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Dedicated to my mother for her 78th birthday

In this article we introduce a new conformal invariant and we prove a conformal rigidity theorem which has no restriction on the size of the codimension. We also prove an isometric rigidity theorem whose assumptions are less restrictive than in Allendoerfer's theorem.

Introduction.

Let $f, g\colon\ M^n \to \mathbb{R}^{n+d}$ be two immersions of an n-dimensional differentiable manifold into Euclidean space. That g is conformal (isometric) to f means that the metrics induced on M^n by f and g are conformal (isometric). We say that f is *conformally* (*isometrically*) *rigid* if given any other conformal (isometric) immersion g there exists a conformal (isometric) diffeomorphism Υ from an open subset of $\mathbb{R}^{n+d}$ to an open subset of $\mathbb{R}^{n+d}$ such that $g = \Upsilon \circ f$. In this case, we say that f and g are *conformally* (*isometrically*) *congruent.* It is then an interesting problem to determine conditions on f which imply conformal (isometric) rigidity.

E. Cartan ([**Ca1**] , see also [**Da**]) showed that when $n \geq 5$ a hypersurface $f\colon\ M^n \to \mathbb{R}^{n+1}$ is "generically" conformally rigid. To be more specific, he proved that f is conformally rigid when the maximal dimension of an umbilical subspace is at most $n-3$ at any point. Later, do Carmo and Dajczer ([**C-D**]) introduced a conformal invariant for immersions of arbitrary codimension, namely, the *conformal s-nullity* ν_s^c, and generalized Cartan's result. More precisely, they showed that conformal rigidity holds whenever $d \leq 4$, $n \geq 2d+3$ and $\nu_s^c \leq n-2s-1$ for $1 \leq s \leq d$. As far as we know, it is still an open problem whether this result remains true for any codimension d. In this paper, we introduce a new conformal invariant, namely, the *conformal type number* τ_f^c, and prove the following result which has no restriction on the size of the codimension.

Theorem 1.1. *Let $f\colon\ M^n \to \mathbb{R}^{n+d}$ be an immersion. Assume that everywhere $\tau_f^c(p) \geq 3$ and that $\nu_s^c(p) \leq n-2s-1$ for $1 \leq s \leq 3$. Suppose further that $n \geq 2d+3$ if $d = 1, 2$. Then f is conformally rigid.*

In relation to the above result see also Theorem 1.3 and Corollary 1.1.

Allendoerfer ([**Al**]) showed that an isometric immersion with type number at least 3 everywhere is isometrically rigid. By using the notions of k^{th} *type number* $\tau_f^k(p)$, $1 \leq k \leq d$, and *s-nullity* ν_s, we obtain the following result whose assumptions are less restrictive (see Remark 2.1) than in Allendoerfer's theorem.

Theorem 1.2. *Let* $f\colon\ M^n \to \mathbb{R}^{n+d}$, $d \geq 2$, *be an immersion. Assume that everywhere* $\tau_f^{d-1}(p) \geq 3$ *and* $\nu_s(p) \leq n - 2s - 1$, $1 \leq s \leq 3$, *then* f *is isometrically rigid.*

This work is part of my Doctoral Thesis at IMPA. I would like to thank my research advisor, Professor Marcos Dajczer, for suggesting the problem and valuable remarks. I also would like to thank Professor Luis A. Florit and Professor Ruy Tojeiro for a number of helpful comments.

1. Proof of Theorem 1.1.

For a symmetric bilinear form $\beta\colon\ V \times V \to W$ we denote by $S(\beta)$ the subspace of W given by

$$S(\beta) = \text{span}\{\beta(X,Y) : X, Y \in V\},$$

and by $N(\beta)$ the nullity space of β defined as

$$N(\beta) = \{\, n \in V : \beta(X,n) = 0,\ \forall\ X \in V\}.$$

Definition 1.1. Assume that V and W are endowed with positive definite inner products. We define the k^{th} *type number* of β, $1 \leq k \leq \dim W$, as being the largest integer r for which there are k vectors $\xi_1, \ldots, \xi_k \in W$ and r vectors $X_1, \ldots, X_r \in V$ necessarily linearly independent such that the vectors $B_{\xi_i} X_j$, $1 \leq i \leq k$, $1 \leq j \leq r$, are linearly independent. Here $B_{\xi_i}\colon\ V \to V$ is given by $\langle B_{\xi_i} X, Y\rangle = \langle \beta(X,Y), \xi_i\rangle$. We point out that when $k = \dim W$ the k^{th} type number does not depend on the basis of W.

