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We construct canonical bundles for Hamiltonian loop group actions with proper moment maps. As an application, we show that for certain moduli spaces of flat connections on Riemann surfaces with boundary, the first Chern class is a multiple of the cohomology class of the symplectic form.

1. Introduction.

One of the simplest invariants of a symplectic manifold is the isomorphism class of the canonical line bundle. Suppose (M, ω) is a symplectic manifold. For any ω-compatible almost complex structure J one defines the canonical line bundle K_M as the dual to the top exterior power of the tangent bundle TM,

$$K_M = \det_{\mathbb{C}}(TM)^*.$$

Since the space of ω-compatible almost complex structures on M is contractible, the isomorphism class of K_M is independent of this choice. If a compact Lie group G acts by symplectomorphisms on M, we can take J to be G-invariant, and K_M is a G-equivariant line bundle.

The canonical bundle behaves well under symplectic quotients. If the G-action is Hamiltonian, with moment map $\Phi : M \to \mathfrak{g}^*$, the symplectic quotient of M is defined by

$$M/\!\!/G := \Phi^{-1}(0)/G.$$

We assume that 0 is a regular value, so that $M/\!\!/G$ is a symplectic orbifold. The canonical line bundle for the reduced space (symplectic quotient) $M/\!\!/G = \Phi^{-1}(0)/G$ is related to the canonical bundle on M by

$$K_{M/\!\!/G} = K_M/\!\!/G := (K_M|\Phi^{-1}(0))/G. \tag{1}$$

The canonical bundle also behaves well under inductions. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Suppose that N is a Hamiltonian T-manifold with moment map $\Psi : N \to \mathfrak{t}^*$. The symplectic induction $M := G \times_T N$ has a unique closed two-form and moment map extending the given data on N. If the image of Ψ is contained in the interior of a positive chamber $\mathfrak{t}_+^*$, then M is symplectic and K_M is induced from K_N, after a ρ-shift:

$$K_M \cong G \times_T (K_N \otimes \mathbb{C}_{-2\rho}).$$

Here $\mathbb{C}_{-2\rho}$ is the T-representation with weight given by the sum -2ρ of the negative roots.

In this paper we develop a notion of canonical line bundle for (infinite-dimensional) Hamiltonian loop group manifolds with proper moment maps. The idea is to use the property of the canonical bundle under inductions as the definition in the infinite-dimensional setting. Just as in the finite dimensional situation, the canonical bundle of the (finite dimensional) reduced spaces are obtained from the canonical bundle K_M upstairs. For the fundamental homogeneous space $\Omega G = LG/G$, our definition agrees with Freed's computation [**4**] of the regularized first Chern class of ΩG.

As an application, we prove the following fact about moduli spaces of flat G-connections on compact oriented surfaces Σ. Suppose G is simple and simply connected, and let c be the dual Coxeter number. Suppose Σ has b boundary components $B_1, \dots, B_b$, and let $\mathcal{C}_1, \dots, \mathcal{C}_b$ be a collection of conjugacy classes. Let $\mathcal{M}(\Sigma, \mathcal{C})$ be the (finite dimensional) moduli space of flat G-connections on Σ with holonomy around B_j contained in $\mathcal{C}_j$. The subset $\mathcal{M}(\Sigma, \mathcal{C})_{\mathrm{irr}}$ of irreducible connections is a smooth symplectic manifold. Let $[\omega]$ be the cohomology class of the basic symplectic form on $\mathcal{M}(\Sigma, \mathcal{C})_{\mathrm{irr}}$.

Theorem 1.1. *If the conjugacy classes $\mathcal{C}_j$ consist of central elements, then the first Chern class of $K_{\mathcal{M}(\Sigma,\mathcal{C})_{\mathrm{irr}}}$ is equal to $-2c[\omega]$.*

This was first proved in the special case of $SU(2)$ by Ramanan [**11**]. In general it is a consequence of the local family index theorem (Quillen [**10**], Zograf and Takhtadzhyan [**13**]). See also Beauville, Laszlo, and Sorger [**3**], and Kumar and Narasimhan [**5**]. Our application, Theorem 4.2 below, expands the list of conjugacy classes for which this result holds. It would be interesting to know which of these are Kähler-Einstein. Our main application of the canonical bundle will be given in a forthcoming paper [**2**], where it enters a fixed point formula for Hamiltonian loop group actions.

