^{\mathbb{N}}$.

Furthermore, we consider the dynamics of (F_n) in the stable domain $\mathcal{A}_{(c_n)}(0)$ provided that $(c_n) \in K_{1/4}^{\mathbb{N}}$. We will show that $\mathcal{A}_{(c_n)}(0)$ is a *contracting domain*, that is a stable domain U such that all limit functions of (F_n) in U are constant. This property is equivalent to $\operatorname{diam} F_n(K) \to 0$ as $n \to \infty$ for every compact set $K \subset U$.

Theorem 4.6. *Let $(c_n) \in K_{1/4}^{\mathbb{N}}$. Then $\mathcal{A}_{(c_n)}(0)$ is a contracting domain.*

Proof. Let $K \subset \mathcal{A}_{(c_n)}(0)$ be a compact set. We first assume that $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta < \frac{1}{4}$, and we choose $r \in \left(s_\delta, \frac{1}{2}\right)$. Then by Theorem 4.1 there exists $N \in \mathbb{N}$ such that $F_N(K) \subset D_r$. If $z_1, z_2 \in K$, then $w_1 := F_N(z_1)$, $w_2 := F_N(z_2) \in D_r$ and thus $|f_{c_k}(w_1) - f_{c_k}(w_2)| = |w_1 + w_2||w_1 - w_2| \leq 2r|w_1 - w_2|$ which implies $|F_{N+k}(z_1) - F_{N+k}(z_2)| \leq (2r)^k |w_1 - w_2|$. Therefore, we obtain $\operatorname{diam} F_{N+k}(K) \le (2r)^k \operatorname{diam} F_N(K) \to 0$ as $k \to \infty$, and the assertion follows.

Now, let $|c_n| \le \frac{1}{4}$ for all $n \in \mathbb{N}$. Again, by Theorem 4.1 there exists $N \in \mathbb{N}$ such that $F_N(K) \subset K_{1/2}$, and we obtain as above $\operatorname{diam} F_{N+k}(K) \le \operatorname{diam} F_{N+k-1}(K)$ so that the sequence $(\operatorname{diam} F_{N+k}(K))$ is monotonically decreasing and thus convergent. In order to deduce $\operatorname{diam} F_{N+k}(K) \to 0$ as $k \to \infty$ we need a better estimate. If $w_1,\, w_2 \in F_N(K)$, we obtain

$$|f_{c_k}(w_1) - f_{c_k}(w_2)| \le 2 \left| \int_{w_1}^{w_2} |z|\, |dz| \right|.$$

For the estimate of the right hand side we consider the worst case which can happen, that is $|w_1| = |w_2| = \frac{1}{2}$. For simplicity, we may assume that $w_2 = \overline{w}_1$, and we set $\varrho := \operatorname{Re} w_1 = \operatorname{Re} w_2 \in \left[0, \frac{1}{2}\right)$. Then with $d := \frac{1}{2}|w_1 - w_2|$ we get $\varrho^2 + d^2 = \frac{1}{4}$ and thus

$$\begin{aligned} 2 \left| \int_{w_1}^{w_2} |z|\, |dz| \right| &\le 4 \int_0^d |\varrho + it|\, dt = 4 \int_0^d \sqrt{\varrho^2 + t^2}\, dt \\ &= d + 2\varrho^2 \log \frac{2d+1}{2\varrho} \\ &= \frac{1}{2}|w_1 - w_2| + \frac{1}{4}(1 - |w_1 - w_2|^2) \log \frac{1 + |w_1 - w_2|}{1 - |w_1 - w_2|}. \end{aligned}$$

This implies with $d_n := \operatorname{diam} F_n(K)$

$$d_{N+k} \le \frac{1}{2} d_{N+k-1} + \frac{1}{4}(1 - d_{N+k-1}^2) \log \frac{1 + d_{N+k-1}}{1 - d_{N+k-1}}.$$

Setting $\alpha := \lim_{k\to\infty} \operatorname{diam} F_{N+k}(K)$ we see that

$$\alpha \le \frac{1}{2}\alpha + \frac{1}{4}(1 - \alpha^2) \log \frac{1+\alpha}{1-\alpha},$$

and an elementary argument shows that this is possible only for $\alpha = 0$ which gives the assertion. □

If $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta \le \frac{1}{4}$, we denote by $L_{(c_n)}$ the set of (constant) limit functions of (F_n) in $\mathcal{A}_{(c_n)}(0)$, that is the set of all $\zeta \in \mathbb{C}$ such that for some subsequence (F_{n_k}) of (F_n) there holds $F_{n_k} \to \zeta$ as $k \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(0)$. It is easy to see that $L_{(c_n)}$ is a compact set, and from the proof of Theorem 4.6 it follows that $L_{(c_n)} \subset K_{s_\delta} \subset K_{1/2}$. From Theorem 1.6 in [**BBR**] we know that the case $L_{(c_n)} = K_{s_\delta}$ may occur. Moreover, this phenomenon happens almost surely, that means that the product measure (cf. Section 5) of the set of these sequences (c_n) in $K_\delta^{\mathbb{N}}$ is one. In a similar way it is possible to construct sequences $(c_n) \in K_\delta^{\mathbb{N}}$ such that $L_{(c_n)} = \partial K_{s_\delta}$.

On the other hand, if $L_{(c_n)}$ consists of a single point ζ, then $F_n \to \zeta$ as $n \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(0)$, and since $F_{n+1}(z) = (F_n(z))^2 + c_n$ we obtain $c_n \to c \in K_\delta$ as $n \to \infty$, where $c = \zeta - \zeta^2$. Therefore, the set C_δ of all these points ζ is the component of the preimage of K_δ under the map $z \mapsto z - z^2$ which is contained in K_{s_δ}. Therefore, C_δ is a proper subset of K_{s_δ} and $C_\delta \cap \partial K_{s_\delta} = \{s_\delta\}$. It would be of interest to characterize those compact sets $K \subset K_{s_\delta}$ such that $K = L_{(c_n)}$ for some sequence $(c_n) \in K_\delta^{\mathbb{N}}$.

