

# *Pacific Journal of Mathematics*

THE SPECTRUM OF AN INTEGRAL OPERATOR IN  
WEIGHTED  $L_2$  SPACES

MOURAD E.H. ISMAIL AND PLAMEN C. SIMEONOV

# THE SPECTRUM OF AN INTEGRAL OPERATOR IN WEIGHTED $L_2$ SPACES

MOURAD E.H. ISMAIL AND PLAMEN C. SIMEONOV

We find the spectrum of the inverse operator of the  $q$ -difference operator  $D_{q,x}f(x) = (f(x) - f(qx))/(x(1 - q))$  on a family of weighted  $L_2$  spaces. We show that the spectrum is discrete and the eigenvalues are the reciprocals of the zeros of an entire function. We also derive an expansion of the eigenfunctions of the  $q$ -difference operator and its inverse in terms of big  $q$ -Jacobi polynomials. This provides a  $q$ -analogue of the expansion of a plane wave in Jacobi polynomials. The coefficients are related to little  $q$ -Jacobi polynomials, which are described and proved to be orthogonal on the spectrum of the inverse operator. Explicit representations for the little  $q$ -Jacobi polynomials are given.

## 1. Introduction.

The  $q$ -difference operator  $D_{q,x}$  is defined by

$$(1.1) \quad D_{q,x}f(x) := \frac{f(x) - f(qx)}{x(1 - q)}.$$

We shall use the following notations for finite and infinite products:

$$(z; q)_0 := 1, \quad (z; q)_n := \prod_{j=0}^{n-1} (1 - q^j z), \quad n > 0 \quad \text{or} \quad n = \infty,$$

$$(z_1, z_2, \dots, z_s; q)_n := \prod_{k=1}^s (z_k; q)_n, \quad n \geq 0 \quad \text{or} \quad n = \infty.$$

The infinite product is defined for  $|q| < 1$ .

A basic hypergeometric series is defined by

$$(1.2) \quad {}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q, z \right) = {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z)$$

$$:= \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} z^n \left( (-1)^n q^{n(n-1)/2} \right)^{1+s-r}.$$

Let  $\mathbf{t} = (t_1, t_2) \in \mathbf{R}^2$ . The big  $q$ -Jacobi polynomials of Andrews and Askey, [2], are defined by

$$(1.3) \quad p_n(x, \mathbf{t}) = p_n(x; t_1, t_2) := {}_3\phi_2 \left( \begin{matrix} q^{-n}, at_1t_2q^{n-1}, t_1x \\ t_1, at_1 \end{matrix} \middle| q, q \right), \quad 0 < q < 1$$

and are orthogonal with respect to the measure  $\mu(x, \mathbf{t})$ , [3, p. 594], [6], defined by

$$(1.4) \quad \mu(x, \mathbf{t}) := \frac{\mu^{(a)}(x)}{(t_1x, t_2x; q)_\infty},$$

where for  $a < 0$ ,  $\mu^{(a)}$  is the discrete probability measure

$$(1.5) \quad \mu^{(a)} := \sum_{n=0}^{\infty} \left[ \frac{q^n}{(q, q/a; q)_n(a; q)_\infty} \delta_{q^n} + \frac{q^n}{(q, aq; q)_n(1/a; q)_\infty} \delta_{aq^n} \right].$$

In (1.5),  $\delta_z$  denotes the unit measure supported on  $\{z\}$ . The orthogonality relation is, [3], [6],

$$(1.6) \quad \int_{\mathbf{R}} p_m(x, \mathbf{t}) p_n(x, \mathbf{t}) d\mu(x, \mathbf{t}) = \delta_{m,n} \xi_n(\mathbf{t}),$$

where

$$(1.7) \quad \xi_n(\mathbf{t}) = \frac{(q, t_2, at_2, at_1t_2q^{n-1}; q)_n (at_1t_2q^{2n}; q)_\infty}{(t_1, at_1, t_2, at_2; q)_\infty (t_1, at_1; q)_n} (-at_1^2)^n q^{n(n-1)/2}.$$

Furthermore, (1.3), (1.6), (1.7), and the symmetry of  $\mu$  in  $t_1$  and  $t_2$  imply the symmetry relation

$$(1.8) \quad p_n(x; t_1, t_2) = \frac{t_1^n(t_2, at_2; q)_n}{t_2^n(t_1, at_1; q)_n} p_n(x; t_2, t_1).$$

We shall use the following  $q$ -analogue of the Chu-Vandermonde sum, [4, (II.6)],

$$(1.9) \quad {}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b; q)_n b^n}{(c; q)_n},$$

and its special case ( $b = t_2x$ ,  $c = t_2q^{1-n}/t_1$ )

$$(1.10) \quad (t_1x; q)_n = (t_1/t_2; q)_n {}_2\phi_1(q^{-n}, t_2x; t_2q^{1-n}/t_1; q, q).$$

Letting  $t_1 \rightarrow \infty$  in (1.10) with  $t_2 = t$  we obtain

$$(1.11) \quad t^n x^n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2} - j(n-1)} (tx; q)_j,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_j (q; q)_{n-j}}, \quad j = 0, \dots, n$$

are the so called  $q$ -binomial coefficients.

We shall use Euler's identities, [6],

$$(1.12) \quad e_q(z) := \frac{1}{(z; q)_\infty} = \sum_{j=0}^{\infty} \frac{z^j}{(q; q)_j}, \quad |z| < 1,$$

$$(1.13) \quad E_q(z) := (z; q)_\infty = \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(-z)^j}{(q; q)_j},$$

and the terminating version of the  $q$ -binomial theorem, [6],

$$(1.14) \quad (z; q)_n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}}(-z)^j.$$

We shall also use the following identity

$$(1.15) \quad (q^{1-n}/A; q)_k = (-1/A)^k q^{\binom{k}{2} + k(1-n)} \frac{(A; q)_n}{(A; q)_{n-k}}.$$

The following theorem of H. Schwartz, [13], plays an important role in the spectral analysis in Section 2.

**Theorem 1.1.** *Let  $\{p_{n,\nu}(x)\}$  be a family of monic polynomials generated by*

$$(1.16) \quad p_{0,\nu}(x) = 1, \quad p_{1,\nu}(x) = x + B_\nu,$$

$$(1.17) \quad p_{n+1,\nu}(x) = (x + B_{n+\nu})p_{n,\nu}(x) + C_{n+\nu}p_{n-1,\nu}(x), \quad n \geq 1.$$

*If both*

$$(1.18) \quad \sum_{n=0}^{\infty} |B_{n+\nu} - \beta| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |C_{n+\nu}| < \infty$$

*hold, then  $x^n p_{n,\nu}(-\beta + 1/x)$  converges uniformly on compact subsets of the complex plane to an entire function of  $x$ .*

The paper is organized in six sections. In Section 2, we study the eigenfunctions of the right inverse operator of  $D_{q,x}$  on the space  $L_2(\mu(\cdot, q\mathbf{t}))$ . This right inverse operator will be denoted by  $\mathcal{T}_{\mathbf{t}}$ . The operator  $\mathcal{T}_{\mathbf{t}}$  is defined first on the big  $q$ -Jacobi polynomials and then extended by linearity to  $L_2(\mu(\cdot, q\mathbf{t}))$ , in which the big  $q$ -Jacobi polynomials are complete. The operator  $\mathcal{T}_{\mathbf{t}}$  is also a discrete integral operator. The rest of the paper is devoted to the study of the properties of  $\mathcal{T}_{\mathbf{t}}$  and its eigenfunctions. It turns out that the matrix representation of  $\mathcal{T}_{\mathbf{t}}$  in the basis formed by the big  $q$ -Jacobi polynomials is tridiagonal. This gives a three-term recurrence relation for the coefficients in the expansion of the eigenfunctions of  $\mathcal{T}_{\mathbf{t}}$  in big  $q$ -Jacobi polynomials.

In Section 3, we find the polynomial solution of the recurrence relation for the coefficients in the expansion of the eigenfunctions of the operator  $\mathcal{T}_{\mathbf{t}}$

from Section 2. The recurrence relation is identified with that of the little  $q$ -Jacobi polynomials.

In Section 4, we find the expansion of the formal eigenfunctions of the operator  $D_{q,x}$  on the space  $L_2(\mu(\cdot, \mathbf{t}))$  in terms of  $\{p_n(x, \mathbf{t})\}$ . The first coefficient of the expansion is used to determine the spectrum of the inverse operator  $\mathcal{T}_{\mathbf{t}}$ . This eigenfunction expansion gives a discrete  $q$ -analogue (on a  $q$ -linear lattice) of the well-known expansion of a plane wave  $\exp(i\lambda x)$  in Jacobi polynomials.

Section 5 contains asymptotic properties of the orthogonal polynomials found in Section 3 and a formula for the Stieltjes transform of the measure of orthogonality. We prove that the measure of orthogonality is purely discrete and we identify the location of its masses with the discrete zero-set of an entire function and show how it is related to the spectrum of the operator  $\mathcal{T}_{\mathbf{t}}$ .

The paper concludes with Section 6, where we find the connection coefficients between families of big  $q$ -Jacobi polynomials corresponding to different values of the parameter  $\mathbf{t}$ .

## 2. The Spectral Analysis.

Let  $\mathbf{t} = (t_1, t_2)$ . By  $q\mathbf{t}$  we shall denote the pair  $(qt_1, qt_2)$ . The Hilbert space  $L_2(\mu(\cdot, \mathbf{t}))$  is defined through the standard dot-product

$$\langle f, g \rangle = \int_{\mathbf{R}} f(x) \overline{g(x)} d\mu(x, \mathbf{t}),$$

and

$$L_2(\mu(\cdot, \mathbf{t})) = \left\{ f : \|f\|_{L_2(\mu(\cdot, \mathbf{t}))} = \left( \int_{\mathbf{R}} |f(x)|^2 d\mu(x, \mathbf{t}) \right)^{1/2} < \infty \right\}.$$

Let  $\mathcal{T}_{\mathbf{t}}$  denote the right inverse operator of  $D_{q,x}$  on  $L_2(\mu(x, \mathbf{t}))$ , that is,  $\mathcal{T}_{\mathbf{t}}$  is a linear operator from  $L_2(\mu(x, q\mathbf{t}))$  to  $L_2(\mu(x, \mathbf{t}))$  such that  $D_{q,x}\mathcal{T}_{\mathbf{t}}$  is the identity operator on the range of  $D_{q,x}$  acting on  $L_2(\mu(x, \mathbf{t}))$ . From (1.3) and (1.1) we find

$$\begin{aligned} (2.1) \quad D_{q,x}p_n(x, \mathbf{t}) &= \frac{t_1 q^{1-n}(1-q^n)(1-at_1t_2q^{n-1})}{(1-q)(1-t_1)(1-at_1)} p_{n-1}(x, q\mathbf{t}) \\ &=: \sigma_n(\mathbf{t}) p_{n-1}(x, q\mathbf{t}). \end{aligned}$$

Thus we require  $\mathcal{T}_{\mathbf{t}}$  to satisfy

$$(2.2) \quad \mathcal{T}_{\mathbf{t}}g(x) \sim \sum_{n=0}^{\infty} (g_n/\sigma_{n+1}(\mathbf{t})) p_{n+1}(x, \mathbf{t})$$

if

$$g(x) \sim \sum_{n=0}^{\infty} g_n p_n(x, q\mathbf{t}).$$

The operator  $\mathcal{T}_{\mathbf{t}}$  can be expressed as an integral operator as well. From the orthogonality relation (1.6) we get

$$g_n = \frac{1}{\xi_n(q\mathbf{t})} \int_{\mathbf{R}} g(u) p_n(u, q\mathbf{t}) d\mu(u, q\mathbf{t}).$$

Substituting in (2.2) and formally interchanging the order of integration and summation we get

$$(2.3) \quad \mathcal{T}_{\mathbf{t}} g(x) = \int_{\mathbf{R}} g(u) \left\{ \sum_{n=0}^{\infty} \frac{p_n(u, q\mathbf{t}) p_{n+1}(x, \mathbf{t})}{\xi_n(q\mathbf{t}) \sigma_{n+1}(\mathbf{t})} \right\} d\mu(u, q\mathbf{t}).$$

The sum in (2.3) is the kernel of the integral operator  $\mathcal{T}_{\mathbf{t}}$ .

We now consider the eigenvalue problem for the operator  $\mathcal{T}_{\mathbf{t}}$ , namely

$$(2.4) \quad \mathcal{T}_{\mathbf{t}} g = \lambda g, \quad g(x) = \sum_{n=0}^{\infty} a_n(\lambda, \mathbf{t}) p_n(x, \mathbf{t}),$$

with  $g \in L_2(\mu(\cdot, \mathbf{t})) \cap L_2(\mu(\cdot, q\mathbf{t}))$ . The function  $g$  can be expanded in terms of the polynomials  $\{p_n(x, \mathbf{t})\}$  since they are dense in  $L_2(\mu(\cdot, \mathbf{t}))$ . Furthermore, (2.2) implies  $a_0(\lambda, \mathbf{t}) = 0$  since  $g$  is in the range of  $\mathcal{T}_{\mathbf{t}}$ .

We will need a connection coefficient formula of the form

$$p_n(x, \mathbf{t}) = \sum_{j=0}^n c_{n,j}(\mathbf{t}) p_j(x, q\mathbf{t}).$$

Such formula we can get using a simple duality theorem, [15, Theorem 2.5]. Let  $\mu$  be a measure,  $w$  and  $\rho$ , weight functions, and  $\{p_n\}$  and  $\{q_n\}$ , polynomials orthogonal with respect to  $w\mu$  and  $\rho\mu$ , respectively. Let  $\alpha_n = \int |p_n|^2 w d\mu$  and  $\beta_n = \int |q_n|^2 \rho d\mu$ . If

$$w(x) p_n(x) \sim \rho(x) \sum_{j=n}^{\infty} c_{n,j} q_j(x), \quad \text{then} \quad q_n(x) = \sum_{k=0}^n (\beta_n / \alpha_k) c_{k,n} p_k(x).$$

Indeed, if  $q_n = \sum_{k=0}^n d_{n,k} p_k$ , then  $d_{n,k} = (1/\alpha_k) \int q_n p_k w d\mu = (\beta_n / \alpha_k) c_{k,n}$ .

