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In this paper, we consider the zeta function $Z(P,\chi,s)$ associated with a polynomial $P(X) \in \mathbb{R}[X_1,\dots,X_r]$ and $\chi = (\chi_1,\dots,\chi_r)$ with χ_j non-trivial Dirichlet characters, defined by

$$Z(P,\chi,s) = \sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\chi_1(n_1)\cdots\chi_r(n_r)P(n_1,\dots,n_r)^{-s},$$

which is absolutely convergent for sufficiently large Re s under some conditions on $P(X)$. We shall prove that the special value $Z(P,\chi,-m)$ is completely determined by $P^m(X)$ in a simple way. As an immediate application, we give a closed expression for sums of products of any number of generalized Bernoulli numbers.

1. Introduction and Notation.

As usual, $\mathbb{N}$ denotes the set of positive numbers, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$ and $\mathbb{R}$ denotes the field of real numbers. Let χ be a non-trivial Dirichlet character with conductor N. The L-series attached to χ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}, \quad \operatorname{Re} s > 1.$$

It is well known [**14**] that $L(s,\chi)$ may be continued analytically to the whole complex s-plane. Furthermore, the special values at non-positive integers $s = -m$ $(m = 0,1,2,\dots)$ can be expressed by the generalized Bernoulli numbers B_χ^n $(n = 0,1,2,\dots)$ defined by

$$\sum_{a=1}^{N}\frac{\chi(a)te^{at}}{e^{Nt}-1} = \sum_{n=0}^{\infty}\frac{B_\chi^n t^n}{n!}, \ \ |t| < \frac{2\pi}{N}.$$

Indeed, $L(-m,\chi) = -\frac{B_\chi^{m+1}}{m+1}$ as given on Page 30 of [**14**]. The generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_\chi^n = N^{n-1}\sum_{a=1}^{N}\chi(a)B_n\left(\frac{a}{N}\right)$$

where the Bernoulli polynomials $B_n(X)$ are defined by

$$\frac{te^{Xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(X)\frac{t^n}{n!},\quad |t|<2\pi.$$

Also

$$B_n(X)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}X^k$$

where the Bernoulli numbers B_n $(n=0,1,2,\dots)$ are defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad |t|<2\pi.$$

Consequently, we can express the generalized Bernoulli numbers in terms of Bernoulli numbers as follows:

$$B_\chi^n=\sum_{a=1}^{N}\chi(a)\sum_{k=0}^{n}\binom{n}{k}B_k a^{n-k}N^{k-1}.$$

Let $P(X)=P(X_1,\dots,X_r)$ be a polynomial of r variables with non-negative real coefficients such that $P(n)>0$ for all $n\in\mathbb{N}^r$ and the series

$$\sum_{n\in\mathbb{N}^r}P(n)^{-s}=\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}P(n_1,\dots,n_r)^{-s}$$

is absolutely convergent for $\operatorname{Re}s>\sigma>0$. $\chi_1,\dots,\chi_r$ are non-trivial Dirichlet characters with conductors $N_1,\dots,N_r$, respectively. Consider the zeta function associated with P and $\chi=(\chi_1,\dots,\chi_r)$ defined by

$$Z(P,\chi,s)=\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\chi_1(n_1)\cdots\chi_r(n_r)P(n_1,\dots,n_r)^{-s},\quad \operatorname{Re}s>\sigma.$$

It is the main purpose of this paper to prove the following result.

Theorem. *$Z(P,\chi,s)$ defined above has a meromorphic analytic continuation to the whole complex s-plane. For any integer $m\geq 0$, if*

$$P^m(X)=\sum_{|\alpha|=0}^{mp}C_\alpha X_1^{\alpha_1}\cdots X_r^{\alpha_r},\qquad p=\deg P,$$

then

$$\begin{aligned}Z(P,\chi,-m)&=\sum_{|\alpha|=0}^{mp}C_\alpha L(-\alpha_1,\chi_1)\cdots L(-\alpha_r,\chi_r)\\&=(-1)^r\sum_{|\alpha|=0}^{mp}C_\alpha\prod_{j=1}^{r}\frac{B_{\chi_j}^{\alpha_j+1}}{\alpha_j+1}.\end{aligned}$$

Another zeta function $Z(P,\xi,s)$ defined by

$$Z(P,\xi,s)=\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\xi_1^{n_1}\cdots\xi_r^{n_r}P(n_1,\ldots,n_r)^{-s}$$

was considered by P. Cassou-Noguès in [**2**]. Her result for the special values of $Z(P,\xi,s)$ can be restated as

$$Z(P,\xi,-m)=\lim_{t\to 0^+}\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\xi_1^{n_1}\cdots\xi_r^{n_r}P^m(n)e^{-(n_1+\cdots+n_r)t}.$$

Here we also have the same formula for the special values of $Z(P,\chi,s)$, i.e.,

$$Z(P,\chi,-m)=\lim_{t\to 0^+}\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\chi_1(n_1)\cdots\chi_r(n_r)P^m(n)e^{-(n_1+\cdots+n_r)t}.$$

However $\chi(n)=\prod_{j=1}^{r}\chi_j(n_j)$ is a multiplicative character while $\xi^n=\prod_{j=1}^{r}\xi_j^{n_j}$ is an additive character. Hence the treatments are different in some respect. As shown in Section 4, P. Cassou-Noguès' formula for the special values of $Z(P,\xi,s)$ follows from our formula for the special values of $Z(P,\chi,s)$. In additon we have another explicit expression for the special values of $Z(P,\xi,s)$.