The stable domain $\mathcal{A}_{(c_n)}(\infty)$ may be viewed as a Böttcher domain. If it is simply connected, then there exists a conformal map ϕ of $\mathcal{A}_{(c_n)}(\infty)$ onto Δ_1 normalized at infinity by

$$\phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots . \tag{4.7}$$

Note that the capacity of $\mathcal{K}_{(c_n)}$ (cf. Section 8) is equal to one. Like in the iteration of a fixed polynomial we show that ϕ may be described dynamically.

Theorem 4.7. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$ such that $\mathcal{A}_{(c_n)}(\infty)$ is simply connected. Then the conformal map ϕ of $\mathcal{A}_{(c_n)}(\infty)$ onto Δ_1 with the normalization (4.7) is given by*

$$\phi(z) = \lim_{k\to\infty} \sqrt[2^k]{F_k(z)} = z \lim_{k\to\infty} \sqrt[2^k]{\frac{F_k(z)}{z^{2^k}}}$$

with locally uniform convergence in $\mathcal{A}_{(c_n)}(\infty)$, and where the branch of the root is determined by $\sqrt[2^k]{1} = 1$.

Proof. Let $R > R_\delta$ such that $R^2 \geq 2\delta$ and $U_m := F_m^{-1}(\Delta_R)$ for $m \in \mathbb{N}$. Then we have $U_m \subset U_{m+1} \subset \mathcal{A}_{(c_n)}(\infty)$ and $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{m=1}^{\infty} U_m$. For $k \in \mathbb{N}$ we define

$$\phi_k(z) := \sqrt[2^k]{F_k(z)} = z \sqrt[2^k]{\frac{F_k(z)}{z^{2^k}}}.$$

Then ϕ_k maps U_k conformally onto Δ_{R_k}, where $R_k := \sqrt[2^k]{R}$. For $z \in U_m$ and $k \geq m$ we have

$$\left|\frac{c_k}{(F_k(z))^2}\right| \leq \frac{\delta}{R^2} \leq \frac{1}{2},$$

and the elementary inequality

$$|\sqrt[p]{1+u} - 1| \leq \tfrac{1}{p} \qquad (u \in K_{1/2})$$

yields

$$\left|\frac{\phi_{k+1}(z)}{\phi_k(z)} - 1\right| = \left|\sqrt[2^{k+1}]{\frac{F_{k+1}(z)}{(F_k(z))^2}} - 1\right| = \left|\sqrt[2^{k+1}]{1 + \frac{c_k}{(F_k(z))^2}} - 1\right| \leq \frac{1}{2^{k+1}}.$$

Therefore, the limit

$$\phi(z) := \lim_{k\to\infty} \phi_k(z) = z \prod_{k=0}^{\infty} \frac{\phi_{k+1}(z)}{\phi_k(z)}$$

exists uniformly in U_m, and ϕ is the desired conformal map. □

5. Lebesgue measure of Julia sets.

From a result of Lyubich [**L2**] (see also [**CG**, p. 90] or [**St**, p. 144]) it follows that the Julia set of a hyperbolic rational function has two-dimensional Lebesgue measure (which we denote by m_2) zero. In particular, this is true for $\mathcal{J}(f_c)$ provided that c is contained in a hyperbolic component of the interior of the Mandelbrot set $\mathcal{M}$ or $c \notin \mathcal{M}$. In this section we show that this is true to a certain extent in our situation.

We begin with $\delta < \frac{1}{4}$. Then by Section 4 we know that if $(c_n) \in K_\delta^{\mathbb{N}}$, then $\mathcal{J}_{(c_n)}$ is a quasicircle, and from the differentiability properties of quasiconformal maps it follows that quasicircles always have two-dimensional Lebesgue measure zero (see for example [**LV**, p. 165]).

Corollary 5.1. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $m_2(\mathcal{J}_{(c_n)}) = 0$.*

Now, we will show that $m_2(\mathcal{J}_{(c_n)})$ is almost surely zero provided that the c_n are randomly chosen in K_δ for some $\delta > \frac{1}{4}$. To be more precise, let λ_δ denote the two-dimensional Lebesgue measure on K_δ normalized by $\lambda_\delta(K_\delta) = 1$. Then the product space $K_\delta^{\mathbb{N}}$ carries the usual product measure $\tilde{\lambda}_\delta := \bigotimes_{k=1}^{\infty} \lambda_\delta$. We set

$$\mathfrak{N}_\delta := \{ (c_n) \in K_\delta^{\mathbb{N}} : m_2(\mathcal{J}_{(c_n)}) = 0 \}. \tag{5.1}$$

Then the goal is to show that $\tilde{\lambda}_\delta(\mathfrak{N}_\delta) = 1$. In order to do this we recall:

Theorem 5.2 ([**BBR**]). *Let $\delta > \frac{1}{4}$ and $R > 0$. Then for every $z \in \widehat{\mathbb{C}}$ there exists an open set $\mathfrak{U}_z \subset K_\delta^{\mathbb{N}}$ with the following properties*:

(a) $\tilde{\lambda}_\delta(\mathfrak{U}_z) = 1$,
(b) *for every $(c_n) \in \mathfrak{U}_z$ there holds $|F_k(z)| > R$ for all sufficiently large k.*