Moreover, if  $w/\rho$  is a polynomial of degree  $s$ , then  $c_{n,j} = 0$  for  $j > n + s$ . In our case with  $w(x) = 1/(xqt_1, xqt_2; q)_{\infty}$ ,  $\rho(x) = 1/(xt_1, xt_2; q)_{\infty}$ , and  $\mu = \mu^{(a)}$  we have  $w(x)/\rho(x) = (1 - t_1 x)(1 - t_2 x)$ . Thus

$$(2.5) \quad p_n(x, \mathbf{t}) = \sum_{m=n-2}^n c_{m,n}(\mathbf{t}) (\xi_n(\mathbf{t}) / \xi_m(q\mathbf{t})) p_m(x, q\mathbf{t}), \quad n \geq 0,$$

where the coefficients  $\{c_{m,n}(\mathbf{t})\}$  satisfy the equation

$$(2.6) \quad (1 - t_1 x)(1 - t_2 x)p_n(x, q\mathbf{t}) = \sum_{k=n}^{n+2} c_{n,k}(\mathbf{t})p_k(x, \mathbf{t}), \quad n \geq 0.$$

The coefficients in (2.6) can be computed explicitly. Comparing the coefficients of  $x^{n+2}$  in (2.6) we get

$$(2.7) \quad c_{n,n+2}(\mathbf{t}) = \frac{(q^{-n}, at_1 t_2 q^{n+1}; q)_n}{(q^{-n-2}, at_1 t_2 q^{n+1}; q)_{n+2}} \frac{(t_1, at_1, q; q)_{n+2}}{(qt_1, aqt_1, q; q)_n} \frac{q^n (-qt_1)^n q^{\binom{n}{2}}}{q^{n+2} (-t_1)^{n+2} q^{\binom{n+2}{2}}} t_1 t_2$$

$$= \frac{(1 - t_1)(1 - at_1)(1 - t_1 q^{n+1})(1 - at_1 q^{n+1})}{(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2})} (t_2/t_1) q^n.$$

From (1.3) and (1.8) we have  $p_m(1/t_1; t_1, t_2) = 1$  and

$$p_m(1/t_2; t_1, t_2) = t_1^m(t_2, at_2; q)_m / (t_2^m(t_1, at_1; q)_m).$$

Substituting in (2.6)  $x$  first by  $1/t_1$  and then by  $1/t_2$  we obtain

$$(2.8) \quad c_{n,n}(\mathbf{t}) + c_{n,n+1}(\mathbf{t}) + c_{n,n+2}(\mathbf{t}) = 0,$$

$$(2.9) \quad c_{n,n}(\mathbf{t}) + \alpha_n c_{n,n+1}(\mathbf{t}) + \alpha_n \alpha_{n+1} c_{n,n+2}(\mathbf{t}) = 0,$$

where

$$\alpha_n := \frac{t_1(1 - t_2 q^n)(1 - at_2 q^n)}{t_2(1 - t_1 q^n)(1 - at_1 q^n)}.$$

From (2.8) and (2.9) we obtain

$$(2.10) \quad c_{n,n}(\mathbf{t}) = \frac{\alpha_n(\alpha_{n+1} - 1)}{\alpha_n - 1} c_{n,n+2}(\mathbf{t})$$

$$= \frac{(1 - t_1)(1 - at_1)(1 - t_2 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})} q^n.$$

Applying  $D_{q,x}$  to both sides of (2.4) and using (2.1) and (2.5) we obtain

$$D_{q,x}(\lambda g(x)) = \lambda \sum_{n=1}^{\infty} a_n(\lambda, \mathbf{t}) \sigma_n(\mathbf{t}) p_{n-1}(x, q\mathbf{t}) = g(x)$$

$$= \sum_{m=1}^{\infty} a_m(\lambda, \mathbf{t}) p_m(x, \mathbf{t})$$

$$= \sum_{m=1}^{\infty} a_m(\lambda, \mathbf{t}) \sum_{j=m-2}^m (\xi_m(\mathbf{t})/\xi_j(q\mathbf{t})) c_{j,m}(\mathbf{t}) p_j(x, q\mathbf{t}).$$

The above identity implies that the coefficients  $\{a_n(\lambda, \mathbf{t})\}$  are generated by

$$(2.11) \quad \lambda a_n(\lambda, \mathbf{t}) \sigma_n(\mathbf{t}) = \sum_{m=n-1}^{n+1} (\xi_m(\mathbf{t})/\xi_{n-1}(q\mathbf{t})) c_{n-1,m}(\mathbf{t}) a_m(\lambda, \mathbf{t}), \quad n \geq 1,$$

$a_0(\lambda, \mathbf{t}) = 0$  and  $a_1(\lambda, \mathbf{t}) \neq 0$  is arbitrary. Formula (2.11) and the initial conditions show that  $a_n(\lambda, \mathbf{t})/a_1(\lambda, \mathbf{t})$  is a polynomial of degree  $n - 1$ . We set

$$\tilde{a}_{m-1}(\lambda, \mathbf{t}) := \xi_m(\mathbf{t}) a_m(\lambda, \mathbf{t}) / (\xi_1(\mathbf{t}) a_1(\lambda, \mathbf{t})), \quad m \geq 1,$$

$\tilde{a}_{-1}(\lambda, \mathbf{t}) := 0$  and  $c_{n,m}(\mathbf{t}) := (1 - t_1)(1 - at_1)\tilde{c}_{n,m}(\mathbf{t})$ . Since

$$\frac{\sigma_n(\mathbf{t})\xi_{n-1}(q\mathbf{t})}{\xi_n(\mathbf{t})} = \frac{(1 - t_1)(1 - at_1)}{-at_1(1 - q)},$$

formula (2.11) can be written in the form

$$(2.12) \quad \lambda \tilde{a}_n(\lambda, \mathbf{t}) = -at_1(1 - q) \sum_{m=n}^{n+2} \tilde{c}_{n,m}(\mathbf{t}) \tilde{a}_{m-1}(\lambda, \mathbf{t}), \quad n \geq 0.$$

In terms of the variable  $\eta$  and the functions  $\tilde{b}_m(\eta, \mathbf{t})$  defined by

$$\eta := -\lambda/(at_1(1 - q)) \quad \text{and} \quad \tilde{b}_m(\eta, \mathbf{t}) := \tilde{a}_m(-at_1(1 - q)\eta, \mathbf{t}),$$

formula (2.12) can be written in the form

$$(2.13) \quad \eta \tilde{b}_n(\eta, \mathbf{t}) = \sum_{m=n}^{n+2} \tilde{c}_{n,m}(\mathbf{t}) \tilde{b}_{m-1}(\eta, \mathbf{t}), \quad n \geq 0$$

with the initial conditions

$$\tilde{b}_{-1}(\eta, \mathbf{t}) = 0, \quad \tilde{b}_0(\eta, \mathbf{t}) = 1.$$

From (2.10), (2.7) and (2.8) for the coefficients we obtain

$$(2.14) \quad \tilde{c}_{n,n} = \tilde{c}_{n,n}(\mathbf{t}) = \frac{(1 - t_2 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})} q^n,$$

$$(2.15) \quad \tilde{c}_{n,n+2} = \tilde{c}_{n,n+2}(\mathbf{t}) = \frac{(1 - t_1 q^{n+1})(1 - at_1 q^{n+1})t_2}{(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2})t_1} q^n,$$

$$(2.16) \quad \tilde{c}_{n,n+1} = \tilde{c}_{n,n+1}(\mathbf{t}) = -(\tilde{c}_{n,n}(\mathbf{t}) + \tilde{c}_{n,n+2}(\mathbf{t})).$$

It is convenient to have (2.13) written in monic form. Let

$$\tilde{b}_m(\eta, \mathbf{t}) := G_m(\mathbf{t}) b_m(\eta, \mathbf{t}).$$



From (2.13) we have

$$\begin{aligned} \eta b_n(\eta, \mathbf{t}) &= \frac{\tilde{c}_{n,n} G_{n-1}(\mathbf{t})}{G_n(\mathbf{t})} b_{n-1}(\eta, \mathbf{t}) + \tilde{c}_{n,n+1} b_n(\eta, \mathbf{t}) \\ &\quad + \frac{\tilde{c}_{n,n+2} G_{n+1}(\mathbf{t})}{G_n(\mathbf{t})} b_{n+1}(\eta, \mathbf{t}), \quad n \geq 0, \end{aligned}$$

which is a monic equation if the coefficient of  $b_{n+1}(\eta, \mathbf{t})$  is 1, that is, if  $G_{n+1}(\mathbf{t}) = G_n(\mathbf{t})/\tilde{c}_{n,n+2}$ . In this case

$$G_n(\mathbf{t}) = G_0(\mathbf{t}) / \prod_{j=0}^{n-1} \tilde{c}_{j,j+2}, \quad n \geq 1, \quad G_0(\mathbf{t}) \neq 0,$$

and (2.13) takes the form

$$(2.17) \quad \eta b_n(\eta, \mathbf{t}) = b_{n+1}(\eta, \mathbf{t}) + \tilde{c}_{n,n+1} b_n(\eta, \mathbf{t}) + \tilde{c}_{n,n} \tilde{c}_{n-1,n+1} b_{n-1}(\eta, \mathbf{t}), \quad n \geq 0.$$

We set  $b_0(\eta, \mathbf{t}) = G_0(\mathbf{t}) = 1$  and  $b_{-1}(\eta, \mathbf{t}) = 0$ . When the coefficients in (2.17) are written explicitly in terms of  $t_1$ ,  $t_2$ , and  $n$ , we obtain

$$(2.18) \quad \begin{aligned} b_{n+1}(\eta, \mathbf{t}) &= \left( \eta + \frac{(1 - t_2 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})} q^n \right. \\ &\quad \left. + \frac{(1 - t_1 q^{n+1})(1 - at_1 q^{n+1}) t_2}{(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2}) t_1} q^n \right) b_n(\eta, \mathbf{t}) \\ &\quad - \frac{(1 - t_1 q^n)(1 - at_1 q^n)(1 - t_2 q^n)(1 - at_2 q^n) t_2}{(1 - at_1 t_2 q^{2n-1})(1 - at_1 t_2 q^{2n})^2 (1 - at_1 t_2 q^{2n+1}) t_1} q^{2n-1} b_{n-1}(\eta, \mathbf{t}). \end{aligned}$$

We now apply Schwartz's theorem (Theorem 1.1). With  $B_n(\mathbf{t}) = -t_1 \tilde{c}_{n,n+1}(\mathbf{t})$ ,  $C_n(\mathbf{t}) = -t_1^2 \tilde{c}_{n,n}(\mathbf{t}) \tilde{c}_{n-1,n+1}(\mathbf{t})$ , and  $\hat{b}_n(\eta, \mathbf{t}) = t_1^n b_n(\eta/t_1, \mathbf{t})$ , (2.18) takes the form

$$\hat{b}_{n+1}(\eta, \mathbf{t}) = (\eta + B_n(\mathbf{t})) \hat{b}_n(\eta, \mathbf{t}) + C_n(\mathbf{t}) \hat{b}_{n-1}(\eta, \mathbf{t}), \quad n \geq 0.$$

Furthermore, by (2.14)-(2.16) we have  $B_n(q^\nu \mathbf{t}) = B_{n+\nu}(\mathbf{t})$ ,  $C_n(q^\nu \mathbf{t}) = C_{n+\nu}(\mathbf{t})$ ,

$$\sum_{n=0}^{\infty} |B_n(\mathbf{t})| < \infty, \quad \text{and} \quad \sum_{n=0}^{\infty} |C_n(\mathbf{t})| < \infty.$$

Then by Theorem 1.1,  $\eta^n \hat{b}_n(1/\eta, \mathbf{t}) = (t_1 \eta)^n b_n(1/(t_1 \eta), \mathbf{t})$ , or equivalently,  $\eta^n b_n(1/\eta, \mathbf{t})$  converges locally uniformly in the complex plane to an entire function of  $\eta$ .

### 3. The Polynomial Solution of the Recurrence Equation for the Coefficients.

In this section we solve recurrence equation (2.18):

$$(3.1) \quad b_{n+1}(\eta, \mathbf{t}) = (\eta + \beta_n q^n) b_n(\eta, \mathbf{t}) - \gamma_n q^{2n-1} b_{n-1}(\eta, \mathbf{t}), \quad n \geq 0,$$

with  $b_{-1}(\eta, \mathbf{t}) = 0$ ,  $b_0(\eta, \mathbf{t}) = 1$ , and

$$(3.2) \quad \beta_n = \frac{(1 - t_2 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})} + \frac{(1 - t_1 q^{n+1})(1 - at_1 q^{n+1})t_2}{(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2})t_1},$$

$$(3.3) \quad \gamma_n = \frac{(1 - t_1 q^n)(1 - at_1 q^n)(1 - t_2 q^n)(1 - at_2 q^n)t_2}{(1 - at_1 t_2 q^{2n-1})(1 - at_1 t_2 q^{2n})^2(1 - at_1 t_2 q^{2n+1})t_1}.$$

The coefficients  $\beta_n$  can be simplified. We have

$$\begin{aligned} t_1(at_1 t_2 q^{2n}; q)_3 \beta_n &= t_1(1 - t_2 q^n)(1 - at_2 q^n)(1 - at_1 t_2 q^{2n+2}) \\ &\quad + t_2(1 - t_1 q^{n+1})(1 - at_1 q^{n+1})(1 - at_1 t_2 q^{2n}) \\ &= t_1 \left( 1 - (1 + a)t_2 q^n + at_2^2 q^{2n} - at_1 t_2 q^{2n+2} \right. \\ &\quad \left. + a(1 + a)t_1 t_2^2 q^{3n+2} - a^2 t_1 t_2^3 q^{4n+2} \right) \\ &\quad + t_2 \left( 1 - (1 + a)t_1 q^{n+1} + at_1^2 q^{2n+2} - at_1 t_2 q^{2n} \right. \\ &\quad \left. + a(1 + a)t_1^2 t_2 q^{3n+1} - a^2 t_1^3 t_2 q^{4n+2} \right) \\ &= (t_1 + t_2)(1 - a^2 t_1^2 t_2^2 q^{4n+2}) - (1 + a)t_1 t_2 q^n(1 + q) \\ &\quad + a(1 + a)t_1^2 t_2^2 q^{3n+1}(1 + q) \\ &= (1 - at_1 t_2 q^{2n+1}) \left( (t_1 + t_2)(1 + at_1 t_2 q^{2n+1}) \right. \\ &\quad \left. - (1 + a)(1 + q)t_1 t_2 q^n \right). \end{aligned}$$

Therefore,

$$(3.4) \quad \beta_n = \frac{(t_1 + t_2)(1 + at_1 t_2 q^{2n+1}) - (1 + a)(1 + q)t_1 t_2 q^n}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+2})t_1}.$$

Recurrence relation (3.1) can be identified with the recurrence relation of the associated little  $q$ -Jacobi polynomials, [5]. The latter work gets the little  $q$ -Jacobi polynomials as limiting cases of the associated big  $q$ -Jacobi polynomials and does not give an explicit representation for the polynomials. Theorem 4.1 below provides an explicit representation for the little  $q$ -Jacobi polynomials.