A well-known relation among the Bernoulli numbers is

$$\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}=-(2n+1)B_{2n},\qquad \text{for } n\geq 2.$$

This was found by many authors, including Euler (ref. [**5**], [**8**]). Dilcher [**5**] generalized the formula for sums of products of any number of both Bernoulli and Euler numbers. Bernoulli and Euler numbers are special cases of the generalized Bernoulli numbers B_χ^n belonging to a residue class character χ. However it is not easy to get the generalized formula for generalized Bernoulli numbers. At the end of this paper, we give a closed expression for the case as an immediate application of our main theorem.

2. Some Basic Results.

We need some classical results reproduced in [**15**].

Proposition 1. *Suppose that* $\varphi(s)=\sum_{\lambda>0}a_\lambda\lambda^{-s}$ *(*λ *ranges over a sequence of positive real numbers tending* $+\infty$*) is a Dirichlet series converging for sufficiently large* Re s*.* $f(t)=\sum_{\lambda>0}a_\lambda e^{-\lambda t}$ *is the corresponding exponential series. Suppose that at* $t=0$*,* $f(t)$ *has the asymptotic expansion*

$$\sum_{n\geq 0}C_n t^{n/p}$$

where p *is a fixed positive number. Then:*

(1) *$\varphi(s)$ has a meromorphic continuation to the whole complex plane.*
(2) *$\varphi(s)$ has possible simple poles at $s=-n/p$, where n is not a multiple of p , with residue $C_n/\Gamma(-n/p)$, and has no other poles.*
(3) *$\varphi(-n)=(-1)^n n! C_{np}$.*

Note that the above proposition is different from Proposition 2 of [**15**]. However, it follows from

$$\begin{aligned}\varphi(s)\Gamma(s) &= \int_0^\infty t^{s-1} f(t)dt, \qquad \operatorname{Re} s > \sigma \\ &= \int_0^\delta t^{s-1}\sum_{n=0}^\infty C_n t^{n/p} dt + \int_\delta^\infty t^{s-1} f(t)dt \\ &= \sum_{n=0}^\infty C_n \frac{\delta^{s+\frac{n}{p}}}{s+\frac{n}{p}} + \int_\delta^\infty t^{s-1} f(t)dt,\end{aligned}$$

where δ is a small positive number so that $f(t)=\sum_{n=0}^\infty C_n t^{n/p}$. From the above, we get our assertions.

A function $f(x)$ is called a rapidly decreasing function if it belongs to $C^\infty(\mathbb{R}^n)$ and satisfies

$$\lim_{|x|\to\infty} |x|^k |D^\alpha f(x)| = 0$$

for any α and any integer $k>0$ (ref. [**10**], or page 245 in [**11**]). The following is a consequence of the Euler-Maclaurin summation formula which is also reproduced in [**15**].

Proposition 2. *Suppose that f is a rapidly decreasing function on $[0,\infty)$ and at $t=0$, f has the power series expansion*

$$f(t) = \sum_{r=0}^\infty \frac{f^{(r)}(0)}{r!} t^r.$$

Suppose that $g(t)=\sum_{n=1}^\infty f(nt)$. Then at $t=0$, $g(t)$ has the asymptotic expansion

$$\frac{C}{t} + \sum_{r=0}^\infty (-1)^r \frac{B_{r+1}}{(r+1)!} f^{(r)}(0) t^r \qquad \text{with} \qquad C = \int_0^\infty f(t)dt.$$

To find the special value at $s=-m$ of the zeta function

$$Z(P,\chi,s) = \sum_{n_1=1}^\infty \cdots \sum_{n_r=1}^\infty \chi_1(n_1)\cdots\chi_r(n_r) P(n)^{-s},$$

by Proposition 1, it is equivalent to find the coefficient of t^m in the asymptotic expansion at $t=0$ of the function

$$\sum_{n_1=1}^\infty \cdots \sum_{n_r=1}^\infty \chi_1(n_1)\cdots\chi_r(n_r)\exp\{-P(n)t\}.$$

It is also equivalent to find the constant term in the asymptotic expansion at $t = 0$ of the function

$$g(t) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) \exp\{-P(n)t\}.$$

For the given polynomial

$$P(X) = \sum_{|\alpha|=0}^{p} A_\alpha X^\alpha, \qquad p = \deg P,$$

we let

$$Q(X,Y) = \sum_{|\alpha|=0}^{p} A_\alpha X^\alpha Y^{p-|\alpha|}$$

be the corresponding homogeneous polynomial in $r+1$ variables. Obviously, $Q(nt,t) = P(n)t^p$ and so

$$\begin{aligned} g(t^p) &= \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) \exp\{-P(n)t^p\} \\ &= \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) \exp\{-Q(nt,t)\} \\ &= \sum_{|\alpha|=0}^{mp} C_\alpha \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) n^\alpha \exp\{-Q(nt,t)\} \end{aligned}$$

where

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_\alpha X^\alpha \qquad \text{and} \qquad n^\alpha = n_1^{\alpha_1} \cdots n_r^{\alpha_r}.$$

In the next section, we shall compute the asymptotic expansion at $t = 0$ of the function

$$f_\beta(t) = \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) n^\beta \exp\{-Q(nt,t)\}.$$

3. The Proof of the Theorem.

First we shall prove the case $r = 1$. Indeed this special case plays an important role in our proof of the theorem.

