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We consider the prescribing scalar curvature equation

$$-\Delta u + \frac{n(n-2)}{4}u = \frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}} \tag{1}$$

on S^n for $n \geq 3$. In the case R is rotationally symmetric, the well-known Kazdan–Warner condition implies that a necessary condition for (1) to have a solution is:

$R > 0$ somewhere and $R'(r)$ changes signs.

Then,

(a) is this a sufficient condition?

(b) If not, what are the necessary and sufficient conditions?

These have been open problems for decades.

In Chen & Li, 1995, we gave question (a) a negative answer. We showed that a necessary condition for (1) to have a solution is:

(2) **$R'(r)$ changes signs in the region *where R is positive.***

Now is this also a sufficient condition? In this paper, we prove that if $R(r)$ satisfies the 'flatness condition', then (2) is the necessary and sufficient condition for (1) to have a solution. This essentially answers question (b). We also generalized this result to non-symmetric functions R. Here the additional 'flatness condition' is a standard assumption which has been used by many authors to guarantee the existence of a solution. In particular, for $n = 3$, 'non-degenerate' functions satisfy this condition.

Based on Theorem 3 in Chen & Li, 1995, we also show that for some rotationally symmetric R, (1) is solvable while none of the solutions is rotationally symmetric. This is interesting in the studying of symmetry breaking.

1. Introduction.

Given a function $R(x)$ on S^2, the well-known Nirenberg problem is to find conditions on $R(x)$, so that it can be realized as the Gaussian curvature of some conformally related metric. This is equivalent to solving the following nonlinear elliptic equation

$$-\Delta u + 1 = R(x)e^{2u} \quad x \in S^2. \tag{3}$$

On a higher dimensional sphere S^n, a similar problem was raised by Kazdan and Warner:

Which functions $R(x)$ can be realized as the scalar curvature of some conformally related metrics? It is equivalent to consider the existence of a solution to the following nonlinear elliptic equation

$$-\Delta u+\frac{n(n-2)}{4}u=\frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}},\ \ u>0,\ \ x\in S^n. \tag{4}$$

Both equations are so-called 'critical' in the sense that lack of compactness occurs. Besides the obvious necessary condition that $R(x)$ be positive somewhere, there are well-known obstructions found by Kazdan and Warner [26] and later generalized by Bourguignon and Ezin [5]. The conditions are:

$$\int_{S^n} X(R)dV_g=0 \tag{5}$$

where dV_g is the volume element of the conformal metric g and X is any conformal vector field associated to the standard metric g_0. We call these Kazdan-Warner type conditions.

These conditions give rise to many examples of $R(x)$ for which (3) and (4) have no solution. In particular, a monotone rotationally symmetric function R admits no solution.

In the last two decades, numerous studies were dedicated to these problems and various sufficient conditions were found (please see the articles [1], [3], [11], [19], [12], [13], [10], [20], [7], [8], [9], [6], [21], [23], [24], [27], [28], [29], [30], [31], [32] and the references therein). However, among others, one problem of common concern was left open, namely, were those Kazdan-Warner type necessary conditions also sufficient? In the case where R is rotationally symmetric, the conditions become:

$$R>0 \text{ somewhere and } R' \text{ changes signs.} \tag{6}$$

Then,

(a) is (6) a sufficient condition?

(b) If not, what are the necessary and sufficient conditions?

Kazdan listed these as open problems in his CBMS Lecture Notes [25].

Recently, we answered question (a) negatively [15] [16]. We found some stronger obstructions. Our results imply that for a rotationally symmetric function R, if it is monotone in the region *where it is positive*, then problems (3) and (4) admit no solution unless R is a constant. In other words, a necessary condition to solve (3) or (4) is that

$$R'(r) \text{ changes signs in the region(s) } \textit{where } R \textit{ is positive}\ . \tag{7}$$

Now is this a sufficient condition?

For Equation (3) on S^2, Xu and Yang [32] showed that if R is 'nondegenerate', then (7) is a sufficient condition.

For Equation (4) on higher dimensional spheres, a major difficulty is that a multiple blow-ups may occur when approaching a solution by a minimax sequence of the functional or by solutions of subcritical equations.

In dimensions higher than 3, the 'non-degeneracy' condition is no longer sufficient to guarantee the existence of a solution. It was illustrated by a counter-example in [**4**] constructed by Bianchi. He found some positive rotationally symmetric function R on S^4, which is non-degenerate and non-monotone, and for which the problem admits no solution. In this situation, a more proper condition is called the 'flatness condition'. Roughly speaking, it requires that at every critical point of R, the derivatives of R up to order $(n-2)$ vanish, while some higher order ($< n$) derivatives are distinct from 0. For $n = 3$, the 'non-degeneracy' condition is a special case of the 'flatness condition'. Although, up to the present, people wonder if the 'flatness condition' is necessary, it is still used widely (see [**9**], [**28**], and [**31**]) as a standard assumption to guarantee the existence of a solution. The above mentioned Bianchi's counter-example seems to suggest that it is somewhat sharp.

