Pacific Journal of Mathematics

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George A. Elliott and Dan Kucerovsky

Volume 198 No. 2 April 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 198, No. 2, 2001

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A common generalization is given of what are often referred to as the Weyl–von Neumann theorems of Voiculescu, Kasparov, Kirchberg, and, more recently, Lin. (These in turn extend a result of Brown, Douglas, and Fillmore.)

More precisely, an intrinsic characterization is obtained of those extensions of one separable C*-algebra by another—the first, i.e., the ideal, assumed to be stable, so that Brown-Douglas-Fillmore addition of extensions can be carried out—which are absorbing in a certain natural sense related to this addition, a sense which reduces to that considered by earlier authors if either the ideal or the quotient is nuclear. The specific absorption theorems referred to above can be deduced from this characterization.

1. Let B be a C*-algebra, and let C be a C*-algebra containing B as a closed two-sided ideal. Let us say that C is purely large with respect to B if for every element c of C which is not in B, the C*-algebra $\overline{cBc^*}$ (the intersection with B of the hereditary sub-C*-algebra of C generated by cc^*) contains a sub-C*-algebra which is stable (i.e., isomorphic to its tensor product with the C*-algebra $\mathcal{K}$ of compact operators on an infinite-dimensional separable Hilbert space) and is full in B (i.e., not contained in any proper closed two-sided ideal of B).

2. Let A and B be C*-algebras, and let

$$0 \to B \to C \to A \to 0$$

be an extension of B by A (i.e., a short exact sequence of C*-algebras). Let us say that the extension is purely large if the C*-algebra of the extension, C, is purely large with respect to the image of B in it, in the sense described above.

Note that, if B is non-zero, a purely large extension of B by A is essential (that is, the image of B in the C*-algebra of the extension is an essential closed two-sided ideal—every non-zero closed two-sided ideal has non-zero intersection with it).

3. Let A and B be C*-algebras, with A unital. An extension $0 \to B \to C \to A \to 0$ will be said to be unital if C is unital.

In this paper we shall consider primarily the context of unital extensions (although we shall indicate how to modify our main result, Theorem 6, to be valid in the non-unital setting).

4. Recall that an extension of B by A is determined by its Busby map—the naturally associated map from A to the quotient multiplier algebra, or corona, of B, $\mathrm{M}(B)/B$. (The C*-algebra of the extension is the pullback of the Busby map and the canonical quotient map $\mathrm{M}(B) \to \mathrm{M}(B)/B$.)

Recall (see e.g., [6]) that, if B is stable, so that the Cuntz algebra $\mathcal{O}_2$ may be embedded unitally in $\mathrm{M}(B)$, then the Brown-Douglas-Fillmore addition of extensions, defined by

$$\tau_1 \oplus \tau_2 \ := \ s_1 \tau_1 s_1^* + s_2 \tau_2 s_2^*,$$

where τ_1 and τ_2 are (the Busby maps of) two extensions of B by A, and s_1 and s_2 are (the images in $\mathrm{M}(B)/B$ of) the canonical generators of $\mathcal{O}_2$ (which are isometries with range projections summing to 1), is compatible with Brown-Douglas-Fillmore equivalence (defined as unitary equivalence with respect to the unitary group of $\mathrm{M}(B)$—or, rather, the image of this group in $\mathrm{M}(B)/B$), and the resulting binary operation on equivalence classes is independent of the embedding of $\mathcal{O}_2$.

With respect to this operation, the equivalence classes of extensions of the stable C*-algebra B by the C*-algebra A form an abelian semigroup.

Recall that an extension of B by A is said to be trivial if, considered as a short exact sequence of C*-algebra maps, it splits. In other words, the map $C \to A$ in the sequence $0 \to B \to C \to A \to 0$ should have a left inverse, $C \leftarrow A$. (Equivalently, the Busby map $A \to \mathrm{M}(B)/B$ should lift to a C*-algebra homomorphism $A \to \mathrm{M}(B)$.)

In the setting of unital extensions, we shall understand triviality of an extension to mean that the splitting can be chosen to be unital.

Recall, furthermore, that, in [8], Kasparov called an extension absorbing if, in the Brown-Douglas-Fillmore semigroup, it is equal to its sum with any trivial extension. (Briefly, if it absorbs every trivial extension.) Of course, a unital extension cannot be absorbing in this sense (unless the quotient algebra is zero); let us say that a unital extension is absorbing if—in the subsemigroup of unital extensions—it is equal to its sum with any trivial unital extension. (Trivial in the sense of admitting a unital splitting.)

5. In order to be able to formulate our main result (Theorem 6, below) for arbitrary (separable) C*-algebras A and B (with B stable and A unital)—i.e., without assuming A or B to be nuclear—we must restrict the notion of trivial extension as follows.

Let us say that an extension of C*-algebras $0 \to B \to C \to A \to 0$ is trivial in the nuclear sense if the splitting homomorphism $A \to C$ may be chosen to be weakly nuclear as defined by Kirchberg in [**9**]: The splitting homomorphism $\pi\colon\ A \to C$ will be said to be weakly nuclear if, for every $b \in B \subseteq C$, the map

$$A \ni a \ \mapsto\ b\pi(a)b^* \in B \subseteq C$$

is nuclear. (Recall that a C*-algebra map is said to be nuclear if it factors approximately through finite-dimensional C*-algebras, by means of completely positive contractions, in the sense of convergence in norm.)

Let us say, correspondingly, that an extension is absorbing in the nuclear sense if it absorbs every extension which is trivial in the nuclear sense. Again, let us say that a unital extension is absorbing in the nuclear sense to mean that this holds within the semigroup of (equivalence classes of) unital extensions. (With triviality in the nuclear sense the existence of a unital weakly nuclear splitting.)

6.

Theorem. *Let A and B be separable* C*-*algebras, with B stable and A unital. A unital extension of B by A is absorbing, in the nuclear sense, if, and only if, it is purely large.*

7. Purely large algebras have an approximation property similar to that of purely infinite algebras. (This is the fundamental ingredient in the proof of our main result, that an extension that is purely large is absorbing—either in the unital setting, as in Theorem 6, or, if the extension is non-unital, as in Corollary 16.)

Lemma. *Let C be a C^*-algebra that is purely large with respect to a closed two-sided ideal B, in the sense of Section 1. Then, for any positive element c of C which is not in B, any $\epsilon > 0$, and any positive element b of B, there exists $b_0 \in B$ with*

$$\|b - b_0 c b_0^*\| < \epsilon.$$

If b is of norm one, and if the image of c in C/B is of norm one, then b_0 may be chosen to have norm one.

Proof. Let $c \in C^+ \setminus B$, $b \in B^+$, and $\epsilon > 0$ be given. Multiplying c by a positive element of the sub-C*-algebra it generates, and changing notation, we may suppose that the hereditary sub-C*-algebra C_c of C on which c acts as a unit is not contained in B.

By hypothesis, there exists a full, stable sub-C*-algebra D of B contained in C_c.

Since D is full in B, there exist $d \in D^+$ and $b_1, \dots, b_n$ in B such that

$$\left\| b - \sum b_i d b_i^* \right\| < \epsilon.$$

(The set S of such elements $\sum b_i d b_i^*$ is closed under the map $x \mapsto yxy^*$ for any $y \in B$, and therefore the closure of S is a hereditary subset of B^+—recall that if $0 \leq s \leq t$ in B then $s^{\frac{1}{2}} = \lim y_n t^{\frac{1}{2}}$ with $y_n = s^{\frac{1}{2}}(t + \frac{1}{n})^{-\frac{1}{2}} \in B$, so that $s = \lim y_n t y_n^*$. In particular, the closure of S is a subcone of B^+, as the sum of two elements of S, associated with d_1 and d_2, say, is majorized by an element associated with the single element $d_1 + d_2$, and therefore is a limit of elements associated with $d_1 + d_2$. The closure of S is thus a closed subcone of B^+ closed under the map $x \mapsto yxy^*$ for any $y \in B$. Such a subset is known to be the positive part of a closed two-sided ideal: as above it must be a hereditary subset, and it is then the positive part of an ideal by Theorem 2.7(ii) of [4].)