Lemma 1. *Let P be a polynomial with real coefficients such that $P(n) > 0$ for all $n \in \mathbb{N}$ and Q be the corresponding polynomial defined above. Let*

$h(x,t)=x^\beta\exp\{-Q(xt,t)\}$, N a positive integer and $1\le j\le N$. Furthermore, denote

$$\begin{aligned}f_j(t)&=\sum_{k=0}^\infty h(kN+j,t)\\&=\sum_{k=0}^\infty (kN+j)^\beta\exp\{-Q((kN+j)t,t)\}.\end{aligned}$$

Then

$$f_j(t)=h(j,t)+\frac{1}{N}\int_j^\infty h(x,t)dx+\sum_{r=0}^\infty\frac{(-1)^rB_{r+1}}{(r+1)!}h^{(r)}(j,t)N^r$$

where $h^{(r)}(x,t)$ is the r-th partial derivative with respect to x.

Proof. It follows from the Euler-Maclaurin summation formula that

$$\begin{aligned}\sum_{k=1}^\infty h(kN+j,t)&=\int_0^\infty (Nx+j)^\beta\exp\{-Q((Nx+j)t,t)\}dx\\&\quad+\sum_{r=0}^\infty\frac{(-1)^rB_{r+1}}{(r+1)!}h^{(r)}(j,t)N^r\\&=\frac{1}{N}\int_j^\infty h(x,t)dx+\sum_{r=0}^\infty\frac{(-1)^rB_{r+1}}{(r+1)!}h^{(r)}(j,t)N^r.\end{aligned}$$

□

Proposition 3. *Let χ be a non-trivial character with conductor N. Let $\beta\ge 0$ be an integer, and P, Q polynomials as given in the previous lemma. Suppose that*

$$f(t)=\sum_{n=1}^\infty\chi(n)n^\beta\exp\{-Q(nt,t)\}.$$

Then

$$\begin{aligned}f(t)=\sum_{j=1}^N\chi(j)h(j,t)-\frac{1}{N}\sum_{j=1}^N\chi(j)\int_0^j h(x,t)dx\\+\sum_{j=1}^N\chi(j)\sum_{r=0}^\infty\frac{(-1)^rB_{r+1}}{(r+1)!}h^{(r)}(j,t)N^r.\end{aligned}$$

In particular at $t=0$, $f(t)$ has an asymptotic expansion of the form

$$\sum_{n=0}^\infty d_nt^n$$

with the constant term d_0 given by

$$d_0 = -\frac{B_\chi^{\beta+1}}{\beta+1} = L(-\beta,\chi).$$

Proof. Note that

$$\begin{aligned} f(t) &= \sum_{j=1}^{N}\chi(j)\sum_{k=0}^{\infty}(Nk+j)^{\beta}\exp\{-Q((Nk+j)t,t)\} \\ &= \sum_{j=1}^{N}\chi(j)f_j(t). \end{aligned}$$

So the first assertion follows from Lemma 1 by noting that

$$\int_j^\infty h(x,t)dx = \int_0^\infty h(x,t)dx - \int_0^j h(x,t)dx$$

and

$$\sum_{j=1}^{N}\chi(j)\int_j^\infty h(x,t)dx = -\sum_{j=1}^{N}\chi(j)\int_0^j h(x,t)dx$$

since $\sum_{j=1}^{N}\chi(j) = 0$. Also, from this expression of $f(t)$ we have a power series expansion of the form

$$\sum_{n=0}^{\infty} d_n t^n$$

with

$$\begin{aligned} d_0 &= \sum_{j=1}^{N}\chi(j)h(j,0) - \frac{1}{N}\sum_{j=1}^{N}\chi(j)\int_0^j h(x,0)dx \\ &\quad + \sum_{j=1}^{N}\chi(j)\sum_{r=0}^{\infty}\frac{(-1)^r B_{r+1}}{(r+1)!}h^{(r)}(j,0)N^r \\ &= \sum_{j=1}^{N}\chi(j)j^{\beta} - \frac{1}{N}\sum_{j=1}^{N}\chi(j)\frac{j^{\beta+1}}{\beta+1} + \sum_{j=1}^{N}\chi(j)\sum_{r=0}^{\infty}\frac{(-1)^r B_{r+1}}{(r+1)!}h^{(r)}(j,0)N^r. \end{aligned}$$

Now it remains to compute $h^{(r)}(j,0)$. The Leibniz rule for differentiation yields that

$$\begin{aligned} D_x^r h(x,t) &= D_x^r[x^{\beta}\exp\{-Q(xt,t)\}] \\ &= \sum_{u=0}^{r}\binom{r}{u}D_x^u(x^{\beta})D_x^{r-u}\exp\{-Q(xt,t)\}. \end{aligned}$$