We shall use (3.1)–(3.4) to find the coefficient  $\beta_{n,n-1}$  of  $\eta^{n-1}$  in  $b_n(\eta, \mathbf{t})$  and to guess the structure of the polynomials  $b_n(\eta, \mathbf{t})$ . From (3.1) we get

$\beta_{n+1,n} = \beta_n q^n + \beta_{n,n-1}$  which implies  $\beta_{n,n-1} = \sum_{j=0}^{n-1} \beta_j q^j$ . From (3.4) we have

$$a(1-q)t_1\beta_j q^j = a(t_1+t_2)q^j \left( \frac{1}{1-at_1t_2q^{2j}} - \frac{q}{1-at_1t_2q^{2j+2}} \right) \\ - (1+a) \left( \frac{1}{1-at_1t_2q^{2j}} - \frac{1}{1-at_1t_2q^{2j+2}} \right) =: R(q^j) - R(q^{j+1}),$$

where

$$R(z) = \frac{a(t_1+t_2)z - (1+a)}{1-at_1t_2z^2}.$$

Then  $\beta_{n,n-1}$  becomes a telescoping sum and we find

$$a(1-q)t_1\beta_{n,n-1} \\ = (R(1) + 1) - (R(q^n) + 1) \\ = -\frac{a(1-t_1)(1-t_2)}{(1-at_1t_2)} + \frac{a(1-t_1q^n)(1-t_2q^n)}{(1-at_1t_2q^{2n})} \\ = \frac{a(1-t_1q^n)(1-t_2q^n)}{(1-at_1t_2q^{2n})} \left( 1 - \frac{(1-t_1)(1-t_2)(1-at_1t_2q^{2n})}{(1-at_1t_2)(1-t_1q^n)(1-t_2q^n)} \right).$$

The coefficient  $\beta_{n,n-1}$  can be written in a form that resembles similar formulas for the Wimp polynomials from [16] and their  $q$ -analogue from [8]:

$$(3.5) \quad \beta_{n,n-1} = -\frac{(1-q^{-n}/t_1)(1-q^{-n}/t_2)}{(1-q)(1-q^{-2n}/(at_1t_2))} (at_1)^{-1} \\ \times \left( 1 + \frac{(q^{-1}, t_1, t_2, at_1t_2q^{2n}; q)_1 q}{(at_1t_2, t_1q^n, t_2q^n, q; q)_1} \right).$$

The analogue of these polynomials that solves (3.1) is defined below.

**Theorem 3.1.** *The polynomials  $b_n(\eta, \mathbf{t})$  defined by*

$$(3.6) \quad b_n(\eta, \mathbf{t}) = \sum_{j=0}^n \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_j}{(q, q^{-2n}/(at_1t_2); q)_j} (-at_1)^{-j} \eta^{n-j} \\ \times {}_4\phi_3 \left( \begin{matrix} q^{-j}, t_1, t_2, at_1t_2q^{2n+1-j} \\ t_1q^{n+1-j}, t_2q^{n+1-j}, at_1t_2 \end{matrix} \middle| q, q \right), \quad n \geq 0$$

are the solutions of the recurrence relation (3.1)-(3.4).

*Proof.* The proof is similar to the proof of Theorem 4.1 in [9]. Let  $\tilde{b}_n(\eta, \mathbf{t})$  denote the polynomial on the right-hand side of (3.6). We shall demonstrate that  $\tilde{b}_n(\eta, \mathbf{t})$  satisfies (3.1).

The polynomials  $\tilde{b}_n(\eta, \mathbf{t})$  can be written in the form

$$\begin{aligned}
 (3.7) \quad \tilde{b}_n(\eta, \mathbf{t}) &= \sum_{j=0}^n \eta^{n-j} \sum_{k=0}^j \frac{(-1)^j}{(at_1)^j} \frac{(t_1, t_2; q)_k}{(q, at_1 t_2; q)_k} \\
 &\quad \times \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_j}{(t_1 q^{n+1-j}, t_2 q^{n+1-j}; q)_k} \frac{(at_1 t_2 q^{2n+1-j}; q)_k}{(q^{-2n}/(at_1 t_2); q)_j} \frac{(q^{-j}; q)_k q^k}{(q; q)_j} \\
 &= \sum_{j=0}^n \eta^{n-j} \sum_{k=0}^j \frac{(-1)^j}{(at_1)^j} \frac{(t_1, t_2; q)_k}{(q, at_1 t_2; q)_k} a^k \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{j-k}}{(q, q^{-2n}/(at_1 t_2); q)_{j-k}} \\
 &=: \sum_{j=0}^n \sum_{k=0}^j A_k B_{j-k}^{(n)} (-1/t_1)^j a^{k-j} \eta^{n-j},
 \end{aligned}$$

where we applied formula (1.15) with  $A = 1/(t_1 q^n)$ ,  $1/(t_2 q^n)$ ,  $1/(at_1 t_2 q^{2n})$ , and  $q$ , and we defined

$$\begin{aligned}
 A_k &:= \frac{(t_1, t_2; q)_k}{(q, at_1 t_2; q)_k}, \\
 B_s^{(n)} &:= \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_s}{(q, q^{-2n}/(at_1 t_2); q)_s}.
 \end{aligned}$$

We separate the leading term  $\eta^{n+1}$  and write  $\tilde{b}_{n+1}(\eta, \mathbf{t})$  in the form

$$\begin{aligned}
 (3.8) \quad \tilde{b}_{n+1}(\eta, \mathbf{t}) &= \eta \left( \tilde{b}_n(\eta, \mathbf{t}) - \sum_{j=1}^n \sum_{k=0}^j A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n-j} \right) \\
 &\quad + \sum_{j=1}^{n+1} \sum_{k=0}^j A_k B_{j-k}^{(n+1)} a^{k-j} (-1/t_1)^j \eta^{n+1-j} \\
 &= (\eta + \beta_n q^n) \tilde{b}_n(\eta, \mathbf{t}) - \tilde{r}_n(\eta, \mathbf{t}),
 \end{aligned}$$

where

$$\begin{aligned}
 (3.9) \quad \tilde{r}_n(\eta, \mathbf{t}) &:= \sum_{j=1}^n \sum_{k=0}^j A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n+1-j} \\
 &\quad - \sum_{j=1}^{n+1} \sum_{k=0}^j A_k B_{j-k}^{(n+1)} a^{k-j} (-1/t_1)^j \eta^{n+1-j} \\
 &\quad + \beta_n q^n \sum_{j=0}^n \sum_{k=0}^j A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n-j} \\
 &= \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n)} a^{k-j-1} (-1/t_1)^{j+1} \eta^{n-j} \\
 &\quad - \sum_{j=0}^n \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n+1)} a^{k-j-1} (-1/t_1)^{j+1} \eta^{n-j} \\
 &\quad + \beta_n q^n \sum_{j=0}^n \sum_{k=0}^j A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n-j}.
 \end{aligned}$$

The coefficient of  $\eta^n$  in  $\tilde{r}_n(\eta, \mathbf{t})$  equals

$$\begin{aligned}
 &\sum_{k=0}^1 A_k \left( B_{1-k}^{(n)} - B_{1-k}^{(n+1)} \right) a^{k-1} (-1/t_1) + \beta_n q^n \\
 &= - \left\{ B_1^{(n)} - B_1^{(n+1)} \right\} / (at_1) + \beta_n q^n
 \end{aligned}$$

since  $A_0 = 1$  and  $B_0^{(n)} = 1$ . Furthermore,

$$B_1^{(n)} - B_1^{(n+1)} = -\frac{a}{1-q} \left( \frac{(1-t_1 q^n)(1-t_2 q^n)}{1-at_1 t_2 q^{2n}} - \frac{(1-t_1 q^{n+1})(1-t_2 q^{n+1})}{1-at_1 t_2 q^{2n+2}} \right)$$

and then

$$\begin{aligned}
 &- (at_1 t_2 q^{2n}; q^2)_2 ((1-q)/a) \left\{ B_1^{(n)} - B_1^{(n+1)} \right\} \\
 &= (1-t_1 q^n)(1-t_2 q^n)(1-at_1 t_2 q^{2n+2}) \\
 &\quad - (1-t_1 q^{n+1})(1-t_2 q^{n+1})(1-at_1 t_2 q^{2n}) \\
 &= -q^n \left( (t_1 + t_2)(1-q)(1+at_1 t_2 q^{2n+1}) - (1+a)(1-q^2)t_1 t_2 q^n \right).
 \end{aligned}$$

Hence in view of (3.4) we get

$$(3.10) \quad - \left\{ B_1^{(n)} - B_1^{(n+1)} \right\} / (at_1) = -\beta_n q^n,$$

which shows that the coefficient of  $\eta^n$  in  $\tilde{r}_n(\eta, \mathbf{t})$  is zero. Then in (3.9) we can replace the lower bound of the range of  $j$  by 1 and then replace  $j$  by

$j + 1$ . We obtain

$$\begin{aligned}
 (3.11) \quad \tilde{r}_n(\eta, \mathbf{t}) &= \beta_n q^n \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n)} a^{k-j-1} (-1/t_1)^{j+1} \eta^{n-j-1} \\
 &\quad + \sum_{j=0}^{n-2} \sum_{k=0}^{j+2} A_k B_{j+2-k}^{(n)} a^{k-j-2} (-1/t_1)^{j+2} \eta^{n-j-1} \\
 &\quad - \sum_{j=0}^{n-1} \sum_{k=0}^{j+2} A_k B_{j+2-k}^{(n+1)} a^{k-j-2} (-1/t_1)^{j+2} \eta^{n-j-1}.
 \end{aligned}$$

Note that in (3.11) we can first separate the two constant terms, and then in the last two double sums we can replace the upper bound  $j + 2$  of the range of  $k$  by  $j + 1$  using that  $B_{j+2-k}^{(n)} - B_{j+2-k}^{(n+1)} = 0$  if  $k = j + 2$ . Then we can write

$$(3.12) \quad \tilde{r}_n(\eta, \mathbf{t}) = K_n + \sum_{j=0}^{n-2} \sum_{k=0}^{j+1} A_k a^{k-j-2} (-1/t_1)^{j+2} \Delta_{j+2-k}^{(n)} \eta^{n-j-1},$$

where

$$(3.13) \quad \Delta_s^{(n)} = B_s^{(n)} - B_s^{(n+1)} - at_1 \beta_n q^n B_{s-1}^{(n)}, \quad s = 1, \dots, n,$$

and  $K_n$  is the constant term of  $\tilde{r}_n(\eta, \mathbf{t})$ .

From (3.10) we get  $\Delta_1^{(n)} = 0$ .

For  $s \in \{2, \dots, n\}$  we have

$$\begin{aligned}
 (3.14) \quad \Delta_s^{(n)} &= \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_s}{(q, q^{-2n}/(at_1 t_2); q)_s} - \frac{(q^{-n-1}/t_1, q^{-n-1}/t_2; q)_s}{(q, q^{-2n-2}/(at_1 t_2); q)_s} \\
 &\quad - \frac{q^{-n-1}}{t_1 t_2} \frac{((t_1 + t_2)(1 + q^{-2n-1}/(at_1 t_2)) - (1 + a)(1 + q)q^{-n-1}/a)}{(1 - q^{-2n}/(at_1 t_2))(1 - q^{-2n-2}/(at_1 t_2))} \\
 &\quad \times \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{s-1}}{(q, q^{-2n}/(at_1 t_2); q)_{s-1}} \\
 &= \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{s-1}}{(q; q)_s (q^{-2n-2}/(at_1 t_2); q)_{s+2} (1 - q^{-2n}/(at_1 t_2))} \\
 &\quad \times \left\{ [(1 - q^{-n-1+s}/t_1)(1 - q^{-n-1+s}/t_2)(q^{-2n-2}/(at_1 t_2); q)_2 \right. \\
 &\quad \left. - (1 - q^{-n-1}/t_1)(1 - q^{-n-1}/t_2)(q^{-2n-2+s}/(at_1 t_2); q)_2] \right. \\
 &\quad \times (1 - q^{-2n}/(at_1 t_2)) - (q^{-n-1}/(t_1 t_2))(1 - q^s) \\
 &\quad \times ((t_1 + t_2)(1 + q^{-2n-1}/(at_1 t_2)) - (1 + a)(1 + q)q^{-n-1}/a) \\
 &\quad \left. \times (1 - q^{-2n-1}/(at_1 t_2))(1 - q^{-2n-1+s}/(at_1 t_2)) \right\}.
 \end{aligned}$$

We set  $x = q^{-n-1}/t_1$ ,  $y = q^{-n-1}/t_2$ ,  $\alpha = q^s$ ,  $\beta = 1/a$ ,  $S = x + y$ , and  $P = xy$ . Let  $E_1$  and  $E_2$  denote the expressions inside  $\{ \}$  and  $[ ]$  in (3.14), respectively. Then

$$E_1 = E_2(1 - q^2\beta P) \\ - (1 - \alpha)(S(1 + q\beta P) - (1 + \beta)(1 + q)P)(1 - q\beta P)(1 - q\alpha\beta P)$$

and

$$E_2 = (1 - \alpha x)(1 - \alpha y)(1 - \beta P)(1 - q\beta P) \\ - (1 - x)(1 - y)(1 - \alpha\beta P)(1 - q\alpha\beta P).$$

We now simplify  $E_2$ . We have

$$(3.15) \\ E_2 = 1 - \alpha S + \alpha^2 P - (1 + q)\beta P(1 - \alpha S + \alpha^2 P) + q\beta^2 P^2(1 - \alpha S + \alpha^2 P) \\ - 1 + S - P + (1 + q)\alpha\beta P(1 - S + P) - q\alpha^2\beta^2 P^2(1 - S + P) \\ = (1 - \alpha)[S - (1 + \alpha)P - (1 + q)\beta P \\ + (1 + q)\alpha\beta P^2 + q(1 + \alpha)\beta^2 P^2 - q\alpha\beta^2 P^2 S] \\ = (1 - \alpha)[S(1 - q\alpha\beta^2 P^2) - (1 + \alpha)(1 - q\beta^2 P)P - (1 + q)\beta(1 - \alpha P)P].$$