Now let $f\colon\ M^n \to \tilde{M}^{n+d}$ be an immersion into a Riemannian manifold with vector valued second fundamental form $\alpha^f\colon\ TM \times TM \to T_f^{\perp} M$. The k^{th} *type number* $\tau_f^k(p)$, $1 \leq k \leq d$, of f at p is defined as the k^{th} type number of α^f at p. Observe that $\tau_f^d(p)$ is exactly the type number introduced by Allendoerfer.

Definition 1.2. We define the *conformal type number* $\tau_f^c(p)$ of f at $p \in M^n$ as being the integer

$$\tau_f^c(p) = \max_{\eta \in T^{\perp}_{f(p)} M} \tau_\eta^d$$

where τ_η^d denote the d^{th} type number of $\alpha^f - \langle\ ,\ \rangle\eta$ at p.

We claim that the conformal type number is a conformal invariant. In fact, let Υ be a conformal diffeomorphism of $\tilde{M}^{n+d}$ with conformal factor ρ, that is, $\langle \Upsilon_* X, \Upsilon_* Y\rangle = \rho^2\langle X, Y\rangle$. For $h = \Upsilon \circ f$, one easily verifies that

$$\alpha^h = \Upsilon_*\alpha^f - \frac{1}{2(\rho\circ f)^2}\langle\ ,\ \rangle \Upsilon_*\left(\nabla\rho^2\right)^{\perp}_f ,$$

where ∇ is the gradient operator. Thus the claim follows.

Given an s-dimensional subspace $U^s \subseteq T^{\perp}_{f(p)}M$, $1 \le s \le d$, consider the bilinear form

$$\alpha^f_{U^s}\colon\ T_pM \times T_pM \to U^s$$

defined as $\alpha^f_{U^s} = P\circ\alpha^f$, where P denotes the orthogonal projection of $T^{\perp}_{f(p)}M$ onto U^s. Endow M^n with the induced metric. The *conformal s-nullity* $\nu^c_s(p)$ of f at p (see [**C-D**]) is the integer

$$\nu^c_s(p) = \max_{U^s\subseteq T^{\perp}_{f(p)}M\,,\,\eta\in U^s}\left\{\dim N\left(\alpha^f_{U^s} - \langle\ ,\ \rangle\eta\right)\right\}.$$

The following result relate conformal type number and conformal s-nullity.

Proposition 1.1. *Let $f\colon\ M^n \to \tilde{M}^{n+d}$ be an isometric immersion. If $\tau^c_f(p) \ge r$, then $\nu^c_s(p) \le n-(s-1)r$ for $1\le s\le d$.*

Proof. Suppose $r \ge 1$ and $s\ge 2$. In any other case the result is immediate. Since $\tau^c_f(p)\ge r$, there exists $\eta\in T^{\perp}_{f(p)}M$ such that $\alpha^f - \langle\ ,\ \rangle\eta$ has d^{th} type number at least r. Consequently, for all basis $\xi_1,\dots,\xi_d$ of $T^{\perp}_{f(p)}M$ there exist r vectors $X_1,\dots,X_r$ tangent at p such that the vectors

$$(A_{\xi_i} - \langle\eta,\xi_i\rangle I)X_j,\quad 1\le i\le d,\quad 1\le j\le r,$$

are linearly independent. Let $U^s\subseteq T^{\perp}_{f(p)}M$, $2\le s\le d$, be an s-dimensional subspace and $\xi\in U^s$ an arbitrary vector. For the subspace $W = U^s\cap(\text{span}\{\eta-\xi\})^{\perp}$ it holds that $\dim W \ge s-1$. Take a basis $\xi_1,\dots,\xi_d$ of $T^{\perp}_{f(p)}M$ such that $\xi_1,\dots,\xi_s$ span U^s and $\xi_1,\dots,\xi_{s-1}$ are in W. Let L be the subspace of T_pM with dimension $(s-1)r$ given by

$$L = \text{span}\{\,(A_{\xi_i}-\langle\eta,\xi_i\rangle I)X_j,\ 1\le i\le s-1,\ 1\le j\le r\,\}.$$