From the above, we see that

$$D_x^r h(x,t)\Big|_{x=j,t=0} = \begin{cases} \frac{\beta!}{(\beta-r)!} j^{\beta-r}, & \text{if } r \leq \beta; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$d_0 = \sum_{j=1}^{N} \chi(j) j^{\beta} - \frac{1}{N} \sum_{j=1}^{N} \chi(j) \frac{j^{\beta+1}}{\beta+1} + \sum_{j=1}^{N} \chi(j) \sum_{r=0}^{\beta} \frac{(-1)^r \beta!}{(r+1)!(\beta-r)!} B_{r+1} N^r j^{\beta-r}.$$

Note that $B_1 = -\frac{1}{2}$ and $(-1)^r B_{r+1} = -B_{r+1}$ if $r \geq 1$. So

$$\begin{aligned} d_0 &= -\frac{1}{N} \sum_{j=1}^{N} \chi(j) \frac{j^{\beta+1}}{\beta+1} - \sum_{j=1}^{N} \chi(j) \sum_{r=0}^{\beta} \frac{\beta!}{(r+1)!(\beta-r)!} B_{r+1} N^r j^{\beta-r} \\ &= -\frac{B_{\chi}^{\beta+1}}{\beta+1}. \end{aligned}$$

□

Our theorem is a direct consequence of the following proposition.

Proposition 4. *Let $\chi = (\chi_1, \dots, \chi_r)$, $\beta = (\beta_1, \dots, \beta_r)$, P and Q as given in Section 2. Suppose that*

$$f_{\beta}(t) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) n^{\beta} \exp\{-Q(nt,t)\}.$$

Then $f_{\beta}(t)$ has an asymptotic expansion of the form

$$\sum_{n=0}^{\infty} d_n t^n$$

with the constant term d_0 given by

$$\begin{aligned} d_0 &= L(-\beta_1, \chi_1) \cdots L(-\beta_r, \chi_r) \\ &= (-1)^r \prod_{j=1}^{r} \frac{B_{\chi_j}^{\beta_j+1}}{\beta_j+1}. \end{aligned}$$

Proof. We prove the assertion by induction on r. The case $r = 1$ was already proved in the previous proposition. Suppose that $r \geq 2$ and the assertion is true for the case of $r-1$ variables. Consider the case of r variables. Applying the previous proposition to the first summation of $f_\beta(t)$, where n_1 ranges over all positive integers, we obtain

$$f_\beta(t) = \sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} h(j,t) - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \int_0^j x^{\beta_1} h(x,t)dx + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} \tilde{h}_j^{(r)}(0,t)$$

where

$$h(x,t) = \sum_{n_2=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_2(n_2) \cdots \chi_r(n_r) n_2^{\beta_2} \cdots n_r^{\beta_r} \cdot \exp\{-Q(xt, n_2 t, \dots, n_r t, t)\},$$

and

$$\begin{aligned} \tilde{h}_j(x,t) &= (N_1 x + j)^{\beta_1} \sum_{n_2=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_2(n_2) \cdots \chi_r(n_r) n_2^{\beta_2} \cdots n_r^{\beta_r} \\ &\quad \cdot \exp\{-Q((N_1 x + j)t, n_2 t, \dots, n_r t, t)\} \\ &= (N_1 x + j)^{\beta_1} h(N_1 x + j, t). \end{aligned}$$

Note that

$$Q(\alpha t, n_2 t, \dots, n_r t, t) = P(\alpha, n_2, \dots, n_r) t^{p'}, \qquad p' = \deg P(\alpha, X_2, \dots, X_r)$$

for any fixed number $\alpha > 0$. Applying our induction hypothesis to $h(j,t)$, $h(x,t)$, and $\tilde{h}_j(x,t)$, we get the asymptotic expansion of $f_\beta(t)$, and the constant term d_0 is

$$\begin{aligned} d_0 &= f_\beta(0) \\ &= \sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} h(j,0) - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \int_0^j x^{\beta_1} h(x,0)dx \\ &\quad + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} \tilde{h}_j^{(r)}(0,0). \end{aligned}$$

To compute $\tilde{h}_j^{(r)}(0,0)$, we use a trick similar to the one in Proposition 3 for computing $h^{(r)}(j,0)$. The Leibniz rule for differentiation yields that

$$D_x^r \tilde{h}_j(x,t) = \sum_{u=0}^{r} \binom{r}{u} D_x^u[(N_1 x + j)^{\beta_1}] D_x^{r-u}[h(N_1 x + j, t)].$$

From the above, we see that

$$D_x^r \tilde{h}_j(x,t)\Big|_{x=0,t=0} = \begin{cases} \frac{\beta_1! N_1^r}{(\beta_1-r)!} j^{\beta_1-r} h(j,0), & \text{if } r \le \beta_1; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$d_0 = \sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} h(j,0) - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \int_0^j x^{\beta_1} h(x,0)dx + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\beta_1} \frac{(-1)^r B_{r+1} \beta_1! N_1^r j^{\beta_1-r}}{(r+1)!(\beta_1-r)!} h(j,0).$$

