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A certain finiteness result for special values of character twists of Koecher-Maass series attached to Siegel cusp of genus *g* is proved.

1. Introduction.

Let f be an elliptic cusp form of even integral weight k on $\Gamma_1 := SL_2(\mathbf{Z})$. Let χ be a primitive Dirichlet character modulo a positive integer N and denote by $L(f,\chi,s)\,(s \in \mathbf{C})$ the Hecke L-function of f twisted with χ, defined by analytic continuation of the series

$$\sum_{n\geq 1} \chi(n)a(n)n^{-s} \qquad (\mathrm{Re}\,(s) \gg 0;\ a(n) = n\text{-th Fourier coefficient of } f).$$

Let $g(\chi)$ be the Gauss sum attached to χ. As is well-known, there exists a $\mathbf{Z}$-module $M_f \subset \mathbf{C}$ (depending only on f) of finite rank such that all the special values

$$i^{s+1}(2\pi)^{-s}g(\overline{\chi})L(f,\chi,s)$$

$$(s \in \mathbf{N},\ 1 \leq s \leq k-1; \quad \chi \ \text{a primitive Dirichlet character modulo } N,\ N \in \mathbf{N})$$

lie in $M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\overline{\chi}]$, where $\mathbf{Z}[\overline{\chi}]$ is the $\mathbf{Z}$-module obtained from $\mathbf{Z}$ by adjoining the values of $\overline{\chi}$. In fact, if f is a Hecke eigenform, one has $\mathrm{rk}_{\mathbf{Z}}\, M_f \leq 2$ [**1**, **7**, **8**, **10**].

The purpose of this paper is to give a generalization of the above result to the case of a Siegel cusp form f, where now $L(f,\chi,s)$ is replaced by an appropriate χ-twist of the Koecher-Maass series attached to f.

More precisely, let f be a cusp form of even integral weight $k \geq g+1$ w.r.t. the Siegel modular group $\Gamma_g := Sp_g(\mathbf{Z})$ of genus g and write $a(T)$ (T a positive definite half-integral matrix of size g) for its Fourier coefficients. For χ as above we set

$$L(f,\chi,s) := \sum_{\{T>0\}/GL_{g,N}(\mathbf{Z})} \frac{\chi(\mathrm{tr}\,T)a(T)}{\epsilon_N(T)(\det T)^s} \qquad (\mathrm{Re}\,(s) \gg 0), \tag{1}$$

where the summation extends over all positive definite half-integral (g,g)-matrices T modulo the action $T \mapsto T[U] := U^tTU$ of the group $GL_{g,N}(\mathbf{Z}) := \{U \in GL_g(\mathbf{Z}) \,|\, U \equiv E_g \pmod N\}$ and $\epsilon_N(T) := \#\{U \in GL_{g,N}(\mathbf{Z}) \,|\, T[U] = T\}$ is the order of the corresponding unit group of T (note that $\epsilon_N(T) = 1$ whenever $N > 2$ by a classical result of Minkowski). Furthermore, $\mathrm{tr}\, T$ denotes the trace of T. Note that $\chi(\mathrm{tr}\, T)$ depends only on the $GL_{g,N}(\mathbf{Z})$-class of T.

In §2 (Thm. 1) we shall prove that the series $L(f,\chi,s)$ have holomorphic continuations to $\mathbf{C}$ and satisfy functional equations under $s \mapsto k-s$. The proof is fairly standard and follows the same pattern as in [**6**] for the case $N=1$ (compare also [**5**]) and [**9**, §3.6] for $g=1$.

The main result of the paper (Thm. 2) which will be proved in §3, states that all the special values

$$i^{gs+\frac{g(g+1)}{2}}\pi^{\frac{g(g-1)}{4}+[\frac{g}{2}]}\,(2\pi)^{-gs}\,g(\overline{\chi})\,L(f,\chi,s)$$

$$\Big(s \in \mathbf{N},\ \frac{g+1}{2} \le s \le k - \frac{g+1}{2};\quad \chi \text{ a primitive Dirichlet character modulo } N,\ N \in \mathbf{N}\Big)$$

are contained in $M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\overline{\chi}]$ where $M_f \subset \mathbf{C}$ is a finite $\mathbf{Z}$-module depending only on f. Its rank is bounded by the rank of a certain singular relative homology group of a toroidal compactification of a quotient space of $\mathcal{H}_g \times \mathbf{C}^{gw}$, where $\mathcal{H}_g$ is the Siegel upper half-space of genus g and $w := k-(g+1)$. See §3 for details.