Now a natural question is:

Under the 'flatness condition', is (7) a sufficient condition?

In this paper, we answer the question affirmatively. We prove that, in this situation, (7) is a necessary and sufficient condition for (4) to have a solution. This is true in all dimensions $n \geq 3$ and it applies to functions R with changing signs. Thus we essentially answer the open question (b) posed by Kazdan.

There are many versions of 'flatness conditions', a general one was presented in [**28**] by Y. Li. Here to better illustrate the idea, in the statement of the following theorem, we only list a typical and easy-to-verify one.

Theorem 1. *Let $n \geq 3$. Let $R = R(r)$ be rotationally symmetric and satisfy the following flatness condition near every positive critical point r_o:*

$$R(r) = R(r_o) + a|r - r_o|^\alpha + h(|r - r_o|), \text{ with } a \neq 0 \text{ and } n-2 < \alpha < n, \tag{8}$$

where $h'(s) = o(s^{\alpha-1})$.

Then a necessary and sufficient condition for Equation (4) to have a solution is that

$$R'(r) \text{ changes signs in the region(s) where } R > 0.$$

The Theorem is proved by a variational approach. We blend in our new ideas with other ideas in [**32**], [**1**], [**28**] and [**31**]. We use the 'center of mass' to define neighborhoods of 'critical points at infinity', obtain some quite sharp and clean estimates in those neighborhoods, and construct a maxmini variational scheme at sub-critical levels and then approach the critical level.

In one of our previous papers [**16**], we listed a family of rotationally symmetric functions R for which Equation (4) has no rotationally symmetric solutions. This kind of phenomenon is called 'symmetry breaking' and is of independent interest. We can modify that family of functions R in [**16**] so that they satisfy the conditions in Theorem 1 here, and hence prove:

Theorem 2. *There exists a family of rotationally symmetric functions R for which Equation* (4) *is solvable, however none of the solutions are rotationally symmetric.*

Using a similar approach as for Theorem 1, we can generalize the existence result to non-symmetric functions $R(x)$ as stated in the following Theorem.

Theorem 3. *Assume that $R(x)$ has at least two positive local maxima and satisfies the flatness condition:*

For any positive critical point x_o of R, there exists $\alpha = \alpha(x_o) \in (n-2, n)$, such that in some geodesic normal coordinate system centered at x_o,

$$R(x) = R(0) + \sum_{i=1}^{n} a_i |x^i|^\alpha + h(x)$$

where $a_i = a_i(x_o) \neq 0, \Sigma a_i \neq 0,$ and $\bigtriangledown h(x) = o(|x|^{\alpha-1})$.

Further assume that for any positive critical point x_o below the two least positive local maxima, holds

$$\sum_{i=1}^{n} a_i(x_o) > 0.$$

Then Equation (4) *has at least one solution.*

This Theorem complements the known existence results (see [**2**] [**3**], [**17**], [**7**], [**8**], [**28**], [**29**] and [**31**]).

Outline of the proof of Theorem 1.

Let $\gamma_n = \frac{n(n-2)}{4}$ and $\tau = \frac{n+2}{n-2}$. We first find a positive solution u_p of the subcritical equation

$$-\Delta u + \gamma_n u = R(r) u^p, \tag{9}$$

for each $p < \tau$ and close to τ. Then let $p \to \tau$, take the limit.

To find the solution of Equation (9), we construct a maxmini variational scheme. Let

$$J_p(u) := \int_{S^n} R u^{p+1} dV$$

and

$$E(u) := \int_{S^n} (|\bigtriangledown u|^2 + \gamma_n u^2) dV.$$

We seek critical points of $J_p(u)$ under the constraint

$$S = \{u \in H^1(S^n) \mid E(u) = \gamma_n |S^n|, u \geq 0\},$$

where $|S^n|$ is the volume of S^n.

If R has only one positive local maximum, then by condition (7), it must have local minima on both poles. Then similar to the approach in [**11**], we seek a solution by minimizing the functional in a family of rotationally symmetric functions in S.

In the following, we assume that R has at least two positive local maxima. In this case, the solutions we seek are not necessarily symmetric.

Our scheme is based on the following key estimates on the values of the functional $J_p(u)$ in a neighborhood of the 'critical points at infinity' associated with each local maximum of R. Let r_1 be a positive local maximum of R. We prove that there is an open set $G_1 \subset S$ (independent of p), such that on the boundary ∂G_1 of G_1, we have

$$J_p(u) \leq R(r_1)|S^n| - \delta, \tag{10}$$

while there is a function $\psi_1 \in G_1$, such that

$$J_p(\psi_1) > R(r_1)|S^n| - \frac{\delta}{2}. \tag{11}$$

Here $\delta > 0$ is independent of p. Roughly speaking, we have some kind of 'mountain pass' associated to each local maximum of R. The set G_1 is defined by using the 'center of mass'.