Since the constant term in the asymptotic expansion of $h(j,t)$ or $h(x,t)$ is

$$(-1)^{r-1} \prod_{j=2}^{r} \frac{B_{\chi_j}^{\beta_j+1}}{\beta_j+1},$$

we have

$$\begin{aligned} d_0 &= (-1)^{r-1} \prod_{j=2}^{r} \frac{B_{\chi_j}^{\beta_j+1}}{\beta_j+1} \left[\sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \frac{j^{\beta_1+1}}{\beta_1+1} \right. \\ &\qquad \left. + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\beta_1} \frac{(-1)^r \beta_1!}{(r+1)!(\beta_1-r)!} B_{r+1} N^r j^{\beta_1-r} \right] \\ &= (-1)^{r-1} \prod_{j=2}^{r} \frac{B_{\chi_j}^{\beta_j+1}}{\beta_j+1} \cdot \left(-\frac{B_{\chi_1}^{\beta_1+1}}{\beta_1+1} \right) \\ &= (-1)^r \prod_{j=1}^{r} \frac{B_{\chi_j}^{\beta_j+1}}{\beta_j+1}. \end{aligned}$$

This proves our assertions. □

Corollary. *Suppose that*

$$F(t) = \sum_{n\in\mathbb{N}^r} \chi_1(n_1)\cdots\chi_r(n_r) P^m(n) e^{-(n_1+\cdots+n_r)t}, \qquad t>0$$

then

$$Z(P,\chi,-m) = \lim_{t\to 0^+} F(t).$$

Proof. From the notation in our main theorem it follows that

$$F(t) = \sum_{|\beta|=0}^{mp} C_\beta F_\beta(t)$$

with

$$F_\beta(t) = \sum_{n\in\mathbb{N}^r} \chi_1(n_1)\cdots\chi_r(n_r)n^\beta e^{-(n_1+\cdots+n_r)t}$$
$$= \prod_{j=1}^{r}\left[\sum_{n=1}^{\infty}\chi_j(n)n^{\beta_j}e^{-nt}\right].$$

From

$$\sum_{n=1}^{\infty}\chi_j(n)e^{-nt} = \sum_{k=0}^{\infty}\sum_{a=1}^{N_j}\chi_j(a)e^{-(a+kN_j)t}$$
$$= \sum_{a=1}^{N_j}\frac{\chi_j(a)e^{-at}}{1-e^{-N_jt}}$$
$$= \sum_{n=1}^{\infty}\frac{-B^n_{\chi_j}(-t)^{n-1}}{n!}$$

and differentiating term-by-term β_j times with respect to t, we get

$$\sum_{n=1}^{\infty}\chi_j(n)n^{\beta_j}e^{-nt} = \sum_{n=\beta_j+1}^{\infty}\frac{-B^n_{\chi_j}(-t)^{n-\beta_j-1}}{n\cdot(n-\beta_j-1)!}\ .$$

Consequently we have

$$\lim_{t\to 0^+}F_\beta(t) = \prod_{j=1}^{r}\left(-\frac{B^{\beta_j+1}_{\chi_j}}{\beta_j+1}\right)$$

and hence our assertion follows. □

4. A Consequence.

Let $P(X)\in\mathbb{R}[X_1,\dots,X_r]$ be a polynomial as given before and $\xi=(\xi_1,\dots,\xi_r)\in\mathbb{C}^r$ such that $|\xi_j|=1$ and $\xi_j\neq 1$ for all j. In 1982, P. Cassou-Noguès considered the zeta function

$$Z(P,\xi,s) = \sum_{n\in\mathbb{N}^r}\xi^nP(n)^{-s}$$
$$= \sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\xi_1^{n_1}\cdots\xi_r^{n_r}P(n)^{-s},\ \operatorname{Re}s>\sigma,$$

and she proved that

$$Z(P,\xi,-m) = R(P^m)(\xi)$$

where

$$R(P^m)(T) = \sum_{n\in\mathbb{N}^r}P^m(n)T^n$$

which is a power series and can be realized as a rational function in T.

Here we change the dummy variable n and reformulate the above result so that we can use our theorem to give a new proof.

Theorem (P. Cassou-Noguès)**.** *Suppose that*

$$Z(P,\xi,s)=\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}\xi_1^{n_1}\cdots\xi_r^{n_r}P(n_1,\dots,n_r)^{-s},\ \operatorname{Re}s>\sigma.$$

Then $Z(P,\xi,s)$ has a meromorphic continuation to the whole complex s-plane and for any integer $m\geq 0$,

$$Z(P,\xi,-m)=\lim_{t\to 0^+}\sum_{n\in\mathbb{N}^r}\xi^n P^m(n)e^{-(n_1+\cdots+n_r)t}.$$

Proof. Recall that in the proof of our result, we use only the following two properties of the Dirichlet characters $\chi_1,\dots,\chi_r$.

(1) χ_j is a periodic function, $\chi_j(n+N_j)=\chi_j(n)$ for all $n\in\mathbb{N}$.
(2) χ_j is non-trivial and $\sum_{a=1}^{N_j}\chi_j(a)=0$.

Thus, in particular, it works for the case $\chi_j(n)=e^{2\pi in/N_j}$ or in general $\chi_j(n)=e^{2\pi in\eta_j}$, η_j is a positive rational number such that $0<\eta_j<1$.