(Alternatively, to obtain the assertion of the preceding paragraph, approximate $b^{\frac{1}{2}}$ by $\sum b_{1i} d_0 b_{2i}^*$ for some $d_0 \in D^+$ and $b_{1i}, b_{2i} \in B$. Replacing $\sum b_{1i} d_0 b_{2i}^*$ by its self-adjoint part (which has a similar form), and changing notation, we may suppose that $\sum b_{1i} d_0 b_{2i}^*$ is self-adjoint. (In any case, this element is almost self-adjoint, which is sufficient.) Write

$$\sum b_{1i} d_0 b_{2i}^* \;=\; b_1(d_0)b_2^*$$

where b_1 and b_2 denote the row vectors (b_{1i}) and (b_{2i}), and (d_0) denotes the square matrix of appropriate size with d_0 repeated down the diagonal and 0 elsewhere. Note that $b_1(d_0)b_2^* \;=\; b_2(d_0)b_1^*$. We then have

$$\begin{aligned}
&\|b - b_1(d_0)b_2^* b_2(d_0)b_1^*\| \\
&= \|b - b^{\frac{1}{2}} b_1(d_0)b_2^* + b^{\frac{1}{2}} b_2(d_0)b_1^* - b_1(d_0)b_2^* b_2(d_0)b_1^*\| \\
&\leq \|b^{\frac{1}{2}}\| \, \|b^{\frac{1}{2}} - b_1(d_0)b_2^*\| + \|b^{\frac{1}{2}} - b_1(d_0)b_2^*\| \, \|b_2(d_0)b_1^*\|,
\end{aligned}$$

and the right side is arbitrarily small. Finally, noting that $(d_0)b_2^* b_2(d_0)$ belongs to the hereditary sub-C*-algebra generated by $(d_0^{\frac{1}{2}})$, we may approximate this element by the element

$$((d_0^{\frac{1}{2}})c(d_0^{\frac{1}{2}}))((d_0^{\frac{1}{2}})c(d_0^{\frac{1}{2}}))^* \;=\; (d_0^{\frac{1}{2}})c(d_0)c^*(d_0^{\frac{1}{2}})$$

for some matrix c over B, and then with $b_1' = b_1(d_0^{\frac{1}{2}})c$, the element

$$b - b_1'(d_0)b_1'^*$$

is small, i.e., $b - \sum b_{1i}' d_0 b_{1i}'^*$ is small, as desired.)

Since D is stable, we may suppose, changing d by a small amount, that there exists a multiplier projection e of D such that $ed = d$ and such that for multipliers $u_1, \ldots, u_n$,

$$u_i u_j^* \;=\; \delta_{ij} e.$$

Hence with $d_i = d^{\frac{1}{2}} u_i$,

$$d_i d_j^* \;=\; \delta_{ij} d.$$

Set

$$\sum b_i d_i \;=\; b_0.$$

Then, on the one hand,

$$b_0 b_0^* \;=\; \sum b_i d_i d_j^* b_j^* \;=\; \sum b_i d b_i^*,$$

and, on the other hand, as $d_i c = d_i$ and so $b_0 c = b_0$,

$$b_0 b_0^* \;=\; b_0 c b_0^*.$$

We now have

$$\|b - b_0 c b_0^*\| \;=\; \|b - b_0 b_0^*\| \;=\; \left\| b - \sum b_i d b_i^* \right\| < \epsilon.$$

Now suppose that

$$\|b\| \;=\; \|c + B\| \;=\; 1,$$

and let us show that b_0 may be chosen with norm one. The modification of c in the above construction may then be arbitrarily small, and so, as b_0 will be chosen with norm one (see below), we may again suppose that the hereditary sub-C*-algebra C_c of C on which c acts as a unit is not contained in B. Repeating the construction above with $\epsilon/2$ in place of ϵ, we have $\|b - b_0 b_0^*\| \le \epsilon/2$, and so (as $\|b\| = 1$),

$$1 - \frac{\epsilon}{2} \le \|b\| - \|b - b_0 b_0^*\| \le \|b_0 b_0^*\| \;\le\; \|b\| + \|b_0 b_0^* - b\| \le 1 + \frac{\epsilon}{2}.$$

Hence,

$$\left\| b_0 b_0^* \left(1 - \frac{1}{\|b_0 b_0^*\|}\right) \right\| \;=\; \big|\|b_0 b_0^*\| - 1\big| \le \frac{\epsilon}{2},$$

and so

$$\left\| b - \frac{b_0}{\|b_0\|} c \frac{b_0^*}{\|b_0\|} \right\| \;=\; \left\| b - \frac{b_0 b_0^*}{\|b_0 b_0^*\|} \right\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

8. Let us recall the generalization of Glimm's Lemma due to Akemann, Anderson, and Pedersen (Proposition 2.2 of [1]). Because we shall only need the unital case, and that case is much easier, let us give a proof in that case.

Lemma. *Let C be a separable unital C^*-algebra and let ρ be a pure state of C. There exists $c_0 \in C^+$ with $\|c_0\| = 1$ such that $\rho(c_0) = 1$ and*

$$\lim_{n \to \infty} \|c_0^n (c - \rho(c)) c_0^n\| \;=\; 0, \quad c \in C.$$

Proof. With $N_\rho = \{c \in C;\ \rho(c^* c) = 0\}$, recall that, as ρ is pure, $\mathrm{Ker}\rho = N_\rho + N_\rho^*$. Hence,

$$C \;=\; \mathbb{C}1 + N_\rho + N_\rho^*.$$

Choose a strictly positive element h of $N_\rho \cap N_\rho^*$ of norm at most one, and set $1 - h = c_0$. The desired convergence—which is additive—holds obviously for $c \in \mathbb{C}$, and it holds for $c \in N_\rho$ or N_ρ^* because $h c_0^n = h(1-h)^n \to 0$ (as $t(1-t)^n \to 0$ uniformly for $t \in [0,1]$).

9. The following consequence of Lemmas 7 and 8 is the main step in the proof of Theorem 6.

Corollary. *Let C be a separable unital C^*-algebra that is purely large with respect to the closed two-sided ideal B. Let ρ be a pure state of C that is zero on B, let $c_1 = (c_{11}, \ldots, c_{1n})$ be a row vector over C and let $b_1 = (b_{11}, \ldots, b_{n1})$ be a row vector over B. Denote the tensor product of ρ with the identity on $\mathrm{M}_n(\mathbb{C})$,*

$$\mathrm{id} \otimes \rho\colon\ \mathrm{M}_n(C) = \mathrm{M}_n(\mathbb{C}) \otimes C \to \mathrm{M}_n(\mathbb{C}),$$

by ρ_n. The map

$$\begin{aligned} C &\to B \\ c &\mapsto b_1 \rho_n(c_1^* c c_1) b_1^* \end{aligned}$$

can be approximated on finite sets by the maps

$$c \ \mapsto\ bcb^*, \quad b \in B.$$

Proof. It is immediate to reduce to the case $n = 1$. (Considering b_1 and c_1 as elements of $\mathrm{M}_n(C)$, and C as the subalgebra of upper left corner matrices, extend ρ to a pure state of $\mathrm{M}_n(C)$—necessarily unique, and concentrated in the upper left corner—and denote this again by ρ. If $(b_{ij}) \in \mathrm{M}_n(B)$ gives an approximating map for the map

$$\begin{aligned} \mathrm{M}_n(C) &\to \mathrm{M}_n(B) \\ c &\mapsto b_1 \rho(c_1^* c c_1) b_1^*, \end{aligned}$$

which fulfils the hypotheses of the Corollary with $n = 1$ and with C and B replaced by $\mathrm{M}_n(C)$ and $\mathrm{M}_n(B)$ (note that in this case $\rho_1 = \rho$), then $b_{11} \in B$ gives an approximating map for the given map.)

Let a finite subset $F \subseteq C$ be given. By Lemma 8, there exists $c_0 \in C^+$ such that $\|c_0\| = 1$, $\rho(c_0) = 1$, and $c_0 c_1^* c c_1 c_0$ is arbitrarily close to $c_0 \rho(c_1^* c c_1) c_0$ for each $c \in F$. Namely, c_0 may be taken to be a power of the c_0 of Lemma 8; note that $0 \leq c_0 \leq 1$ and $\rho(c_0) = 1$ imply that $\rho(c_0^k) = 1$ for any k.

Since $\rho(c_0^2) = 1$ and $\rho(B) = 0$, the element c_0^2 does not belong to B and so by Lemma 7 there exists $b_0 \in B$ such that $b_0 c_0^2 b_0^*$ is arbitrarily close to any given positive element of B. In particular, approximating an approximate unit for B, we may choose $b_0 \in B$ such that $b_1(b_0 c_0^2 b_0^*)$ is arbitrarily close to b_1. Since the image of c_0 in C/B is of norm one, by Lemma 7 we may suppose that $\|b_0\| = 1$. Then, with $b = b_1 b_0 c_0 c_1^*$, for each $c \in F$, the element

$$bcb^* \ =\ b_1 b_0 (c_0 c_1^* c c_1 c_0) b_0^* b_1^*$$

is (by choice of c_0), arbitrarily close to

$$b_1 b_0 c_0 \rho(c_1^* c c_1) c_0 b_0^* b_1^* \ =\ b_1 b_0 c_0^2 b_0^* b_1^* \rho(c_1^* c c_1),$$

which in turn (by choice of b_0) is arbitrarily close to

$$b_1 b_1^* \rho(c_1^* c c_1) \;=\; b_1 \rho(c_1^* c c_1) b_1^*.$$

In other words, the desired approximation holds.