2. Hamiltonian loop group manifolds.

2.1. Notation. Let $\mathfrak{g}$ be a simple Lie algebra, and G the corresponding compact, connected, simply connected Lie group. Choose a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t}$, and let $\Lambda \subset \mathfrak{t}$ resp. $\Lambda^* \subset \mathfrak{t}^*$ denote the integral resp. (real) weight lattice. Let $\mathfrak{R}$ be the set of roots and $\mathfrak{R}_+$ the subset of positive roots, for some choice of positive Weyl chamber $\mathfrak{t}_+$. We will identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{t} \cong \mathfrak{t}^*$, using the normalized inner product $\cdot$ for which the long roots have length $\sqrt{2}$. The highest root is denoted α_0, and the half-sum of positive roots $\rho = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\alpha$. The integer

$$c = 1 + \rho\cdot\alpha_0$$

is called the dual Coxeter number of G. The fundamental alcove for G is the simplex

$$\mathfrak{A} = \{\xi\in\mathfrak{t}_+,\ \alpha_0\cdot\xi\le 1\}\subset\mathfrak{t}\subset\mathfrak{g}. \tag{2}$$

It parametrizes the set of conjugacy classes of G, in the sense that every conjugacy class contains an element $\exp(\xi)$ for a unique $\xi\in\mathfrak{A}$. The centralizer $G_{\exp(\xi)}$ depends only on the open face σ containing ξ and will be denoted G_σ. Introduce a partial ordering on the set of open faces of $\mathfrak{A}$ by setting $\sigma\prec\tau$ if $\sigma\subset\overline{\tau}$. Then $\sigma\prec\tau\Rightarrow G_\sigma\supset G_\tau$.

A similar discussion holds for semi-simple simply-connected groups, with the alcove replaced by the product of the alcoves for the simple factors.

2.2. Loop groups.

Let LG denote the loop group of maps $S^1\to G$ of some fixed Sobolev class $s>1$, $L\mathfrak{g}=\Omega^0(S^1,\mathfrak{g})$ its Lie algebra, and $L\mathfrak{g}^*\in\Omega^1(S^1,\mathfrak{g})$ the space of Lie algebra valued 1-forms of Sobolev class $s-1$. Integration over S^1 defines a non-degenerate pairing between $L\mathfrak{g}^*$ and $L\mathfrak{g}$. One defines the (affine) coadjoint action of LG on $L\mathfrak{g}^*\in\Omega^1(S^1,\mathfrak{g})$ by

$$g\cdot\mu = \mathrm{Ad}_g\,\mu - \mathrm{d}g\,g^{-1} \tag{3}$$

where $\mathrm{d}g\,g^{-1}$ is the pull-back of the right-invariant Maurer-Cartan form on G. Let $\widehat{LG}$ be the basic central extension [9] of LG, defined infinitesimally by the cocycle $(\xi_1,\xi_2)\mapsto\oint\mathrm{d}\xi_1\cdot\xi_2$ on $L\mathfrak{g}$. The adjoint action of $\widehat{LG}$ on $\widehat{L\mathfrak{g}}$ descends to an action of LG since the central circle acts trivially, and for the coadjoint action of LG on $\widehat{L\mathfrak{g}}^* = \Omega^1(S^1,\mathfrak{g})\oplus\mathbb{R}$ one finds

$$g\cdot(\mu,\lambda) = (\mathrm{Ad}_g(\mu)+\lambda\mathrm{d}gg^{-1},\lambda). \tag{4}$$

This identifies $L\mathfrak{g}^*$ with the affine hyperplane $\Omega^1(S^1,\mathfrak{g})\times\{1\}\subset\widehat{L\mathfrak{g}}^*$.

There is a natural smooth map $\mathrm{Hol}: L\mathfrak{g}^*\to G$ sending $\mu\in L\mathfrak{g}^*$, viewed as a connection on the trivial bundle $S^1\times G$, to its holonomy around S^1. This map sets up a 1-1 correspondence between the sets of G-conjugacy classes and coadjoint LG-orbits, hence both are parametrized by points in the alcove.