Theorem 5.3. *Let $\delta > \frac{1}{4}$, and let $\mathfrak{N}_\delta \subset K_\delta^{\mathbb{N}}$ be defined by (5.1). Then $\tilde{\lambda}_\delta(\mathfrak{N}_\delta) = 1$.*

Proof. Let $M = \Delta_R$ be an invariant domain and

$$\widetilde{\mathfrak{E}} := \{ ((c_n), z) \in K_\delta^{\mathbb{N}} \times \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \}.$$

By Theorem 5.2 we have $\tilde{\lambda}_\delta(\widetilde{\mathfrak{E}}_z) = 1$ for $z \in \widehat{\mathbb{C}}$, where

$$\widetilde{\mathfrak{E}}_z := \{ (c_n) \in K_\delta^{\mathbb{N}} : ((c_n), z) \in \widetilde{\mathfrak{E}} \}.$$

If σ denotes the two-dimensional Lebesgue measure on $\widehat{\mathbb{C}}$ normalized by $\sigma(\widehat{\mathbb{C}}) = 1$, it follows

$$(\tilde{\lambda}_\delta \otimes \sigma)(\widetilde{\mathfrak{E}}) = \int_{\widehat{\mathbb{C}}} \tilde{\lambda}_\delta(\widetilde{\mathfrak{E}}_z)\, d\sigma(z) = 1.$$

Now, let

$$\mathfrak{E} := \{\, (c_n) \in K_\delta^{\mathbb{N}} : \sigma(\widetilde{\mathfrak{E}}_{(c_n)}) = 1 \,\},$$

where

$$\widetilde{\mathfrak{E}}_{(c_n)} := \{\, z \in \widehat{\mathbb{C}} : ((c_n), z) \in \widetilde{\mathfrak{E}} \,\}.$$

Since

$$1 = (\tilde{\lambda}_\delta \otimes \sigma)(\widetilde{\mathfrak{E}}) = \int_{K_\delta^{\mathbb{N}}} \sigma(\widetilde{\mathfrak{E}}_{(c_n)})\, d\tilde{\lambda}_\delta((c_n))$$

we obtain $\tilde{\lambda}_\delta(\mathfrak{E}) = 1$. If $(c_n) \in \mathfrak{E}$, then

$$\sigma(\mathcal{A}_{(c_n)}(\infty)) = \sigma(\{\, z \in \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \,\}) = \sigma(\widetilde{\mathfrak{E}}_{(c_n)}) = 1$$

which implies $\sigma(\mathfrak{J}_{(c_n)}) = 0$. □

It would be of interest whether Theorem 5.3 remains valid for $\delta = \frac{1}{4}$. Concerning the question whether there exists a sequence $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta \geq \frac{1}{4}$ such that $m_2(\mathfrak{J}_{(c_n)}) > 0$, the referee mentioned that, recently, a group of mathematicians around P.W. Jones at Yale University have constructed such an example. More precisely, there exists a sequence (c_n) with $c_n \in \left\{0, \pm\frac{1}{4}, \frac{1}{2}\right\}$ such that $m_2(\mathfrak{J}_{(c_n)}) > 0$. This result was communicated to the author by P.W. Jones. The author is grateful to both for bringing this information to his attention.

Finally, we prove:

Theorem 5.4. *Let $\delta > 2$ (which is equivalent to $\delta > R_\delta$), and let $\varepsilon > 0$ such that $R_\delta + \varepsilon \leq |c_n| \leq \delta$ for all $n \in \mathbb{N}$. Then $m_2(\mathfrak{J}_{(c_n)}) = 0$.*

Proof. We choose R such that $R_\delta < R < R_\delta + \varepsilon$ and $\eta := R_\delta + \varepsilon - R > 0$. Then we have $\mathfrak{J}_{(c_n)} \subset \overline{D}_{R_\delta} \subset D := D_R$, and D is backward invariant, that is $f_{c_n}^{-1}(D) \subset D$ for all $n \in \mathbb{N}$. Furthermore, there holds $|f_{c_n}(0)| = |c_n| \geq R_\delta + \varepsilon = R + \eta$ and thus $|(f_{c_m} \circ \cdots \circ f_{c_{k+1}})(0)| \geq R + \eta$ for $k = 0, 1, \ldots, m-1$ and all $m \in \mathbb{N}$. Therefore, D does not contain any critical value of (F_n) so that in D there exist 2^k analytic branches of the inverse function of F_k which we denote by $G_{j,k}$ for $j = 1, \ldots, 2^k$ and $k \in \mathbb{N}$. Furthermore, we set $D_{j,k} := G_{j,k}(D) \subset D$ and $D_k := \bigcup_{j=1}^{2^k} D_{j,k}$. Then $D_{1,k}, \ldots, D_{2^k,k}$ are mutually disjoint simply connected domains, and $\mathfrak{J}_{(c_n)} \subset D_{k+1} \subset D_k$. Finally, we set $U_k := D_k \setminus \overline{D}_{k+1}$ and $U_{j,k} := D_{j,k} \setminus \overline{D}_{k+1}$ so that $U_{1,k}, \ldots, U_{2^k,k}$ are mutually disjoint multiply connected domains, and $U_k = \bigcup_{j=1}^{2^k} U_{j,k}$.

Now, we prove that there exists a constant $q > 0$ such that

$$\frac{m_2(U_k)}{m_2(D_k)} \geq q \tag{5.2}$$

for all $k \in \mathbb{N}$. For that purpose it is enough to show that

$$\frac{m_2(U_{j,k})}{m_2(D_{j,k})} \geq q \tag{5.3}$$

for $j = 1, \ldots, 2^k$ and all $k \in \mathbb{N}$.