10. The following lemma, incorporating techniques of Kirchberg, brings Corollary 9 to bear in the nuclear setting.

Lemma. *Let C be a separable unital C^*-algebra and let B be a closed two-sided ideal of C. Suppose that C is purely large with respect to B. Let ψ be a completely positive map from C to B which is zero on B. If the map from the quotient C/B to B determined by ψ is nuclear, and if B is stable, then ψ can be approximated on finite sets by the maps*

$$c \;\mapsto\; b^* c b, \quad b \in B.$$

Proof. First, without assuming that C is purely large, let us show, using ideas of Kirchberg presented in [**9**], that if the map $C/B \to B$ determined by ψ is nuclear then ψ can be approximated on finite sets by sums of maps of the kind considered in Corollary 9 (each one corresponding to a row vector over C, a pure state of C zero on B, and a column vector over B).

By the nuclearity hypothesis, which implies that ψ is the limit of a sequence of products of two completely positive maps, the first from C to M_k for some k, and zero on B, and the second from M_k to B, we may suppose that ψ itself is the product of two such maps—i.e., a completely positive map $C \to \mathrm{M}_k$, zero on B, and a completely positive map $\mathrm{M}_k \to B$.

As B is stable, so that $\mathcal{O}_k$ is unitally contained in $\mathrm{M}(B)$—unless $B = 0$ in which case the assertion is vacuous—, by Lemma 1.1 of [**9**] a completely positive map $\mathrm{M}_k \to B$ is necessarily of the form $x \mapsto RxR^*$ where R is a row vector over B. (As shown in [**9**] this holds with R the transpose of the matrix $(e_1, \ldots, e_k)^*$—i.e., for $R = (e_1^*, \ldots, e_k^*)$, where

$$(e_1, \ldots, e_k) \;=\; (s_1, \ldots, s_k) G^{\frac{1}{2}}$$

with $s_1, \ldots, s_k$ the canonical generating isometries of $\mathcal{O}_k$ and G the image in $\mathrm{M}_k \otimes B$ of the positive element (e_{ij}) of $\mathrm{M}_k(\mathrm{M}_k)$ corresponding to the canonical system of matrix units for M_k.)

It remains to show—in order to verify the assertion above—that a completely positive map $C \to \mathrm{M}_k$, zero on B, can be approximated on finite sets by sums of maps of the form

$$c \mapsto \rho(F^* c F)$$

where F is a row vector over C of length k and ρ is a pure state of C zero on B. Replacing C by C/B, we see that it is enough to establish this in the case $B = 0$. In this case, we may proceed as follows (in a way somewhat similar to the proof of Lemma 1.2 of [**9**]—which concerns the special case that C is simple and not elementary). By the Krein-Milman theorem, we may suppose (since we are allowing sums) that the given completely positive map $C \to \mathrm{M}_k$ belongs to an extremal ray within the cone of all such maps. (Consider a compact base for this cone.) By Stinespring's theorem, the given map may be expressed as a representation of C on a Hilbert space, followed by cutting down to a generating subspace of dimension k—with a specified orthonormal basis identifying the operators on this subspace with the elements of M_k. By extremality of the ray containing the given map (just as in the case of a positive linear functional), this representation must be irreducible. By the Kadison transitivity theorem the specified basis then has the form $F\eta$ where η is an arbitrary nonzero vector in the space of the representation and F is a row vector over C. With ρ the pure state of C determined by η, the given completely positive map is now equal to the map $c \mapsto \rho(F^*cF)$.

This completes the proof that ψ can be approximated on finite sets by sums of maps of the kind considered in Corollary 9, say in particular, on the given finite subset S of C, by the sum

$$\psi_1 + \cdots + \psi_n$$

where each ψ_i is as in Corollary 9 (and in particular is zero on B, which of course is no longer necessarily zero). Then—as C is purely large—by Corollary 9, on a given finite family of elements of C, say S, the map ψ_1 can be approximated by the map $c \mapsto b_1cb_1^*$ for some $b_1 \in B$. By Corollary 9 again, the map ψ_2 can be approximated by the map $c \mapsto b_2cb_2^*$ for some $b_2 \in B$, not only on S but on any larger finite subset of C, and in particular on the set

$$S_2 \ := \ S \cup \{cb_1^*b_1c^*; \quad c \in S \cup S^*\}.$$

Since ψ_2 is zero on $cb_1^*b_1c^*$, $c \in S \cup S^*$, it follows that $b_2cb_1^*b_1c^*b_2^*$ is small for $c \in S \cup S^*$, i.e., the norms $\|b_2cb_1^*\| = \|b_2cb_1^*b_1c^*b_2^*\|^{\frac{1}{2}}$ and $\|b_1cb_2^*\| = \|b_2c^*b_1^*b_1cb_2^*\|^{\frac{1}{2}}$ are small for each $C \in S$. Hence, for each $c \in S$,

$$(b_1 + b_2)c(b_1 + b_2)^*$$

is close to $b_1cb_1^* + b_2cb_2^*$, and so to $(\psi_1 + \psi_2)(c)$. Proceeding in this way (as, for instance, in [**2**]), we obtain $b_1, \cdots, b_n \in B$ such that $\psi_1 + \cdots + \psi_n$—and hence ψ—is approximated (arbitrarily closely) on S by the map

$$c \ \mapsto \ (b_1 + \cdots + b_n)c(b_1 + \cdots + b_n)^*.$$

11. The following technique is basic in some form to all earlier absorption results. It was formulated more or less explicitly in special cases in [**2**], [**8**], and [**9**], and expressed in the following abstract form in a later version of the preprint [**9**].

Lemma (Kirchberg). *Let C be a unital separable C^*-algebra and let B be an essential closed two-sided ideal of C, so that we may view C as a unital subalgebra of* $\mathrm{M}(B)$*:*

$$B \subseteq C \subseteq \mathrm{M}(B); \quad 1 \in C.$$

Let $\phi : C \to \mathrm{M}(B)$ be a completely positive map which is zero on B, and suppose that, for every $b_0 \in B$, the map

$$\begin{aligned} b_0^*\phi b_0 : C &\to B \\ c &\mapsto b_0^*\phi(c)b_0 \end{aligned}$$

can be approximated (*on finite sets*) *by the maps*

$$c \mapsto b^*cb, \quad b \in B.$$

It follows that there exists $v \in \mathrm{M}(B)$ such that

$$\phi(c) - v^*cv \in B, \quad c \in C.$$

The element v may be chosen so that the map $c \mapsto v^*cv$ also approximates ϕ on a given finite subset of C.

Proof. Let us recall, for the convenience of the reader, the argument of (the extended version of) [**9**].

First, by a slight reformulation of Theorem 2 of [**2**] (and its proof), there exist positive elements $w_1, w_2, \ldots$ of B of norm one such that the series $\sum w_i x_i w_i$ converges strictly in $\mathrm{M}(B)$ for any bounded sequence (x_i) in $\mathrm{M}(B)$, and such that the sum $\sum w_i^2\phi(c)w_i^2 \in \mathrm{M}(B)$ is equal to $\phi(c)$ modulo B for every c, and approximately equal to $\phi(c)$ in $\mathrm{M}(B)$ (in norm) for each c in a given finite subset of C. The sequence $w_1, w_2, \ldots$ may be chosen furthermore such that the sequence $(\sum_1^n w_i^4)$ is an approximate unit for B, and such that w_{n+2}^4 is orthogonal to $\sum_1^n w_i^4$ for each n.

One now proceeds very much as in the proof of Lemma 10 above (which dealt with a finite sum of maps from C to B) to show that the infinite sum $w_1^2\phi w_1^2 + w_2^2\phi w_2^2 + \cdots$ of maps from C to B (convergent pointwise in the strict topology of $\mathrm{M}(B)$ to a map from C to $\mathrm{M}(B)$—equal to ϕ modulo B and equal to ϕ approximately on the given finite set), each of which is zero on B and is determined approximately by an element of B, is determined approximately on the given finite set by a strictly converging sum of elements of B, and determined by this multiplier exactly modulo B.

More explicitly, one chooses $b_1 \in B$ such that $b_1^*cb_1$ is close to $w_1\phi(c)w_1$ for c in a finite set S_1, to be specified, one then chooses $b_2 \in B$ such that $b_2^*cb_2$ is close to $w_2\phi(c)w_2$ for c in a finite set S_2, also to be specified—and depending in addition on the choice of b_1, as in the proof of Lemma 10—and one continues in this way. As we shall show, with suitable choices of the finite sets $S_1, S_2, \cdots$ and of the approximations at each stage, the series

$$b_1w_1 + b_2w_2 + \cdots$$

converges strictly in $\mathrm{M}(B)$ to an element v with the desired properties (it determines the sum of maps $w_1^2\phi w_1^2 + w_2^2\phi w_2^2 + \cdots$, and hence also the map ϕ, to within a specified approximation on a given finite set, and exactly, modulo B, on all of C).