More explicitly this parametrization is given as follows. View $\mathfrak{A}$ as a subset of $L\mathfrak{g}^*$ by the embedding $\xi\mapsto\xi\mathrm{d}\theta/(2\pi)$. Then every coadjoint LG-orbit passes through a unique point $\xi\in\mathfrak{A}$. The stabilizer group $(LG)_\xi$ depends only on the open face $\sigma\subset\mathfrak{A}$ containing ξ and will be denoted $(LG)_\sigma$. The evaluation map $LG\to G$, $g\mapsto g(1)$ restricts to an isomorphism $(LG)_\sigma\cong G_\sigma$; in particular $(LG)_\sigma$ is compact and connected. If $\sigma\prec\tau$ then $(LG)_\sigma\supset(LG)_\tau$. In particular, every $(LG)_\sigma$ contains $T=(LG)_{\mathrm{int}\,\mathfrak{A}}$.

2.3. Hamiltonian LG-manifolds. We begin by reviewing the definition of a symplectic Banach manifold. A two-form ω on a Banach manifold M is weakly non-degenerate if the map $\omega^\sharp : T_m^M \to T_m^* M$ is injective, for all $m \in M$. A *Hamiltonian LG-manifold* is a Banach manifold M together with an LG-action, an invariant, weakly non-degenerate closed two-form ω and an equivariant moment map $\Phi : M \to L\mathfrak{g}^*$. Equivalently, one can think of M has a Hamiltonian $\widehat{LG}$-manifold, where the central circle acts trivially with constant moment map $+1$.

Example 2.1. 1) For any $\mu \in L\mathfrak{g}^*$, the coadjoint orbit $LG \cdot \mu$ is a Hamiltonian LG-manifold, with moment map the inclusion.

2) Let Σ be a compact oriented surface with boundary $\partial\Sigma \cong (S^1)^b$. Let $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, G)$ be the gauge group, and $\mathcal{G}_\partial(\Sigma)$ be the gauge transformation that are trivial on the boundary.

The space $\Omega^1(\Sigma, \mathfrak{g})$ of connections carries a natural symplectic structure, and the action of $\mathcal{G}_\partial(\Sigma)$ is Hamiltonian with moment map the curvature. The symplectic quotient $\mathcal{M}(\Sigma)$ is the moduli space of flat connection up to based gauge transformations. It carries a residual action of LG^b, with moment map induced by the pull-back of connections to the boundary.

2.4. Symplectic cross-sections. In the case where the moment map Φ is *proper*, a Hamiltonian LG-space with proper moment map behaves very much like a compact Hamiltonian space for a compact group.[1] The reason for this is that the coadjoint LG-action on $L\mathfrak{g}^*$ has finite dimensional slices, and the pre-images of these slices are finite dimensional symplectic submanifolds. To describe these slices, we view the alcove as a subset of $L\mathfrak{g}^*$ as explained above. Let

$$\mathfrak{A}_\sigma := \bigcup_{\tau \succeq \sigma} \tau.$$

Then the flow-out under the action of the compact group $(LG)_\sigma$,

$$U_\sigma = (LG)_\sigma \cdot \mathfrak{A}_\sigma \subset L\mathfrak{g}^*$$

is a slice for the LG-action at points in σ.

For example, if $G = SU(2)$, then the alcove may be identified with the interval

$$\mathfrak{A} = [0, 1/2].$$

For the three faces $\{0\}, (0, 1/2), \{1/2\}$ we have

$$\mathfrak{A}_{\{0\}} = [0, 1/2), \quad \mathfrak{A}_{(0,1/2)} = (0, 1/2), \quad \mathfrak{A}_{\{1/2\}} = (0, 1/2].$$

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[1] In fact, there is a 1-1 correspondence between Hamiltonian LG-spaces with proper moment map and compact Hamiltonian G-spaces with G-valued moment maps [1].

The slice $Y_{(0,1/2)} = (0,1/2)$, since $LG_{(0,1/2)} = T$. The other slices $Y_{\{0\}}, Y_{\{1/2\}}$ are open balls of radius 1/2 in $L\mathfrak{g}^*_{\{0\}}$, resp. $L\mathfrak{g}^*_{\{1/2\}}$. Note that although $L\mathfrak{g}^*_{\{0\}}, L\mathfrak{g}^*_{\{1/2\}}$ are isomorphic as G-modules to the Lie algebra $\mathfrak{g}$, the intersection $L\mathfrak{g}^*_{\{0\}} \cap L\mathfrak{g}^*_{\{1/2\}} = L\mathfrak{g}^*_{(0,1/2)}$.