For the proof one represents the functions $L(f,\chi,s)$ (similar as in the case $g=1$) as finite linear combinations of integrals of certain differential forms attached to f along certain $\frac{g(g+1)}{2}$-dimensional real subcycles of $\Gamma_g\backslash\mathcal{H}_g$. Our assertion then can be deduced if we use results of Hatada given in [**2**, **3**]. More precisely, in [**2**] it is shown that the space of cusp forms of weight $k \ge g+1$ w.r.t. a torsion-free congruence subgroup $\Gamma \subset \Gamma_g$ is canonically isomorphic to the space of holomorphic differential forms of highest degree on a compactification of $\Gamma \propto \mathbf{Z}^{2gw}\backslash\mathcal{H}_g \times \mathbf{C}^{gw}$, and in [**3**] using [**2**] a certain finiteness statement for a certain family of integrals of Siegel cusp forms is derived. (Actually, as we think, some of the assertions of [**3**] have to be slightly modified, for complete correctness' purposes; cf. §3.)

Inspecting the proof of Thm. 2, it is quite suggestive or even more or less clear that a similar finiteness statement as given there can be proved for special values of Dirichlet series of a much more general type. In fact, such a result essentially seems to be true for finite linear combinations of all the partial series

$$\sum_{\{T>0\}/GL_g^{(S)}(\mathbf{Z})} \frac{e^{2\pi i\, \mathrm{tr}\,(TS)}a(T)}{\epsilon^{(S)}(T)(\det T)^s} \qquad (\mathrm{Re}\,(s) \gg 0),$$

where S is any rational symmetric matrix of size g, $GL_g^{(S)}(\mathbf{Z})$ is the subgroup $\{U \in GL_g(\mathbf{Z}) \,|\, S[U^t] \equiv S \pmod{\mathbf{Z}}\}$ and $\epsilon^{(S)}(T) := \#\{U \in GL_g^{(S)}(\mathbf{Z}) \,|\, T[U] = T\}$. However, we do not want to pursue this point further.

We finally remark that in [4] the Koecher-Maass series of a Siegel-Eisenstein series of genus g is explicitly expressed in terms of "elementary" zeta functions. In particular, if g is odd it is shown to be a sum of products of Riemann zeta functions. It would be interesting to investigate if a similar statement as given in Thm. 2 would also hold in this case. In fact, it is suggestive that such an assertion can be derived directly from the explicit formulas given in [4].

One can also ask similar questions in the case of a Klingen-Siegel-Eisenstein series.

Notations. If A and B are complex matrices of appropriate sizes, we put $A[B] := B^tAB$. We simply write $E = E_g$ resp. $0 = 0_g$ for the unit resp. zero matrix of size g if there is no confusion.

We often write elements of the group $GSp_g^+(\mathbf{R}) \subset GL_{2g}(\mathbf{R})$ consisting of real symplectic similitudes of size $2g$ with positive scale in the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, understanding that A, B, C and D are real (g, g)-matrices.

If $Y \in \mathbf{R}^{(g,g)}$, we write $Y > 0$ if Y is symmetric and positive definite. The group $GL_g(\mathbf{R})$ operates on $\mathcal{P}_g := \{Y \in \mathbf{R}^{(g,g)} \,|\, Y > 0\}$ in the usual way from the right by $Y \mapsto Y[U]$.

If $f(Z)$ is a complex-valued function on $\mathcal{H}_g$, k a positive integer and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_g^+(\mathbf{R})$, we set

$$(f|_k\gamma)(Z) := \det(CZ + D)^{-k} f((AZ + B)(CZ + D)^{-1}) \qquad (Z \in \mathcal{H}_g).$$

We often write $f|\gamma$ instead of $f|_k\gamma$ if there is no misunderstanding.