From (3.15) and the definition of  $E_1$  and  $E_2$  we obtain

$$(3.16)$$

$$E_1/(1 - \alpha) = S[(1 - q\alpha\beta^2 P^2)(1 - q^2\beta P) - (1 - q^2\beta^2 P^2)(1 - q\alpha\beta P)] \\ - P[(1 + \alpha)(1 - q\beta^2 P) + (1 + q)\beta(1 - \alpha P)](1 - q^2\beta P) \\ - (1 + q)(1 + \beta)(1 - q\beta P)(1 - q\alpha\beta P) \\ =: B_1 S - B_2 P,$$

where  $B_1$  and  $B_2$  denote the expressions inside the brackets. We factor  $B_1$  and  $B_2$ :

$$(3.17) \\ B_1 = 1 - q\alpha\beta^2 P^2 - q^2\beta P + q^3\alpha\beta^3 P^3 - 1 + q^2\beta^2 P^2 + q\alpha\beta P - q^3\alpha\beta^3 P^3 \\ = q\beta P(\alpha + q\beta P - q - \alpha\beta P) = q\beta(\alpha - q)(1 - \beta P)P,$$

(3.18)

$$\begin{aligned}
B_2 &= (1 + \alpha + (1 + q)\beta - \beta(q\beta(1 + \alpha) + (1 + q)\alpha)P)(1 - q^2\beta P) \\
&\quad - (1 + q)(1 + \beta)(1 - q\beta(1 + \alpha)P + q^2\alpha\beta^2P^2) \\
&= \alpha - q + \beta((1 + \alpha)(q(1 + q)(1 + \beta) - q\beta - q^2) - (1 + q)(\alpha + q^2\beta))P \\
&\quad + q^2\beta^2(q(1 + \alpha)\beta + (1 + q)\alpha - (1 + q)\alpha(1 + \beta))P^2 \\
&= \alpha - q + \beta((1 + \alpha)(q + \beta q^2) - (1 + q)(\alpha + \beta q^2))P + q^2(q - \alpha)\beta^3P^2 \\
&= -(q - \alpha) + \beta(q - \alpha)(1 - \beta q^2)P + q^2(q - \alpha)\beta^3P^2 \\
&= -(q - \alpha)(1 - \beta P)(1 + q^2\beta^2P).
\end{aligned}$$

From (3.16)-(3.18) we obtain

$$\begin{aligned}
(3.19) \quad \frac{E_1}{(1 - \alpha)(q - \alpha)(1 - \beta P)} &= (-q\beta S + (1 + q^2\beta^2P))P \\
&= (-q\beta(x + y) + 1 + q^2\beta^2xy)xy = (1 - q\beta x)(1 - q\beta y)xy.
\end{aligned}$$

We recall that  $\alpha = q^s$ ,  $\beta = 1/a$ ,  $x = q^{-n-1}/t_1$ ,  $y = q^{-n-1}/t_2$ , and  $s \in \{2, \dots, n\}$ .

From (3.14)-(3.19) we get

(3.20)

$$\begin{aligned}
&\triangle_s^{(n)} \\
&= \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{s-1}q}{(1 - \alpha)(q - \alpha)(q; q)_{s-2}(q^{-2n-2}/(at_1t_2); q)_{s+2}(1 - q^{-2n}/(at_1t_2))} \\
&\quad \times (1 - \alpha)(q - \alpha)(1 - \beta xy)(1 - q\beta x)(1 - q\beta y)xy \\
&= \frac{(q^{-n+1}/t_1, q^{-n+1}/t_2; q)_{s-2}}{(q, q^{-2n+2}/(at_1t_2); q)_{s-2}} \\
&\quad \times \frac{(1 - q^{-n}/t_1)(1 - q^{-n}/t_2)(1 - q^{-n}/(at_1))(1 - q^{-n}/(at_2))q^{-2n-1}/(t_1t_2)}{(q^{-2n-1}/(at_1t_2); q)_3(1 - q^{-2n}/(at_1t_2))} \\
&= a^2t_1^2\gamma_nq^{2n-1}B_{s-2}^{(n-1)}.
\end{aligned}$$

At the end we used formula (3.3) for  $\gamma_n$ .

From (3.12) and (3.20) we obtain

$$(3.21) \quad \tilde{r}_n(\eta, \mathbf{t}) = \gamma_nq^{2n-1} \sum_{j=0}^{n-2} \sum_{k=0}^j A_k B_{j-k}^{(n-1)} a^{k-j} (-1/t_1)^j \eta^{n-1-j} + K_n,$$

where by (3.8),  $K_n$  is the constant term of

$$(3.22) \quad \tilde{r}_n(\eta, \mathbf{t}) = (\eta + \beta_nq^n)\tilde{b}_n(\eta, \mathbf{t}) - \tilde{b}_{n+1}(\eta, \mathbf{t}).$$

We recall that  $\tilde{b}_n(\eta, \mathbf{t})$  denotes the polynomial on the right-hand side of (3.6).



To complete the proof it remains to show that  $K_n$  equals  $\gamma_n q^{2n-1}$  times the constant term of  $\tilde{b}_{n-1}(\eta, \mathbf{t})$ . Let  $f_n$  denotes the constant term of  $\tilde{b}_n(\eta, \mathbf{t})$ . From (3.22) we have  $K_n = \beta_n q^n f_n - f_{n+1}$ . Therefore, it is enough to verify that

$$(3.23) \quad f_{n+1} = \beta_n q^n f_n - \gamma_n q^{2n-1} f_{n-1}.$$

From (3.6) and (1.15) we have

$$(3.24) \quad f_n = \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_n (-1)^n}{(q, q^{-2n}/(at_1 t_2); q)_n a^n t_1^n} \phi_n = \frac{(t_1 q, t_2 q; q)_n q^{\binom{n}{2}}}{(q, at_1 t_2 q^{n+1}; q)_n t_1^n} \phi_n,$$

with

$$(3.25) \quad \phi_n = {}_4\phi_3 \left( \begin{matrix} q^{-n}, at_1 t_2 q^{n+1}, t_1, t_2 \\ qt_1, qt_2, at_1 t_2 \end{matrix} \middle| q, q \right).$$

We shall use a recurrence formula for the Askey-Wilson polynomials (see [8] or [12]). The Askey-Wilson polynomials  $p_n(x; A, B, C, D | q)$  are defined by

$$(3.26) \quad p_n(x; A, B, C, D | q) := {}_4\phi_3 \left( \begin{matrix} q^{-n}, ABCDq^{n-1}, Ae^{i\theta}, Ae^{-i\theta} \\ AB, AC, AD \end{matrix} \middle| q, q \right),$$

where  $x = \cos \theta = (e^{i\theta} + e^{-i\theta})/2$ . They satisfy the recurrence equation

$$(3.27) \quad xp_n(x; A, B, C, D | q) = \frac{A_n}{2} p_{n+1}(x; A, B, C, D | q) + \frac{B_n}{2} p_n(x; A, B, C, D | q) + \frac{C_n}{2} p_{n-1}(x; A, B, C, D | q), \quad n \geq 0,$$

with  $p_{-1}(x; A, B, C, D | q) = 0$ ,  $p_0(x; A, B, C, D | q) = 1$ , and coefficients

$$(3.28) \quad A_n = \frac{(1 - ABq^n)(1 - ACq^n)(1 - ADq^n)(1 - ABCDq^{n-1})}{A(1 - ABCDq^{2n-1})(1 - ABCDq^{2n})},$$

$$(3.29) \quad C_n = \frac{A(1 - q^n)(1 - BCq^{n-1})(1 - BDq^{n-1})(1 - CDq^{n-1})}{(1 - ABCDq^{2n-2})(1 - ABCDq^{2n-1})},$$

$$(3.30) \quad B_n = A + 1/A - A_n - C_n.$$

As in Section 2 recurrence equation (3.27) can be written in monic form in terms of the polynomials

$$q_m = q_m(x; A, B, C, D | q) = p_m(x; A, B, C, D | q) 2^{-m} \prod_{j=0}^{m-1} A_j.$$

The monic equation is

$$(3.31) \quad xq_n = q_{n+1} + (B_n/2)q_n + (C_n A_{n-1}/4)q_{n-1}, \quad n \geq 0,$$

$$q_{-1} = 0, q_0 = 1.$$

We select  $A = \sqrt{t_1 t_2}$ ,  $B = q\sqrt{t_1/t_2}$ ,  $C = q\sqrt{t_2/t_1}$ ,  $D = a\sqrt{t_1 t_2}$  and  $e^{i\theta} = \sqrt{t_1/t_2}$ . Then  $x = \cos \theta = (t_1 + t_2)/(2\sqrt{t_1 t_2})$ , and from (3.24)-(3.31) we obtain

$$(3.32) \quad A_n = \frac{(1 - t_1 q^{n+1})(1 - t_2 q^{n+1})(1 - at_1 t_2 q^n)(1 - at_1 t_2 q^{n+1})}{\sqrt{t_1 t_2}(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2})},$$

$$(3.33) \quad C_n = \frac{\sqrt{t_1 t_2}(1 - q^n)(1 - q^{n+1})(1 - at_1 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})},$$

$$(3.34) \quad B_n = \sqrt{t_1 t_2} + 1/\sqrt{t_1 t_2} - A_n - C_n,$$

and

$$(3.35) \quad q_n = 2^{-n} \left( \prod_{j=0}^{n-1} A_j \right) \\ \times p_n \left( (t_1 + t_2)/(2\sqrt{t_1 t_2}); \sqrt{t_1 t_2}, q\sqrt{t_1/t_2}, q\sqrt{t_2/t_1}, a\sqrt{t_1 t_2} \mid q \right) \\ = \frac{(t_1 q, t_2 q, at_1 t_2, at_1 t_2 q; q)_n}{2^n (t_1 t_2)^{n/2} (at_1 t_2 q; q)_{2n}} \phi_n = \frac{(t_1 q, t_2 q, at_1 t_2; q)_n}{2^n (t_1 t_2)^{n/2} (at_1 t_2 q^{n+1}; q)_n} \phi_n \\ = \frac{(q, at_1 t_2; q)_n (t_1/t_2)^{n/2}}{2^n q^{\binom{n}{2}}} f_n.$$

Furthermore, from (3.31) and (3.34) we have

$$(3.36) \quad q_{n+1} = \left( \frac{t_1 + t_2}{2\sqrt{t_1 t_2}} - \frac{t_1 t_2 + 1}{2\sqrt{t_1 t_2}} + \frac{A_n}{2} + \frac{C_n}{2} \right) q_n - \frac{C_n A_{n-1}}{4} q_{n-1} \\ =: (R_n/(2\sqrt{t_1 t_2})) q_n - (C_n A_{n-1}/4) q_{n-1}.$$

From (3.35), (3.36), (3.32), (3.33), and (3.3) we get

$$(3.37) \quad f_{n+1} = \frac{R_n q^n}{(1 - q^{n+1})(1 - at_1 t_2 q^n) t_1} f_n - \frac{4q^{2n-1}}{(q^n, at_1 t_2 q^{n-1}; q)_{2t_1/t_2}} \\ \times \frac{1}{4} \frac{(1 - q^n)(1 - q^{n+1})(1 - at_1 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})} \\ \times \frac{(1 - t_1 q^n)(1 - t_2 q^n)(1 - at_1 t_2 q^{n-1})(1 - at_1 t_2 q^n)}{(1 - at_1 t_2 q^{2n-1})(1 - at_1 t_2 q^{2n})} f_{n-1} \\ = \frac{R_n q^n}{(1 - q^{n+1})(1 - at_1 t_2 q^n) t_1} f_n - \gamma_n q^{2n-1} f_{n-1}.$$

To complete the proof of (3.23) and the theorem we have to show that the coefficient of  $f_n$  in (3.37) equals  $\beta_n q^n$ . From (3.36), (3.32), and (3.33) we

get

$$\begin{aligned} (at_1t_2q^{2n}; q)_3 R_n &= -(1-t_1)(1-t_2)(at_1t_2q^{2n}; q)_3 \\ &+ (1-t_1q^{n+1})(1-t_2q^{n+1})(1-at_1t_2q^n)(1-at_1t_2q^{n+1})(1-at_1t_2q^{2n}) \\ &+ (1-q^n)(1-q^{n+1})(1-at_1q^n)(1-at_2q^n)(1-at_1t_2q^{2n+2})t_1t_2. \end{aligned}$$

Setting  $S = t_1 + t_2$ ,  $P = t_1t_2$ , and  $\alpha = q^n$  we obtain

$$\begin{aligned} (3.38) \quad (at_1t_2q^{2n}; q)_3 R_n &= -(1-S+P)(1-a\alpha^2P)(1-aq\alpha^2P)(1-aq^2\alpha^2P) \\ &+ (1-q\alpha S + q^2\alpha^2P)(1-a\alpha P)(1-aq\alpha P)(1-a\alpha^2P) \\ &+ (1-\alpha)(1-q\alpha)(1-a\alpha S + a^2\alpha^2P)(1-aq^2\alpha^2P)P \\ &=: E_1S + E_2P + E_3P. \end{aligned}$$