Now we suppose that $\xi=(\xi_1,\dots,\xi_r)=(e^{2\pi iq_1},\dots,e^{2\pi iq_r})$ with $0<q_j<1$. Let $\eta_k=(\eta_1^{(k)},\dots,\eta_r^{(k)})$ be a sequence of r-tuples of rational numbers such that

(1) $0<\eta_j^{(k)}<1$ for all $1\leq j\leq r$, $k\geq 1$,
(2) $\lim_{k\to\infty}\eta_k=\xi$.

Consider the sequence of zeta functions $\{Z_k\}$ defined by

$$Z_k(P,\eta_k,s)=\sum_{n\in\mathbb{N}^r}\eta_k^n P(n)^{-s},\ \operatorname{Re}s>\sigma.$$

On the half-plane $\operatorname{Re}s>\sigma$, we have

$$\lim_{k\to\infty}Z_k(P,\eta_k,s)=Z(P,\xi,s).$$

Also all the zeta function $Z_k(P,\eta_k,s)$ and $Z(P,\xi,s)$ have analytic continuation to the whole complex s-plane. So that

$$\lim_{k\to\infty}Z_k(P,\eta_k,-m)=Z(P,\xi,-m).$$

By our result

$$Z_k(P,\eta_k,-m)=\lim_{t\to 0^+}\sum_{n\in\mathbb{N}^r}\eta_k^n P^m(n)e^{-(n_1+\cdots+n_r)t},$$

it follows that

$$Z(P,\xi,-m)=\lim_{t\to 0^+}\sum_{n\in\mathbb{N}^r}\xi^n P^m(n)e^{-(n_1+\cdots+n_r)t}.$$

□

The special values $Z(P, \xi, -m)$ can be expressed in terms of special values of the L-series

$$L_q(s) = \sum_{n=1}^{\infty} e^{2\pi i n q} n^{-s}, \qquad \operatorname{Re} s > 1,\ 0 < q < 1.$$

From

$$\begin{aligned} L_q(s)\Gamma(s) &= \sum_{n=1}^{\infty} e^{2\pi i n q} \int_0^{\infty} t^{s-1} e^{-nt} dt \\ &= \int_0^{\infty} \frac{e^{2\pi i q} t^{s-1}}{e^t - e^{2\pi i q}}\, dt\ , \qquad \operatorname{Re} s > 1, \end{aligned}$$

we conclude that

$$L_q(-m) = (-1)^m m! \times \left\{ \text{the coefficient of } t^m \text{ in the power series expansion at } t = 0 \text{ of } \frac{e^{2\pi i q}}{e^t - e^{2\pi i q}} \right\}.$$

In other words,

$$\sum_{n=1}^{\infty} e^{2\pi i n q} e^{-nt} = \frac{e^{2\pi i q}}{e^t - e^{2\pi i q}} = \sum_{m=0}^{\infty} \frac{(-1)^m L_q(-m) t^m}{m!}, \qquad |t| < 2\pi q.$$

Differentiating the above equality β times with respect to t, we obtain

$$\sum_{n=1}^{\infty} e^{2\pi i n q} n^{\beta} e^{-nt} = \sum_{m=\beta}^{\infty} \frac{L_q(-m)(-t)^{m-\beta}}{(m-\beta)!}.$$

Proposition 5. *Suppose that*

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_\alpha X^\alpha.$$

Then

$$Z(P, \xi, -m) = \sum_{|\alpha|=0}^{mp} C_\alpha L_{q_1}(-\alpha_1) \cdots L_{q_r}(-\alpha_r).$$

Proof. Note that

$$\begin{aligned} \sum_{n \in \mathbb{N}^r} \xi^n P^m(n) e^{-(n_1 + \cdots + n_r)t} &= \sum_{|\alpha|=0}^{mp} C_\alpha \sum_{n \in \mathbb{N}^r} \xi^n n^\alpha e^{-(n_1 + \cdots + n_r)t} \\ &= \sum_{|\alpha|=0}^{mp} C_\alpha \prod_{j=1}^{r} \left\{ \sum_{n=1}^{\infty} e^{2\pi i n q_j} n^{\alpha_j} e^{-nt} \right\}. \end{aligned}$$

From

$$\lim_{t\to 0^+}\sum_{n=1}^{\infty} e^{2\pi i n q_j} n^{\alpha_j} e^{-nt} = L_{q_j}(-\alpha_j),$$

we get our assertion by the previous theorem. □

Now we give expressions for $L_q(-m)$. From the power series expansion

$$\frac{e^{2\pi i q}}{e^t - e^{2\pi i q}} = \sum_{n=0}^{\infty} \frac{(-t)^n \varepsilon_n(e^{2\pi i q})}{n!(1-e^{2\pi i q})^{n+1}}\ , \qquad |t| < 2\pi q,$$

where $\varepsilon_n(p) = \sum_{k=1}^{n} A_{n,k} p^k$ is the Eulerian polynomials, the coefficients $A_{n,k}$ are the Eulerian numbers which are the numbers of permutations of the chain $\{1 < 2 < \cdots < n\}$ with precisely $k-1$ descents (see, e.g., [**4**]), we have

$$L_q(-m) = \frac{\varepsilon_m(e^{2\pi i q})}{(1-e^{2\pi i q})^{m+1}}.$$

Meanwhile, we have the following

Proposition 6. *Suppose that*

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_\alpha X^\alpha,$$

then

$$Z(P,\xi,-m) = \sum_{|\alpha|=0}^{mp} C_\alpha \prod_{j=1}^{r} \frac{\varepsilon_{\alpha_j}(\xi_j)}{(1-\xi_j)^{\alpha_j+1}}.$$