If k is a positive integer, Γ is a subgroup of Γ_g and χ is a character of Γ of finite order, we denote by $S_k(\Gamma, \chi)$ the space of Siegel cusp forms of weight k and character χ w.r.t. Γ. If $\chi = 1$ we simply write $S_k(\Gamma)$.

2. Character twists of Koecher-Maass series.

For N a natural number we define

$$\Gamma^*_{g,0}(N^2) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,|\, C \equiv 0 \pmod{N^2}, \right.$$
$$\left. D \equiv \lambda E \pmod{N} \text{ for some } \lambda \in \mathbf{Z} \right\}$$

(note that λ must necessarily satisfy $(\lambda, N) = 1$).

It is easy to see that $\Gamma^*_{g,0}(N^2)$ is a subgroup of Γ_g. If χ is a Dirichlet character modulo N, we extend χ to a character of $\Gamma^*_{g,0}(N^2)$ by putting $\chi(\gamma) := \chi(\lambda)$ if $\gamma \equiv \begin{pmatrix} * & * \\ 0 & \lambda E \end{pmatrix} \pmod{N}$.

Lemma 1. *Let $f \in S_k(\Gamma_g)$ with Fourier coefficients $a(T)$ ($T > 0$ half-integral). Let χ be a primitive Dirichlet character modulo N. Then the function*

$$f_\chi(Z) := \sum_{T>0} \chi(\operatorname{tr} T) a(T) e^{2\pi i \operatorname{tr}(TZ)} \qquad (Z \in \mathcal{H}_g)$$

belongs to $S_k(\Gamma^*_{g,0}(N^2), \chi^2)$.

Proof. Let

$$g(\overline{\chi}) := \sum_{\nu \pmod{N}} \overline{\chi}(\nu) e^{2\pi i \nu/N}$$

be the Gauss sum attached to $\overline{\chi}$. Since

$$\sum_{\nu \pmod{N}} \overline{\chi}(\nu) e^{2\pi i \operatorname{tr}(T) \frac{\nu}{N}} = \chi(\operatorname{tr} T) g(\overline{\chi}),$$

we obtain

$$f_\chi = \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod{N}} \overline{\chi}(\nu)\, f|\alpha_\nu, \tag{2}$$

where

$$\alpha_\nu := \begin{pmatrix} E & \frac{\nu}{N} E \\ 0 & E \end{pmatrix} \qquad (\nu \in \mathbf{Z}).$$

Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^*_{g,0}(N^2)$ and put

$$A' := A + \frac{\nu}{N} C,$$

$$B' := B + \frac{\nu}{N}(E - AD^t)D - \frac{\nu^2}{N^2} CD^t D,$$
$$D' := D - \frac{\nu}{N} CD^t D.$$

Then A', B' and D' are integral matrices, one has $D' \equiv D \pmod N$ and

$$\alpha_\nu \gamma = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} E & \frac{\nu}{N} D^t D \\ 0 & E \end{pmatrix};$$

in particular $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma^*_{g,0}(N^2)$, and it follows that

$$\begin{aligned} f_\chi|\gamma &= \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod N} \overline{\chi}(\nu)\, f| \begin{pmatrix} E & \frac{\nu}{N} D^t D \\ 0 & E \end{pmatrix} \\ &= \chi(\lambda^2) \cdot \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod N} \overline{\chi}(\nu)\, f|\alpha_\nu \qquad (D \equiv \lambda E \pmod N) \\ &= \chi^2(\gamma) f. \end{aligned}$$

This proves the claim.

Lemma 2. *Let the notations be as in Lemma 1 and put*

$$W_{N^2} := \begin{pmatrix} 0 & -E \\ N^2 E & 0 \end{pmatrix}.$$

Then

$$f_\chi | W_{N^2} = g(\chi)^2 N^{-gk-1} f_{\overline{\chi}}.$$

Proof. For $(\nu, N) = 1$ determine $\lambda, \mu \in \mathbf{Z}$ with $\lambda N - \mu\nu = 1$. Then

$$\alpha_\nu W_{N^2} = N \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \begin{pmatrix} NE & -\mu E \\ -\nu E & \lambda E \end{pmatrix} \alpha_\mu.$$