The sets $S_1, S_2, \ldots$ should of course all contain the given finite set, say S, and their union should be dense in C. They should also all contain the unit of C ($1 \in \mathrm{M}(B)$); then for each i the element $b_i^* b_i$ ($= b_i^* 1 b_i$) is close to $w_i\phi(1)w_i$ and in particular the sequence $b_1, b_2, \ldots$ is bounded. In order for the series $b_1w_1 + b_2w_2 + \cdots$ to be strictly convergent, it would be sufficient in view of the properties of the sequence $w_1, w_2, \ldots$ and the boundedness of the sequence $b_1, b_2, \ldots$ to ensure that

$$\sum \|b_i - w_i^4 b_i\| \ < \ \infty,$$

as then convergence of the series $b_1w_1 + b_2w_2 + \cdots$ (in the strict topology) follows from convergence of the series $w_1^4b_1w_1 + w_2^4b_2w_2 + \cdots$, which holds as the sequence $w_1^3b_1, w_2^3b_2, \ldots$ is bounded.

It would also be sufficient to arrange that, instead of convergence of $\sum \|b_i - w_i^4 b_i\|$, one has convergence of the series

$$\sum \|b_i - z_i^4 b_i\|$$

where $z_1, z_2, \ldots$ is some other sequence of positive elements of B with the last property mentioned for (w_i) (namely, that $\sum_1^n z_i^4$ is an approximate unit for B, and $z_{n+2}^4 \sum_1^n z_i^4 = 0$ for each n). Indeed, this property (for both (w_i) and (z_i)) is enough for the series

$$\sum z_i x_i w_i$$

to converge strictly in $\mathrm{M}(B)$ for any bounded sequence x_i in $\mathrm{M}(B)$. While z_i^4 may be taken to be the sum of a consecutive group of elements w_j^4, it would not appear to be possible to choose $z_i = w_i$.

Let us now elaborate on the choice of the finite sets $S_1, S_2, \ldots$, and on the choice of a partition of $\mathbf{N}$ into consecutive subsets $J_1, J_2, \ldots$ such that, with

$$\sum_{j \in J_i} w_j^4 \ =: \ z_i^4,$$

the necessary approximations can be made. (Namely, for $\sum b_i w_i$ to exist and have the desired properties; note that the introduction of z_i is purely to ensure convergence of the sum.)

The finite set S_i should contain, as well as the given finite set S and the unit, $1 \in C \subseteq \mathrm{M}(B)$, the first i elements of a fixed dense sequence $(c_1, c_2, \ldots)$ in C. Let us choose

$$S_1 \ = \ S \cup \{c_1\} \cup \{1\}.$$

In order to ensure convergence of $v = \sum b_i w_i$, and negligibility of the cross terms in the product v^*cv, for $c \in S$ or, when working modulo B, for $c \in C$ (it is enough to consider $c \in \{c_1, c_2, \dots\}$), we must choose

$$S_2 \;=\; (S_1 \cup \{c_2\}) \;\cup\; \left\{\left(\sum_1^{n_1} w_k^4\right)^2\right\} \cup\; \{cb_1b_1^*c^*; \quad c \in S_1 \cup S_1^*\},$$

where b_1 is such that $b_1^*cb_1$ is close to $w_1\phi(c)w_1$ for $c \in S_1$, and n_1 is such that the difference

$$\left(\sum_1^{n_1} w_k^4\right) b_1 - b_1$$

is small; proceeding in this way, for each $i \geq 2$ we must choose

$$S_{i+1} \;=\; (S_i \cup \{c_{i+1}\}) \;\cup\; \left\{\left(\sum_1^{n_i} w_k^4\right)^2\right\} \;\cup\; \{cb_ib_i^*c^*; \quad c \in S_i \cup S_i^*\},$$

where b_i is such that $b_i^*cb_i$ is close to $w_i\phi(c)w_i$ for $c \in S_i$, and n_i is such that the difference

$$\left(\sum_1^{n_i} w_k^4\right) b_i - b_i$$

is small. By "close", and "small", we mean that the sum of all the tolerances in question should be finite, and smaller than a certain single number (small enough that the desired approximation of ϕ occurs on the set S).

Note that, as $\phi(B) = 0$, the element

$$b_{i+1}^* \left(\sum_1^{n_i} w_k^4\right)^2 b_{i+1},$$

is small, i.e., $(\sum_1^{n_i} w_k^4)b_{i+1}$ is small. As $(\sum_1^{n_{i+1}} w_k^4)b_{i+1} - b_{i+1}$ is small (by the choice of n_{i+1}), also

$$\left(\sum_{n_i+1}^{n_{i+1}} w_k^4\right) b_{i+1} - b_{i+1} \quad \text{is small.}$$

In other words, with

$$\{n_i + 1, \dots, n_{i+1}\} \;=\; J_{i+1},$$

$i = 1, 2, \dots$, and with, say, $J_1 = \{1, \dots, n_i\}$, setting $\sum_{j \in J_i} w_j^4 = z_i^4$ (with $z_i \geq 0$), we have a sequence (z_i) with the desired properties (including that $z_i^4 b_i - b_i$ is small, in the sense of being summable).

For each i, $(b_iw_i)^*c(b_iw_i)$ is close to $w_i^2\phi(c)w_i^2$ for $c \in S \cup \{1, c_1, \dots, c_i\}$—in the summable sense described above. The cross terms in the expression

$$v^*cv \;=\; \left(\sum b_iw_i\right)^* c \left(\sum b_iw_i\right)$$

are negligible in the sense described above by the choice of the sequence $S_1, S_2, \ldots$ (to correlate with the choice of $b_1, b_2, \ldots$; cf. proof of Lemma 10).

12. In order to prove that an arbitrary extension (of a stable separable C*-algebra by a separable C*-algebra) which is absorbing in the nuclear sense is purely large, we must first establish the existence of some purely large extension, and in fact one which is trivial in the nuclear sense—so that we can use the absorbing hypothesis. (It follows from the other implication of Theorem 6 that, in the unital setting, such an extension is necessarily unique—up to equivalence.)

An extension with these properties (purely large, and trivial in the nuclear sense) was constructed by Kasparov in [**8**]—although Kasparov did not establish these properties. (What Kasparov proved, in terms of our terminology, was that his extension was absorbing in the nuclear sense.) Let us now verify the asserted properties.

Lemma. *Let A and B be separable C^*-algebras, with B stable and A unital. There exists a purely large unital extension of B by A which is trivial in the nuclear sense* (*as a unital extension*).

Proof. We may suppose that both A and B are non-zero. Kasparov in [**8**] considered the extension of $B \otimes \mathcal{K}(H)$ by A with splitting

$$A \hookrightarrow 1 \otimes B(H) \hookrightarrow \mathrm{M}(B \otimes \mathcal{K}(H)),$$

where $A \hookrightarrow B(H)$ is a faithful unital representation of A on the separable infinite-dimensional Hilbert space H. Choosing such a representation π of A, and choosing an isomorphism of $B \otimes \mathcal{K}(H)$ with B, we obtain an extension of B by A—obviously trivial (but a priori depending on the choices made). Let us denote this extension by τ_0.

To show that τ_0 is trivial in the nuclear sense, it is sufficient to show that the given splitting,

$$A \xrightarrow{\pi} 1 \otimes B(H) \hookrightarrow \mathrm{M}(B \otimes \mathcal{K}(H)) \cong \mathrm{M}(B),$$

is weakly nuclear. In other words, given $d \in B \otimes \mathcal{K}(H)$, it is enough to show that the map

$$d\pi d^* : \quad A \ni a \mapsto d\pi(a)d^* \in B \otimes \mathcal{K}(H)$$

is nuclear, i.e., factorizes approximately through a finite-dimensional C*-algebra by means of completely positive maps. With (e_n) an approximate unit for $\mathcal{K}(H)$ consisting of projections of finite rank, note that for each n the completely positive map

$$(e_n d e_n)\pi(e_n d e_n)^*,$$

where we write e_n again for $1 \otimes e_n \in \mathrm{M}(B \otimes \mathcal{K}(H))$, factors through the finite-dimensional C*-algebra $e_n \mathcal{K}(H) e_n$ (as the composition of the completely positive maps $e \mapsto e_n\pi(a)e_n \in e_n\mathcal{K}(H)e_n$ and $x \mapsto (e_nde_n)x(e_nde_n)^* \in B \otimes \mathcal{K}(H)$). Since $e_n = 1 \otimes e_n$ converges to 1 in $\mathrm{M}(B \otimes \mathcal{K}(H))$ in the strict topology, in the topology of pointwise convergence

$$(e_nde_n)\pi(e_nde_n)^* \ \to \ d\pi d^*.$$

We shall prove below, in Theorem 17(iii), that a considerably more general construction than Kasparov's also gives rise to a purely large extension (trivial, but not necessarily in the nuclear sense). Therefore, rather than duplicating this proof—or omitting it in the more general case, which includes the interesting class of extensions considered by Lin in [**10**]—, we shall omit it in the present case.