If M is a symplectic Hamiltonian LG-space with proper moment map Φ, the *symplectic cross-sections*

$$Y_\sigma = \Phi^{-1}(U_\sigma)$$

are finite-dimensional symplectic submanifolds. In fact, they are Hamiltonian $(\widehat{LG})_\sigma$-manifolds, where the central S^1 acts trivially. The moment maps are the restrictions $\Phi_\sigma = \Phi|Y_\sigma : Y_\sigma \to U_\sigma \subset (L\mathfrak{g})^*_\sigma \subset \widehat{L\mathfrak{g}}^*$. Here $(L\mathfrak{g})^*_\sigma$ is identified with the unique $(LG)_\sigma$-invariant complement to the annihilator of $(L\mathfrak{g})_\sigma$ in $L\mathfrak{g}^*$, or equivalently with the span of U_σ.

For a proof of the symplectic cross-section theorem for loop group actions, see [8]. The flowouts $LG \cdot Y_\sigma = LG \times_{(LG)_\sigma} Y_\sigma$ form an open covering of M. Therefore, the Hamiltonian LG-space (M, ω, Φ) can be reconstructed from its collection of symplectic cross-sections $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$ and the inclusions $Y_\tau \hookrightarrow Y_\sigma$ for $\sigma \prec \tau$.

3. Construction of the canonical bundle.

Suppose (M, ω, Φ) is a Hamiltonian LG-manifold with proper moment map. In this section we contruct an $\widehat{LG}$-equivariant line bundle $K_M \to M$ which will play the role of a canonical line bundle.

For any $\widehat{LG}$-equivariant line bundle $L \to M$, the (locally constant) weight of the action of the central circle $S^1 \subset \widehat{LG}$ is called the *level* of L. Any $\widehat{LG}$-bundle $L \to M$ is determined by the collection of $(\widehat{LG})_\sigma$-equivariant line bundles $L_\sigma \to Y_\sigma$ over the cross-sections, together with $(LG)_\tau$-equivariant isomorphisms $\varphi_{\sigma,\tau} : L_\sigma|Y_\tau \cong L_\tau$ for all $\sigma \prec \tau$, such that

$$\varphi_{\sigma,\tau} \circ \varphi_{\tau,\nu} = \varphi_{\sigma,\nu} \tag{5}$$

if $\sigma \preceq \tau \preceq \nu$.

Let $K_\sigma \to Y_\sigma$ be the canonical line for some invariant compatible almost complex (a.c.) structure on Y_σ. There exist $(LG)_\tau$-equivariant isomorphisms

$$K_\sigma|Y_\tau \cong K_\tau \otimes \det_{\mathbb{C}}(\nu^\sigma_\tau)^* \tag{6}$$

where $\nu^\sigma_\tau \to Y_\tau$ is the symplectic normal bundle to Y_τ inside Y_σ. We will therefore begin by describing the complex structure on ν^σ_τ.

3.1. The normal bundle of Y_τ in Y_σ.

Suppose $\sigma \prec \tau$ so that Y_τ is an $(LG)_\tau$-invariant submanifold of $(LG)_\sigma$. Since $(LG)_\sigma \times_{(LG)_\tau} Y_\tau$ is an open subset of Y_σ, the normal bundle of Y_τ in Y_σ is $(LG)_\tau$-equivariantly isomorphic to the trivial bundle $(L\mathfrak{g})_\sigma/(L\mathfrak{g})_\tau$. It carries a unique $(LG)_\tau$-invariant complex structure compatible with the symplectic structure. In terms of the root space decomposition this complex structure is given as follows. Given a face σ of $\mathfrak{A}$, define the positive Weyl chamber $\mathfrak{t}_{+,\sigma}$ for $(LG)_\sigma$ as the cone over $\mathfrak{A}-\mu$, for any $\mu\in\sigma$. Similarly define $\mathfrak{t}_{+,\tau}$. Let $\mathfrak{R}_{+,\sigma}\supset\mathfrak{R}_{+,\tau}$ the corresponding collections of positive roots.