The expressions  $\{E_j\}_{j=1}^3$  are defined and factored below:

$$\begin{aligned} (3.39) \quad E_1 &:= (1-a\alpha^2P)(1-aq\alpha^2P)(1-aq^2\alpha^2P) \\ &\quad - q\alpha(1-a\alpha P)(1-aq\alpha P)(1-a\alpha^2P) \\ &\quad - a\alpha(1-\alpha)(1-q\alpha)(1-aq^2\alpha^2P)P \\ &= (1-a\alpha^2P) [1-aq\alpha^2P-aq^2\alpha^2P+a^2q^3\alpha^4P^2 \\ &\quad -q\alpha+aq\alpha^2P+aq^2\alpha^2P-a^2q^2\alpha^3P^2] \\ &\quad - a\alpha(1-\alpha)(1-q\alpha)(1-aq^2\alpha^2P)P \\ &= (1-q\alpha) ((1-a\alpha^2P)(1-a^2q^2\alpha^3P^2) \\ &\quad -a\alpha(1-\alpha)(1-aq^2\alpha^2P)P) \\ &= (1-q\alpha)(1+a^3q^2\alpha^5P^3-a\alpha P-a^2q^2\alpha^4P^2) \\ &= (1-q\alpha)(1-a\alpha P)(1-a^2q^2\alpha^4P^2), \end{aligned}$$

$$\begin{aligned} (3.40) \quad E_2 &:= (-(1-a\alpha^2P)(1-aq\alpha^2P)(1-aq^2\alpha^2P) \\ &\quad + (1-a\alpha P)(1-aq\alpha P)(1-a\alpha^2P)) / P \\ &= (1-a\alpha^2P)a\alpha(\alpha q + \alpha q^2 - aq^3\alpha^3P - 1 - q + aq\alpha P) \\ &= a\alpha(1-q\alpha)(1-a\alpha^2P) (-(1+q) + (1+q\alpha)aq\alpha P) \\ &= -a\alpha(1-q\alpha) [1+q-a\alpha((1+q)\alpha + q(1+q\alpha))P + a^2q\alpha^3(1+q\alpha)P^2], \end{aligned}$$

and

(3.41)

$$\begin{aligned}
 E_3 &:= -(1 - a\alpha^2 P)(1 - aq\alpha^2 P)(1 - aq^2\alpha^2 P) \\
 &\quad + q^2\alpha^2(1 - a\alpha P)(1 - aq\alpha P)(1 - a\alpha^2 P) \\
 &\quad + (1 - \alpha)(1 - q\alpha)(1 + a^2\alpha^2 P)(1 - aq^2\alpha^2 P) \\
 &= (1 - a\alpha^2 P) \left( -1 + aq(1 + q)\alpha^2 P + q^2\alpha^2 - aq^2(1 + q)\alpha^3 P \right) \\
 &\quad + (1 - \alpha)(1 - q\alpha)(1 + a(a - q^2)\alpha^2 P - a^3 q^2 \alpha^4 P^2) \\
 &= (1 - q\alpha)(1 - a\alpha^2 P) \left( -1 - q\alpha + aq(1 + q)\alpha^2 P \right) \\
 &\quad + (1 - q\alpha)(1 - \alpha) \left( 1 + a(a - q^2)\alpha^2 P - a^3 q^2 \alpha^4 P^2 \right) \\
 &= (1 - q\alpha) \left[ -(1 + q)\alpha + a\alpha^2 (q(1 + q) + 1 + q\alpha + (1 - \alpha)(a - q^2)) P \right. \\
 &\quad \left. - a^2 q \alpha^4 (1 + q + aq(1 - \alpha)) P^2 \right].
 \end{aligned}$$

Combining (3.40) and (3.41) we obtain

(3.42)

$$\begin{aligned}
 (E_2 + E_3)/(1 - q\alpha) &= -(1 + q)(1 + a)\alpha \\
 &\quad + a\alpha^2 (a\alpha + aq\alpha + aq + aq^2\alpha + q \\
 &\quad + 1 + q\alpha + a - a\alpha + q^2\alpha) P \\
 &\quad - a^2 q \alpha^4 (a(1 + q\alpha) + 1 + q + aq(1 - \alpha)) P^2 \\
 &= -(1 + q)(1 + a)\alpha + a\alpha^2(1 + q)(aq\alpha + a + 1 + q\alpha)P \\
 &\quad - a^2 q \alpha^4 (1 + q)(1 + a)P^2 \\
 &= -(1 + q)(1 + a)\alpha (1 - a\alpha(1 + q\alpha)P + a^2 q \alpha^3 P^2) \\
 &= -(1 + q)(1 + a)\alpha(1 - a\alpha P)(1 - aq\alpha^2 P).
 \end{aligned}$$

Then from (3.38), (3.39), and (3.42) we get

$$\begin{aligned}
 (3.43) \quad (at_1 t_2 q^{2n}; q)_3 R_n &= (1 - q\alpha)(1 - a\alpha P)(1 - aq\alpha^2 P) \\
 &\quad \times ((1 + aq\alpha^2 P)S - (1 + q)(1 + a)\alpha P).
 \end{aligned}$$

In terms of  $\alpha = q^n$ ,  $S = t_1 + t_2$ , and  $P = t_1 t_2$ ,  $(at_1 t_2 q^{2n}; q)_3 = (a\alpha^2 P; q)_3$ , hence (3.43) implies

$$\begin{aligned}
 (3.44) \quad \frac{R_n}{(1 - q^{n+1})(1 - at_1 t_2 q^n)t_1} &= \frac{R_n}{(1 - q\alpha)(1 - a\alpha P)t_1} \\
 &= \frac{(1 + aq\alpha^2 P)S - (1 + q)(1 + a)\alpha P}{(1 - a\alpha^2 P)(1 - aq^2\alpha^2 P)t_1} = \beta_n
 \end{aligned}$$

in view of (3.4). The proof of Theorem 3.1 is complete.  $\square$

#### 4. The Spectrum of the Inverse Operator.

To find the spectrum of the operator  $\mathcal{T}_{\mathbf{t}}$ , that is, to solve eigenvalue problem (2.4) we consider the eigenfunctions of the  $q$ -difference operator  $D_{q,x}$ . It is easy to see that for every  $\lambda$  the equation  $D_{q,x}f(x) = \lambda f(x)$  has solution  $f_{\lambda}(x) = 1/(\lambda(1-q)x; q)_{\infty} = e_q(\lambda(1-q)x)$ . The eigenfunctions of  $\mathcal{T}_{\mathbf{t}}$  are also eigenfunctions for  $D_{q,x}$ , in fact if  $g_{\lambda}$  is such that  $\mathcal{T}_{\mathbf{t}}g_{\lambda} = \lambda g_{\lambda}$ , then  $g_{\lambda} = D_{q,x}\mathcal{T}_{\mathbf{t}}g_{\lambda} = \lambda D_{q,x}g_{\lambda}$  and therefore,  $g_{\lambda}(x) = f_{1/\lambda}(x) = e_q((1-q)x/\lambda)$ . Hence, the eigenvalues of  $\mathcal{T}_{\mathbf{t}}$  are the reciprocals of the numbers  $\lambda$  such that

$$(4.1) \quad f_{\lambda}(x) = \sum_{n=0}^{\infty} c_n(\lambda, \mathbf{t}) p_n(x, \mathbf{t}) \in L_2(\mu(\cdot, \mathbf{t})) \cap L_2(\mu(\cdot, q\mathbf{t})).$$

From equations  $\mathcal{T}_{\mathbf{t}}g_{\lambda} = \lambda g_{\lambda}$ ,  $g_{\lambda} = f_{1/\lambda}$ , (2.2), and (4.1) we get  $c_0(1/\lambda, \mathbf{t}) = 0$ . It turns out that the condition  $c_0(1/\lambda, \mathbf{t}) = 0$  completely characterizes the spectrum of the operator  $\mathcal{T}_{\mathbf{t}}$ .

We proceed with computing the coefficients  $\{c_n(\lambda, \mathbf{t})\}$  in (4.1). Let  $\lambda$  be small enough so that  $|\lambda(1-q)| < 1$ . Using the orthogonality relation (1.6) and applying Euler's identity (1.12) to  $f_{\lambda}(x) = e_q(\lambda(1-q)x)$  we obtain

$$(4.2) \quad c_n(\lambda, \mathbf{t}) \xi_n(\mathbf{t}) = \int_{\mathbf{R}} f_{\lambda}(x) p_n(x, \mathbf{t}) d\mu(x, \mathbf{t}) = \sum_{s=0}^{\infty} \frac{(\lambda(1-q))^s}{(q; q)_s} d_{n,s}(\lambda, \mathbf{t}),$$

where we have defined

$$(4.3) \quad d_{n,s}(\lambda, \mathbf{t}) := \int_{\mathbf{R}} x^s p_n(x, \mathbf{t}) d\mu(x, \mathbf{t}).$$

By (1.6),  $d_{n,s} = 0$  if  $s < n$ , hence we may assume that  $s \geq n$ .

From (1.11) and (1.3) we have

$$x^s = \frac{1}{t_2^s} \sum_{j=0}^s a_{s,j}(t_2x; q)_j, \quad a_{s,j} := (-1)^j q^{\binom{j}{2} - j(s-1)} \begin{bmatrix} s \\ j \end{bmatrix}_q,$$

$$p_n(x, \mathbf{t}) = \sum_{k=0}^n b_{n,k}(t_1x; q)_k, \quad b_{n,k} := \frac{(q^{-n}, at_1t_2q^{n-1}; q)_k}{(t_1, at_1, q; q)_k} q^k.$$

Then for  $d_{n,s}(\lambda, \mathbf{t})$  we get

$$(4.4) \quad d_{n,s}(\lambda, \mathbf{t}) = \sum_{j=0}^s \sum_{k=0}^n a_{s,j} b_{n,k} \frac{1}{t_2^s} \int_{\mathbf{R}} \frac{d\mu^{(a)}(x)}{(q^k t_1 x, q^j t_2 x; q)_{\infty}}$$

$$= \sum_{j=0}^s \sum_{k=0}^n a_{s,j} b_{n,k} \frac{1}{t_2^s} \frac{(at_1 t_2 q^{k+j}; q)_{\infty}}{(q^k t_1, q^j t_2, a q^k t_1, a q^j t_2; q)_{\infty}},$$

where (6.4) was used to evaluate the integrals. From the formulas for  $a_{s,j}$  and  $b_{n,k}$ , and (4.4) we obtain

$$(4.5) \quad \begin{aligned} d_{n,s} &= \frac{(at_1t_2; q)_\infty (q; q)_s}{(t_1, at_1, t_2, at_2; q)_\infty t_2^s} \\ &\times \sum_{j=0}^s \sum_{k=0}^n \frac{(-1)^j q^{\binom{j}{2} - j(s-1) + k} (q^{-n}, at_1t_2q^{n-1}; q)_k (t_2, at_2; q)_j}{(q; q)_j (q; q)_{s-j} (q; q)_k (at_1t_2; q)_{k+j}} \\ &= c^{(1)} \sum_{j=0}^s \frac{(-1)^j q^{\binom{j}{2} - j(s-1)} (t_2, at_2; q)_j}{(q; q)_j (q; q)_{s-j} (at_1t_2; q)_j} {}_2\phi_1 \left( \begin{matrix} q^{-n}, at_1t_2q^{n-1} \\ at_1t_2q^j \end{matrix} \middle| q, q \right), \end{aligned}$$

where  $c^{(1)}$  denotes the coefficient of the double sum. By (1.9) the  ${}_2\phi_1$  sum is equal to

$$\frac{(q^{j+1-n}; q)_n (at_1t_2q^{n-1})^n}{(at_1t_2q^j; q)_n},$$

which is 0 for  $j < n$ . Then for  $s \geq n$  we obtain

$$(4.6) \quad d_{n,s} = c^{(2)} \sum_{j=n}^s \frac{(-1)^j q^{\binom{j}{2} - j(s-1)} (t_2, at_2; q)_j}{(q; q)_{j-n} (q; q)_{s-j} (at_1t_2; q)_{n+j}},$$

where  $c^{(2)} = (at_1t_2q^{n-1})^n c^{(1)}$  and we used the identity  $(q^{j+1-n}; q)_n = (q; q)_j / (q; q)_{j-n}$ . Replacing  $j$  by  $n+l$  we get

$$(4.7) \quad d_{n,s} = c^{(3)} \sum_{l=0}^{s-n} \frac{(-1)^l q^{\binom{n+l}{2} - (n+l)(s-1)} (t_2q^n, at_2q^n; q)_l}{(q; q)_l (q; q)_{(s-n)-l} (at_1t_2q^{2n}; q)_l}, \quad s \geq n,$$

with  $c^{(3)} = (-1)^n (t_2, at_2; q)_n / (at_1t_2; q)_{2n} c^{(2)}$ . Next with  $p = s - n$  we have

$$\frac{(q; q)_p}{(q; q)_{p-l}} = \prod_{j=p-l+1}^p (1 - q^j) = (-1)^l q^{\binom{p+1}{2} - \binom{p-l+1}{2}} (q^{-p}; q)_l$$

and

$$\binom{n+l}{2} - (n+l)(s-1) + \binom{p+1}{2} - \binom{p-l+1}{2} = \binom{n}{2} - n(s-1) + l.$$

Substituting these identities in (4.7) we obtain

$$(4.8) \quad d_{n,s} = c^{(4)} {}_3\phi_2 \left( \begin{matrix} q^{-(s-n)}, t_2q^n, at_2q^n \\ at_1t_2q^{2n}, 0 \end{matrix} \middle| q, q \right), \quad s \geq n,$$

where

$$(4.9) \quad c^{(4)} = c_{n,s}^{(4)} = \frac{(at_1t_2q^{2n}; q)_\infty (t_2, at_2; q)_n (at_1t_2)^n (q; q)_s q^{\binom{n}{2} + n(n-s)} (-1)^n}{(t_1, at_1, t_2, at_2; q)_\infty (q; q)_{s-n} t_2^s}.$$

From formula (1.7) for  $\xi_n(\mathbf{t})$  and (4.9) we get

$$(4.10) \quad c_{n,s}^{(4)} = \frac{(t_1, at_1; q)_n (q; q)_s q^{n(n-s)} t_2^{n-s}}{(at_1 t_2 q^{n-1}, q; q)_n (q; q)_{s-n} t_1^n} \xi_n(\mathbf{t}).$$

Then from (4.2), (4.8), (4.9), and (4.10) we get

$$(4.11) \quad c_n(\lambda, \mathbf{t}) = \sum_{s=n}^{\infty} \frac{((1-q)\lambda)^s}{(q; q)_s} (d_{n,s}/\xi_n(\mathbf{t})) = \frac{(t_1, at_1; q)_n (1-q)^n}{(at_1 t_2 q^{n-1}, q; q)_n t_1^n} \lambda^n \\ \times \sum_{k=0}^{\infty} \frac{((1-q)/(t_2 q^n))^k}{(q; q)_k} \lambda^k {}_3\phi_2 \left( \begin{matrix} q^{-k}, t_2 q^n, at_2 q^n \\ at_1 t_2 q^{2n}, 0 \end{matrix} \middle| q, q \right),$$

with  $k = s - n$ . Applying the transformation, [6],

$$(4.12) \quad {}_3\phi_2 \left( \begin{matrix} q^{-k}, a, b \\ c, 0 \end{matrix} \middle| q, q \right) = \frac{(b; q)_k a^k}{(c; q)_k} {}_2\phi_1 \left( \begin{matrix} q^{-k}, c/b \\ q^{1-k}/b \end{matrix} \middle| q, q/a \right)$$

to the  ${}_3\phi_2$  expression in (4.11) we get

$${}_3\phi_2 \left( \begin{matrix} q^{-k}, t_2 q^n, at_2 q^n \\ at_1 t_2 q^{2n}, 0 \end{matrix} \middle| q, q \right) \\ = \frac{(at_2 q^n; q)_k (t_2 q^n)^k}{(at_1 t_2 q^{2n}; q)_k} {}_2\phi_1 \left( \begin{matrix} q^{-k}, t_1 q^n \\ q^{1-k-n}/(at_2) \end{matrix} \middle| q, q^{1-n}/t_2 \right).$$