13.

Lemma. *The sum of any two C*-algebra extensions one of which is purely large is again purely large.*

Proof. Recall that, by definition, an extension is purely large when the associated C*-algebra is purely large, with respect to the canonical closed two-sided ideal. Recall also, that, in this case, the canonical closed two-sided ideal is essential—so that the C*-algebra of the extension may be considered as a subalgebra of the multiplier algebra of the ideal. It is sufficient to show, then, that if B is a C*-algebra, if C is a sub-C*-algebra of $\mathrm{M}(B)$ containing B, and if there exists a projection e in $\mathrm{M}(B)$ commuting with C modulo B, such that the C*-algebra eCe is purely large with respect to the ideal eBe, such that if $c \in C$ and $ece \in B$ then $c \in B$, and such that eBe is full in B, then the C*-algebra C is purely large with respect to the ideal B.

With B and C (and e) as above, let c be an element of C not contained in B, and let us show that $\overline{cBc^*}$ contains a stable sub-C*-algebra which is full in B.

Since $c \notin B$, by hypothesis $ece \notin eBe$, and so $(eceBec^*e)^-$ contains a stable sub-C*-algebra which is full in eBe, and hence also full in B. Hence, as

$$eceBec^*e \ \subseteq \ ecBc^*e,$$

the sub-C*-algebra $(ecBc^*e)^-$ of B contains a stable sub-C*-algebra which is full in B.

While $(ecBc^*e)^-$ may not be contained in $(cBc^*)^-$, there is a natural isomorphism of the C*-algebra $(ecBc^*e)^-$ with $(c^*eBec)^-$, which is contained in the algebra $(c^*Bc)^-$. Furthermore, as this isomorphism consists of the restriction to $(ecBc^*e)^-$ of the map

$$B^{**} \ni b \ \mapsto \ w^*bw \in B^{**},$$

where w denotes the partially isometric part of $ec \in C \subseteq B^{**}$, and its inverse is the restriction to $(c^*eBec)^-$ of the map

$$B^{**} \ni b \;\mapsto\; wbw^* \in B^{**},$$

the subalgebras $(ecBc^*e)^-$ and $(c^*eBec)^-$ of B generate the same closed two-sided ideal. This shows that $(c^*Bc)^-$ contains a stable sub-C*-algebra which is full in B. It follows by a similar argument (or just by replacing c by c^*) that $(cBc^*)^-$ does, too.

14.

Lemma. *Any C^*-algebra extension equivalent to a purely large one is purely large.*

Proof. The property in question is, by definition, a property of the C*-algebra of the extension, together with the distinguished ideal, not of the extension itself. Equivalence of extensions preserves the isomorphism class of the associated C*-algebra, with its canonical ideal.

15. *Proof of Theorem* 6. Let τ be a unital extension of B by A. (We shall identify τ with its Busby map $A \to \mathrm{M}(B)/B$.)

Suppose that τ is purely large, and let us show that τ is absorbing in the nuclear sense.

Given a unital extension τ' of B by A which is trivial in the nuclear sense, i.e., which has a unital weakly nuclear splitting, we must show that

$$\tau \;\sim\; \tau \oplus \tau',$$

i.e., that τ and $\tau \oplus \tau'$, considered as maps from A to $\mathrm{M}(B)/B$, are unitarily equivalent, by means of the image in $\mathrm{M}(B)/B$ of a unitary element of $\mathrm{M}(B)$.

As in the case of earlier absorption theorems, it is sufficient to prove (for arbitrary τ' as above) that

$$\tau \;\sim\; \sigma \oplus \tau'$$

for some unital extension σ, not necessarily equal to τ. Indeed, as in [**2**] (which systematizes [**13**], and is the model for later absorption proofs, including the present one)—see also below—one may construct a trivial extension τ''—trivial also in the nuclear sense, and as a unital extension—such that

$$\tau'' \oplus \tau' \;\sim\; \tau''.$$

Hence, with σ such that

$$\tau \sim \sigma \oplus \tau'',$$

it follows that

$$\begin{aligned}\tau \oplus \tau' &\sim (\sigma \oplus \tau'') \oplus \tau' \\ &\sim \sigma \oplus (\tau'' \oplus \tau') \\ &\sim \sigma \oplus \tau'' \\ &\sim \tau.\end{aligned}$$

A unital extension τ'' such that $\tau'' \oplus \tau' \sim \tau''$, which is trivial in the nuclear sense—as a unital extension—, is obtained by forming the infinite multiplicity sum $(\pi')^\infty$ of a unital weakly nuclear splitting π' of τ' (cf. [2]). This is defined first as just the map

$$\begin{aligned}\pi' \oplus 1 : \quad A &\to \mathrm{M}(B \otimes \mathcal{K}) \\ a &\mapsto \pi'(a) \otimes 1.\end{aligned}$$

This map then is transformed into a (unital) map

$$\pi'' : \quad A \to \mathrm{M}(B)$$

by identifying B with $B \otimes e_{11} \subseteq B \otimes \mathcal{K}$, and then transforming $B \otimes e_{11}$ onto $B \otimes \mathcal{K}$ by means of an isometry in $\mathrm{M}(B \otimes \mathcal{K})$, which we shall denote by s_2, such that $s_2 s_2^* = 1 \otimes e_{11}$. (Such an isometry exists because B is stable; more explicitly, with $B = B_0 \otimes \mathcal{K}$, we may choose $s_2 = 1 \otimes t_2 \in \mathrm{M}(B_0 \otimes (\mathcal{K} \otimes \mathcal{K}))$ where t_2 is an isometry in $\mathrm{M}(\mathcal{K} \otimes \mathcal{K})$ with range $1 \otimes e_{11}$.) Choose an isometry t_1 in $\mathrm{M}(\mathcal{K})$ with range $1 - e_{11}$, and set $1 \otimes t_1 = s_1$. The (desired) equivalence

$$\pi'' \oplus \pi' \sim \pi''$$

(unitary equivalence of maps from A to $\mathrm{M}(B)$) then reduces (by transformation by s_2) to the equivalence

$$(\pi' \otimes 1) \oplus s_2^*(\pi' \otimes e_{11})s_2 \sim \pi' \otimes 1$$

(unitary equivalence of maps from A to $\mathrm{M}(B \otimes \mathcal{K})$), which may be seen by using the Cuntz isometries s_1 and s_2 to compute the left-hand side:

$$\begin{aligned}s_1(\pi' \otimes 1)s_1^* + s_2(s_2^*(\pi' \otimes e_{11})s_2)s_2^* &= \pi' \otimes (1 - e_{11}) + \pi' \otimes e_{11} \\ &= \pi' \otimes 1.\end{aligned}$$

With τ'' the unital extension with splitting π'', we then have $\tau'' \oplus \tau' \sim \tau''$; it remains only to note that τ'' is trivial in the nuclear sense, as $\pi' \otimes 1$ and hence π'' are weakly nuclear.

To show that $\tau \sim \sigma \oplus \tau'$, for some unital extension σ, with τ' as given—a unital extension with a weakly nuclear unital splitting—we shall in fact not use that this splitting is a C*-algebra homomorphism, but only that it is completely positive (and unital, and weakly nuclear, in the sense described for a homomorphism).

Since the C*-algebra, C, of the extension τ is purely large with respect to the closed two-sided ideal B (canonically contained in it), in particular B is essential, and so we may write

$$B \subseteq C \subseteq \mathrm{M}(B),$$

and aim to apply Lemma 11 to the completely positive map $\phi : C \to \mathrm{M}(B)$ obtained by composing the canonical quotient map from C to A with a weakly nuclear, unital, completely positive map from A to $\mathrm{M}(B)$ lifting τ'. (Note that the existence of such a map is clearly equivalent to the existence of a splitting map with these properties from A to the C*-algebra of the extension τ', namely, the pullback of A and its preimage in $\mathrm{M}(B)$.)

In order to apply Lemma 11, we must verify that for every $b_0 \in B$, the map

$$b_0^*\phi b_0 : \quad C \ni c \mapsto b_0^*\phi(c)b_0 \in B$$

can be approximated by the maps

$$c \mapsto b^*cb, \quad b \in B.$$

Fix $b_0 \in B$, and set $b_0^*\phi b_0 = \psi$. Since, by construction, the map ψ from C to B is zero on B, and the associated map from C/B to B is nuclear, the approximibility of ψ by maps $c \mapsto b^*cb$ with $b \in B$ is ensured by Lemma 10.