Let $V_{1,k}$ and $V_{2,k}$ denote the two components of $f_{c_{k+1}}^{-1}(D)$, and let $W_k := D \setminus (\overline{V}_{1,k} \cup \overline{V}_{2,k})$. Then $U_{j,k} = G_{j,k}(W_k)$, and we obtain

$$\frac{m_2(U_{j,k})}{m_2(D_{j,k})} = \frac{\int_{W_k} |G'_{j,k}(z)|^2 \, dm_2(z)}{\int_D |G'_{j,k}(z)|^2 \, dm_2(z)} \geq \frac{|G'_{j,k}(z_{j,k})|^2 m_2(W_k)}{|G'_{j,k}(\zeta_{j,k})|^2 m_2(D)},$$

where $z_{j,k} \in W_k \subset D$ and $\zeta_{j,k} \in D$ such that $|G'_{j,k}(z_{j,k})| = \min_{z \in W_k} |G'_{j,k}(z)|$ and $|G'_{j,k}(\zeta_{j,k})| = \max_{z \in D} |G'_{j,k}(z)|$. By the Koebe distortion theorem (see for example [**P2**, p. 9]) applied to the disk $D_{R+\eta}$ there holds

$$\left| \frac{G'_{j,k}(z)}{G'_{j,k}(\zeta)} \right| \geq \left(\frac{\eta}{\eta + 2R} \right)^4$$

for all z, $\zeta \in D$. Therefore, it remains to show that there exists a constant $\gamma > 0$ such that

$$\frac{m_2(W_k)}{m_2(D)} \geq \gamma$$

for all $k \in \mathbb{N}$.

For simplicity we write $c = c_{k+1}$, and let $V \in \{V_{1,k}, V_{2,k}\}$. Then

$$m_2(V) = \frac{1}{4} \int_D \frac{dm_2(z)}{|z - c|} = \frac{1}{4} \int_0^R \int_0^{2\pi} \frac{\varrho}{|\varrho e^{it} - c|} \, dt \, d\varrho.$$

By the Cauchy-Schwarz inequality we get

$$\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|} \leq \sqrt{2\pi} \left(\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|^2} \right)^{1/2},$$

and the Poisson integral formula yields

$$\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|^2} = \frac{2\pi}{|c|^2 - \varrho^2}.$$

Therefore, we arrive at

$$m_2(V) \leq \frac{\pi}{2} \int_0^R \frac{\varrho}{\sqrt{|c|^2 - \varrho^2}} \, d\varrho = \frac{\pi}{2}(|c| - \sqrt{|c|^2 - R^2}) \leq \frac{1}{2}\pi R.$$

This implies $m_2(V_{1,k} \cup V_{2,k}) \le \pi R$ and thus

$$\frac{m_2(W_k)}{m_2(D)} \ge 1 - \frac{1}{R} \ge \frac{1}{2}$$

which proves (5.3).

Finally, (5.2) gives $m_2(\mathcal{J}_{(c_n)}) \le m_2(D_{k+1}) = m_2(D_k) - m_2(U_k) \le (1 - q)m_2(D_k)$ so that $m_2(\mathcal{J}_{(c_n)}) \le (1-q)^k m_2(D) \to 0$ as $k \to \infty$ which completes the proof. □

6. Hausdorff dimension of Julia sets.

We first recall the notion of Hausdorff dimension. Let $E \subset \mathbb{C}$ be a non-empty compact set, and denote by $(D_j)_\varepsilon$ any covering of E by finitely many open sets D_j with $\operatorname{diam} D_j < \varepsilon$. Then for $t \in (0, 2]$

$$m_t(E) := \sup_{\varepsilon > 0} \inf_{(D_j)_\varepsilon} \sum_j (\operatorname{diam} D_j)^t$$

is called the *t-dimensional Hausdorff measure* of E. Obviously, $m_t(E) < \infty$ implies $m_s(E) = 0$ for $s > t$, and conversely, $m_t(E) > 0$ implies $m_s(E) = \infty$ for $s < t$. Hence, there exists a unique $\tau \in [0, 2]$ such that $m_s(E) = 0$ and $m_t(E) = \infty$ for $0 < t < \tau < s \le 2$. This number τ is called the *Hausdorff dimension* of E and is denoted by $\dim_{\mathrm{H}} E$.

It is well-known (cf. [**G**], see also [**Be**, p. 251] or [**St**, p. 169]) that the Hausdorff dimension of the Julia set of any fixed rational function f is positive. More precisely, if $\infty \notin \mathcal{J}(f)$ and if d denotes the degree of f, then

$$\dim_{\mathrm{H}} \mathcal{J}(f) \ge \frac{\log d}{\log \max_{z \in \mathcal{J}(f)} |f'(z)|}.$$

We show that this estimate holds true in a certain sense in our situation.

Theorem 6.1. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} > 0$. More precisely, there holds*

$$\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} \ge \frac{\log 2}{\log (2R_\delta)} = \frac{\log 2}{\log (1 + \sqrt{1 + 4\delta})}.$$

Proof. We show that the Green function g of $\mathcal{A}_{(c_n)}(\infty)$ is Hölder continuous with exponent

$$\alpha = \frac{\log 2}{\log 2 + \log (2R - R_\delta)}$$

for any $R > R_\delta$. Then a result of Carleson [**C**] gives $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} \ge \alpha$. For that purpose, it suffices to show that there exists a constant $\gamma > 0$ such that $g(z) \le \gamma (d(z))^\alpha$ for all $z \in \mathcal{A}_{(c_n)}(\infty)$, where $d(z) := \operatorname{dist}(z, \mathcal{J}_{(c_n)})$. Of course, we may assume that $d(z)$ is small.