As complex $(\widehat{LG})_\tau$-representations,

$$(L\mathfrak{g})_\sigma/(L\mathfrak{g})_\tau=\bigoplus_{\alpha\in\mathfrak{R}_{+,\sigma}\setminus\mathfrak{R}_{+,\tau}}\mathbb{C}_\alpha.$$

In particular,

$$\det{}_{\mathbb{C}}(\nu^\sigma_\tau)^*=\bigotimes_{\alpha\in\mathfrak{R}_{+,\sigma}\setminus\mathfrak{R}_{+,\tau}}\mathbb{C}_\alpha=\mathbb{C}_{-2(\rho_\sigma-\rho_\tau)},\tag{7}$$

where ρ_σ,ρ_τ are the half-sums of positive roots of $\mathfrak{R}_{+,\sigma},\mathfrak{R}_{+,\tau}$ respectively.

3.2. Compatibility condition. Our candidate for $L_\sigma=(K_M)|Y_\sigma$ will be of the form $K_\sigma\otimes\mathbb{C}_{\gamma_\sigma}$, for suitable weights $\gamma_\sigma\in\Lambda^*\times\mathbb{Z}$. The key point which makes the problem non-trivial is that in order for $\mathbb{C}_{\gamma_\sigma}$ to give $\widehat{LG}_\sigma$-representations, the weight γ_σ should be fixed under the $(\widehat{LG})_\sigma$-action on $\widehat{L\mathfrak{g}}^*$. According to (6) and (7) these weights should satisfy

$$\gamma_\sigma-\gamma_\tau=2(\rho_\sigma-\rho_\tau)$$

for all faces $\sigma\prec\tau$.

The following Lemma gives a solution to this system of equations.

Lemma 3.1. *For all faces $\sigma\subset\mathfrak{A}$, the difference $2\rho-2\rho_\sigma\in\Lambda^*$ is the orthogonal projection of 2ρ to the affine span of the dilated face $2c\sigma$. In particular the weight*

$$\gamma_\sigma:=-(2\rho-2\rho_\sigma,2c)\in\Lambda^*\times\mathbb{Z}$$

is fixed under $(\widehat{LG})_\sigma$.

Proof. The weight $2\rho_\sigma$ is characterized by the property

$$2\rho_\sigma\cdot\alpha=\alpha\cdot\alpha$$

for every simple root α of $(LG)_\sigma$. Letting $\{\alpha_1,\ldots,\alpha_l\}$ be the simple roots for G, the simple roots for $(LG)_\sigma$ are precisely those roots in the collection $\{\alpha_1,\ldots,\alpha_l,-\alpha_0\}$ which are perpendicular to the span of $\sigma-\mu$ (where $\mu\in\sigma$). In particular $-\alpha_0$ is a simple root for $(LG)_\sigma$ precisely if $0\notin\overline{\sigma}$.

If $\alpha\in\{\alpha_1,\ldots,\alpha_l\}$ is a simple root of $(LG)_\sigma$ then $2\rho\cdot\alpha=2\rho_\sigma\cdot\alpha=\alpha\cdot\alpha$ so that $(2\rho-2\rho_\sigma)\cdot\alpha=0$. If $0\notin\overline{\sigma}$ so that $-\alpha_0$ is among the set of simple roots for $(LG)_\sigma$, we also have

$$(2\rho-2\rho_\sigma)\cdot\alpha_0=2(c-1)+\alpha_0\cdot\alpha_0=2c,$$

as required. $\square$

The solution given by the lemma is unique, since for $\sigma = \{0\}$ the group $LG_\sigma = G$ has the unique fixed point $\gamma_0 = (0, -2c)$.

3.3. Gluing. Let $L_\sigma = K_\sigma \otimes \mathbb{C}_{\gamma_\sigma}$. We still have to construct isomorphisms $\varphi_{\sigma,\tau} : L_\sigma | Y_\tau \to L_\tau$ satisfying the cocycle condition. If the compatible a.c. structures on Y_σ can be chosen in such a way that for $\sigma \prec \tau$, Y_τ is an a.c. submanifold of Y_σ, the isomorphisms would be canonically defined and the cocycle condition would be automatic. Unfortunately, it is in general impossible to choose the a.c. structures to have this property.

To get around this difficulty we replace the sets Y_σ with smaller open subsets. The compact set M/LG is covered by the collection of sets $Y_\sigma/(LG)_\sigma$ with σ a vertex of $\mathfrak{A}$, since $\mathfrak{A}$ is covered by the (relative) open subsets $\mathfrak{A}_\sigma$. It is therefore possible to choose for each vertex σ of $\mathfrak{A}$, an $(LG)_\sigma$-invariant, open subset $Y'_\sigma \subset Y_\sigma$, such that the collection of these subsets has the following two properties:

a. The collection of all $Y'_\sigma/(LG)_\sigma$ covers M/LG.

b. The closure of Y'_σ is contained in Y_σ.