For an arbitrary vector $v\in N\left(\alpha^f_{U^s}-\langle\ ,\ \rangle\xi\right)$ and $1\le i\le s$, we obtain that

$$\begin{aligned} 0 &= \left\langle \alpha^f(v,X_j) - \langle v,X_j\rangle\xi, \xi_i\right\rangle \\ &= \left\langle \alpha^f(v,X_j) - \langle v,X_j\rangle\eta + \langle v,X_j\rangle(\eta-\xi),\xi_i\right\rangle. \end{aligned}$$

Thus, we have that $\langle(A_{\xi_i}-\langle\eta,\xi_i\rangle I)X_j, v\rangle = 0$, $1\le i\le s-1$, $1\le j\le r$, that is, $N\left(\alpha^f_{U^s}-\langle\ ,\ \rangle\xi\right)\subseteq L^{\perp}$. Since $U^s\subseteq T^{\perp}_{f(p)}M$ is arbitrary, the proof follows.

Before proving Theorem 1.1 we recall some basic facts; from [**Da**] and [**D-T**]. Consider the Lorentz space $\mathbb{L}^{n+d+2}$, that is, Euclidean space $\mathbb{R}^{n+d+2}$ endowed with the metric $\langle\ ,\ \rangle$ defined by

$$\langle X, X\rangle = -x_1^2 + x_2^2 + \cdots + x_{n+d+2}^2$$

for $X = (x_1, x_2, \dots, x_{n+d+2})$. The *light cone* is the degenerate totally umbilical hypersurface of $\mathbb{L}^{n+d+2}$ defined by

$$\mathbb{V}^{n+d+1} = \{ X \in \mathbb{L}^{n+d+2} : \langle X, X\rangle = 0,\ X \neq 0\}.$$

Given $\zeta \in \mathbb{V}^{n+d+1}$ consider the hyperplane

$$H_\zeta = \{ X \in \mathbb{L}^{n+d+2} : \langle X, \zeta\rangle = 1\}$$

and the $(n+d)$-dimensional submanifold $H_\zeta \cap \mathbb{V}^{n+d+1}$. It is easy to see that the normal space to $H_\zeta \cap \mathbb{V}^{n+d+1}$ in $\mathbb{L}^{n+d+2}$ at p is the Lorentzian plane $\mathbb{L}^2$ generated by p and ζ. Therefore, the metric induced by $\mathbb{L}^{n+d+2}$ on $H_\zeta \cap \mathbb{V}^{n+d+1}$ is riemannian. The second fundamental form of this intersection is given by

$$\alpha = -\langle\ ,\ \rangle\zeta.$$

Using the Gauss equation, it follows that $H_\zeta \cap \mathbb{V}^{n+d+1}$ is an embedded flat riemannian submanifold of $\mathbb{L}^{n+d+2}$. Indeed, it can be checked that it is the image of an isometric embedding $J_\zeta\colon\ \mathbb{R}^{n+d} \to \mathbb{V}^{n+d+1}$.

The light cone is a very useful tool in the study of conformal immersions. Given any conformal immersion $g\colon\ M^n \to \mathbb{R}^{n+d}$ such that $\langle g_*X, g_*Y\rangle = \phi_g^2\langle X, Y\rangle$, where $\phi_g > 0$ is the conformal factor of g, we associate to g an isometric immersion $G\colon\ M^n \to \mathbb{V}^{n+d+1} \subset \mathbb{L}^{n+d+2}$ by setting

$$G = \frac{1}{\phi_g} J_\zeta \circ g$$

for an arbitrary $\zeta \in \mathbb{V}^{n+d+1}$.