Let r_1 and r_2 be two smallest positive local maxima of R. Let ψ_1 and ψ_2 be two functions defined by (11) associated to r_1 and r_2. Let γ be a path in S connecting ψ_1 and ψ_2, and let Γ be the family of all such paths.

Define

$$c_p = \sup_{\gamma \in \Gamma} \min_{\gamma} J_p(u). \tag{12}$$

For each $p < \tau$, by compactness, there is a critical point u_p of $J_p(\cdot)$, such that

$$J_p(u_p) = c_p.$$

Obviously, a constant multiple of u_p is a solution of (9). Moreover, by (10),

$$J_p(u_p) \leq R(r_i)|S^n| - \delta, \tag{13}$$

for any positive local maximum r_i of R.

To find a solution of (4), we let $p \to \tau$, take the limit. To show the convergence of a subsequence of $\{u_p\}$, we established apriori bounds for the solutions in the following order.

(i) In the region where $R < 0$: This is done by the 'Kelvin Transform' and a maximum principle.

(ii) In the region where R is small: This is mainly due to the boundedness of the energy $E(u_p)$.

(iii) In the region where R is positively bounded away from 0: First due to the energy bound, $\{u_p\}$ can only blow up at most finitely many points. Using a Pohozaev type identity, we show that the sequence can only blow up at one point and the point must be a local maximum of R. Finally, we use (13) to argue that even one point blow up is impossible and thus establish an apriori bound for a subsequence of $\{u_p\}$.

In Section 2, we carry on the maxmini variational scheme and obtain a solution u_p of (9) for each p.

In Section 3, we establish a priori estimates on the solution sequence $\{u_p\}$.

2. The Variational Approach

In this section, we construct a maxmini variational scheme to find a solution of

$$-\Delta u + \gamma_n u = R(r) u^p \tag{14}$$

for each $p < \tau := \frac{n+2}{n-2}$.

Let

$$J_p(u) := \int_{S^n} R u^{p+1} dV$$

and

$$E(u) := \int_{S^n} (|\bigtriangledown u|^2 + \gamma_n u^2) dV.$$

Let

$$\|u\| := \sqrt{E(u)}$$

be the norm in the Hilbert space $H^1(S^n)$.

We seek critical points of $J_p(u)$ under the constraint

$$S = \{u \in H^1(S^n) \mid E(u) = E(1) = \gamma_n |S^n|, u \geq 0\},$$

where $|S^n|$ is the volume of S^n.

One can easily see that a critical point of J_p in S multiplied by a constant is a solution of (14).

We divide the rest of the section into two parts.

In part I, we establish the key estimates (10) and (11).

In part II, we carry on the maxmini variational scheme.

Part I. Estimate the values of the functional.

To construct a maxmini variational scheme, we first show that there is some kind of 'mountain pass' associated to each positive local maximum of R. Unlike the classical ones, these 'mountain passes' are in the neighborhood of the 'critical points at infinity.' (See Proposition 2.1 and 2.2 below.)

Choose a coordinate system in R^{n+1}, so that the south pole of S^n is at the origin O and the center of the ball B^{n+1} is at $(0, \cdots, 0, 1)$. As usual, we use $|\cdot|$ to denote the distance in R^{n+1}.

Define the center of mass of u as

$$q(u) =: \frac{\int_{S^n} x u^{\tau+1}(x) dV}{\int_{S^n} u^{\tau+1}(x) dV}. \tag{15}$$

We recall some well-known facts in conformal geometry.

Let ϕ_q be the standard solution with its 'center of mass' at $q \in B^{n+1}$, that is, $\phi_q \in S$ and satisfies

$$-\Delta u + \gamma_n u = \gamma_n u^\tau. \tag{16}$$

We may also regard ϕ_q as depending on two parameters λ and $\tilde{q}$, where $\tilde{q}$ is the intersection of S^n with the ray passing the center and the point q. When $\tilde{q} = O$ (the south pole of S^n), we can express, in the spherical polar coordinates $x = (r, \theta)$ of S^n centered at the south pole ($0 \le r \le \pi, \theta \in S^{n-1}$),

$$\phi_q = \phi_{\lambda, \tilde{q}} = \left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{\frac{n-2}{2}}, \tag{17}$$

with $0 < \lambda \le 1$.