Then the second sum in (4.11) can be written in the form

$$(4.13) \quad \sum_{k=0}^{\infty} \frac{(at_2 q^n; q)_k (1-q)^k}{(at_1 t_2 q^{2n}, q; q)_k} \lambda^k \sum_{j=0}^k \frac{(q^{-k}; q)_j (t_1 q^n; q)_j (q^{1-n}/t_2)^j}{(q^{1-k-n}/(at_2); q)_j (q; q)_j}.$$

Using (1.15) we obtain

$$\frac{(q^{-k}; q)_j}{(q^{1-k-n}/(at_2); q)_j} = \frac{(q; q)_k (at_2 q^n; q)_{k-j}}{(q; q)_{k-j} (at_2 q^n; q)_k} (at_2 q^{n-1})^j.$$

Hence the double sum in (4.13) equals

$$(4.14) \quad \sum_{k=0}^{\infty} \frac{((1-q)\lambda)^k}{(at_1 t_2 q^{2n}, q; q)_k} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q (t_1 q^n; q)_j (at_2 q^n; q)_{k-j} a^j.$$

The above formulas hold when  $|(1-q)\lambda| < 1$  since in this range (1.12) can be applied. To extend the formulas to arbitrary  $\lambda$  we need a meromorphic continuation of the function in (4.14). We set

$$(4.15) \quad \alpha_k(a, t_1, t_2) := \frac{1}{(at_1 t_2, q; q)_k} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q (t_1; q)_j (at_2; q)_{k-j} a^j,$$

$$(4.16) \quad A(z; a, t_1, t_2) := \sum_{k=0}^{\infty} \alpha_k(a, t_1, t_2) z^k, \quad |z| < 1.$$

Note that the sum in (4.14) equals  $A((1-q)\lambda; a, q^n \mathbf{t})$ . For  $|z| < 1$  we consider the product of the functions  $(z, az; q)_\infty$  and  $A(z; a, t_1, t_2)$ . Using (1.13) we get

$$(4.17) \quad (z; q)_\infty A(z; a, t_1, t_2) = \sum_{n=0}^{\infty} z^n \left( \sum_{k=0}^n \alpha_k(a, t_1, t_2) \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(q; q)_{n-k}} \right).$$

The coefficient of  $z^n$  in (4.17) equals

$$(4.18) \quad \begin{aligned} & \sum_{k=0}^n \left( \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(t_1; q)_j (at_2; q)_{k-j} a^j}{(at_1 t_2, q; q)_k} \right) \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(q; q)_{n-k}} \\ &= \sum_{j=0}^n \frac{(t_1; q)_j a^j}{(at_1 t_2, q; q)_j (q; q)_{n-j}} \left( \sum_{\nu=0}^{n-j} \begin{bmatrix} n-j \\ \nu \end{bmatrix}_q \frac{(at_2; q)_\nu (-1)^{n-j-\nu} q^{\binom{n-j-\nu}{2}}}{(at_1 t_2 q^j; q)_\nu} \right) \end{aligned}$$

with  $\nu = k - j$ . The sum over  $\nu$  in (4.18) has the form

$$\begin{aligned} & \sum_{\nu=0}^m \begin{bmatrix} m \\ \nu \end{bmatrix}_q \frac{(\alpha; q)_\nu}{(\beta; q)_\nu} (-1)^{m-\nu} q^{\binom{m-\nu}{2}} \\ &= (-1)^m q^{\binom{m}{2}} \sum_{\nu=0}^m \frac{(q^{-m}, \alpha; q)_\nu}{(\beta, q; q)_\nu} q^\nu \\ &= (-1)^m q^{\binom{m}{2}} {}_2\phi_1 \left( \begin{matrix} q^{-m}, \alpha \\ \beta \end{matrix} \middle| q, q \right) \\ &= (-1)^m q^{\binom{m}{2}} \frac{(\beta/\alpha; q)_m}{(\beta; q)_m} \alpha^m, \end{aligned}$$

where we first used the identity

$$(4.19) \quad (q; q)_m / (q; q)_{m-\nu} = (-1)^\nu q^{\binom{m+1}{2} - \binom{m-\nu+1}{2}} (q^{-m}; q)_\nu,$$

and then (1.9). Then the coefficient of  $z^n$  in (4.17) given with (4.18) equals

$$(4.20) \quad \begin{aligned} & \sum_{j=0}^n \frac{(t_1; q)_j a^j}{(at_1 t_2, q; q)_j (q; q)_{n-j}} (-1)^{n-j} q^{\binom{n-j}{2}} \frac{(t_1 q^j; q)_{n-j} (at_2)^{n-j}}{(at_1 t_2 q^j; q)_{n-j}} \\ &= \frac{(t_1; q)_n a^n}{(at_1 t_2, q; q)_n} \sum_{j=0}^n \begin{bmatrix} n \\ n-j \end{bmatrix}_q q^{\binom{n-j}{2}} (-t_2)^{n-j} = \frac{(t_1, t_2; q)_n a^n}{(at_1 t_2, q; q)_n}, \end{aligned}$$

where we applied (1.14). From (4.17) and (4.20) we get

$$(4.21) \quad (z; q)_\infty A(z; a, t_1, t_2) = {}_2\phi_1 \left( \begin{matrix} t_1, t_2 \\ at_1 t_2 \end{matrix} \middle| q, az \right), \quad |az| < 1.$$



From (1.13) and (4.21) for  $z$  such that  $\max\{|z|, |az|\} < 1$  we have

$$\begin{aligned}
 (4.22) \quad & (z, az; q)_\infty A(z; a, t_1, t_2) \\
 &= \sum_{m=0}^{\infty} \left( \sum_{k=0}^m \frac{(t_1, t_2; q)_k}{(at_1 t_2, q; q)_k} \frac{(-1)^{m-k} q^{\binom{m-k}{2}}}{(q; q)_{m-k}} \right) a^m z^m \\
 &= \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}}{(q; q)_m} {}_3\phi_2 \left( \begin{matrix} q^{-m}, t_1, t_2 \\ at_1 t_2, 0 \end{matrix} \middle| q, q \right) a^m z^m \\
 &= \sum_{m=0}^{\infty} \frac{(-t_1)^m q^{\binom{m}{2}} (t_2; q)_m}{(at_1 t_2, q; q)_m} {}_2\phi_1 \left( \begin{matrix} q^{-m}, at_1 \\ q^{1-m}/t_2 \end{matrix} \middle| q, q/t_1 \right) a^m z^m,
 \end{aligned}$$

where we applied (4.19) and (4.12). Next, from (1.15) we have

$$\frac{(q^{-m}; q)_j}{(q^{1-m}/t_2; q)_j} = \frac{(q; q)_m (t_2; q)_{m-j} (t_2/q)^j}{(q; q)_{m-j} (t_2; q)_m},$$

which combined with (4.22) yields

$$\begin{aligned}
 (4.23) \quad & (z, az; q)_\infty A(z; a, t_1, t_2) = \sum_{m=0}^{\infty} \frac{(-t_1)^m q^{\binom{m}{2}}}{(at_1 t_2; q)_m} \\
 & \quad \times \left( \sum_{j=0}^m \frac{(at_1; q)_j (t_2; q)_{m-j} (t_2/t_1)^j}{(q; q)_j (q; q)_{m-j}} \right) a^m z^m.
 \end{aligned}$$

Clearly  $(z, az; q)_\infty$  is an entire function of  $z$ . Furthermore, the right-hand side of (4.23) is an entire function of  $z$ , and in an open neighborhood of  $z = 0$  it coincides with the function  $(z, az; q)_\infty A(z; a, t_1, t_2)$ . Hence a meromorphic extension of  $A(z; a, t_1, t_2)$  to the complex plane can be found by dividing the right-hand side of (4.23) by  $(z, az; q)_\infty$ .

The main results of this section can be described with the following two theorems.

**Theorem 4.1.** *The coefficients in the expansion formula for the eigenfunction  $f_\lambda(x) = e_q(\lambda(1-q)x)$  in terms of big  $q$ -Jacobi polynomials  $\{p_n(x, \mathbf{t})\}$  are given by*

$$\begin{aligned}
 (4.24) \quad & c_n(\lambda, \mathbf{t}) = \frac{1}{((1-q)\lambda, a(1-q)\lambda; q)_\infty} \frac{(t_1, at_1; q)_n (1-q)^n}{(at_1 t_2 q^{n-1}, q; q)_n t_1^n} \lambda^n \\
 & \times \sum_{m=0}^{\infty} \frac{(-t_1 q^n)^m q^{\binom{m}{2}}}{(at_1 t_2 q^{2n}; q)_m} \left( \sum_{j=0}^m \frac{(at_1 q^n; q)_j (t_2 q^n; q)_{m-j} \left(\frac{t_2}{t_1}\right)^j}{(q; q)_j (q; q)_{m-j}} \right) a^m (1-q)^m \lambda^m.
 \end{aligned}$$

Furthermore, the coefficients  $\{c_n(\lambda, \mathbf{t})\}$  satisfy the recurrence equation

(4.25)

$$\sigma_n(\mathbf{t})c_n(\lambda, \mathbf{t}) = \lambda \sum_{m=n-1}^{n+1} (\xi_m(\mathbf{t})/\xi_{n-1}(q\mathbf{t})) c_{n-1,m}(\mathbf{t})c_m(\lambda, \mathbf{t}), \quad n \geq 1,$$

where  $\{c_{n,m}(\mathbf{t})\}_{m=n}^{n+2}$  are the coefficients defined with (2.7)-(2.10).

The spectrum of the inverse operator  $\mathcal{T}_{\mathbf{t}}$  is described in Theorem 4.2.

**Theorem 4.2.** *The function  $f_{\lambda}(x)$  belongs to the space  $L_2(\mu(\cdot, \mathbf{t}))$  for every  $\lambda$  that is not a zero of  $((1-q)\lambda, a(1-q)\lambda; q)_{\infty}$ .*

*The spectrum of  $\mathcal{T}_{\mathbf{t}}$ , the inverse operator of the  $q$ -difference operator  $D_{q,x}$ , acting on the space  $L_2(\mu(\cdot, \mathbf{t})) \cap L_2(\mu(\cdot, q\mathbf{t}))$  is the set of the reciprocals of the zeros of the meromorphic function  $c_0(\lambda, \mathbf{t})$ .*

*Proof.* From (1.6), (1.7), (4.1), and (4.24) it immediately follows that

$$\|f_{\lambda}\|_{L_2(\mu(\cdot, \mathbf{t}))}^2 = \sum_{n=0}^{\infty} c_n(\lambda, \mathbf{t})^2 \xi_n(\mathbf{t}) < \infty$$

for all  $\lambda$  such that  $1/((1-q)\lambda) \notin \text{supp}(\mu^{(a)})$  and all parameters  $\mathbf{t}$  for which the function  $c_0(\lambda, \mathbf{t})$  is well-defined. This is due to the presence of the factor  $q^{(n)}_{(2)}$  in  $\xi_n(\mathbf{t})$ . Furthermore,  $\mathcal{T}_{\mathbf{t}}g_{\lambda} = \lambda g_{\lambda}$  implies  $g_{\lambda} = f_{1/\lambda}$  and  $c_0(1/\lambda, \mathbf{t}) = 0$ . Hence  $c_0(1/\lambda, \mathbf{t}) = 0$  is a necessary and sufficient condition for  $\lambda$  to be in the spectrum of the operator  $\mathcal{T}_{\mathbf{t}}$ .  $\square$

## 5. Asymptotic Properties of the Polynomials $\{b_n(\eta, \mathbf{t})\}$ .

In Section 2 we applied Schwartz's theorem to prove that the sequence  $\{\eta^n b_n(1/\eta, \mathbf{t})\}$  converges locally uniformly in the complex plane to an entire function. The recurrence relation (2.17) has bounded coefficients, hence the polynomials  $\{b_n(\eta, \mathbf{t})\}$  are orthogonal with respect to a unique measure  $\varphi(\cdot, \mathbf{t})$  with compact support, [1], [14]. From Markov's theorem, [15], the Stieltjes transform of  $\varphi(\cdot, \mathbf{t})$  is given by

$$(5.1) \quad \int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \lim_{n \rightarrow \infty} \frac{b_n^*(z, \mathbf{t})}{b_n(z, \mathbf{t})}, \quad z \notin \mathbf{R},$$

where  $\{b_n^*(\eta, \mathbf{t})\}$  is the solution of (2.17) or equivalently, (3.1) satisfying the initial conditions

$$b_0^*(\eta, \mathbf{t}) = 0, \quad b_1^*(\eta, \mathbf{t}) = 1.$$

We observe that  $\beta_n(\mathbf{t})$  and  $\gamma_n(\mathbf{t})$  defined with (3.4) and (3.3) have the property

$$(5.2) \quad \beta_{n-1}(q\mathbf{t}) = \beta_n(\mathbf{t}) \quad \text{and} \quad \gamma_{n-1}(q\mathbf{t}) = \gamma_n(\mathbf{t}).$$

From (3.1) with  $n$ ,  $\mathbf{t}$  and  $\eta$  replaced by  $n-1$ ,  $q\mathbf{t}$  and  $\eta/q$ , respectively, we get

$$(5.3) \quad b_n(\eta/q, q\mathbf{t}) = (\eta/q + \beta_{n-1}(q\mathbf{t})q^{n-1}) b_{n-1}(\eta/q, q\mathbf{t}) \\ - \gamma_{n-1}(q\mathbf{t})q^{2n-3}b_{n-2}(\eta/q, q\mathbf{t}).$$

Multiplying (5.3) by  $q^n$  and using (5.2) we see that

$$(5.4) \quad b_n^*(\eta, \mathbf{t}) = q^{n-1}b_{n-1}(\eta/q, q\mathbf{t}), \quad n \geq 0.$$