Conversely, any isometric immersion $G\colon\ M^n \to \mathbb{V}^{n+d+1}$ arises this way. In fact, choose $\zeta \in \mathbb{V}^{n+d+1}$ such that $\langle G, \zeta\rangle > 0$. Define $g\colon\ M^n \to \mathbb{R}^{n+d}$ by setting

$$J_\zeta \circ g = \frac{G}{\langle G, \zeta\rangle}.$$

It is not difficult to verify that g is a conformal immersion with conformal factor given by $1/\langle G, \zeta\rangle$.

Now, let g, $f\colon\ M^n \to \mathbb{R}^{n+d}$ be conformal immersions and like previously discussed consider isometric immersions G, $F\colon\ M^n \to \mathbb{V}^{n+d+1}$ associated to them. If there exists an isometry $T\colon\ \mathbb{V}^{n+d+1} \to \mathbb{V}^{n+d+1}$ such that $F = T \circ G$, then T induces a conformal diffeomorphism Υ from an open subset of $\mathbb{R}^{n+d}$ to an open subset of $\mathbb{R}^{n+d}$ defined by

$$J_\zeta \circ \Upsilon = \frac{T \circ J_\zeta}{\langle T \circ J_\zeta, \zeta\rangle}$$

which satisfies $f = \Upsilon \circ g$. In order to obtain such T it suffices to construct a vector bundle isomorphism $\hat{T}\colon\ T_G^\perp M \to T_F^\perp M$ preserving metrics, second fundamental forms and normal connections. Here, $T_G^\perp M$ and $T_F^\perp M$ stand for the normal bundles of G and F, respectively, in $\mathbb{L}^{n+d+2}$. From the fundamental theorem for isometric immersions adapted to the Lorentzian case we conclude that there exists an isometry $\overline{T}\colon\ \mathbb{L}^{n+d+2} \to \mathbb{L}^{n+d+2}$ such that $F = \overline{T} \circ G$. Then we take T as the restriction of $\overline{T}$ to $\mathbb{V}^{n+d+1}$.

Proof of Theorem 1.1. We only have to deal with the case of codimension $d \geq 5$. If $\tau_f^c(p) \geq 3$ then $n \geq 3d$ and can be easily deduced from Proposition 1.1 that $\nu_s^c(p) \leq n - 2s - 1$ when $s \geq 4$. Consequently, under our assumptions we always have that $n \geq 2d + 3$ and $\nu_s^c(p) \leq n - 2s - 1$. Thus, the following result already reported in the introduction applies for $d \leq 4$.

Theorem 1.3 ([**C-D**])**.** *Let $f\colon\ M^n \to \mathbb{R}^{n+d}$ be an immersion where $d \leq 4$ and $n \geq 2d + 3$. Assume that $\nu_s^c(p) \leq n - 2s - 1$ for all $p \in M^n$ and every integer s, $1 \leq s \leq d$. Then f is conformally rigid.*

Let $g\colon\ M^n \to \mathbb{R}^{n+d}$ be any immersion conformal to f and $G\colon\ M^n \to \mathbb{V}^{n+d+1}$ its associated isometric immersion. We may assume that M^n is endowed with the metric induced by f. Taking the derivative of $\langle G, G\rangle = 0$, we see that the null vector field G is normal to the immersion G. The normal field G also satisfies $A_G^G = -I$. The normal bundle of G is given by the orthogonal direct sum

$$T_G^\perp M = T_g^\perp M \oplus \mathbb{L}^2$$

where $T_g^\perp M$ is identified with $(J_\zeta)_* T_g^\perp M$ and $\mathbb{L}^2$ is a Lorentzian plane bundle which contains G. We can easily see that there exists a unique orthogonal frame $\{\xi, \eta\}$ of $\mathbb{L}^2$ with $|\xi|^2 = -1$ such that

$$G = \xi + \eta.$$

Writing α^G in terms of this orthogonal frame we obtain

$$\alpha^G = -\left\langle \alpha^G, \xi \right\rangle \xi + \left\langle \alpha^G, \eta \right\rangle \eta + \left(\alpha^G\right)^*$$

where $\left(\alpha^G\right)^* = (1/\phi_g)(J_\zeta)_* \alpha^g$ is the $T_g^\perp M$-component of α^G.