Correspondingly, there is a family of conformal transforms

$$T_q : S \longmapsto S; \quad (T_q u) := u(h_\lambda(x)) [\det(dh_\lambda)]^{\frac{n-2}{2n}} \tag{18}$$

with

$$h_\lambda(r, \theta) = \left(2 \tan^{-1} \left(\lambda \tan \frac{r}{2} \right), \theta \right).$$

It is well-known that this family of conformal transforms leaves the Equation (16), the energy $E(\cdot)$, and the integral $\int_{S^n} u^{\tau+1} dV$ invariant. In particular, we have

$$T_q \phi_q = 1.$$

The relations between q and λ, $\tilde{q}$ and T_q for $\tilde{q} \neq O$ can be expressed in a similar way.

We now carry on the estimates near the south pole $(0, \theta)$ which we assume to be a positive local maximum. The estimates near other positive local maxima are similar. Our conditions on R implies that

$$R(r) = R(0) - a r^\alpha, \text{ for some } a > 0, \ n - 2 < \alpha < n, \tag{19}$$

in a small neighborhood of O.

Define

$$\Sigma = \left\{ u \in S \mid |q(u)| \le \rho_o, \|v\| := \min_{t,q} \|u - t\phi_q\| \le \rho_o \right\}. \tag{20}$$

Notice that the 'centers of mass' of functions in Σ are near the south pole O. This is a neighborhood of the 'critical points at infinity' corresponding to O. We will estimate the functional J_p in this neighborhood.

We first notice that the supremum of J_p in Σ approaches $R(0)|S^n|$ as $p \to \tau$. More precisely, we have:

Proposition 2.1. *For any $\delta_1 > 0$, there is a $p_1 \leq \tau$, such that for all $\tau \geq p \geq p_1$,*

$$\sup_{\Sigma} J_p(u) > R(0)|S^n| - \delta_1. \tag{21}$$

Then we show that on the boundary of Σ, J_p is bounded away from $R(0)|S^n|$.

Proposition 2.2. *There exist positive constants ρ_o, p_o, and δ_o, such that for all $p \geq p_o$ and for all $u \in \partial\Sigma$, holds*

$$J_p(u) \leq R(0)|S^n| - \delta_o. \tag{22}$$

Proof of Proposition 2.1.

Through a straight forward calculation, one can show that

$$J_\tau(\phi_{\lambda,O}) \to R(0)|S^n|, \quad \text{as } \lambda \to 0. \tag{23}$$

For a given $\delta_1 > 0$, by (23), one can choose λ_o, such that $\phi_{\lambda_o,O} \in \Sigma$, and

$$J_\tau(\phi_{\lambda_o,O}) > R(0)|S^n| - \frac{\delta_1}{2}. \tag{24}$$

It is easy to see that for a fixed function $\phi_{\lambda_o,O}$, J_p is continuous with respect to p. Hence, by (24), there exists a p_1, such that for all $p \geq p_1$,

$$J_p(\phi_{\lambda_o,O}) > R(0)|S^n| - \delta_1.$$

This completes the proof of Proposition 2.1.

The proof of Proposition 2.2 is rather complex. We first estimate J_p for a family of standard functions in Σ.

Lemma 2.1. *For p sufficiently close to τ, and for λ and $|\tilde{q}|$ sufficiently small, we have*

$$J_p(\phi_{\lambda,\tilde{q}}) \leq (R(0) - C_1|\tilde{q}|^\alpha)|S^n|(1 + o_p(1)) - C_1\lambda^{\alpha+\delta_p}, \tag{25}$$

where $\delta_p := \tau - p$ and $o_p(1) \to 0$ as $p \to \tau$.

Proof of Lemma 2.1.

Let $\epsilon > 0$ be small such that (19) holds in $B_\epsilon(O) \subset S^n$. We express

$$J_p(\phi_{\lambda,\tilde{q}}) = \int_{S^n} R(x)\phi_{\lambda,\tilde{q}}^{p+1} dV = \int_{B_\epsilon(O)} \cdots + \int_{S^n \setminus B_\epsilon(O)} \cdots. \tag{26}$$

From the definition of $\phi_{\lambda,\tilde{q}}$ (17) and the boundedness of R, we obtain the smallness of the second integral:

$$\int_{S^n \setminus B_\epsilon(O)} R(x)\phi_{\lambda,\tilde{q}}^{p+1} dV \leq C_2 \lambda^{n-\delta_p \frac{n-2}{2}} \quad \text{for } \tilde{q} \in B_{\frac{\epsilon}{2}}(O). \tag{27}$$

To estimate the first integral, we work in a local coordinate centered at O.

$$\begin{aligned}
\int_{B_\epsilon(O)} R(x)\phi_{\lambda,\tilde{q}}^{p+1} dV &= \int_{B_\epsilon(O)} R(x+\tilde{q})\phi_{\lambda,O}^{p+1} dV \\
&\leq \int_{B_\epsilon(O)} [R(0) - a|x+\tilde{q}|^\alpha]\phi_{\lambda,O}^{p+1} dV \\
&\leq \int_{B_\epsilon(O)} [R(0) - C_3|x|^\alpha - C_3|\tilde{q}|^\alpha]\phi_{\lambda,O}^{p+1} dV \\
&\leq [R(0) - C_3|\tilde{q}|^\alpha]|S^n|[1+o_p(1)] - C_4\lambda^{\alpha+\delta_p}.
\end{aligned} \tag{28}$$

Here we have used the fact that $|x+\tilde{q}|^\alpha \geq c(|x|^\alpha + |\tilde{q}|^\alpha)$ for some $c > 0$ in one half of the ball $B_\epsilon(O)$ and the symmetry of $\phi_{\lambda,O}$.