Let $R > R_\delta$ and $U_k := F_k^{-1}(\Delta_R)$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(\infty)$ and $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{k=1}^{\infty} U_k$. The Green function g_k of U_k with pole at infinity is given by

$$g_k(z) = \frac{1}{2^k} \log \frac{|F_k(z)|}{R} \qquad (z \in U_k).$$

There holds $g_k(z) \le g_{k+1}(z) \le g(z)$ for $z \in U_k$ and $g_k \to g$ as $k \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(\infty)$.

We will show that there exists some constant $C > 0$ such that $g(z) \le g_k(z) + \frac{C}{2^k}$ for $z \in U_k$. There holds $|F_{k+1}(z)| = |(F_k(z))^2 + c_{k+1}| \le |F_k(z)|^2 + \delta$ and this gives

$$g_{k+1}(z) \le \frac{1}{2^{k+1}} \log \frac{|F_k(z)|^2 + \delta}{R}.$$

If a, $b > 0$, then $\log^+(a+b) \le \log^+ a + \log^+ b + \log 2$, and thus

$$\begin{aligned} g_{k+1}(z) &\le \frac{1}{2^{k+1}} \left(\log \frac{|F_k(z)|^2}{R} + \log^+ \frac{\delta}{R} + \log 2 \right) \\ &= \frac{1}{2^{k+1}} \left(2 \log \frac{|F_k(z)|}{R} + \log^+ \frac{\delta}{R} + \log(2R) \right) \\ &= g_k(z) + \frac{C}{2^{k+1}}, \end{aligned}$$

where $C := \log^+ \frac{\delta}{R} + \log(2R)$. From this we obtain by induction

$$g_{k+m}(z) \le g_k(z) + C \left(\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+m}} \right) \le g_k(z) + \frac{C}{2^k}$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$ we get

$$g(z) \le g_k(z) + \frac{C}{2^k} \qquad (z \in U_k).$$

Now, let $z \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$. Then $|F_{k-1}(z)| \le R$ which implies $|F_k(z)| \le R^2 + \delta$. Hence, we have $g_k(z) \le \frac{1}{2^k} \log \left(R + \frac{\delta}{R}\right)$ and thus

$$g(z) \le \frac{\Gamma}{2^k} \qquad (z \in U_k \setminus U_{k-1}), \tag{6.1}$$

where $\Gamma := C + \log\left(R + \frac{\delta}{R}\right)$.

Finally, we prove a lower estimate for $d(z)$, if $z \in U_k$ for some $k \in \mathbb{N}$. We set $w := F_k(z)$ and $\eta := |w| - R_\delta$. If U denotes the component of $F_k^{-1}(D_\eta(w))$ containing z, there holds $U \subset \mathcal{A}_{(c_n)}(\infty)$. Let $\varrho > 0$ such that $D_\varrho(z) \subset U$. Then $F_k(D_\varrho(z)) \subset D_\eta(w) \subset D_{|w|+\eta}$ which implies $F_j(D_\varrho(z)) \subset D_{|w|+\eta}$ for

$j = 0, 1, \dots, k$ and thus $|F_k'(t)| \le 2^k(|w|+\eta)^k$ for all $t \in D_\varrho(z)$. If $z' \in D_\varrho(z)$ and $w' := F_k(z')$, then

$$w' - w = \int_z^{z'} F_k'(t)\, dt,$$

where we integrate over the line segment joining z and z'. This yields

$$|w' - w| \le 2^k(|w| + \eta)^k |z' - z| < 2^k(|w| + \eta)^k \varrho.$$

Setting

$$\varrho := \frac{\eta}{2^k(|w| + \eta)}$$

we obtain $D_\varrho(z) \subset U$ and thus

$$d(z) \ge \frac{\eta}{2^k(|w| + \eta)} = \frac{|w| - R_\delta}{2^k(2|w| - R_\delta)} \ge \frac{R - R_\delta}{2^k(2R - R_\delta)} \qquad (z \in U_k).$$

We choose $q := R - R_\delta$ and

$$\alpha := \frac{\log 2}{\log 2 + \log (R + q)}$$

and arrive at

$$(d(z))^\alpha \ge \frac{q^\alpha}{2^k} \qquad (z \in U_k). \tag{6.2}$$

Finally, putting (6.1) and (6.2) together we get

$$g(z) \le \frac{\Gamma}{q^\alpha} (d(z))^\alpha \qquad (z \in U_k \setminus U_{k-1})$$

which completes the proof. □

Gehring and Väisälä [**GV**] have shown that quasicircles always have Hausdorff dimension less than two and thus by Corollary 4.5 we obtain:

Corollary 6.2. *Let* $\delta < \frac{1}{4}$ *and* $(c_n) \in K_\delta^{\mathbb{N}}$. *Then* $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} < 2$.

If $0 < \delta \le \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$, then the Julia set $\mathcal{J}_{(c_n)}$ is connected so that its Hausdorff dimension is at least one. Moreover, Sullivan [**Su**] has shown, that if $c \ne 0$ is in the interior of the main cardioid of the Mandelbrot set, then $\dim_{\mathrm{H}} \mathcal{J}(f_c) > 1$. Furthermore, it follows by a result of Shishikura [**Sh**] that $\dim_{\mathrm{H}} \mathcal{J}(f_{1/4}) = 2$. It would be of interest whether $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ is almost surely (in the sense of Section 5) greater than one if $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta < \frac{1}{4}$. In our general setting, it is clear that we can only expect such an almost surely statement.