Hence

$$\begin{aligned} g(\overline{\chi}) \cdot f_\chi | W_{N^2} &= N^{-gk} \sum_{\nu \pmod N, (\nu,N)=1} \overline{\chi}(\nu) f|\alpha_\mu \\ &= \chi(-1) N^{-gk} \sum_{\mu \pmod N, (\mu,N)=1} \chi(\mu) f|\alpha_\mu \\ &= \chi(-1) g(\chi) N^{-gk} f_{\overline{\chi}}. \end{aligned}$$

Since $g(\chi)g(\overline{\chi}) = \chi(-1)N$, we obtain our claim.

Theorem 1. *Let k be even and let $f \in S_k(\Gamma_g)$. Let χ be a primitive Dirichlet character modulo N and define $L(f, \chi, s)$ $(\mathrm{Re}\,(s) \gg 0)$ by (1). Let*

$$\gamma_g(s) := (2\pi)^{-gs} \prod_{\nu=1}^{g} \pi^{(\nu-1)/2} \Gamma\left(s - \frac{\nu - 1}{2}\right) \qquad (s \in \mathbf{C})$$

and set

$$L^*(f, \chi, s) := N^{gs} \gamma_g(s) L(f, \chi, s) \qquad (\mathrm{Re}\,(s) \gg 0).$$

Then $L^*(f,\chi,s)$ extends to a holomorphic function on $\mathbf{C}$, and the functional equation

$$L^*(f,\chi,k-s) = (-1)^{\frac{gk}{2}} g(\chi)^2 \frac{1}{N} L^*(f,\overline{\chi},s)$$

holds, where $g(\chi)$ is the Gauss sum attached to χ.

Proof. Since

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \,|\, U \in GL_{g,N}(\mathbf{Z}) \right\} \subset \Gamma^*_{g,0}(N^2)$$

and k is even, the function $f_\chi(iY)$ $(Y>0)$ is invariant under $Y \mapsto Y[U]$ $(U \in GL_{g,N}(\mathbf{Z}))$. Hence it follows in the usual way that

$$(3) \qquad L^*(f,\chi,s) = \frac{1}{2} N^{gs} \int_{\mathcal{F}_{g,N}} f_\chi(iY)(\det\, Y)^s dv \qquad (\mathrm{Re}\,(s) \gg 0),$$

where $\mathcal{F}_{g,N}$ is any fundamental domain for the action of $GL_{g,N}(\mathbf{Z})$ on $\mathcal{P}_g$ and $dv = (\det Y)^{-(g+1)/2} dY$ is the $GL_g(\mathbf{R})$-invariant volume element on $\mathcal{P}_g$.

We fix a set of representatives $U_1, \ldots, U_r$ for $GL_g(\mathbf{Z})/GL_{g,N}(\mathbf{Z})$ and now take

$$(4) \qquad \mathcal{F}_{g,N} = \bigcup_{\nu=1}^{r} \mathcal{R}_g[U_\nu],$$

where $\mathcal{R}_g$ is Minkowski's fundamental domain for the action of $GL_g(\mathbf{Z})$.

Since $GL_{g,N}(\mathbf{Z})$ is closed under transposition, also $\mathcal{F}_{g,N}^{-1}$ is a fundamental domain for $GL_{g,N}(\mathbf{Z})$.

We let

$$\mathcal{P}_{g,+} := \{Y \in \mathcal{P}_g \,|\, \det Y \geq N^{-g}\}, \quad \mathcal{P}_{g,-} := \{Y \in \mathcal{P}_g \,|\, \det Y \leq N^{-g}\},$$

write

$$\mathcal{F}_{g,N} = \big(\mathcal{F}_{g,N} \cap \mathcal{P}_{g,+}\big) \cup \big(\mathcal{F}_{g,N} \cap \mathcal{P}_{g,-}\big)$$

and observe that $\mathcal{F}_{g,N} \cap \mathcal{P}_{g,-}$ under the map $Y \mapsto (N^2Y)^{-1}$ is transformed bijectively onto $\mathcal{F}_{g,N}^{-1} \cap \mathcal{P}_{g,+}$. We also observe that both $\mathcal{F}_{g,N} \cap \mathcal{P}_{g,+}$ and $\mathcal{F}_{g,N}^{-1} \cap \mathcal{P}_{g,+}$ are fundamental domains for the induced action of $GL_{g,N}(\mathbf{Z})$ on $\mathcal{P}_{g,+}$, the integral in (3) is absolutely convergent and the integrand is invariant under $GL_{g,N}(\mathbf{Z})$.