Noticing that $\alpha < n$ and $\delta_p \to 0$, we conclude that (26), (27) and (28) imply (25). This completes the proof of the Lemma.

To estimate J_p for all $u \in \partial\Sigma$, we also need the following two lemmas that describe some useful properties of the set Σ.

Lemma 2.2. (On the 'center of mass')**.**

(i) *Let q, λ, and $\tilde{q}$ be defined by* (17)*. Then for sufficiently small q,*

$$|q|^2 \leq C(|\tilde{q}|^2 + \lambda^4). \tag{29}$$

(ii) *Let ρ_o, q, and v be defined by* (20)*. Then for ρ_o sufficiently small,*

$$\rho_o \leq |q| + C\|v\|. \tag{30}$$

Lemma 2.3 (Orthogonality)**.** *Let $u \in \Sigma$ and $v = u - t_o\phi_{q_o}$ be defined by* (20)*. Let T_{q_o} be the conformal transform such that*

$$T_{q_o}\phi_{q_o} = 1. \tag{31}$$

Then

$$\int_{S^n} T_{q_o} v dV = 0 \tag{32}$$

and

$$\int_{S^n} T_{q_o} v x^i dV = 0, \quad i = 1, 2, \ldots, n, \tag{33}$$

where x^i are coordinate functions on S^n.

Proof of Lemma 2.2.

(i) From the definition that $\phi_q = \phi_{\lambda,\tilde{q}}$, and (17), and by an elementary calculation, one arrives at

$$|q - \tilde{q}| \sim \lambda^2, \quad \text{for } \lambda \text{ sufficiently small.} \tag{34}$$

Draw a perpendicular line segment from O to $\overline{q\tilde{q}}$, then one can see that (29) is a direct consequence of the triangle inequality and (34).

(ii) For any $u = t\phi_q + v \in \partial\Sigma$, by the definition, we have either $\|v\| = \rho_o$ or $|q(u)| = \rho_o$. If $\|v\| = \rho_o$, then we are done. Hence we assume that $|q(u)| = \rho_o$. It follows from the definition of the 'center of mass' that

$$\rho_o = \left| \frac{\int_{S^n} x(t\phi_q + v)^{\tau+1}}{\int_{S^n}(t\phi_q + v)^{\tau+1}} \right|. \tag{35}$$

Noticing that $\|v\| \leq \rho_o$ is very small, we can expand the integrals in (35) in terms of $\|v\|$:

$$\begin{aligned}\rho_o &= \left| \frac{t^{\tau+1}\int x\phi_q^{\tau+1} + (\tau+1)t^\tau \int x\phi_q^\tau v + o(\|v\|)}{t^{\tau+1}\int \phi_q^{\tau+1} + (\tau+1)t^\tau \int \phi_q^\tau v + o(\|v\|)} \right| \\ &\leq \left| \frac{\int x\phi_q^{\tau+1}}{\int \phi_q^{\tau+1}} \right| + C\|v\| \leq |q| + C\|v\|.\end{aligned}$$

Here we have used the fact that v is small and that t is close to 1. This completes the proof of Lemma 2.2.

Proof of Lemma 2.3.

Write $L = \Delta + \gamma_n$. We use the fact that $v = u - t_o\phi_{q_o}$ is the minimum of $E(u - t\phi_q)$ among all possible values of t and q.

(i) By a variation with respect to t, we have

$$\int_{S^n} vL\phi_{q_o} dV = 0. \tag{36}$$

It follows from (16) that

$$\int_{S^n} v\phi_{q_o}^\tau dV = 0.$$

Now, apply the conformal transform T_{q_o} to both v and ϕ_{q_o} in the above integral. By the invariant property of the transform, we arrive at (32).

(ii) To prove (33), we make a variation with respect to q.

$$0 = \bigtriangledown_q \int vL\phi_q \mid_{q=q_o} = \gamma_n \bigtriangledown_q \int v\phi_q^\tau \mid_{q=q_o}$$

$$\bigtriangledown_q \int T_{q_o} v (T_{q_o}\phi_q)^\tau \mid_{q=q_o} = \bigtriangledown_q \int T_{q_o} v (\phi_{q-q_o})^\tau \mid_{q=q_o}$$

$$\int T_{q_o} v \phi_{q-q_o}^{\tau-1} \bigtriangledown_q \phi_{q-q_o} \mid_{q=q_o}$$

$$= \int T_{q_o} v x.$$

This completes the proof of the Lemma.