By Lemma 11, there exists $v \in \mathrm{M}(B)$ such that

$$\phi(c) - v^*cv \in B, \quad c \in C,$$

and such that also v^*cv is close to $\phi(c)$ for c belonging to any given finite set, and in particular for $c = 1$. As ϕ is unital, v^*v is close to 1, and equal to 1 modulo B. Hence, replacing v by $v(v^*v)^{-\frac{1}{2}}$, we may suppose that $v^*v = 1$.

The first property of v may be rewritten as

$$\tau' = v^*\tau v$$

(i.e., $\tau'(a) = v^*\tau(a)v, \quad a \in A$, where v denotes the image of $v \in \mathrm{M}(B)$ in $\mathrm{M}(B)/B$).

Since τ' is multiplicative this in particular implies that the projection $vv^* \in \mathrm{M}(B)/B$ commutes with $\tau(A)$. (As v is an isometry, also $v\tau'v^*$ is multiplicative, and therefore also $(vv^*)\tau(vv^*)$; with $vv^* = e$ we then have $e\tau(a^*)\tau(a)e = e\tau(a^*a)e = e\tau(a^*)e\tau(a)e$, whence $e\tau(a^*)(1-e)\tau(a)e = 0$, i.e., $(1-e)\tau(a)e = 0$; since a is arbitrary, also $(1-e)\tau(a^*)e = 0$, and so $\tau(a)e = e\tau(a)$.)

Since Brown-Douglas-Fillmore addition of (equivalence classes of) extensions is independent of the choice of the unital copy of $\mathcal{O}_2$ in $\mathrm{M}(B)$ (cf. above), to show that

$$\tau \sim \sigma \oplus \tau'$$

it would be sufficient to know that the projection $1 - vv^*$ is Murray-von Neumann equivalent to 1 in $\mathrm{M}(B)$. Indeed, with s_1 an isometry with range

$1 - vv^*$, and $s_2 = v$,

$$\begin{aligned}\tau &= (1 - vv^*)\tau + vv^*\tau \\ &= (1 - vv^*)\tau(1 - vv^*) + vv^*\tau vv^* \\ &= s_1 s_1^* \tau s_1 s_1^* + s_2 s_2^* \tau s_2 s_2^* \\ &= s_1 \sigma s_1^* + s_2 \tau' s_2^* \\ &= \sigma \oplus \tau'\end{aligned}$$

where $\sigma = s_1^* \tau s_1$ (recall that $\tau' = v^* \tau v$).

Instead of showing directly that it is possible to choose v above with $1 - vv^*$ equivalent to 1, let us choose v with respect to $\tau' \oplus \tau'$ instead of τ'—and call this w. (Note that $\tau' \oplus \tau'$ has a unital weakly nuclear completely positive splitting if τ' does.) Then

$$\begin{aligned}\tau &= (1 - ww^*)\tau + ww^*\tau \\ &= (1 - ww^*)\tau + e_1\tau + e_2\tau,\end{aligned}$$

where e_1 and e_2 are projections equivalent to 1 with $e_1 + e_2 = ww^*$, commuting with $\tau(A)$ modulo B, and $e_2\tau$ is equivalent (by means of an isometry with range e_2) to τ'. Provided we show that also $(1 - ww^*) + e_1$ is equivalent to 1, this says that $\tau = \sigma \oplus \tau'$.

Let us show, then, using that B is stable, that if e is a projection in $\mathrm{M}(B)$ equivalent to 1, and f is any projection orthogonal to e, then $e + f$ is equivalent to 1. We shall deduce this from the well known fact that $\mathrm{M}(\mathcal{K})$, and hence $\mathrm{M}(B)$, contains an infinite sequence of mutually orthogonal projections, say $e_1, e_2, \ldots$, equivalent to 1 and with sum 1 (in the strict topology). It follows that any sequence of projections (f_i) in $\mathrm{M}(B)$ with $f_i \leq e_i$ also has convergent sum. Clearly, the projection $e_2 + e_3 + \cdots$ is equivalent to $e_1 + e_2 + \cdots = 1$. If f_1 is any subprojection of e_1, choose a subprojection f_i of e_i for $i \geq 2$ equivalent to f_1, and note that, also, $f_1 + f_2 + \cdots$ is equivalent to $f_2 + f_3 + \cdots$. Therefore, by additivity of equivalence, on adding the single projection $(e_2 - f_2) + (e_3 - f_3) + \cdots$ to both of these projections we obtain that $f_1 + e_2 + e_3 + \cdots$ is equivalent to $e_2 + e_3 + \cdots$, as desired.

(The preceding considerations are superfluous in the case $B = \mathcal{K}$, considered in [**13**] and [**2**].)

Now assume that τ is absorbing, in the nuclear sense, and let us show that τ is purely large.

By Lemma 12, there exists a purely large unital extension τ_0 of B by A which is trivial in the nuclear sense. By hypothesis,

$$\tau \sim \tau \oplus \tau_0.$$

By Lemma 13, $\tau \oplus \tau_0$ is purely large. Hence by Lemma 14, τ is purely large, as desired.

16. It follows immediately from Theorem 6 that, if one considers the non-unital setting (i.e., extensions which are not necessarily unital, or with a non-unital quotient), then one has the following criterion for an extension τ of a stable separable C*-algebra B by a separable C*-algebra A to be absorbing in the nuclear sense:

The unital extension $\tilde{\tau}$ of B by $\tilde{A}$, the C*-algebra A with unit adjoined (i.e., a new unit if A is already unital), naturally corresponding to τ (with Busby map extending that of τ), should be purely large.

(To see this, note that by Theorem 6, the preceding condition is equivalent to the condition that $\tilde{\tau}$ be absorbing in the nuclear sense, as a unital extension. This, on the other hand, is equivalent to the condition that τ be absorbing in the nuclear sense (in the non-unital setting): the extensions of B by A which are trivial in the nuclear sense are in bijective correspondence with the unital extensions of B by $\tilde{A}$ which are trivial in the nuclear sense, in the unital setting, by the map $\sigma \mapsto \tilde{\sigma}$. Finally it is clear that $\tau + \sigma \sim \tau$ if, and only if, $\tilde{\tau} + \tilde{\sigma} \sim \tilde{\tau}$.)

Let us note that, for the extension $\tilde{\tau}$ of B by $\tilde{A}$ to be purely large, it is necessary and sufficient for τ itself to be purely large, and non-unital. (If C is purely large with respect to B, and non-unital, we must show that also $\tilde{C}$ is purely large with respect to B. (Clearly, if $\tilde{C}$ is purely large then C is purely large and non-unital.) In other words, we must show that for any $c \in C$, $((1+c)B(1+c)^*)^-$ contains a full stable sub-C*-algebra of B. If $(1+c)C \subseteq B$, then the image of $-c$ in $\mathrm{M}(B)/B$ is a unit for the image of C in $\mathrm{M}(B)/B$; hence, the image of C in $\mathrm{M}(B)$ contains $1 \in \mathrm{M}(B)$; as τ is essential the map $C \to \mathrm{M}(B)$ is injective, and hence C is unital, contrary to hypothesis. This shows that there exists $c' \in C$ with $(1+c)c'$ not in B. Hence, the subalgebra

$$((1+c)c'B((1+c)c')^*)^- \subseteq ((1+c)B(1+c)^*)^-$$

contains a stable sub-C*-algebra which is full in B.) (As a consequence, Corollary 9 and Lemma 10 hold also in the non-unital case—but we will not use this.)

Let us summarize:

Corollary. *Let B be a stable separable C^*-algebra, and let A be a separable C^*-algebra. Let τ be an extension of B by A.*

The extension τ is absorbing, in the nuclear sense, if and only if τ is purely large and non-unital.

In particular, if A is non-unital (*i.e., does not have a unit element*), *then τ is absorbing if and only if τ is purely large.*

17. Let us now show directly that those extensions previously known to be absorbing (in the nuclear sense) are purely large. (We refer to the absorption theorems of [**3**], [**13**], [**11**], [**8**], [**9**], and [**10**].)

On the one hand, this yields a new proof—via Theorem 6—of the absorption property. On the other hand, as pointed out in Section 12, the proof that an arbitrary extension which is absorbing in the nuclear sense is purely large depends on first knowing the existence of at least one purely large extension—which is also trivial in the nuclear sense. This is proved in Lemma 12, using Theorem 17(iii) below.

Concerning the notion of absorbing extension, note that if an extension is absorbing in the sense that it absorbs every trivial extension (in the class of unital extensions, say), then it is certainly absorbing in the nuclear sense; on the other hand, so far the only known examples of true absorbing extensions are in the case that either the ideal or the quotient is nuclear, so that the true sense and the nuclear sense coincide (every trivial extension is trivial in the nuclear sense).