Therefore, since by Lemma 2

$$f_\chi(i(N^2Y)^{-1}) = (-1)^{\frac{gk}{2}} g(\chi)^2 N^{gk-1} (\det Y)^k f_{\overline{\chi}}(iY),$$

we conclude that

$$L^*(f,\chi,s) = \frac{1}{2}\int_{\mathcal{F}_{g,N}\cap\mathcal{P}_{g,+}} \Big(f_\chi(iY)(N^g\det Y)^s + (-1)^{\frac{gk}{2}} g(\chi)^2 N^{-1} f_{\overline{\chi}}(iY)(N^g\det Y)^{k-s}\Big)dv. \tag{5}$$

Standard arguments and estimates taking into account (4) and properties of $\mathcal{R}_g$ (compare e.g., [**5**, Chap. VI]) now show that the integral on the right of (5) is (absolutely) convergent for all $s \in \mathbf{C}$ and represents a holomorphic function of s.

Since

$$g(\chi)g(\overline{\chi}) = \chi(-1)N,$$

we also easily see the claimed functional equation. This concludes the proof of the Theorem.

3. Special values.

In this section we shall prove:

Theorem 2. *Let k be even, $k \geq g+1$ and let $f \in S_k(\Gamma_g)$. If χ is a primitive Dirichlet character modulo N, define $L(f,\chi,s)$ $(s \in \mathbf{C})$ by holomorphic continuation of the series (1) (Theorem 1). Let $g(\overline{\chi})$ be the Gauss sum attached to $\overline{\chi}$ and let $\mathbf{Z}[\overline{\chi}]$ be the $\mathbf{Z}$-module obtained from $\mathbf{Z}$ by adjoining the values of $\overline{\chi}$.*

Then there exists a $\mathbf{Z}$-module $M_f \subset \mathbf{C}$ depending only on f of finite rank such that all the special values

$$i^{gs+\frac{g(g+1)}{2}}\pi^{\frac{g(g-1)}{4}+[\frac{g}{2}]}(2\pi)^{-gs}g(\overline{\chi})L(f,\chi,s)$$

where $s \in \mathbf{N}$, $\frac{g+1}{2} \leq s \leq k - \frac{g+1}{2}$ and χ runs over all primitive Dirichlet characters modulo all positive integers N, are contained in $M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\overline{\chi}]$.

Proof. From (2) and (3) and the proof of Theorem 1 we find that

$$g(\overline{\chi})\gamma_g(s)L(f,\chi,s) = \frac{1}{2}\sum_{\nu \pmod N}\overline{\chi}(\nu)\int_{\mathcal{F}_{g,N}} f\left(iY + \frac{\nu}{N}E\right)(\det Y)^{s-\frac{g+1}{2}}dY \tag{6}$$

for all $s \in \mathbf{C}$.

Note that the individual integrands on the right of (6) are $GL_{g,N}(\mathbf{Z})$-invariant since $f(Z)$ is invariant under $\left\{\begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1}\end{pmatrix} \mid U \in GL_{g,N}(\mathbf{Z})\right\}$ and under translations. Let $w \in \mathbf{Z}$, $w \geq 0$ and $Sp_g(\mathbf{R}) \propto \mathbf{R}^{2gw}$ be the semi-direct product of $Sp_g(\mathbf{R})$ and $\mathbf{R}^{2gw} \cong (\mathbf{R}^{2g})^w$ with multiplication given by

$$(\gamma,\lambda)(\gamma',\lambda') = (\gamma\gamma', \lambda\gamma'^{\uparrow} + \lambda')$$

where by $\gamma \mapsto \gamma^{\uparrow}$ we denote the diagonal embedding of $Sp_g(\mathbf{R})$ into $GL_{2gw}(\mathbf{R})$.