We now study the limiting behavior of  $\eta^n b_n(1/\eta, \mathbf{t})$  as  $n \rightarrow \infty$ . For each fixed  $j$  the  ${}_4\phi_3$  expression in (3.6) is bounded by  $M_1(\mathbf{t})_1 \phi_0(q^{-j}; -; q, -q)$  and the coefficient of the  ${}_4\phi_3$  is bounded by  $M_2(\mathbf{t})|\eta/t_1|^j q^{j(j-1)/2}/(q; q)_j$ . Here both  $M_1(\mathbf{t})$  and  $M_2(\mathbf{t})$  are positive and depend only on  $\mathbf{t}$ . The following estimate

$$\sum_{j=0}^{\infty} \left| \frac{\eta}{t_1} \right|^j \frac{q^{\binom{j}{2}}}{(q; q)_j} {}_2\phi_1(q^{-j}; -; q, -q) \leq \sum_{j=0}^{\infty} \frac{|\eta/t_1|^j}{(q; q)_j} \sum_{\nu=0}^j \frac{q^{\binom{j-\nu}{2}}}{(q; q)_{j-\nu}} \leq \frac{(-1; q)_{\infty}}{(|\eta/t_1|; q)_{\infty}}$$

holds for  $\eta$  with  $|\eta| < |t_1|$ . In the last inequality we used Euler's identities (1.12) and (1.13). Hence for  $|\eta| < |t_1|$ , Tannery's theorem (the discrete version of the Lebesgue dominated convergence theorem) can be applied. Using formula (3.6) we get

$$(5.5) \quad \lim_{n \rightarrow \infty} \eta^n b_n(1/\eta, \mathbf{t}) = \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(\eta/t_1)^j}{(q; q)_j} {}_3\phi_2 \left( \begin{matrix} q^{-j}, t_1, t_2 \\ at_1 t_2, 0 \end{matrix} \middle| q, q \right) \\ = \sum_{k=0}^{\infty} \frac{(t_1, t_2; q)_k q^k}{(at_1 t_2, q; q)_k} \sum_{j=k}^{\infty} \frac{(q^{-j}; q)_k q^{\binom{j}{2}}(\eta/t_1)^j}{(q; q)_j} \\ = \sum_{k=0}^{\infty} \frac{(t_1, t_2; q)_k (-\eta/t_1)^k}{(at_1 t_2, q; q)_k} \sum_{j=k}^{\infty} \frac{q^{\binom{j-k}{2}}(\eta/t_1)^{j-k}}{(q; q)_{j-k}} \\ = (-\eta/t_1; q)_{\infty} {}_2\phi_1 \left( \begin{matrix} t_1, t_2 \\ at_1 t_2 \end{matrix} \middle| q, -\eta/t_1 \right),$$

where we also used (1.13).

From (5.1), (5.5), and (5.4) for  $z \notin \mathbf{R}$  we get

$$(5.6) \quad \int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \lim_{n \rightarrow \infty} \frac{z^{-n} q^{n-1} b_{n-1}(z/q, q\mathbf{t})}{z^{-n} b_n(z, \mathbf{t})} \\ = \frac{{}_2\phi_1(qt_1, qt_2; at_1 t_2 q^2; q, -1/(t_1 z))}{z {}_2\phi_1(t_1, t_2; at_1 t_2; q, -1/(t_1 z))}, \quad |z| > 1/|t_1|.$$

An analytic continuation of (5.6) can be found using the Heine transformation [4, (III.1)],

(5.7)

$${}_2\phi_1(A, B; C; q, Z) = \frac{(B, AZ; q)_\infty}{(C, Z; q)_\infty} {}_2\phi_1(C/B, Z; AZ; q, B), \quad |Z|, |B| < 1.$$

From (5.6) and (5.7) we obtain

(5.8)

$$\int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \frac{(1 - at_1 t_2)(1 - at_1 t_2 q)}{(1 - t_2)(z + 1)} \frac{{}_2\phi_1(aqt_1, -1/(t_1 z); -q/z; q, qt_2)}{{}_2\phi_1(at_1, -1/(t_1 z); -1/z; q, t_2)}.$$

The Heine transformation also provides an analytic continuation of formula (5.5).

Formulas (4.11)-(4.16), (4.22), and (5.5) imply

$$\begin{aligned} (5.9) \quad c_0(\lambda, \mathbf{t}) &= A((1 - q)\lambda; a, \mathbf{t}) = ((1 - q)\lambda, a(1 - q)\lambda; q)_\infty^{-1} \\ &\times \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}} (-a(1 - q)\lambda)^j}{(q; q)_j} {}_3\phi_2(q^{-j}, t_1, t_2; at_1 t_2, 0; q, q) \\ &= \frac{1}{((1 - q)\lambda; q)_\infty} {}_2\phi_1 \left( \begin{matrix} t_1, t_2 \\ at_1 t_2 \end{matrix} \middle| q, a(1 - q)\lambda \right). \end{aligned}$$

From (5.6) and (5.9) we obtain

$$(5.10) \quad \int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \frac{c_0(-1/(a(1 - q)t_1 z), q\mathbf{t})}{zc_0(-1/(a(1 - q)t_1 z), \mathbf{t})}.$$

The  $c_0$ -functions in (5.10) have no common zeros. This can be seen as follows: Assume that  $c_0(\lambda_0, \mathbf{t}) = c_0(\lambda_0, q\mathbf{t}) = 0$  for some  $\lambda_0$ . Formula (4.11) implies

$$c_n(\lambda, q\mathbf{t}) = \frac{(1 - at_1 t_2 q^n)(1 - q^{n+1})t_1}{(1 - t_1)(1 - at_1)(1 - q)\lambda} c_{n+1}(\lambda, \mathbf{t}).$$

Then our assumption implies  $c_1(\lambda_0, \mathbf{t}) = 0$  and from the three term recurrence equation (4.25) we get  $c_n(\lambda_0, \mathbf{t}) = 0$  for all  $n \geq 0$ . But then by Theorem 4.1,

$$e_q(\lambda_0(1 - q)x) = (\lambda_0(1 - q)x; q)_\infty^{-1} = \sum_{n=0}^{\infty} c_n(\lambda_0, \mathbf{t}) p_n(x, \mathbf{t}) \equiv 0,$$

which is impossible.

The Perron-Stieltjes inversion formula is

$$(5.11) \quad F(z) = \int_{\mathbf{R}} \frac{d\mu(t)}{z - t} \quad \text{if and only if} \\ \mu(x) - \mu(y) = \lim_{\epsilon \rightarrow 0^+} \int_y^x \frac{F(t - i\epsilon) - F(t + i\epsilon)}{2\pi i} dt.$$

This inversion formula shows that  $\varphi$  is a purely discrete measure. It is clear that an isolated point mass  $m$  of  $\varphi$  located at  $x = x_0$  contributes  $m/(z - x_0)$  to the left-hand side of (5.8). Thus the isolated point masses of  $\varphi$  coincide with the isolated poles of the right-hand side of (5.8) and the masses are the corresponding residues. Below we will show that  $x = 0$ , which is the only essential singularity of the right-hand side of (5.8) does not support a discrete mass, so let us assume this for the time being. Formula (5.10), Theorem 4.2, and the above discussion describe the relationship between the support of the measure  $\varphi(\cdot, \mathbf{t})$  and the spectrum of the operator  $\mathcal{T}_{\mathbf{t}}$ .

**Theorem 5.1.** *The support of the measure of orthogonality  $\varphi(\cdot, \mathbf{t})$  is the set of the elements of the spectrum of the operator  $\mathcal{T}_{\mathbf{t}}$  multiplied by  $-1/(a(1 - q)t_1)$ .*

It remains to show that  $x = 0$  is not a mass point for  $\varphi$ . From the theory of the moment problem [1], [14], it is known that when the measure  $\varphi$  is unique then  $x = x_0$  is a mass point for the measure if and only if  $\sum_{n=0}^{\infty} p_n(x_0)^2$  converges,  $\{p_n(x)\}$  being the orthonormal polynomials. From Theorem 3.1 we find

$$(5.12) \quad b_n(0, \mathbf{t}) = \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_n}{(q, q^{-2n}/(at_1t_2); q)_n} (-at_1)^{-n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, at_1t_2q^{n+1}, t_1, t_2 \\ t_1q, t_2q, at_1t_2 \end{matrix} \middle| q, q \right).$$

Applying (1.15) to (5.12) we get

$$(5.13) \quad b_n(0, \mathbf{t}) = \frac{(qt_1, qt_2; q)_n}{(q, at_1t_2q^{n+1}; q)_n} \frac{q^{\binom{n}{2}}}{t_1^n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, at_1t_2q^{n+1}, t_1, t_2 \\ t_1q, t_2q, at_1t_2 \end{matrix} \middle| q, q \right).$$

Ismail and Wilson, [10], determined the asymptotic behavior of the Askey-Wilson polynomials. They proved that

$$(5.14) \quad \lim_{n \rightarrow \infty} (z/A)^n {}_4\phi_3 \left( \begin{matrix} q^{-n}, ABCDq^{n-1}, Az, A/z \\ AB, AC, AD \end{matrix} \middle| q, q \right) \\ = \frac{(Az, Bz, Cz, Dz; q)_{\infty}}{(z^2, AB, AC, AD; q)_{\infty}},$$

for  $|z| < 1$ , and that the left-hand side of (5.14) is bounded if  $|z| = 1$  but  $z \neq \pm 1$ . If  $z = \pm 1$  then the left-hand side of (5.14) is  $O(n)$ . The  ${}_4\phi_3$  quantity in (5.13) corresponds to the  ${}_4\phi_3$  function in (5.14) with  $A = \sqrt{t_1t_2}$ ,  $B = q\sqrt{t_1/t_2}$ ,  $C = q\sqrt{t_2/t_1}$ ,  $D = a\sqrt{t_1t_2}$ , and  $z = \sqrt{t_1/t_2}$  if  $|t_1| \leq |t_2|$  and  $z = \sqrt{t_2/t_1}$  otherwise. The orthonormal polynomials associated with the  $b_n$ 's are

$$p_m(\eta, \mathbf{t}) = b_m(\eta, \mathbf{t}) \xi_0^{-1/2}(\mathbf{t}) \prod_{n=1}^m u_n^{-1/2},$$

where  $-u_n$  denotes the coefficient of  $b_{n-1}(\eta, \mathbf{t})$  in (2.18). From (2.18) we get

$$\prod_{k=1}^n u_k = \frac{(qt_1, aqt_1, qt_2, aqt_2; q)_n}{(at_1t_2q, at_1t_2q^2; q)_{2n}} (t_2/t_1)^n q^{2\binom{n+1}{2}-n}.$$

Combining the above formulas we obtain

$$\begin{aligned} (5.15) \quad q^n \xi_0(\mathbf{t}) z^{2n} p_n(0, \mathbf{t})^2 &= \frac{(qt_1, qt_2; q)_n}{(aqt_1, aqt_2; q)_n} \frac{(at_1t_2q; q)_n^2 (1 - at_1t_2q^{2n+1})}{(q; q)_n^2 (1 - at_1t_2q)} \frac{z^{2n}}{(t_1t_2)^n} \\ &\quad \times {}_4\phi_3 \left( \begin{matrix} q^{-n}, at_1t_2q^{n+1}, t_1, t_2 \\ qt_1, qt_2, at_1t_2 \end{matrix} \middle| q, q \right)^2 \\ &\rightarrow \frac{(qt_1, qt_2; q)_\infty (at_1t_2q; q)_\infty^2}{(aqt_1, aqt_2; q)_\infty (1 - at_1t_2q)} \\ &\quad \times \frac{(z\sqrt{t_1t_2}, qz\sqrt{t_1/t_2}, qz\sqrt{t_2/t_1}, az\sqrt{t_1t_2}; q)_\infty^2}{(z^2, qt_1, qt_2, at_1t_2, q; q)_\infty^2}, \quad n \rightarrow \infty, \end{aligned}$$

if  $|z| \leq 1$  and  $z \neq \pm 1$ . If  $z = 1$  the right-hand side of (5.15) becomes unbounded. Since  $|z| \leq 1$  and  $|q| < 1$ , (5.15) clearly implies that  $\sum_{n=0}^{\infty} p_n(0, \mathbf{t})^2$  diverges.

Equations (3.1)–(3.3) show that the polynomials  $\{b_n(\eta, \mathbf{t})\}$  are constant multiples of birth and death process polynomials associated with a process with birth and death rates

$$\frac{(1 - t_1q^{n+1})(1 - at_1q^{n+1})t_2q^n}{(1 - at_1t_2q^{2n+1})(1 - at_1t_2q^{2n+2})} \quad \text{and} \quad \frac{(1 - t_2q^n)(1 - at_2q^n)t_1q^n}{(1 - at_1t_2q^{2n})(1 - at_1t_2q^{2n+1})},$$

respectively. An exposition of the theory of birth and death processes and orthogonal polynomials can be found in [7].