Theorem. *Let A and B be separable C^*-algebras, with B stable. Let τ be a C^*-algebra extension of B by A. Suppose that τ is essential (i.e., that the Busby map $A \to M(B)/B$ is injective; see Section 2). In each of the following cases, τ is purely large (in the sense of Section 2).*

(i) *$B = \mathcal{K}$. (Cf. [**3**], [**13**].)*

(i)′ *$B = C_0(X) \otimes \mathcal{K}$ where X is a finite-dimensional locally compact Hausdorff space, and the map from A to the canonical quotient $M(\mathcal{K})/\mathcal{K}$ of $M(B)/B$ corresponding to each point of X is injective (in other words, τ is homogeneous in the sense introduced for such a B in [**11**]). (Cf. [**11**].) (In [**11**], X is restricted to be compact.)*

(ii) *B is simple and purely infinite. (Cf. [**9**].)*

(iii) *τ is trivial, with a splitting*

$$\pi : \quad A \to 1 \otimes M(B_1) \hookrightarrow \mathrm{M}(B_0 \otimes B_1) = \mathrm{M}(B)$$

*for some tensor product decomposition $B = B_0 \otimes B_1$, with B_0 stable, such that, for any non-zero $a \in A$, the closed two-sided ideal of B_1 generated by $\pi(a)B_1$ is equal to B_1. (This last property is automatic if B_1 is simple—for instance, as in [**10**], or as in the case $B_1 = \mathcal{K}$ considered in [**8**] and in Lemma 12 above. It is also automatic if A is simple and $\pi(A)B_1$ is dense in B_1—as considered also in [**10**].)*

Proof. As in the proof of Theorem 6, since the map $A \to \mathrm{M}(B)/B$ is injective, we may suppose that the C*-algebra of the extension is a subalgebra of $M(B)$.

Ad (i). For any $c \in \mathrm{M}(\mathcal{K})$ which is not in $\mathcal{K}$, the hereditary sub-C*-algebra $(c\mathcal{K}c^*)^-$ of $\mathcal{K}$ is infinite-dimensional and hence, as it is equal to $e\mathcal{K}e$ for some projection $e \in \mathrm{M}(\mathcal{K})$ ($\mathrm{M}(\mathcal{K})$ being the bidual of $\mathcal{K}$), it is isomorphic to $\mathcal{K}$ and in particular is stable and full.

Ad (i)$'$. By hypothesis, for any $c \in \mathrm{M}(B)$ belonging to the C*-algebra of the extension, but not to B, the hereditary sub-C*-algebra $(cBc^*)^-$ of B is full. (This is a simple reformulation of the hypothesis of homogeneity.)

Let us show, that for any such c the C*-algebra $(cBc^*)^-$ is stable. By hypothesis, for each point of X, not only is the image of $(cBc^*)^-$ in the quotient $\mathcal{K}$ of B at this point non-zero, but (since the C*-algebra of the extension contains B, and this property holds with c replaced by $c+b$ for any $b \in B$) also this image is stable.

Let us show that, more generally, any hereditary sub-C*-algebra of B the image of which in each primitive quotient of B is stable (possibly equal to zero) is itself stable. Here, B is still as above. Let D be such a hereditary sub-C*-algebra of B. Note that D is a C*-algebra with continuous trace—as B has continuous trace, and this property is preserved (as is easily seen) under passage to a hereditary sub-C*-algebra. By Theorem 10.9.5 of [**5**], D is determined up to isomorphism, among the class of all separable C*-algebras with continuous trace, with all primitive quotients equal to $\mathcal{K}$ and with the same spectrum as D (note that this space has finite dimension), by its Dixmier-Douady invariant. By inspection of the construction of this invariant (see 10.7.14 of [**5**]), one sees that it is unchanged by tensoring by $\mathcal{K}$. It follows that D is isomorphic to $D \otimes \mathcal{K}$, as desired.

Ad (ii). For any $c \in \mathrm{M}(B)$ which is not in B, the hereditary sub-C*-algebra $(cBc^*)^-$ of B is non-zero and therefore (by the definition of purely infinite simple C*-algebra that we shall use) contains an infinite projection. In other words, $(cBc^*)^-$ contains a partial isometry v such that $vv^* < v^*v$. The partial isometries $v^n(v^*v - vv^*)$, $n = 1, 2, \ldots$, generate a sub-C*-algebra of $(cBc^*)^-$ isomorphic to $\mathcal{K}$, full in B as B is simple.

(In fact, in the present case, as $(cBc^*)^-$ cannot be unital, by [**14**] this algebra itself is stable.)

Ad (iii). Recall that, as shown by Hjelmborg and Rørdam in [**7**], using the criterion for stability that they established, as B is separable and stable the hereditary sub-C*-algebra $((1+b)B(1+b)^*)^-$ is stable for any $b \in B$.

Let us begin by noting that a similar, but rather simpler, argument shows that, also, the hereditary sub-C*-algebra $((1+b)B(1+b)^*)^-$ is full in B for each $b \in B$. (We are indebted to M. Rørdam for this argument.) With (u_n) a sequence of unitary elements of $\mathrm{M}(B)$ such that

$$b_1 u_n b_2 \ \to \ 0 \quad \text{for all} \quad b_1, b_2 \in B,$$

as exists by [**7**] if B is stable (u_n may be chosen to be $1 \otimes v_n$ with (v_n) such a sequence in $\mathrm{M}(\mathcal{K})$, in particular a sequence of unitaries corresponding to finite permutations of an orthonormal basis), one has for each fixed $b \in B$,

$$u_n(1+b)u_n^* \ \to \ 1 \quad \text{strictly in} \quad \mathrm{M}(B).$$

Hence, for each $b' \in B$,

$$(u_n(1+b)u_n^*)b'(u_n(1+b)u_n^*)^* \ \to \ b'.$$

This shows in particular that the closed two-sided ideal generated by $(1+b)B(1+b)^*$ is dense in B, i.e., $((1+b)B(1+b)^*)^-$ is full in B, as asserted.

Now let us show that for any element c of C, the C*-algebra of the extension, not contained in B, the C*-algebra $(cBc^*)^-$ contains a stable sub-C*-algebra which is full in B. We shall base our argument on the case $c = 1+b$, considered above.

The special nature of the present setting may be expressed as follows:

In a certain decomposition of B as $B_1 \otimes \mathcal{K}$, with B_1 stable (and hence isomorphic to B), the given element $c \in \mathrm{M}(B)$ is decomposed as $c_1 + b$ where $c_1 \in 1 \otimes \mathrm{M}(\mathcal{K})$ and $b \in B_1 \otimes \mathcal{K}$.

This may then be exploited as follows:

Write B_1 as $B_2 \otimes \mathcal{K}$, so that $c = c_1 + b$ with

$$c_1 \in 1 \otimes 1 \otimes \mathrm{M}(\mathcal{K}) \ \subseteq \ \mathrm{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$$

and $b \in B_2 \otimes \mathcal{K} \otimes \mathcal{K}$. As in [7] (see also above), choose a sequence of unitaries (u_n) in $\mathrm{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ with

$$b_1 u_n b_2 \ \to \ 0 \quad \text{for all} \quad b_1, b_2 \in B_2 \otimes \mathcal{K} \otimes \mathcal{K},$$

such that, in addition,

$$u_n \ = \ 1 \otimes v_n \otimes 1 \quad \text{with} \quad v_n \in \mathrm{M}(\mathcal{K}).$$

Then, as $c_1 \in 1 \otimes 1 \otimes \mathrm{M}(\mathcal{K})$,

$$u_n c_1 u_n^* \ = \ c_1.$$

Hence (cf. above),

$$u_n c u_n^* \ \to \ c_1 \quad \text{strictly in} \quad \mathrm{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K}) \ = \ \mathrm{M}(B).$$

(This holds as $c = c_1 + b$ with $u_n c_1 u_n^* = c_1$ and $u_n b u_n^* \ \to \ 0$ strictly.)

Note also that $(c_1 b c_1^*)^-$ is stable, and full in B, as $c_1 \in 1 \otimes \mathrm{M}(\mathcal{K}) \subseteq \mathrm{M}(B_1 \otimes \mathcal{K}) = \mathrm{M}(B)$ and $c_1 \notin B$. (See proof of Case (i).)

Let us first show that $(cBc^*)^-$ is full in B—this is the simpler step. Since

$$(u_n c u_n^*)b'(u_n c u_n^*)^* \ \to \ c_1 b' c_1^* \quad \text{for all} \quad b' \in B,$$

the closed two-sided ideal of B generated by cBc^* contains $(c_1 B c_1^*)^-$, and hence is equal to B, as desired.

We are unable to prove that $(cBc^*)^-$ is stable, for arbitrary c as above, i.e., for c equal to $c_1 + b$, with c_1 fixed as above, and b arbitrary in B. Nevertheless, we shall show that, for arbitrary such c, the algebra $(cBc^*)^-$ contains a stable sub-C*-algebra which is full in B, which is all that is required. (The subalgebra will be constructed to be $(c'Bc'^*)^-$ for some $c' \in C \setminus B$; such a subalgebra is full in B by the preceding paragraph.)