Proof of Proposition 2.2.

Make a perturbation of $R(x)$:

$$\bar{R}(x) = \begin{cases} R(x) & x \in B_{2\rho_o}(O) \\ m & x \in S^n \setminus B_{2\rho_o}(O), \end{cases} \tag{37}$$

where $m = R \mid_{\partial B_{2\rho_o}(O)}$.

Define

$$\bar{J}_p(u) = \int_{S^n} \bar{R}(x) u^{p+1} dV.$$

The estimate is divided into two steps.

In Step 1, we show that the difference between $J_p(u)$ and $\bar{J}_\tau(u)$ is very small. In Step 2, we estimate $\bar{J}_\tau(u)$.

Step 1.

First we show

$$\bar{J}_p(u) \le \bar{J}_\tau(u)(1 + o_p(1)), \tag{38}$$

where $o_p(1) \to 0$ as $p \to \tau$.

In fact, by the Holder inequality

$$\int \bar{R} u^{p+1} \le \left(\int \bar{R}^{\frac{\tau+1}{p+1}} u^{\tau+1} \right)^{\frac{p+1}{\tau+1}} |S^n|^{\frac{\tau-p}{\tau+1}}$$

$$\le \left(\int \bar{R} u^{\tau+1} \right)^{\frac{p+1}{\tau+1}} \mid R(0)|S^n| \mid^{\frac{\tau-p}{\tau+1}} .$$

This implies (38).

Now estimate the difference between $J_p(u)$ and $\bar{J}_p(u)$.

$$\mid J_p(u) - \bar{J}_p(u) \mid \le C_1 \int_{S^n \setminus B_{2\rho_o}(O)} u^{p+1}$$

(39)

$$\leq C_2 \int_{S^n \setminus B_{2\rho_o}(O)} (t\phi_q)^{p+1} + C_2 \int_{S^n \setminus B_{2\rho_o}(O)} v^{p+1} \leq C_3 \lambda^{n-\delta_p \frac{n-2}{2}} + C_3 \|v\|^{p+1}.$$

Step 2.

By virtue of (38) and (39), we now only need to estimate $\bar{J}_\tau(u)$. For any $u \in \partial\Sigma$, write $u = v + t\phi_q$. (36) implies that v and ϕ_q are orthogonal with respect to the inner product associated to $E(\cdot)$, and hence

$$t^2 = 1 - \frac{\|v\|^2}{\gamma_n |S^n|}. \tag{40}$$

It follows that

$$\begin{aligned} \int_{S^n} \bar{R}(x) u^{\tau+1} &\leq t^{\tau+1} \int_{S^n} \bar{R}(x) \phi_q^{\tau+1} + (\tau+1) \int_{S^n} \bar{R}(x) \phi_q^\tau v \\ &\quad + \frac{\tau(\tau+1)}{2} \int_{S^n} \phi_q^{\tau-1} v^2 + o(\|v\|) \\ &= I_1 + (\tau+1) I_2 + \frac{\tau(\tau+1)}{2} I_3 + o(\|v\|^2). \end{aligned} \tag{41}$$

To estimate I_1, we use (40) and (25),

(42)

$$I_1 \leq \left(1 - \frac{\tau+1}{2} \frac{\|v\|^2}{\gamma_n |S^n|}\right) R(0)|S^n|(1 - c_1|\tilde{q}|^\alpha - c_1 \lambda^\alpha) + O(\lambda^n) + o(\|v\|^2),$$

for some positive constant c_1.

To estimate I_2, we use the L_2 orthogonality between v and ϕ_q (see (36)),

$$I_2 = \int_{S^n} (\bar{R}(x) - m)\phi_q^\tau v = \int_{B_{2\rho_o}(O)} (\bar{R}(x) - m)\phi_q^\tau v \leq C_4 \rho_o^\alpha \|v\|. \tag{43}$$

To estimate I_3, we use (32) and (33). It is well-known that the first and second eigenspaces of $-\Delta$ on S^n are constants and coordinates functions x^i corresponding to eigenvalues 0 and n respectively. Now (32) and (33) imply that $T_q v$ is orthogonal to these two eigenspaces and consequently

$$\|v\|^2 = \|T_q v\|^2 \geq (\gamma_n + n + c_2) \int_{S^n} (T_q v)^2,$$

for some positive constant c_2. It follows that

$$I_3 \leq R(0) \int_{S^n} \phi_q^{\tau-1} v^2 = R(0) \int_{S^n} (T_q v)^2 \leq \frac{R(0)}{\gamma_n + n + c_2} \|v\|^2. \tag{44}$$

Now (42), (43), and (44) imply that there is a $\beta > 0$ such that

$$\bar{J}_\tau(u) \leq R(0)|S^n|[1 - \beta(|\tilde{q}|^\alpha + \lambda^\alpha + \|v\|^2)]. \tag{45}$$

Here we have used the fact that $\|v\|$ is very small and $\rho_o^\alpha \|v\|$ can be controlled by $|\tilde{q}|^\alpha + \lambda^\alpha$ (see Lemma 2.2). Notice that the positive constant c_2 in (44) has played a key role, and without it, the coefficient of $\|v\|^2$ in (45) would be 0.