Given such a collection of subsets $\{Y'_\sigma\}$ we define, for any open face τ of $\mathfrak{A}$,

$$Y'_\tau = \bigcap_{\sigma \preceq \tau,\ \dim \sigma = 0} Y'_\sigma.$$

Then Y'_τ is an $(LG)_\tau$-invariant open subset of Y_τ, with the property that its closure in M is contained in Y_τ.

Lemma 3.2. *There exists a collection of $(LG)_\sigma$-invariant compatible a.c. structures on the collection of Y'_σ, with the property that for all $\sigma \preceq \tau$, the embedding $Y'_\tau \hookrightarrow Y'_\sigma$ is a.c.. Moreover, any two a.c. structures on the disjoint union $\coprod_\sigma Y'_\sigma$ with the required properties are homotopic.*

Proof. We construct a.c. structures J_σ on Y'_σ with the required properties by induction over dimension of the faces σ, starting from the interior of the alcove $\mathfrak{A}$ and ending at vertices.

Given $k \geq 0$, suppose that we have constructed compatible a.c. structures on all Y_σ with $\dim \sigma > \dim \mathfrak{t} - k$, in such a way that if $\sigma \preceq \tau$, the embedding $Y_\tau \hookrightarrow Y_\sigma$ is a.c. on some open neighborhood of the closure of Y'_τ. Let ν be a face of dimension $\dim \mathfrak{t} - k$. Each of the a.c. structures on Y_τ with $\tau \succ \nu$ defines an invariant compatible a.c. structure on Y_ν, and by hypothesis these complex structures match on some open neighborhood of $\bigcup_{\nu \prec \tau} (LG)_\nu \cdot \overline{Y'_\tau}$. We choose an invariant a.c. structure on Y_ν such that it matches with the given a.c. structures over a possibly smaller open neighborhood of $\bigcup_{\nu \prec \tau} (LG)_\nu \cdot \overline{Y'_\tau}$. This can be done by choosing a Riemannian metric on Y_ν which matches the given one in a possibly smaller neighborhood, and taking the compatible almost complex structure defined by the metric in the standard way (see e.g. [6]).

Now let $\{J^0_\sigma\}, \{J^1_\sigma\}$ be two collections of a.c. structures with the required properties. They define Riemannian metrics g^0_σ, g^1_σ. Let $g^t_\sigma = (1-t)g^0_\sigma + t\, g^1_\sigma$, and let J^t_σ be the compatible a.c. structure which it defines. For $\sigma \prec \tau$, the metric g^t_τ on Y'_τ is the restriction of g^t_σ and the symplectic normal bundle of Y'_τ in Y'_σ coincides with the Riemannian normal bundle. This implies that the embedding $Y'_\tau \to Y'_\sigma$ is a.c.. ☐

Choose a.c. structures on Y_σ as in the Lemma, and define $(\widehat{LG})_\sigma$-equivariant line bundles $L'_\sigma = K'_\sigma \otimes \mathbb{C}_{\gamma_\sigma}$. We then have canonical isomorphisms

$$\phi_{\sigma,\tau} : L'_\sigma | Y'_\tau = L'_\tau$$

and they automatically satisfy the cocycle condition. It follows that there is a unique $\widehat{LG}$-equivariant line bundle $K_M \to M$ with $K_M | Y'_\sigma = L'_\sigma$. By construction, the collection of line bundles L'_σ, hence also K_M, is independent of the choice of a.c. structures up to homotopy.

Lemma 3.3. *The isomorphism class of K_M is independent of the choice of "cover" Y'_σ.*

Proof. Given two choices Y^1_σ and Y^2_σ labeled by the vertices of $\mathfrak{A}$, let $Y^3_\sigma = Y^1_\sigma \cup Y^2_\sigma$. Given a.c. structures J^j_σ on Y^j_σ and the canonical line bundles K^j_M constructed from them, we have an equivariant homotopy $K^1_M \sim K^3_M \sim K^2_M$ (because J^3_σ restricts to a.c. structures on Y^1_σ and Y^2_σ). ☐

This completes our construction of the canonical bundle. The central circle in $\widehat{LG}$ acts with weight $-2c$, that is, K_M is a line bundle at level $-2c$.