Note that for any $x \in \mathrm{M}(B)$ the sub-C*-algebra $(xBx^*)^-$ is equal to $(xx^*Bxx^*)^-$, so that the problem reduces to considering the case that c is positive—and dividing by B we see that c_1 is then positive, too. Of course, we may also suppose that c and c_1 have norm at most one.

Now, set $c^{\frac{1}{2}}c_1c^{\frac{1}{2}} = c'$ and $c_1^{\frac{1}{2}}cc_1^{\frac{1}{2}} = c''$, and note that, first,

$$0 \le c' \le c, \quad 0 \le c'' \le c_1,$$

so that

$$(c'Bc')^- \subseteq (cBc)^-, \quad (c''Bc'')^- \subseteq (c_1Bc_1)^-,$$

and, second, the hereditary sub-C*-algebras $(c'Bc')^-$ and $(c''Bc'')^-$ are isomorphic. (As shown in the proof of Lemma 13, $(xBx^*)^-$ is isomorphic to $(x^*Bx)^-$ for any $x \in \mathrm{M}(B)$, and applying this with $x = c_1^{\frac{1}{2}}c^{\frac{1}{2}}$ yields

$$(c'Bc')^- \;=\; (xBx^*)^- \;\cong\; (x^*Bx)^- \;=\; (c''Bc'')^-,$$

as asserted.)

It now suffices, to complete the proof, to show that $(c''bc'')^-$ is stable—as then $(c'Bc')^-$ is a stable sub-C*-algebra of $(cBc)^-$, full in B by the first part of the proof.

To simplify notation, let us assume that already $c \le c_1$, and let us show that, at least in this case, $(cBc)^-$ is stable. We shall essentially repeat the proof of Corollary 4.3 of [7].

Recall that $b_1u_nb_2 \to 0$ for all $b_1, b_2 \in B$. Let us verify the criterion (b) of Proposition 2.2 of [7], shown in Proposition 2.2 and Theorem 2.1 of [7] to be equivalent to stability for a C*-algebra with countable approximate unit (in particular, for a separable C*-algebra), with $(cBc)^-$ in place of A. Fix $0 \le a \in (cBc)^-$. Since $(cBc)^- \subseteq (c_1Bc_1)^-$, there exists a continuous function (a root) d_1 of c_1 such that d_1a is arbitrarily close to a. Since $c = c_1 + b$, also $d - d_1 \in B$ where d is the corresponding function of c. Therefore, for large n, $du_na^{\frac{1}{2}}$ is arbitrarily close to $d_1u_na^{\frac{1}{2}} = u_nd_1a^{\frac{1}{2}}$ and hence also to $u_na^{\frac{1}{2}}$. Since

$$(u_na^{\frac{1}{2}})^*(u_na^{\frac{1}{2}}) \;=\; a,$$

with $a_n = du_na^{\frac{1}{2}} \in (cBc)^-$ we have that, if n is sufficiently large, the element $a_n^*a_n$ is close to a, and the product of equivalent elements

$$(a_n^*a_n)(a_na_n^*) \;=\; a_n^*(du_na^{\frac{1}{2}}du_na^{\frac{1}{2}})a_n$$

is close to zero (as $a^{\frac{1}{2}}du_na^{\frac{1}{2}} \to 0$), as required in the criterion 2.2(b) of Hjelmborg and Rørdam.

18. Questions. A number of questions arise naturally in connection with the notion of purely large extension.

For instance, is the obvious stronger form of the property that an extension is purely large in fact the same thing? In other words, if the C*-algebra C is purely large with respect to the closed two-sided ideal B, i.e., if $(cBc^*)^-$ always contains a stable sub-C*-algebra which is full in B for any $c \in C$ not in B, must the subalgebra $(cBc^*)^-$ always be stable itself (for such c)?

Again, is it possible to characterize when an extension is purely large in terms of the image of the Busby map in the corona of B, the quotient $\mathrm{M}(B)/B$? (Remembering also that the extension is essential—equivalently, that the Busby map is injective.) Of course, this must mean by some intrinsic property of the image, which makes sense more generally—perhaps in an arbitrary C*-algebra. (As the image of the Busby map in the corona—if this is given as the corona—is already enough to reconstruct the C*-algebra associated with an essential extension—which by definition contains sufficient information to determine whether the extension is purely large.) For instance, is it sufficient that every non-zero element of the image be full (i.e., not contained in any proper closed two-sided ideal)? This condition is at least necessary—at least in the separable case—as can be seen by Theorem 6, together with the (obvious) fact that Kasparov's extension (17(iii) above) satisfies this condition—and, as shown in Lemma 12, is trivial in the nuclear sense (and so, by Theorem 6, is absorbed by a purely large extension).

Note that as (by Theorem 17(iii)) Kasparov's extension is also absorbing in the nuclear sense, an extension of one separable C*-algebra by another is purely large—equivalently, absorbing in the nuclear sense—precisely when it absorbs Kasparov's extension. One might ask whether this characterization of purely large extensions can be extended to the non-separable case. The difficulty with this is that Kasparov's extension, being based on an extension of $\mathcal{K}$, does not exist if the quotient has too large a cardinality. On the other hand, the characterization of purely large extensions simply as those which are absorbing in the nuclear sense (either among unital extensions, if the extension is unital, or among all extensions if it is not unital—Theorem 6 and Corollary 16), although it is proved using Kasparov's extension, makes sense and could conceivably still hold in the non-separable case.

One thing the notion of purely large extension—or, more precisely, the notion of extension which is absorbing in the nuclear sense (cf. Theorem 6 and Corollary 16)—makes possible is a generalization of Kasparov's semigroup description of $\mathrm{Ext}(A, B)$ in the setting of nuclear (separable) C*-algebras. Namely, for arbitrary (separable) C*-algebras A and B, with B stable, the extensions of B by A which are absorbing in the nuclear sense form, as we have shown, a semigroup with zero element. The invertible elements of this semigroup are seen—on using Kasparov's Stinespring Theorem, [**8**]—to be precisely the weakly nuclear extensions which are absorbing in the nuclear sense.

Here, by a weakly nuclear extension of B by A we mean an extension for which the Busby map $A \to \mathrm{M}(B)/B$ lifts to a completely positive contraction $A \to \mathrm{M}(B)$ which is weakly nuclear, in the sense described in Section 5 for homomorphisms. (Recall that if A is exact, then by Corollary 5.11 of [**9**], any weakly nuclear map with domain A is nuclear.) One should note that the proof of Kasparov's Stinespring theorem preserves weak nuclearity: a weakly nuclear completely positive map dilates to a weakly nuclear homomorphism. The group of invertible elements of this semigroup with zero (the semigroup of absorbing extensions in the nuclear sense, i.e., those extensions absorbing every trivial extension with a weakly nuclear splitting) therefore maps into the group, which we shall denote by $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$, of all Brown-Douglas-Fillmore equivalence classes of weakly nuclear extensions of B by A, modulo extensions trivial in the nuclear sense. Since Kasparov's extension is weakly nuclear, and, what is more, trivial in the nuclear sense, and so zero in $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$, and since the sum of this with any extension is absorbing in the nuclear sense, this mapping is onto $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$. Since any two extensions which are both absorbing and trivial in the nuclear sense are equivalent, this map is injective, and therefore an isomorphism.

It is interesting to consider whether the group $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$ defined above —and realized as a subset of the Brown-Douglas-Fillmore semigroup—is isomorphic in the natural way to the group $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ defined by Skandalis in [**12**]. (With the appropriate dimension shift.) This amounts to the following, perhaps surprising, question:

As pointed out above, any extension which is trivial in the nuclear sense—i.e., has a weakly nuclear splitting—is weakly nuclear. Is every weakly nuclear trivial extension trivial in the nuclear sense?

Acknowledgments. The authors are indebted to Rajarama Bhat and Peter Friis for conversations at an early stage concerning the question addressed in this paper—the possibility of generalizing Kirchberg's Weyl-von Neumann theorem.

The first author is indebted to the Canada Council for the Arts for a Killam Research Fellowship, and to the Natural Sciences and Engineering Research Council of Canada for a Research Grant.

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Received March 30, 1999 and revised March 1, 2000.

Department of Mathematics
University of Copenhagen
Copenhagen, Denmark;

Department of Mathematics
University of Toronto
Toronto, M5S 3G3
Canada

The Fields Institute for Research in Mathematical Sciences
222 College Street
Toronto, M5T 3J1
Canada
E-mail address: elliott@math.toronto.edu

The Fields Institute for Research in Mathematical Sciences
222 College Street
Toronto, M5T 3J1
Canada

Department of Mathematics
University of New Brunswick
Fredericton, New Brunswick E3B 5A3
Canada
E-mail address: dan@math.unb.ca