Now (22) is an immediate consequence of (38), (39), and (45). This completes the proof of the Proposition.

Part II. The Variational Scheme.

In this part, we construct variational schemes to show the existence of a solution to Equation (14) for each $p < \tau$.

Case (i): *R has only one positive local maximum.*

In this case, condition (7) implies that R must have local minima at both poles. Similar to [**11**], we seek a solution by maximizing the functional J_p in a class of rotationally symmetric functions:

$$S_r = \{u \in S \mid u = u(r)\}. \tag{46}$$

Obviously, J_p is bounded from above in S_r and it is well-known that the variational scheme is compact for each $p < \tau$. Therefore any maximizing sequence possesses a converging subsequence in S_r and the limiting function multiplied by a suitable constant is a solution of (14).

Case (ii): *R has at least two positive local maxima.*

Let r_1 and r_2 be the two least positive local maxima of R. By Proposition 2.1 and 2.2, there exist two disjoint open sets $\Sigma_1, \Sigma_2 \subset S, \psi_i \in \Sigma_i, p_o < \tau$ and $\delta > 0$, such that for all $p \geq p_o$,

$$J_p(\psi_i) > R(r_i)|S^n| - \frac{\delta}{2},\ i = 1, 2; \tag{47}$$

and

$$J_p(u) \leq R(r_i)|S^n| - \delta,\ \ \forall u \in \partial\Sigma_i,\ i = 1, 2. \tag{48}$$

Let γ be a path in S joining ψ_1 and ψ_2. Let Γ be the family of all such paths. Define

$$c_p = \sup_{\gamma\in\Gamma} \min_{u\in\gamma} J_p(u). \tag{49}$$

For each fixed $p < \tau$, by the well-known compactness theorem, there exists a critical (saddle) point u_p of J_p with

$$J_p(u_p) = c_p.$$

Moreover, due to (48) and the definition of c_p, we have

$$J_p(u_p) \leq \min_i R(r_i)|S^n| - \delta. \tag{50}$$

One can easily see that a critical point of J_p in S multiplied by a constant is a solution of (14) and for all p close to τ, the constant multiples are bounded from above and bounded away from 0.

3. The Apriori Estimates

In the previous section, we showed the existence of a positive solution u_p to the subcritical Equation (14) for each $p < \tau$. In this section, we prove that as $p \to \tau$, there is a subsequence of $\{u_p\}$, which converges to a solution u_o of (4). The convergence is based on the following apriori estimate.

Theorem 3.1. *Assume that R satisfies condition (8) in Theorem 1, then there exists $p_o < \tau$, such that for all $p_o < p < \tau$, the solution of (14) obtained by the variational scheme are uniformly bounded.*

To prove the theorem, we estimate the solutions in three regions where $R < 0, R$ close to 0, and $R > 0$ respectively.

Part I. In the region where $R < 0$.

The apriori bound of the solutions is stated in the following proposition, which is a direct consequence of Proposition 2.1 in our previous paper [**17**].

Proposition 3.1. *For all $1 < p \leq \tau$, the solutions of (14) are uniformly bounded in the regions where $R \leq -\delta$, for every $\delta > 0$. The bound depends only on δ, $\operatorname{dist}(\{x \mid R(x) \leq -\delta\}, \{x \mid R(x) = 0\})$, and the lower bound of R.*

Part II. In the region where R is small.

In [**17**], we also obtained estimates in this region by the method of moving planes. However, in that paper, we assume that $\bigtriangledown R$ be bounded away from zero. In [**18**], for rotationally symmetric functions R on S^3, we removed this condition by using a blowing up analysis near $R(x) = 0$. That method may be applied to higher dimensions with a few modifications. Now within our variational frame in this paper, the apriori estimate is simpler. It is mainly due to the energy bound.