5. Sums of Products of Generalized Bernoulli Numbers.

A well-known relation among the Bernoulli numbers is

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1)B_{2n}, \qquad \text{for } n \geq 2.$$

This was found by many authors, including Euler (ref. [**5**], [**8**]). Dilcher remarked in [**5**] that it may be of interest to find formulas of the above type for sums of products of generalized Bernoulli numbers. In the following Proposition 7, we give a closed expression for sums of products of generalized Bernoulli numbers.

Proposition 7. *Let r be a positive integer and χ_i be a non-trivial Dirichlet character with conductor N_i, for $i = 1, 2, \ldots, r$. Then for any positive* *integer m,*

$$\sum_{\substack{p_1+\cdots+p_r=m\\ p_1,\dots,p_r\geq 0}}^{m} \binom{m}{p_1,\dots,p_r} \frac{B_{\chi_1}^{p_1+1}}{N_1^{p_1}(p_1+1)} \cdots \frac{B_{\chi_r}^{p_r+1}}{N_r^{p_r}(p_r+1)}$$

$$= \sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \chi_1(a_1)\cdots\chi_r(a_r) \frac{(-1)^{r-1}}{(r-1)!}$$

$$\cdot \sum_{j=0}^{r-1} (-1)^j \left\{ \sum_{k=0}^{j} \binom{r-1-j+k}{k} s(r, r-j+k)\delta^k \right\} \frac{B_{m+r-j}(\delta)}{m+r-j},$$

where $\delta = \frac{a_1}{N_1} + \cdots + \frac{a_r}{N_r}$ *and* $s(n,k)$ *is the Stirling number of the first kind.*

Proof. Consider the zeta function

$$Z_r(s) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1)\cdots\chi_r(n_r) \left(\sum_{j=1}^{r} \left(\prod_{\substack{i=1\\ i\neq j}}^{r} N_i \right) n_j \right)^{-s}.$$

Substitute $n_i = a_i + N_i m_i$ where $a_i = 1, \dots, N_i$ and $m_i \geq 0$ for $i = 1, \dots, r$. Thus $Z_r(s)$ becomes

$$\sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left(\prod_{i=1}^{r} \chi_i(a_i + m_i N_i) N_i^{-s} \right) \left[\sum_{j=1}^{r} \left(m_j + \frac{a_j}{N_j} \right) \right]^{-s}.$$

Now we let

$$Z_B(s) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left(\prod_{i=1}^{r} N_i \right)^{-s} \left[\sum_{j=1}^{r} \left(m_j + \frac{a_j}{N_j} \right) \right]^{-s}.$$

Then we can represent the zeta function $Z_r(s)$ as

$$Z_r(s) = \sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \left(\prod_{i=1}^{r} \chi_i(a_i) \right) Z_B(s).$$

From [**8**] we know that this zeta function $Z_B(s)$ has an analytic continuation to the whole complex plane, and the special values at non-positive integers $s = -m$ are given by

$$Z_B(-m) = \left(\prod_{i=1}^{r} N_i^m \right) \sum_{\substack{p_1+\cdots+p_r=m+r\\ p_1,\dots,p_r\geq 0}} \frac{m!}{p_1!\cdots p_r!} \prod_{j=1}^{r} B_{p_j}\left(\frac{a_j}{N_j} \right).$$

Using the following identity ([**5**], Theorem 3)

$$\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\cdots,j_r\geq 0}} \binom{n}{j_1,\ldots,j_r} B_{j_1}(x_1)\cdots B_{j_r}(x_r) =$$

$$(-1)^{r-1} r\binom{n}{r}\sum_{j=0}^{r-1}(-1)^j\left\{\sum_{k=0}^{j}\binom{r-j-1+k}{k}s(r,r-j+k)y^k\right\}\frac{B_{n-j}(y)}{n-j},$$

where $y = x_1+\cdots+x_r$ and $s(n,k)$ are Stirling numbers of the first kind, and we can rewrite $Z_B(-m)$ as

$$\frac{(\prod_{i=1}^r N_i^m)(-1)^{r-1}}{(r-1)!}\sum_{j=0}^{r-1}(-1)^j \cdot\left\{\sum_{k=0}^{j}\binom{r-1-j+k}{k}s(r,r-j+k)\delta^k\right\}\frac{B_{m+r-j}(\delta)}{m+r-j},$$

where $\delta = \frac{a_1}{N_1}+\cdots+\frac{a_r}{N_r}$. Now applying our theorem, the special values at non-positive integers $s=-m$ of the zeta function $Z_r(s)$ are

$$Z_r(-m) = \sum_{\substack{p_1+\cdots+p_r=m\\ p_1,\cdots,p_r\geq 0}}\binom{m}{p_1,\ldots,p_r}\left(\prod_{i=1}^r\frac{N_i^{m-p_i}B_{\chi_i}^{p_i+1}}{p_i+1}\right).$$

On the other hand, using the equality

$$Z_r(-m) = \sum_{a_1=1}^{N_1}\cdots\sum_{a_r=1}^{N_r}\prod_{i=1}^r\chi_i(a_i)Z_B(-m)$$

and the above values of $Z_r(-m)$ and $Z_B(-m)$, we get our assertion. □

Remark. As special cases we state formulas for sums of products of two, respectively three, generalized Bernoulli numbers.