Given an m-dimensional real vector space W endowed with a non-degenerate inner product $\langle\ ,\ \rangle$ of *index* r, that is, the maximal dimension of a subspace of W where $\langle\ ,\ \rangle$ is negative definite, we say that W is of type (r, q) and we write $W^{(r,q)}$ with $q = m - r$.

At $p \in M^n$, let

$$W = T_{f(p)}^\perp M \oplus \mathrm{span}\{\xi(p)\} \oplus \mathrm{span}\{\eta(p)\} \oplus T_{g(p)}^\perp M$$

be endowed with the natural metric of type $(d + 1, d + 1)$ which is negative definite on $T_{f(p)}^\perp M \oplus \mathrm{span}\{\xi(p)\}$. We also define a symmetric bilinear form

$\beta\colon\ TM \times TM \to W$ setting $\beta = \alpha^f + \alpha^G$, i.e.,

$$\beta = \alpha^f - \langle \alpha^G, \xi \rangle \xi + \langle \alpha^G, \eta \rangle \eta + \left(\alpha^G\right)^*.$$

The Gauss equations for f and G imply that β is *flat*, i.e.,

$$\langle \beta(X,Y), \beta(Z,U) \rangle = \langle \beta(X,U), \beta(Y,Z) \rangle, \quad \forall\, X,Y,Z,U \in TM.$$

Observe also that $\beta(X,X) \neq 0$ for all $X \neq 0$, because $A^G_{\xi+\eta} = -I$.

Lemma 1.1. *The bilinear form β is null, that is,*

$$\langle \beta(X,Y), \beta(Z,U) \rangle = 0, \quad \forall\, X,Y,Z,U \in TM.$$

Proof. Fixed $p \in M^n$, set $V\colon = T_pM$ and for each $X \in V$ define the linear map

$$\beta(X)\colon V \to W$$

by setting $\beta(X)(v) = \beta(X,v)$ for all $v \in V$. For simplicity of notation, we omit the p. The kernel and image of $\beta(X)$ are denoted by $\ker \beta(X)$ and $\beta(X,V)$, respectively. We say that X is a *regular element* of β if

$$\dim \beta(X,V) = \max_{Z \in V} \dim \beta(Z,V).$$

The set of regular elements of β is denoted by $RE(\beta)$. For each $X \in V$, set $U(X) = \beta(X,V) \cap \beta(X,V)^\perp$ and define

$$RE^*(\beta) = \{\, Y \in RE(\beta) : \dim U(Y) = d_0\}$$

where $d_0 = \min\{\dim U(Y) : Y \in RE(\beta)\}$.

We will need the following from [**Da**].

Sublemma 1.1. The set $RE^*(\beta)$ is open and dense in V and

$$\beta(\ker \beta(X), V) \subseteq U(X), \quad \forall\, X \in RE(\beta).$$

Now recall that a vector subspace L of W is said to be *degenerate* when satisfies $L \cap L^\perp \neq \{0\}$ and *isotropic* if $\langle L, L \rangle = 0$. We also have that

$$\dim L + \dim L^\perp = \dim W \quad \text{and} \quad L^{\perp\perp} = L. \tag{1.1}$$

It follows easily from (1.1), $\dim W = 2d+2$ and the definition of $U(X)$ that $d_0 \leq d+1$. We separate the proof in two cases, namely, $d_0 = d+1$ and $d_0 \leq d$.

Case I. $d_0 = d+1$. In this case, $\dim U(X) = d+1$ for all $X \in RE^*(\beta)$. Then, $U(X) = \beta(X,V) = \beta(X,V)^\perp$ due to (1.1). Using the density of $RE^*(\beta)$, we get

$$\langle \beta(X,Y), \beta(X,Z) \rangle = 0, \quad \forall\, X,Y,Z \in V,$$

and the bilinearity of β yields the claim.

Case II. $d_0 \leq d$. To deal with this case we need several facts.

Assertion 1. $\dim S(\beta) \cap S(\beta)^\perp \geq d-2$.