7. Density of repelling fixpoints.

From iteration theory of a fixed rational function it is well-known that the repelling periodic points are dense in the Julia set (cf. [**Be**, p. 148], [**CG**, p. 63] or [**St**, p. 35]). In our setting we consider the set $\mathcal{R}_{(c_n)}$ of *repelling fixpoints* of the sequence of iterates (F_n), i.e.,

$$\mathcal{R}_{(c_n)} := \{ \zeta \in \mathbb{C} : F_k(\zeta) = \zeta \text{ for some } k \in \mathbb{N} \text{ and } |F_k'(\zeta)| > 1 \}.$$

It is not necessarily true that $\mathcal{R}_{(c_n)} \subset \mathcal{J}_{(c_n)}$. But from a result of Fornæss and Sibony [**FS**, Theorem 2.3] it follows that if $\delta > 0$ is sufficiently small and $(c_n) \in K_\delta^{\mathbb{N}}$, then $(\mathcal{R}_{(c_n)})' = \mathcal{J}_{(c_n)}$. More precisely, we show:

Theorem 7.1. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $(\mathcal{R}_{(c_n)})' = \mathcal{J}_{(c_n)}$.*

Proof. Since $\delta < \frac{1}{4}$ we have $f_c(\overline{D}_r) \subset D_r$ for all $c \in K_\delta$ and $s_\delta < r < r_\delta$. This implies that $F_k(z) \neq z$ for all $k \in \mathbb{N}$ and $s_\delta < |z| < r_\delta$. Since $F_k'(z) = 2^k \prod_{j=0}^{k-1} F_j(z)$ and $f_c(K_{1/2}) \subset K_{1/2}$, we have $\mathcal{R}_{(c_n)} \cap K_{1/2} = \varnothing$. Setting $K := K_r$ for some $r \in \left(\frac{1}{2}, r_\delta\right)$, we also have $\mathcal{R}_{(c_n)} \cap K = \varnothing$. We set $U := \mathbb{C} \setminus K$. If $z \in U$ and $F_j(z) \in U$ for all $j = 1, \ldots, k-1$, then $|F_k'(z)| \geq q^k$ with $q := 2r > 1$.

We first show that $(\mathcal{R}_{(c_n)})' \subset \mathcal{J}_{(c_n)}$. For that purpose let $F_{k_\ell}(z_\ell) = z_\ell$, $|F_{k_\ell}'(z_\ell)| > 1$ and $z_\ell \to \zeta$ as $\ell \to \infty$. If $\zeta \in \mathbb{C} \setminus \mathcal{K}_{(c_n)}$, then $F_{k_\ell} \to \infty$ as $\ell \to \infty$ uniformly in some neighbourhood of ζ. This gives $F_{k_\ell}(z_\ell) \to \infty$ as $\ell \to \infty$ which is a contradiction. Now, assume that $\zeta \in (\mathcal{K}_{(c_n)})^\circ$. If $F_j(\zeta) \in U$ for all $j \in \mathbb{N}_0$, then $|F_k'(\zeta)| \geq q^k \to \infty$ as $k \to \infty$. But this is impossible since (F_k) is normal and bounded in $(\mathcal{K}_{(c_n)})^\circ$. Therefore, we have $F_{k_0}(\zeta) \in K$ for some $k_0 \in \mathbb{N}_0$, and thus $F_k(\zeta) \in K$ for all $k \geq k_0$. By passing to a subsequence we may assume that $F_{k_\ell} \to \phi$ as $\ell \to \infty$ uniformly in some neighbourhood U_ζ of ζ, where ϕ is holomorphic in U_ζ. This implies $z_\ell = F_{k_\ell}(z_\ell) \to \phi(\zeta)$ as $\ell \to \infty$ and thus $z_\ell \in K$ for all ℓ large enough which is again a contradiction.

Now, we show that $\mathcal{J}_{(c_n)} \subset (\mathcal{R}_{(c_n)})'$. Suppose that there exists $\zeta \in \mathcal{J}_{(c_n)}$ and a neighbourhood V of ζ such that $F_k(z) \neq z$ for all $z \in V$ and $k \geq k_0 = k_0(V)$. We set

$$h_k(z) := \frac{1}{2^k} \log |F_k(z) - z|.$$

Then h_k is harmonic and uniformly bounded above in V. By Eq. (2.1) we have $h_k \to g_{(c_n)}$ as $k \to \infty$ in $V \setminus \mathcal{K}_{(c_n)}$, and thus $h_k \to h$ as $k \to \infty$ for some harmonic function h in V. Furthermore, there holds $h_k \to 0$ as $k \to \infty$ in $V \cap \mathcal{K}_{(c_n)}$ so that $h = 0$ in $V \cap \mathcal{K}_{(c_n)}$. But this is a contradiction to the minimum principle for harmonic functions.

Therefore, for every $\zeta \in \mathcal{J}_{(c_n)}$ there exists a strictly increasing sequence (k_ℓ) in $\mathbb{N}$ and $z_\ell \in U$ such that $z_\ell \to \zeta$ as $\ell \to \infty$ and $F_{k_\ell}(z_\ell) = z_\ell$. Then we have $F_j(z_\ell) \in U$ for $j = 1, \dots, k_\ell - 1$ which gives $|F'_{k_\ell}(z_\ell)| \geq q^{k_\ell} > 1$ so that $z_\ell \in \mathcal{R}_{(c_n)}$. □

It would be of interest whether Theorem 7.1 holds for all $\delta > 0$. However, the proof shows that we always have $(\mathcal{R}_{(c_n)})' \subset \mathcal{K}_{(c_n)}$.

8. Asymptotic distribution of predecessors.

If $(c_n) \in K_\delta^{\mathbb{N}}$ and if $a \in \Delta_{R_\delta}$, then the predecessors $F_k^{-1}(a)$ of a are all contained in $\mathcal{A}_{(c_n)}(\infty)$, and they only accumulate on the Julia set $\mathcal{J}_{(c_n)}$. In fact, this follows from the invariance of Δ_R for $R > R_\delta$ and $F_k \to \infty$ as $k \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(\infty)$. We want to study the asymptotic distribution of $F_k^{-1}(a)$ as $k \to \infty$. For iteration of a fixed polynomial this was done by Brolin [**Bro**].