## 6. Connection Coefficients for the Big $q$ -Jacobi Polynomials.

In this section we will compute the connection coefficients in the formula

$$(6.1) \quad p_n(x, \mathbf{t}) = \sum_{l=0}^n a_{n,l}(\mathbf{t}, \mathbf{s}) p_l(x, \mathbf{s}),$$

where  $\mathbf{t} = (t_1, t_2)$  and  $\mathbf{s} = (s_1, s_2)$ . From (6.1), (1.6), and (1.10) we get

$$\begin{aligned}
 (6.2) \quad a_{n,l}(\mathbf{t}, \mathbf{s}) \xi_l(\mathbf{s}) &= \int_{\mathbf{R}} p_n(x, \mathbf{t}) p_l(x, \mathbf{s}) d\mu(x, \mathbf{s}) \\
 &= \int_{\mathbf{R}} \sum_{k=0}^n \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_k q^k}{(t_1, at_1, q; q)_k} \left( (t_1/s_2; q)_k \sum_{\nu=0}^k \frac{(q^{-k}, s_2 x; q)_{\nu} q^{\nu}}{(s_2 q^{1-k}/t_1, q; q)_{\nu}} \right) \\
 &\quad \times \sum_{j=0}^l \frac{(q^{-l}, as_1 s_2 q^{l-1}; q)_j q^j}{(s_1, as_1, q; q)_j} (s_1 x; q)_j d\mu(x, \mathbf{s}).
 \end{aligned}$$

Changing the order of summation in (6.2) we obtain

$$\begin{aligned}
 (6.3) \quad a_{n,l}(\mathbf{t}, \mathbf{s}) \xi_l(\mathbf{s}) &= \sum_{\nu=0}^n \left( \sum_{k=\nu}^n \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_k q^k}{(t_1, at_1, q; q)_k} \frac{(t_1/s_2; q)_k (q^{-k}; q)_{\nu}}{(s_2 q^{1-k}/t_1; q)_{\nu}} \right) \frac{q^{\nu}}{(q; q)_{\nu}} \\
 &\quad \times \sum_{j=0}^l \frac{(q^{-l}, as_1 s_2 q^{l-1}; q)_j q^j}{(s_1, as_1, q; q)_j} \int_{\mathbf{R}} \frac{d\mu^{(a)}(x)}{(x s_1 q^j, x s_2 q^{\nu}; q)_{\infty}}.
 \end{aligned}$$

The last integral is evaluated using the  $q$ -beta integral evaluation from [6]

$$(6.4) \quad \int_{\mathbf{R}} \frac{d\mu^{(a)}(x)}{(x t_1, x t_2; q)_{\infty}} = \frac{(at_1 t_2; q)_{\infty}}{(t_1, at_1, t_2, at_2; q)_{\infty}}.$$

To evaluate the last sum in (6.3) we use (1.9). We get

$$\begin{aligned}
 (6.5) \quad &\frac{(as_1 s_2 q^{\nu}; q)_{\infty}}{(s_1, as_1, s_2 q^{\nu}, as_2 q^{\nu}; q)_{\infty}} \sum_{j=0}^l \frac{(q^{-l}, as_1 s_2 q^{l-1}; q)_j q^j}{(as_1 s_2 q^{\nu}, q; q)_j} q^j \\
 &= \frac{(as_1 s_2 q^{\nu}; q)_{\infty}}{(s_1, as_1, s_2 q^{\nu}, as_2 q^{\nu}; q)_{\infty}} \frac{(q^{\nu+1-l}; q)_l}{(as_1 s_2 q^{\nu}; q)_l} (as_1 s_2 q^{l-1})^l.
 \end{aligned}$$

Since  $(q^{\nu+1-l}; q)_l$  vanishes for  $\nu < l$ , the first sum in (6.3) is over  $\nu \in \{l, \dots, n\}$ .

Let  $S_{n,\nu}(\mathbf{t}, \mathbf{s})$  denote the sum over  $k$  in equation (6.3). Applying (1.15) to  $(q^{-k}; q)_\nu$  and  $(s_2 q^{1-k}/t_1; q)_\nu$  we obtain

(6.6)

$$\begin{aligned} S_{n,\nu}(\mathbf{t}, \mathbf{s}) &= \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_\nu q^\nu}{(t_1, at_1; q)_\nu} \\ &\times \sum_{k=\nu}^n \frac{(q^{-(n-\nu)}, at_1 t_2 q^{n+\nu-1}; q)_{k-\nu} q^{k-\nu}}{(t_1 q^\nu, at_1 q^\nu; q)_{k-\nu}} \frac{q^{-\nu} (t_1/s_2; q)_{k-\nu}}{(s_2/t_1)^\nu (q; q)_{k-\nu}} \\ &= \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_\nu (t_1/s_2)^\nu}{(t_1, at_1; q)_\nu} {}_3\phi_2 \left( \begin{matrix} q^{-(n-\nu)}, at_1 t_2 q^{n+\nu-1}, t_1/s_2 \\ t_1 q^\nu, at_1 q^\nu \end{matrix} \middle| q, q \right). \end{aligned}$$

From (6.3), (6.5), and (6.6) using the identities  $(q^{\nu+1-l}; q)_l / (q; q)_\nu = 1 / (q; q)_{\nu-l}$  and

$$\frac{(as_1 s_2 q^\nu; q)_\infty}{(as_1 s_2 q^\nu; q)_l} = (as_1 s_2 q^{\nu+l}; q)_\infty = \frac{(as_1 s_2 q^{2l}; q)_\infty}{(as_1 s_2 q^{2l}; q)_{\nu-l}}$$

we obtain the following result.

**Theorem 6.1.** *The connection coefficients in the expansion of the polynomial  $p_n(x, \mathbf{t})$  in terms of the polynomials  $\{p_l(x, \mathbf{s})\}$  are given by the formula*

(6.7)

$$\begin{aligned} a_{n,l}(\mathbf{t}, \mathbf{s}) &= \frac{1}{\xi_l(\mathbf{s})} \frac{(as_1 s_2 q^{2l}; q)_\infty (at_1 s_1 q^l)^l}{(s_1, as_1, s_2 q^l, as_2 q^l; q)_\infty} \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_l}{(t_1, at_1; q)_l} \\ &\times \sum_{\nu=l}^n \frac{(q^{-(n-l)}, at_1 t_2 q^{n+l-1}, s_2 q^l, as_2 q^l; q)_{\nu-l} (t_1 q/s_2)^{\nu-l}}{(t_1 q^l, at_1 q^l, as_1 s_2 q^{2l}, q; q)_{\nu-l}} \\ &\times {}_3\phi_2 \left( \begin{matrix} q^{-(n-\nu)}, at_1 t_2 q^{n+\nu-1}, t_1/s_2 \\ t_1 q^\nu, at_1 q^\nu \end{matrix} \middle| q, q \right), \quad l = 0, \dots, n. \end{aligned}$$

The connection coefficient formula (6.7) can be used to find the connection coefficients in certain special cases. In Section 2, we computed these coefficients for the case  $\mathbf{s} = q\mathbf{t}$ .

We will now use (6.7) to give another proof of the fact that  $a_{n,l}(\mathbf{t}, q\mathbf{t}) = 0$  for  $l < n - 2$ . In view of (1.8) it is enough to consider the case  $\mathbf{s} = (qt_2, qt_1)$  since for every  $l \in \mathbf{N}$ ,  $p_l(x; t_1, t_2)$  and  $p_l(x; t_2, t_1)$ , and therefore the coefficients  $a_{n,l}(\mathbf{t}, q\mathbf{t})$  and  $a_{n,l}(\mathbf{t}, q\mathbf{t}^*)$  are linearly dependent, where  $\mathbf{t}^* = (t_2, t_1)$ . So let  $\mathbf{s} = q\mathbf{t}^*$ . In view of (6.7) it is enough to show that  $\tilde{S}_{n,l}(\mathbf{t}) = 0$  for  $l < n - 2$ , where  $\tilde{S}_{n,l}(\mathbf{t})$  denotes the sum in (6.7) with  $\mathbf{s} = q\mathbf{t}^*$ . For this



sum by (6.7) we have

$$\begin{aligned}
 (6.8) \quad (1 - t_1 q^l)(1 - at_1 q^l) \tilde{S}_{n,l}(\mathbf{t}) &= \sum_{\nu=l}^n \frac{(q^{-(n-l)}, at_1 t_2 q^{n+l-1}; q)_{\nu-l}}{(at_1 t_2 q^{2l+2}, q; q)_{\nu-l}} \\
 &\times (1 - t_1 q^\nu)(1 - at_1 q^\nu) \left( 1 - \frac{(1 - q^{-(n-\nu)})(1 - at_1 t_2 q^{n+\nu-1})}{(1 - t_1 q^\nu)(1 - at_1 q^\nu)} \right) \\
 &= \sum_{i=0}^{n-l} \frac{(q^{-(n-l)}, at_1 t_2 q^{n+l-1}; q)_i}{(at_1 t_2 q^{2l+2}, q; q)_i} \\
 &\times \left[ (1 - t_1 q^{l+i})(1 - at_1 q^{l+i}) - (1 - q^{-(n-l)+i})(1 - at_1 t_2 q^{n+l+i-1}) \right] \\
 &= q^l (-t_1 - at_1 + q^{-n} + at_1 t_2 q^{n-1}) {}_2\phi_1 \left( \begin{matrix} q^{-d}, at_1 t_2 q^{d+2l-1} \\ at_1 t_2 q^{2l+2} \end{matrix} \middle| q, q \right) \\
 &\quad + (at_1^2 q^{2l} - at_1 t_2 q^{2l-1}) {}_2\phi_1 \left( \begin{matrix} q^{-d}, at_1 t_2 q^{d+2l-1} \\ at_1 t_2 q^{2l+2} \end{matrix} \middle| q, q^2 \right),
 \end{aligned}$$

with  $d = n - l$ . We now show that the  ${}_2\phi_1$  expressions in (6.8) vanish.

For integer numbers  $d > 0$ ,  $d_1 \geq 0$ , and  $d_2 > 0$  we consider

$$(6.9) \quad A_{d,d_1,d_2}(\alpha, q) := {}_2\phi_1 \left( \begin{matrix} q^{-d}, \alpha q^{d_1} \\ \alpha \end{matrix} \middle| q, q^{d_2} \right).$$

We claim that  $A_{d,d_1,d_2}(\alpha, q) \equiv 0$  if  $d_1 + d_2 < d$ . Indeed we have

$$\begin{aligned}
 (6.10) \quad A_{d,d_1,d_2}(\alpha, q) &= \sum_{j=0}^d \frac{(q^{-d}, \alpha q^{d_1}; q)_j}{(\alpha, q; q)_j} q^{d_2 j} \\
 &= \sum_{j=0}^d \begin{bmatrix} d \\ j \end{bmatrix}_q \frac{(\alpha q^{d_1}; q)_j}{(\alpha; q)_j} \left( -q^{-d+d_2} \right)^j q^{\binom{j}{2}} \\
 &= \sum_{j=0}^d \begin{bmatrix} d \\ j \end{bmatrix}_q \frac{(\alpha q^j; q)_{d_1}}{(\alpha; q)_{d_1}} \left( -q^{-(d-d_2)} \right)^j q^{\binom{j}{2}},
 \end{aligned}$$

where we used the identity

$$(\alpha q^j; q)_{d_1} (\alpha; q)_j = (\alpha; q)_{d_1+j} = (\alpha q^{d_1}; q)_j (\alpha; q)_{d_1}.$$

By the  $q$ -binomial theorem (1.14) we have  $(z; q)_l = \sum_{s=0}^l a_{l,s}(q) z^s$  with coefficients  $a_{l,s}(q) = (-1)^s q^{\binom{s}{2}} \begin{bmatrix} l \\ s \end{bmatrix}_q$ . Then (6.10) can be continued as follows

(6.11)

$$\begin{aligned} A_{d,d_1,d_2}(\alpha, q) &= \frac{1}{(\alpha; q)_{d_1}} \sum_{s=0}^{d_1} a_{d_1,s}(q) \alpha^s \left( \sum_{j=0}^d \begin{bmatrix} d \\ j \end{bmatrix}_q \left( -q^{s-(d-d_2)} \right)^j q^{\binom{j}{2}} \right) \\ &= (1/(\alpha; q)_{d_1}) \sum_{s=0}^{d_1} a_{d_1,s}(q) \alpha^s (q^{s-(d-d_2)}; q)_d = 0, \quad d - d_2 > d_1, \end{aligned}$$

where at the end we used (1.14). From (6.8) and (6.11) we get  $\tilde{S}_{n,l}(\mathbf{t}) = 0$  for  $d = n - l > 2$ , since in this case it is a linear combination of  $A_{d,d-3,1}(\alpha, q)$  and  $A_{d,d-3,2}(\alpha, q)$  with  $\alpha = at_1t_2q^{2l+2}$ . Then  $a_{n,l}(\mathbf{t}, q\mathbf{t}) = 0$  for  $l < n - 2$ .

## References

- [1] N.I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd, Edinburgh, 1965 (English translation).
- [2] G.E. Andrews and R.A. Askey, *Classical Orthogonal Polynomials*, Polynomes Orthogonaux et Applications (C. Breziniski et al., eds.), Lecture Notes in Mathematics, **1171**, Springer-Verlag, Berlin, (1984), 36-63.
- [3] G.E. Andrews, R.A. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [4] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1989.
- [5] D.P. Gupta, M.E.H. Ismail and D.R. Masson, *Contiguous relations, basic hypergeometric functions, and orthogonal polynomials*, II: *Associated big  $q$ -Jacobi polynomials*, J. Math. Anal. Appl., **171** (1992), 477-497.
- [6] M.E.H. Ismail, *Continuous and Discrete  $q$ -Orthogonal Polynomials*, Monograph-manuscript, 1998.
- [7] M.E.H. Ismail, J. Letessier, D.R. Masson and G. Valent, *Birth and Death Processes and Orthogonal Polynomials*, Orthogonal Polynomials: Theory and Practice (P. Nevai, ed.), NATO, ASI on Orthogonal Polynomials and Their Applications, Kluwer Academic Publishers, (1990), 225-229.
- [8] M.E.H. Ismail and M. Rahman, *The associated Askey-Wilson polynomials*, Trans. Amer. Math. Soc., **328** (1991), 201-237.
- [9] M.E.H. Ismail, M. Rahman and R. Zhang, *Diagonalization of certain integral operators II*, J. Comp. Applied Math., **68** (1996), 163-196.
- [10] M.E.H. Ismail and J.A. Wilson, *Asymptotic and generating relations for the  $q$ -Jacobi and  ${}_4\phi_3$  polynomials*, J. Approx. Theory, **36** (1982), 43-54.
- [11] M.E.H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Adv. Math., **109** (1994), 1-33.

- [12] R. Koekoek and R. Swarttouw, *The Askey-scheme of Hypergeometric Orthogonal Polynomials and its  $q$ -analogue*, Technical Report, **94-05**, Delft University of Technology, Delft, The Netherlands, 1994.
- [13] H.M. Schwartz, *A class of continued fractions*, Duke Math. J., **6** (1940), 48-65.
- [14] J. Shohat and J.D. Tamarkin, *The Problem of Moments* (Revised edition), Amer. Math. Soc., Providence, RI, 1950.
- [15] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, RI, 4th edition, 1975.
- [16] J. Wimp, *Some explicit Padé approximants for the function  $\phi'/\phi$  and a related quadrature formula involving Bessel functions*, SIAM J. Math. Anal., **16** (1985), 887-895.

Received May 14, 1999. The first author's research partially was supported by NSF grant DMS 99-70865.

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF SOUTH FLORIDA  
TAMPA, FL 33620-5700  
*E-mail address:* [ismail@math.usf.edu](mailto:ismail@math.usf.edu)

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE  
UNIVERSITY OF HOUSTON-DOWNTOWN  
HOUSTON, TX 77002-1094  
*E-mail address:* [simeonovp@zeus.dh.uh.edu](mailto:simeonovp@zeus.dh.uh.edu)