The group $Sp_g(\mathbf{R}) \propto \mathbf{R}^{2gw}$ acts on $\mathcal{H}_g \times \mathbf{C}^{gw}$ (with $\mathbf{C}^{gw} \cong (\mathbf{C}^g)^w$) from the left by

$$\begin{aligned}(\gamma, \lambda) \circ \big(Z, (\zeta_1, \dots, \zeta_w)\big) \\ = \Bigg((AZ+B)(CZ+D)^{-1}, \Bigg(\zeta_1 + (\mu_1, \nu_1)\begin{pmatrix} Z \\ E_g \end{pmatrix}(CZ+D)^{-1}, \\ \dots, \zeta_w + (\mu_w, \nu_w)\begin{pmatrix} Z \\ E_g \end{pmatrix}(CZ+D)^{-1}\Bigg)\Bigg)\end{aligned}$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\lambda = \big((\mu_1, \nu_1), \dots, (\mu_w, \nu_w)\big)$ with $\mu_j, \nu_j \in \mathbf{R}^g$ for all j. The discrete subgroup $\Gamma_g \propto \mathbf{Z}^{2gw}$ acts properly discontinuously.

Let $\Gamma \subset \Gamma_g$ be any congruence subgroup acting without fixed points on $\mathcal{H}_g$ (e.g., the principal congruence subgroup $\Gamma_g(\ell)$ with $\ell \geq 3$) and view f as an element of $S_k(\Gamma)$.

Put $w := k - (g+1)$. It was shown in [2] that the map

$$h(Z) \mapsto h(Z) dZ d\zeta$$

gives an isomorphism between $S_k(\Gamma)$ and the space of holomorphic differential forms of degree $\frac{g(g+1)}{2} + gw$ of (any) non-singular compactification of the quotient space $\Gamma \propto \mathbf{Z}^{2gw} \backslash \mathcal{H}_g \times \mathbf{C}^{gw}$.

Using toroidal compactifications, in [3] from this a certain finiteness statement for certain cycle integrals attached to h was derived which we now want to describe in the special case we need.

Let S be a given rational symmetric matrix of size g and let n be an integer with $0 \leq n \leq w$. Define

$$\begin{aligned}T_g(S; n) := \bigcup_{Y \in \mathcal{P}_g} \{S + iY\} \\ \times \Big((\mathbf{R}^g)^{w-n} \times \{(\mu_1 iY, \dots, \mu_n iY) \,|\, \mu_1, \dots, \mu_n \in \mathbf{R}^g\}\Big) \\ \subset \mathcal{H}_g \times \mathbf{C}^{gw}.\end{aligned}$$

Then $T_g(S; n)$ is a real submanifold of $\mathcal{H}_g \times \mathbf{C}^{gw}$ of dimension $\frac{g(g+1)}{2} + gw$.

(In the notation of [3, §6] we have taken $a_1 = a_2 = \dots = a_{w-n} \in \{g+1, \dots, 2g\}$ and $a_{w-n+1} = \dots a_w \in \{1, \dots, g\}$. Also note that in the definition of $T_g(a_1, \dots, a_w; X)$ in [3, p. 401] we have replaced the "Z" in $W(a_1, \dots, a_w)[Z]$ by "iY". We think that this is the correct definition, since otherwise the corresponding integrals in [3, Lemma 6.2 and Thm. 5] in general would not be convergent.)

Put

$$\mathcal{U}_g := \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in GL_g(\mathbf{R}) \right\} \subset Sp_g(\mathbf{R}),$$

$$V_{g,n} := \{(\lambda_1, \dots, \lambda_{w-n}, (\mu_1, 0), \dots, (\mu_n, 0)) \mid \lambda_1, \dots, \\ \lambda_{w-n} \in \mathbf{R}^{2g}, \ \mu_1, \dots, \mu_n \in \mathbf{R}^g\}$$

and

$$H_{g,n} := \mathcal{U}_g \propto V_{g,n} \subset Sp_g(\mathbf{R}) \propto \mathbf{R}^{2gw}.$$

Let

$$\alpha^{(S)} := \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}.$$

Then one easily checks that the conjugate subgroup

$$H_{g,n}^{(S)} := (\alpha^{(S)}, 0) \cdot H_{g,n} \cdot (\alpha^{(S)}, 0)^{-1}$$

leaves $T_g(S; n)$ stable.