Since $\tau_f^c(p) \geq 3$, there exists $\eta \in T_{f(p)}^{\perp}M$ such that $\alpha^f - \langle\ ,\ \rangle\eta$ has type number at least 3. Fix a basis $\xi_1, \ldots, \xi_d$ of $T_{f(p)}^{\perp}M$ and vectors $X_1, X_2, X_3 \in V$ so that the $A_{\xi_i}^{\lambda_i, f}X_j = (A_{\xi_i}^f + \lambda_i I)X_j$, $\lambda_i = \langle\eta, \xi_i\rangle$, $1 \leq i \leq d$, $1 \leq j \leq 3$, are linearly independent. Define

$$\tilde{L} = \left(\operatorname{span}\left\{ A_{\xi_i}^{\lambda_i, f}X_j : 1 \leq i \leq d,\ 1 \leq j \leq 3 \right\}\right)^{\perp}.$$

We have that $\dim \tilde{L} = n - 3d$ and that $Z \in \tilde{L}$ if and only if

$$\left\langle \alpha^f(X_j, Z), \xi_i \right\rangle + \lambda_i \langle X_j, Z\rangle = 0, \quad \forall\, i, j.$$

By definition of β we have for $Z \in \ker \beta(X_j)$ that

$$\alpha^f(X_j, Z) = 0 \ \text{ and } \ \left\langle \alpha^G(X_j, Z), \xi \right\rangle = \left\langle \alpha^G(X_j, Z), \eta \right\rangle = 0.$$

Since $A_{\eta+\xi}^G = -I$, we get $\langle X_j, Z\rangle = -\left\langle \alpha^G(X_j, Z), \eta + \xi \right\rangle = 0$. Hence,

$$\bigcap_{h=1}^{3} \ker \beta(X_h) \subseteq \tilde{L}. \tag{1.2}$$

We can assume that $X_1, X_2, X_3 \in RE^*(\beta)$ by Sublemma 1.1. Unless otherwise stated, from now on the indexes $i, j, k \in \{1, 2, 3\}$ are all distinct. Moreover, for simplicity of notation we denote the map $\beta(X_i)$ and its image $\beta(X_i, V)$ by β_i and $\operatorname{Im} \beta_i$, respectively. Take the maps

$$\Gamma_k\colon\ \ker \beta_i \cap \ker \beta_j \to U(X_i) \cap U(X_j)$$

as being the restriction of β_k to $\ker \beta_i \cap \ker \beta_j$ and

$$\Gamma_{ij}\colon\ \ker \beta_j \to U(X_j)$$

as the restriction of β_i to $\ker \beta_j$. By Sublemma 1.1 the maps Γ_k and Γ_{ij} are well defined. Setting $U_j = U(X_j)$ and $U_{ij} = U_i \cap U_j$, we have that

$$\operatorname{Im} \Gamma_k \subseteq \operatorname{Im} \Gamma_{kj} \subseteq U_j \quad \text{and} \quad \operatorname{Im} \Gamma_k \subseteq U_{ij} \subseteq U_j, \ \ \forall\, i, j, k \in \{1, 2, 3\}. \tag{1.3}$$

Define

$$\rho = \dim (\operatorname{Im} \beta_i)^{\perp} - d_0 \quad \text{and} \quad \theta_i^j = d_0 - \dim \operatorname{Im} \Gamma_{ij}. \tag{1.4}$$

A simple calculation shows that

$$\dim \ker \Gamma_{ij} = n - 2d - 2 + \rho + \theta_i^j. \tag{1.5}$$

Setting

$$\gamma_k = d_0 - \dim \operatorname{Im} \Gamma_k \tag{1.6}$$

and using that $\ker \Gamma_{ij} = \ker \beta_i \cap \ker \beta_j$, we obtain that

$$\dim \ker \Gamma_k = n - 2d - d_0 - 2 + \rho + \theta_i^j + \gamma_k.$$

Since

$$\ker \Gamma_k = \bigcap_{h=1}^{3} \ker \beta_h,$$

the last equality shows that the sums $\theta_i^j + \gamma_k$ are independent of the indexes. This and (1.2) imply that $n - 3d \geq n - 2d - d_0 - 2 + \rho + \theta_i^j + \gamma_k$. Hence

$$d - d_0 - 2 + (\rho + \theta_i^j + \gamma_k) \leq 0. \tag{1.7}$$

Since the integers ρ, θ_i^j and γ_k are nonnegative, it follows that $d_0 \geq d - 2$. We have to analyze three possibilities for d_0.