3.4. Examples.

3.4.1. Coadjoint orbits. Let $M = LG \cdot \mu$ be the coadjoint orbit through $\mu \in \mathfrak{A}$, and let $\sigma \subset \mathfrak{A}$ denote the open face containing μ. Thus $M \cong LG/(LG)_\sigma$. Since $Y_\sigma = \{\mu\}$, the canonical line bundle K_M is the associated bundle

$$K_{LG/(LG)_\sigma} := \widehat{LG} \times_{(\widehat{LG})_\sigma} \mathbb{C}_{-2(\rho-\rho_\sigma, c)}. \tag{8}$$

This definition of canonical bundle agrees with Freed's computation [4] of a regularized first Chern class of the fundamental homogeneous space $\Omega G = LG/G$. In this paper, Freed provides further evidence for this being the correct definition of a first Chern class, the simplest being that since $\hat{\rho} = (\rho, c)$ is the sum of fundamental affine weights (cf. [9]), the canonical bundle for LG/T is expected to be $K_{LG/T} = \widehat{LG} \times_{\widehat{T}} \mathbb{C}_{-2\hat{\rho}}$. and that for LG/G should be $\widehat{LG} \times_{\widehat{G}} \mathbb{C}_{-2(0,c)}$.

Since $LG/(LG)_\sigma$ is a homogeneous space the canonical line bundle carries a unique $\widehat{LG}$-invariant connection. Its curvature equals $-2\pi i$ times the symplectic form for the coadjoint orbit (at level $-2c$) through $-2(\rho-\rho_\sigma, c) = -\gamma_\sigma$. Recall that $(\rho-\rho_\sigma)/c \in \mathfrak{A}$ is the orthogonal projection of ρ/c onto the affine subspace spanned by σ. Therefore:

Lemma 3.4. *If* (M, ω) *is the coadjoint* LG*-orbit* (*at level* 1) *through the orthogonal projection* μ *of* ρ/c *onto some face* σ *of* $\mathfrak{A}$*, the curvature of the canonical line bundle is given by* $\frac{i}{2\pi}\operatorname{curv}(K_M) = -2c\omega$*. In particular, this is true for* $\mu = \rho$ *and for* μ *a vertex of* $\mathfrak{A}$*.*

3.4.2. Moduli spaces of flat connections. Let Σ be a compact, oriented surface with boundary $\partial\Sigma \cong (S^1)^b$ and $(\mathcal{M}(\Sigma), \omega)$ the corresponding moduli space. From now on, we assume that $b = 1$, although the more general case is only more difficult notationally. By Corollary 3.12 of [7] there is a unique $\widehat{LG}$-equivariant line bundle at each level, so that every $\widehat{LG}$-equivariant line bundle over $\mathcal{M}(\Sigma)$ at level k is isomorphic to the kth tensor power of the pre-quantum line bundle $L(\Sigma)$.[2] In particular the canonical bundle $K_{\mathcal{M}(\Sigma)} \to \mathcal{M}(\Sigma)$ carries an invariant connection such that $\frac{i}{2\pi}\operatorname{curv}(K_{\mathcal{M}(\Sigma)}) = -2c\omega$.

4. Quotients of canonical bundles.

In this section, we show that the bundles K_M behave well under symplectic quotients, that is, that the symplectic quotient of K_M is the usual canonical bundle on the quotient. For any Hamiltonian LG-space (M, ω, Φ) with proper moment map, and any coadjoint LG-orbit $\mathcal{O} \subset L\mathfrak{g}^*$, the reduced space $M_{\mathcal{O}}$ at level $\mathcal{O}$ is a compact space defined as the quotient

$$M_{\mathcal{O}} := \Phi^{-1}(\mathcal{O})/LG.$$

Let $\mu \in \mathfrak{A}$ is the point of the alcove through which $\mathcal{O}$ passes, σ the open face containing μ, and

$$\mathcal{O}_\sigma := \mathcal{O} \cap U_\sigma = (LG)_\sigma \cdot \mu.$$

Then

$$M_{\mathcal{O}} = \Phi^{-1}(\mu)/(LG)_\sigma = (Y_\sigma)_{\mathcal{O}_\sigma}$$

which identifies $M_{\mathcal{O}}$ as a reduced space of the symplectic cross-section $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$. It follows that the standard theory of symplectic reduction applies: If μ is a regular value then $M_{\mathcal{O}}$ is a finite dimensional symplectic orbifold, and in general it is a finite dimensional stratified symplectic space in the sense of Sjamaar-Lerman [12].