(1) Let χ_1, χ_2 be non-trivial Dirichlet characters with conductors N_1, N_2, respectively. Then for any positive integer m,

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{\chi_1}^{k+1}}{N_1^k(k+1)}\frac{B_{\chi_2}^{m-k+1}}{N_2^{m-k}(m-k+1)}$$
$$=\sum_{a_1=1}^{N_1}\sum_{a_2=1}^{N_2}\chi_1(a_1)\chi_2(a_2)$$
$$\cdot\left[\frac{\frac{a_1}{N_1}+\frac{a_2}{N_2}-1}{m+1}B_{m+1}\left(\frac{a_1}{N_1}+\frac{a_2}{N_2}\right)-\frac{B_{m+2}\left(\frac{a_1}{N_1}+\frac{a_2}{N_2}\right)}{m+2}\right].$$

(2) Let χ_1, χ_2, χ_3 be non-trivial Dirichlet characters with conductors N_1, N_2, N_3, respectively. Then for any positive integer m, we have

$$\sum_{\substack{p+q+r=m\\ p,q,r\geq 0}} \binom{m}{p,q,r} \frac{B_{\chi_1}^{p+1}}{N_1^p(p+1)} \frac{B_{\chi_2}^{q+1}}{N_2^q(q+1)} \frac{B_{\chi_3}^{r+1}}{N_3^r(r+1)}$$
$$= \frac{1}{2} \sum_{a_1=1}^{N_1} \sum_{a_2=1}^{N_2} \sum_{a_3=1}^{N_3} \chi_1(a_1)\chi_2(a_2)\chi_3(a_3)$$
$$\cdot \left[(\delta^2 - 3\delta + 2) \frac{B_{m+1}(\delta)}{m+1} + (3-2\delta) \frac{B_{m+2}(\delta)}{m+2} + \frac{B_{m+3}(\delta)}{m+3} \right],$$

where $\delta = \frac{a_1}{N_1} + \frac{a_2}{N_2} + \frac{a_3}{N_3}$.

As a final example we consider the Euler numbers E_n, $0 \leq n < \infty$. We have $E_{2n+1} = 0$, $n \geq 0$, while E_{2n}, $n \geq 0$, is defined by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, \qquad |x| < \frac{\pi}{2}.$$

The Euler numbers are special cases of the generalized Bernoulli numbers B_χ^n belonging to a residue class character χ. In fact we have

$$E_n = -\frac{2B_\eta^{n+1}}{n+1}, \qquad n \geq 0,$$

where η is the primitive character with conductor 4. If we let $r = 2$ and the characters χ_1 and χ_2 in Proposition 7 be the same character η (the primitive character with conductor 4), then we get an identity which is a special case of Eq. (4.9) in [**5**].

Proposition 8. *For a non-negative integer n, we have the following identity:*

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2n-2k} = (2^{2n+2} - 1) \frac{2^{2n+2} B_{2n+2}}{2n+2}.$$

Proof. Let $r = 2$ and χ_1, χ_2 as indicated above, i.e., $N_1 = N_2 = 4$. Then

$$4^{-m} \sum_{k=0}^{m} \binom{m}{k} \frac{B_\chi^{k+1} B_\chi^{m-k+1}}{(k+1)(m-k+1)}$$
$$= \sum_{a=1}^{4} \sum_{b=1}^{4} \chi(ab) \left[\frac{\frac{a+b}{4} - 1}{m+1} B_{m+1}\left(\frac{a+b}{4}\right) - \frac{B_{m+2}(\frac{a+b}{4})}{m+2} \right]$$
$$= -\frac{B_{m+1}(\frac{1}{2})}{2(m+1)} - \frac{B_{m+2}(\frac{1}{2})}{m+2} + \frac{2B_{m+2}(1)}{m+2} + \frac{B_{m+1}(\frac{3}{2})}{2(m+1)} - \frac{B_{m+2}(\frac{3}{2})}{m+2}.$$

The left-hand side of the above identity is exactly $4^{-m-1}\sum_{k=0}^{m}\binom{m}{k}E_kE_{m-k}$. Using some basic properties of the Bernoulli polynomials:

$$B_n\left(\frac{3}{2}\right) = 2^{1-n}\cdot n + B_n\left(\frac{1}{2}\right) \ ,$$

$$B_n\left(\frac{1}{2}\right) = (2^{1-n}-1)B_n \ ,$$

$$B_n(1-x) = (-1)^nB_n(x),$$

the right-hand side of the above identity becomes

$$2\cdot[1+(-1)^{m+2}-2^{-m-1}]\frac{B_{m+2}}{m+2}.$$

The result follows by setting $m = 2n$. □

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