II.(a). $d_0 = d - 2$. In this case $\rho = \theta_i^j = \gamma_k = 0$ by (1.7). Observe that $\gamma_k = 0$ and (1.3) yield that

$$\operatorname{Im} \Gamma_k = U_j = U_i. \tag{1.8}$$

We show that $\operatorname{Im} \Gamma_i \subseteq S(\beta) \cap S(\beta)^{\perp}$. An arbitrary element in $\operatorname{Im} \Gamma_i$ is given by $\beta(X_i, X_0)$ with $X_0 \in \ker \Gamma_{jk}$. Since $\beta(X_0, Y) \in U_{jk}$ by Sublemma 1.1, we get using (1.8) that

$$\langle \beta(X_i, X_0), \beta(Y, Z) \rangle = \langle \beta(X_i, Z), \beta(X_0, Y) \rangle = 0, \quad \forall\, Y, Z \in V.$$

We conclude that $\dim S(\beta) \cap S(\beta)^{\perp} \geq d_0 = d - 2$.

II.(b). $d_0 = d - 1$. In this case $\rho + \theta_i^j + \gamma_k \leq 1$. We have to consider two sub-cases.

(b).1. There exist indexes such that $U_{ij} = U_{kj}$. Like in II.(a) we conclude that $\operatorname{Im} \Gamma_i \subseteq S(\beta) \cap S(\beta)^{\perp}$ and $\dim S(\beta) \cap S(\beta)^{\perp} \geq d - 2$.

(b).2. Suppose that $U_{ij} \neq U_{kj}$ for all i, j, k. This implies that $U_i \neq U_j$ for all i, j. It follows from (1.3) that $\gamma_k = 1$. In particular,

$$\dim \operatorname{Im} \Gamma_k = d - 2 \quad \text{and} \quad \operatorname{Im} \Gamma_k = U_{ij}. \tag{1.9}$$

Being $\theta_i^j = 0$, then (1.3) gives

$$U_j = \operatorname{Im} \Gamma_{ij} \subset \operatorname{Im} \beta_i. \tag{1.10}$$

Due to $\rho = 0$ and $\operatorname{Im} \beta_i \subseteq U_i^{\perp}$, we deduce that

$$\operatorname{Im} \beta_i = U_i^{\perp}. \tag{1.11}$$

The assumption in (b).2 jointly to (1.9) and (1.11) imply that

$$U_{ki} \not\subset U_{kj} \quad \text{and} \quad \operatorname{Im} \beta_i \not\subset \operatorname{Im} \beta_j. \tag{1.12}$$

It is not difficult to see that $U_j = U_{ij} + U_{kj} \subseteq U_i + U_k$ due to the assumption in (b).2, (1.9) and $\dim U_j = d - 1$. For all i and j, the subspace $U_i + U_j$ has dimension d by the formula

$$\dim (\, L_1 + L_2 \,) = \dim L_1 + \dim L_2 - \dim L_1 \cap L_2. \tag{1.13}$$

Therefore,

$$U_i + U_j = U_i + U_k \tag{1.14}$$

and, consequently,

$$\operatorname{Im}\beta_i \cap \operatorname{Im}\beta_j = \operatorname{Im}\beta_i \cap \operatorname{Im}\beta_k \tag{1.15}$$

by (1.11) and the formula

$$\Big(\sum_h L_h\Big)^\perp = \bigcap_h L_h^\perp, \tag{1.16}$$

valid for any arbitrary finite number of subspaces. If $v \in V$ and $w \in \ker\beta_j$, then

$$\langle\beta(X_k, v), \beta(X_i, w)\rangle = \langle\beta(X_k, X_i), \beta(v, w)\rangle = 0$$

since $\beta(X_k, X_i) \in \operatorname{Im}\beta_j$ by (1.15) and $\beta(v, w) \in U_j$ by Sublemma 1.1. The last equality, (1.10) and (1.11) imply that $\operatorname{Im}\beta_k \subset (\operatorname{Im}\Gamma_{ij})^\perp = U_j^\perp = \operatorname{Im}\beta_j$ which is in contradiction to (1.12).