We first recall some facts from potential theory which are needed in the sequel and which can be found, for example, in the book of Tsuji [**T**]. Let $E \subset \mathbb{C}$ be an infinite compact set, and let D be its outer domain, that is the component of $\widehat{\mathbb{C}} \setminus E$ containing the point ∞. Furthermore, we denote by $\operatorname{cap} E \geq 0$ the *logarithmic capacity* (or *transfinite diameter*) of E. (We do not recall the definition of $\operatorname{cap} E$ because it will not be needed.) We suppose that the Green function g_D of D with pole at infinity exists. Then

$$g_D(z) = \log|z| + V + o(1) \quad \text{as } z \to \infty$$

and $\operatorname{cap} E = e^{-V} > 0$. Note that by Eq. (2.1) this is true for $E = \mathcal{J}_{(c_n)}$ with $\operatorname{cap} E = 1$. Now, let μ be any probability measure on E. Then the *energy integral*

$$I[\mu] := \iint_{E\times E} \log\frac{1}{|\zeta - \omega|}\, d\mu(\zeta)\, d\mu(\omega)$$

is finite, and the *logarithmic potential*

$$p_\mu(z) := \int_E \log\frac{1}{|z-\zeta|}\, d\mu(\zeta)$$

is harmonic in D. Furthermore, there exists a unique probability measure μ^* on E which minimizes the energy integral $I[\mu]$, and there holds

$$g_D(z) - V = -p_{\mu^*}(z) \qquad (z \in D).$$

This measure μ^* is called the *equilibrium measure* on E. In the following μ^* always denotes the equilibrium measure on the Julia set $\mathcal{J}_{(c_n)}$, and $\operatorname{supp}\mu^*$ denotes its support, that is the set of points $z \in \mathcal{J}_{(c_n)}$ such that $\mu^*(D_\varepsilon(z) \cap \mathcal{J}_{(c_n)}) > 0$ for every $\varepsilon > 0$. Note that $\operatorname{supp}\mu^*$ is a closed set.

In order to study the asymptotic distribution of $F_k^{-1}(a)$ for $a \in \Delta_{R_\delta}$ as $k \to \infty$ we consider the following sequence (μ_k^a) of probability measures.

If δ_z denotes the *Dirac measure* concentrated at the point $z \in \mathbb{C}$ (that is $\delta_z(E) = 1$ if $z \in E$ and $\delta_z(E) = 0$ if $z \notin E$), then let

$$\mu_k^a := \frac{1}{2^k} \sum_{F_k(z)=a} \delta_z. \tag{8.1}$$

We will show that (μ_k^a) is weakly convergent to μ^*, that is $\mu_k^a(E) \to \mu^*(E)$ as $k \to \infty$ for every Borel set $E \subset \mathbb{C}$ with $\mu^*(E^\circ) = \mu^*(\overline{E})$. For that purpose we first collect some auxiliary results.

Lemma 8.1 ([**Bro**, Lemma 15.4])**.** *Let $E \subset \mathbb{C}$ be a compact set, and let f be a function defined on E such that for some constant L there holds $|f(z_1) - f(z_2)| \le L|z_1 - z_2|$ for all z_1, $z_2 \in E$. If* $\operatorname{cap} E = 0$*, then* $\operatorname{cap} f(E) = 0$*.*

Lemma 8.2. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then* $\operatorname{cap}(\mathcal{J}_{(c_n)} \setminus \operatorname{supp} \mu^*) = 0$.

Proof. Since $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty)$ and $\operatorname{cap} \mathcal{J}_{(c_n)} > 0$, the assertion immediately follows from Theorem III.31 in [**T**, p. 79]. □

Lemma 8.3. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then* $\operatorname{supp} \mu^* = \mathcal{J}_{(c_n)}$.

Proof. We assume that $\mathcal{J}^* := \mathcal{J}_{(c_n)} \setminus \operatorname{supp} \mu^* \neq \varnothing$. By Lemma 8.2 we have $\operatorname{cap} \mathcal{J}^* = 0$. Since $\mathcal{J}^*$ is an open set in $\mathcal{J}_{(c_n)}$ we may choose $z_0 \in \mathcal{J}^*$ and $\varepsilon > 0$ such that $\mathcal{J}_\varepsilon := \mathcal{J}^* \cap D_\varepsilon(z_0) \subset \mathcal{J}^*$. We also have $\operatorname{cap} \mathcal{J}_\varepsilon = 0$. But by the self-similarity of $\mathcal{J}_{(c_n)}$ (cf. [**Bü1**]) there exists $m \in \mathbb{N}$ such that $F_m(\mathcal{J}_\varepsilon) = F_m(\mathcal{J}_{(c_n)})$. Since $|f_{c_k}(z_1) - f_{c_k}(z_2)| = |z_1 + z_2||z_1 - z_2| \le 2R_\delta |z_1 - z_2|$ for all $k \in \mathbb{N}$ and z_1, $z_2 \in \mathcal{J}_{(c_n)}$, we obtain $\operatorname{cap} F_m(\mathcal{J}_\varepsilon) = 0$ by Lemma 8.1. On the other hand there holds $F_m(\mathcal{J}_{(c_n)}) = \mathcal{J}_{(c_{n+m})}$ and thus $\operatorname{cap} F_m(\mathcal{J}_{(c_n)}) = 1$ which gives a contradiction. □