Note that $H_{g,n}^{(S)}$ consists of all pairs

$$\left(\begin{pmatrix} U & S(U^t)^{-1} - US \\ 0 & (U^t)^{-1} \end{pmatrix}, (\lambda_1, \dots, \lambda_{w-n}, (\mu_1, -\mu_1 S), \dots, (\mu_n, -\mu_n S)) \right)$$

with $\lambda_1, \dots, \lambda_{w-n} \in \mathbf{R}^{2g}$ and $\mu_1, \dots, \mu_n \in \mathbf{R}^g$.

Let

$$H_{g,n,\Gamma}^{(S)} := H_{g,n}^{(S)} \cap \Gamma \propto \mathbf{Z}^{2gw}.$$

Write $M := \Gamma \propto \mathbf{Z}^{2gw} \backslash \mathcal{H}_g \times \mathbf{C}^{gw}$ and denote by $\overline{M}$ a fixed toroidal compactification of M. Let $\partial M = \overline{M} \setminus M$. Then according to [**3**, Lemma 6.1] the closure of the image of $H_{g,n,\Gamma}^{(S)} \backslash T_g(S; n)$ in $\overline{M}$ w.r.t. the usual complex topology is the support of a singular relative $\frac{g(g+1)}{2} + gw$-cycle with integral coefficients w.r.t. $(\overline{M}, \partial M)$.

Since $H_{\frac{g(g+1)}{2}+gw}(\overline{M}, \partial M, \mathbf{Z})$ is of finite rank, one concludes that for any given $h \in S_k(\Gamma)$ all the numbers

$$\int_{H_{g,n,\Gamma}^{(S)} \backslash T_g(S;n)} h(Z) dZ d\zeta \qquad (S \in \mathbf{Q}^{(g,g)}, \ S = S^t)$$

are contained in a finite $\mathbf{Z}$-module (depending only on h) whose rank is bounded by the rank of the above cohomology group ([**3**, Thm. 5], compare our above remark).

On the other hand (compare [**3**, Lemma 6.2]) one has the equality

$$\begin{aligned}(7)\qquad &\int_{H^{(S)}_{g,n,\Gamma}\backslash T_g(S;n)} h(Z)dZd\zeta \\ &= \int_{\alpha^{(S)}\cdot\mathcal{U}_g\cdot(\alpha^{(S)})^{-1}\cap\Gamma\backslash\{S+iY\,|\,Y\in\mathcal{P}_g\}} h(Z)\det{(Z-S)^n}dZ.\end{aligned}$$

In particular, now take $\Gamma=\Gamma_g(\ell)$ with some fixed $\ell\geq 3$. Then the integral on the right of (7) is equal to

$$i^{gn+\frac{g(g+1)}{2}}\int_{\mathcal{P}_g/GL^{(S)}_{g,\ell}(\mathbf{Z})} h(S+iY)\,(\det Y)^n dY,$$

where

$$GL^{(S)}_{g,\ell}(\mathbf{Z}):=\{U\in GL_{g,\ell}(\mathbf{Z})\,|\,S[U^t]\equiv S \pmod{\ell\mathbf{Z}}\}.$$

Let $S=\frac{\nu}{N}E$ with $\nu\in\mathbf{Z}$ (so $\alpha^{(S)}=\alpha_\nu$ in the notation of §2). Then we see that $GL_{g,\ell N}(\mathbf{Z})$ is contained in $GL^{(S)}_{g,\ell}(\mathbf{Z})$. Since the index of $GL_{g,\ell N}(\mathbf{Z})$ in $GL_{g,N}(\mathbf{Z})$ is bounded by a number depending only on ℓ, the assertion of Thm. 2 now follows taking into account (6) and the fact that $\Gamma(\frac{1}{2}+\nu)\in\mathbf{Q}\sqrt{\pi}$ for $\nu=0,1,2\ldots.$

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