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[2] A sketch of the argument is as follows: Two line bundles at the same level differ by a line bundle at level 0, which descends to the quotient $\mathcal{M}(\Sigma)/\Omega G$ by the based loop group. From the holonomy description of the moduli space we have $\mathcal{M}(\Sigma)/\Omega G \cong G^{2g}$. Since $H^2_G(G^{2g})$ is trivial, the descended line bundle is trivial, so the two line bundles are isomorphic.

Over the level set $\Phi^{-1}(\mathcal{O})$ we have two line bundles at level $-2c$, the restriction of the canonical bundle of M and the pull-back by Φ of the canonical bundle $K_{\mathcal{O}}$ on the coadjoint orbit. They differ by an LG-equivariant line bundle (that is an $\widehat{LG}$-bundle at level 0),

$$K_M|_{\Phi^{-1}(\mathcal{O})} \otimes K_{\mathcal{O}}^*.$$

Proposition 4.1. *Suppose $\mathcal{O}$ consists of regular values of Φ. The canonical line bundle for the reduced space $M_{\mathcal{O}}$ is the quotient,*

$$\big(K_M|_{\Phi^{-1}(\mathcal{O})} \otimes \Phi^* K_{\mathcal{O}}^*\big)/LG.$$

Proof. Since

$$K_M = \widehat{LG} \times_{(\widehat{LG})_\sigma} (K_\sigma \otimes \mathbb{C}_{\gamma_\sigma}), \quad K_{\mathcal{O}} = \widehat{LG} \times_{(\widehat{LG})_\sigma} (K_{\mathcal{O}_\sigma} \otimes \mathbb{C}_{\gamma_\sigma})$$

we have

$$K_M|_{\Phi^{-1}(\mathcal{O})} \otimes \Phi^* K_{\mathcal{O}}^* = \widehat{LG} \times_{(\widehat{LG})_\sigma} (K_\sigma \otimes \Phi_\sigma^* K_{\mathcal{O}_\sigma}^*).$$

Taking the quotient by LG we obtain

$$\big(K_M|_{\Phi^{-1}(\mathcal{O})} \otimes \Phi^* K_{\mathcal{O}}^*\big)/LG = \big(K_\sigma|_{\Phi_\sigma^{-1}(\mathcal{O}_\sigma)} \otimes \Phi_\sigma^* K_{\mathcal{O}_\sigma}^*\big)/(LG)_\sigma$$

which is the canonical bundle for the reduced space $(Y_\sigma)_{\mathcal{O}_\sigma} = M_{\mathcal{O}}$. □

Theorem 4.2. *Let $\mathcal{M}(\Sigma)$ be the moduli space of flat connections on a compact oriented surface with boundary, and $\mathcal{C}_\mu$ the conjugacy class corresponding to the projection μ of ρ/c onto σ for some face σ. Suppose μ is a regular value for the moment map $\mathcal{M}(\Sigma)$, so $\mathcal{M}(\Sigma, \mathcal{C}_\mu)$ the moduli space of flat connections with holonomy in $\mathcal{C}_\mu$ is a compact symplectic orbifold. Then the Chern class $c_1(K_M)$ for $M = \mathcal{M}(\Sigma, \mathcal{C}_\mu)$ is $-2c$ times the cohomology class of the reduced symplectic form.*

Proof. Let $\mathcal{O}$ be the a coadjoint orbit through the element ρ_σ/c. By Section 3.4, $K_{M(\Sigma)}$ resp. $K_{\mathcal{O}}$ are isomorphic to the $-2c$-th tensor power of the pre-quantum line bundles on $\mathcal{M}(\Sigma)$ resp. $\mathcal{O}$. By Proposition 4.1, the canonical line bundle on the quotient is isomorphic to the $-2c$-th power of the quotient of the pre-quantum line bundle on the product, which is a pre-quantum line bundle on the quotient. □

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UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
100 ST GEORGE STREET, TORONTO, ONTARIO M5S 3G3
CANADA
E-mail address: mein@math.toronto.edu

MATHEMATICS-HILL CENTER
RUTGERS UNIVERSITY
110 FRELINGHUYSEN ROAD, PISCATAWAY NJ 08854-8019
E-mail address: ctw@math.rutgers.edu