Proposition 3.2. *Let $\{u_p\}$ be solutions of Equation (14) obtained by the variational approach in Section 2. Then there exists a $p_o < \tau$ and a $\delta > 0$, such that for all $p_o < p \leq \tau$, $\{u_p\}$ are uniformly bounded in the regions where $|R(x)| \leq \delta$.*

Proof. It is easy to see that the energy $E(u_p)$ are bounded for all p. Now suppose that the conclusion of the Proposition is violated. Then there exists a subsequence $\{u_i\}$ with $u_i = u_{p_i}, p_i \to \tau$, and a sequence of points $\{x_i\}$, with $x_i \to x_o$ and $R(x_o) = 0$, such that

$$u_i(x_i) \to \infty.$$

We will use a rescaling argument to derive a contradiction. Since x_i may not be a local maximum of u_i, we need to choose a point near x_i, which is almost a local maximum. To this end, let K be any large number and let

$$r_i = 2K[u_i(x_i)]^{-\frac{p_i-1}{2}}.$$

In a small neighborhood of x_o, choose a local coordinate and let

$$s_i(x_i) = u_i(x_i)(r_i - |x - x_i|)^{\frac{2}{p_i-1}}.$$

Let a_i be the maximum of $s_i(x)$ in $B_{r_i}(x_i)$. Let $\lambda_i = [u_i(a_i)]^{-\frac{p_i-1}{2}}$. Then from the definition of a_i, one can easily verify that the ball $B_{\lambda_i K}(a_i)$ is in the interior of $B_{r_i}(x_i)$ and the value of $u_i(x)$ in $B_{\lambda_i K}(a_i)$ is bounded above by a constant multiple of $u_i(a_i)$.

Now we can make a rescaling.

$$v_i(x) = \frac{1}{u_i(a_i)} u_i(\lambda_i x + a_i).$$

Obviously v_i is bounded in $B_K(0)$, and it follows by a standard argument that $\{v_i\}$ converges to a harmonic function v_o in $B_K(0) \subset R^n$, with $v_o(0) = 1$. Consequently for i sufficiently large,

$$\int_{B_K(0)} v_i(x)^{\tau+1} dx \geq cK^n, \tag{51}$$

for some $c > 0$.

On the other hand, the boundedness of the energy $E(u_i)$ implies that

$$\int_{S^n} u_i^{\tau+1} dV \leq C, \tag{52}$$

for some constant C. By a straight forward calculation, one can verify that, for any $K > 0$,

$$\int_{S^n} u_i^{\tau+1} dV \geq \int_{B_K(0)} v_i^{\tau+1} dx. \tag{53}$$

Obviously, (52) and (53) contradict with (51). This completes the proof of the Proposition.

Part III. In the regions where $R > 0$.

Proposition 3.3. *Let $\{u_p\}$ be solutions of Equation (14) obtained by the variational approach in Section 2. Then there exists a $p_o < \tau$, such that for all $p_o < p \leq \tau$ and for any $\delta > 0$, $\{u_p\}$ are uniformly bounded in the regions where $R(x) \geq \delta$.*

Proof. The proof is divided into 3 steps. In Step 1, we argue that the solutions can only blow up at finitely many points because of the energy bound. In Step 2, we show that there is no more than one point blow up and the point must be a local maximum of R. This is done mainly by using a Pohozaeve type identity. In Step 3, we use (50) to conclude that even one point blow up is impossible.

Step 1. The argument is standard. Let $\{x_i\}$ be a sequence of points such that $u_i(x_i) \to \infty$ and $x_i \to x_o$ with $R(x_o) > 0$. Let r_i, $s_i(x)$, and $v_i(x)$ be defined as in Part II. Then similar to the argument in Part II, we see that $\{v_i(x)\}$ converges to a standard function $v_o(x)$ in R^n with

$$-\Delta v_o = R(x_o)v_o^\tau.$$

It follows that

$$\int_{B_{r_i}(x_o)} u_i^{\tau+1} dV \geq c_o > 0.$$

Because the total energy of u_i is bounded, we can have only finitely many such x_o. Therefore, $\{u_i\}$ can only blow up at finitely many points.

Step 2. We have shown that $\{u_i\}$ has only isolated blow up points. As a consequence of a result in [**28**] (also see [**17**] or [**31**]), we have:

Lemma 3.1. *Let R satisfy the 'flatness condition' in Theorem 1, Then $\{u_i\}$ can have at most one simple blow up point and this point must be a local maximum of R. Moreover, $\{u_i\}$ behaves almost like a family of the standard functions $\phi_{\lambda_i,\tilde{q}}$, with $\lambda_i = (\max u_i)^{-\frac{2}{n-2}}$ and with $\tilde{q}$ at the maximum of R.*

Step 3. Finally, we show that even one point blow up is impossible. For convenience, let $\{u_i\}$ be the sequence of critical points of J_{p_i} we obtained in Section 2. From the proof of Proposition 2.2, one can obtain

$$J_\tau(u_i) \leq \min_k R(r_k)|S^n| - \delta, \tag{54}$$

for all positive local maxima r_k of R. Now if $\{u_i\}$ blow up at x_o, then by Lemma 3.1, we have

$$J_\tau(u_i) \to R(x_o)|S^n|.$$

This is a contradiction with (54).

Therefore, the sequence $\{u_i\}$ is uniformly bounded and possesses a subsequence converging to a solution of (4). This completes the proof of Theorem 1.

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