Lemma 8.4 ([**Bro**, Lemma 15.5])**.** *Let E, $H \subset \mathbb{C}$ be compact sets with $E \subset H$ and* $\operatorname{cap} E = e^{-V} > 0$*. Furthermore, let (μ_n) be a sequence of probability measures on H which converges weakly to a probability measure μ on E. If u_n denotes the logarithmic potential with respect to μ_n and μ^* denotes the equilibrium measure on E, then suppose* $\liminf_{n\to\infty} u_n(z) \ge V$ *for $z \in E$ and* $\operatorname{supp} \mu^* = E$*. Then there holds $\mu = \mu^*$.*

Theorem 8.5. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then for any $a \in \Delta_{R_\delta}$ the sequence (μ_k^a) of probability measures defined by (8.1) converges weakly to the equilibrium measure μ^* on $\mathcal{J}_{(c_n)}$.*

Proof. For $k \in \mathbb{N}$ let $z_{1,k}, \ldots, z_{2^k,k}$ be the solutions of the equation $F_k(z) = a$. Then we have $z_{j,k} \in \mathcal{A}_{(c_n)}(\infty)$ and $z_{j,k} \in H := K_{|a|}$ for $j = 1, \ldots, 2^k$ so that $\operatorname{supp} \mu_k^a \subset H$. Since $|F_k(z)| \le R_\delta$ for $z \in \mathcal{J}_{(c_n)}$ and

$$|F_k(z) - a| = \prod_{j=1}^{2^k} |z - z_{j,k}|,$$

we obtain for $z \in \mathcal{J}_{(c_n)}$

$$\sum_{j=1}^{2^k} \log|z - z_{j,k}| = \log|F_k(z) - a| \leq \log(R_\delta + |a|) = C$$

and thus

$$u_k(z) := \frac{1}{2^k} \sum_{j=1}^{2^k} \log \frac{1}{|z - z_{j,k}|} \geq -\frac{C}{2^k}.$$

This can be written as

$$u_k(z) = \int_H \log \frac{1}{|z - \zeta|}\, d\mu_k^a(\zeta) \geq -\frac{C}{2^k}$$

so that

$$\liminf_{k\to\infty} u_k(z) \geq 0 = \log \operatorname{cap} \mathcal{J}_{(c_n)} \qquad (z \in \mathcal{J}_{(c_n)}). \tag{8.2}$$

By the Selection Theorem (cf. [**T**, p. 34]) every sequence of probability measures on H contains a weakly convergent subsequence. Therefore, we only have to show that for every subsequence of (μ_k^a) which converges weakly to some probability measure ν there holds $\nu = \mu^*$. In fact, since the predecessors $F_k^{-1}(a)$ of a do not accumulate in $\mathcal{A}_{(c_n)}(\infty)$ we obtain $\operatorname{supp}\nu \subset \mathcal{J}_{(c_n)}$, and because of (8.2) the assertion follows from Lemma 8.3 and 8.4. □

Remark 8.6. If $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$, then the assertion of Theorem 8.5 also holds for any $a \in D_{r_\delta}$. This requires only a few simple modifications in the proof.

Like in the iteration of a fixed function there holds that for any $a \in \mathcal{J}_{(c_n)}$ the set $\bigcup_{k=1}^{\infty} F_k^{-1}(F_k(a))$ is dense in $\mathcal{J}_{(c_n)}$ (cf. [**Bü1**]). We also want to study the asymptotic distribution of $F_k^{-1}(F_k(a))$ as $k \to \infty$. For that purpose, we consider the following sequence (ν_k^a) of probability measures defined by

$$\nu_k^a := \frac{1}{2^k} \sum_{F_k(z)=F_k(a)} \delta_z. \tag{8.3}$$

Then $\operatorname{supp}\nu_k^a \subset \mathcal{J}_{(c_n)}$, and from iteration theory of a fixed polynomial f_c it is known (cf. [**Bro**], see also [**St**, p. 148]) that (ν_k^a) converges weakly to the equilibrium measure μ^* on $\mathcal{J}(f_c)$. We show that this holds true in our situation.

Theorem 8.7. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then for any $a \in \mathcal{J}_{(c_n)}$ the sequence (ν_k^a) of probability measures defined by (8.3) converges weakly to the equilibrium measure μ^* on $\mathcal{J}_{(c_n)}$.*

Proof. For $k \in \mathbb{N}$ let $z_{1,k}, \dots, z_{2^k,k}$ be the solutions of the equation $F_k(z) = F_k(a)$. Then we have for $z \in \mathcal{A}_{(c_n)}(\infty)$

$$\frac{1}{2^k} \log |F_k(z) - F_k(a)| = \frac{1}{2^k} \sum_{j=1}^{2^k} \log |z - z_{j,k}| = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\nu_k^a(\zeta).$$

Again, we only have to show that every weakly convergent subsequence (λ_ℓ) of (ν_k^a) has the limit μ^*. If $\lambda_\ell \to \lambda$ as $\ell \to \infty$ weakly, then for $z \in \mathcal{A}_{(c_n)}(\infty)$

$$\lim_{\ell \to \infty} \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\lambda_\ell(\zeta) = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\lambda(\zeta).$$

On the other hand we have

$$\frac{1}{2^k} \log |F_k(z) - F_k(a)| = \frac{1}{2^k} \log \left| \frac{F_k(z) - F_k(a)}{F_k(z)} \right| + \frac{1}{2^k} \log |F_k(z)| \to g_{(c_n)}(z) \quad \text{as } k \to \infty.$$

This implies

$$g_{(c_n)}(z) = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\lambda(\zeta) \qquad (z \in \mathcal{A}_{(c_n)}(\infty)),$$

and since μ^* is unique the assertion follows. □

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