II.(c). $d_0 = d$. It follows from (1.7) that $\rho + \theta_i^j + \gamma_k \leq 2$. Like in II.(b), we consider two sub-cases.

(c).1. There exist indexes such that $U_{ij} = U_{kj}$. Proceeding analogous to II.(a) gives $\dim S(\beta) \cap S(\beta)^\perp \geq d - 2$.

(c).2. Suppose that $U_{ij} \neq U_{kj}$ for all i, j, k. This implies that $U_i \neq U_j$ for all i, j. It follows from (1.3) that $1 \leq \gamma_k \leq 2$, then $\rho + \theta_i^j \leq 1$. From (1.6), we conclude that $d-2 \leq \dim\operatorname{Im}\Gamma_k \leq \dim U_{ij} \leq d-1$. If there exist indexes i and j such that $\dim U_{ij} = d - 2$, then $\gamma_k = 2$, $\rho = 0$ and $\theta_i^j = \theta_j^i = 0$. Thus, $U_j = \operatorname{Im}\Gamma_{ij} \subset \operatorname{Im}\beta_i$ and $U_i = \operatorname{Im}\Gamma_{ji} \subset \operatorname{Im}\beta_j$ by (1.3). We have that $\dim(U_i + U_j) = d+2$ by (1.13). Here the subspace $U_i + U_j$ is isotropic since U_i and U_j are isotropic and U_j, being a subset of $\operatorname{Im}\beta_i$, is orthogonal to U_i. But this is not possible due to (1.1). Then we can assume $\dim U_{ij} = d-1$ for all i and j. In this case, $\dim(U_i + U_j) = d + 1$ and (1.14) holds.

(c).2.1. First suppose there exists k with $\gamma_k = 2$. Thus, $\theta_i^j = \rho = 0$. Being $\rho = 0$, (1.11) holds. From (1.14) and (1.16), we have that (1.15) also holds. Like in (b).2, we obtain a contradiction.

(c).2.2. Consider $\gamma_k = 1$ for all k. In this case the θ_i^j's are independent from the indexes. There are three possibilities:

(c).2.2.1. Suppose that $\rho = \theta_i^j = 0$. Similar to (b).2 we have a contradiction.

(c).2.2.2. Suppose that $\rho = 0$ and $\theta_i^j = 1$. Being $\rho = 0$, then (1.11), (1.12) and (1.15) hold. Further,

$$\dim\ker\Gamma_{ij} = n - 2d - 1 \tag{1.17}$$

by (1.5) and

$$\dim \operatorname{Im} \beta_i = d + 2, \quad \text{hence} \quad \dim \ker \beta_i = n - d - 2. \tag{1.18}$$

It holds that $\operatorname{Im}\Gamma_{kj} = \operatorname{Im}\Gamma_k = U_{ij}$ from the assumption in (c).2, $\gamma_k = \theta_k^j = 1$ and (1.3). We claim that $U_i \subset \operatorname{Im}\beta_j$. Otherwise, we have that $U_{ij} = U_i \cap \operatorname{Im}\beta_j$ since $d-1 = \dim U_{ij} \leq \dim(U_i \cap \operatorname{Im}\beta_j) \leq d-1$. Using (1.15), we conclude that

$$U_{ij} = U_i \cap \operatorname{Im}\beta_j = U_i \cap \operatorname{Im}\beta_i \cap \operatorname{Im}\beta_j = U_i \cap \operatorname{Im}\beta_i \cap \operatorname{Im}\beta_k = U_i \cap \operatorname{Im}\beta_k = U_{ik}$$

contradicting (c).2, and the claim follows. From (1.13), (1.17) and (1.18), we deduce that

$$\dim(\ker \beta_i + \ker \beta_j) = n - 3. \tag{1.19}$$

The vector X_i satisfies that $X_i \notin (\ker\beta_i + \ker\beta_j)$. Otherwise, using Sublemma 1.1, we obtain the following contradiction due to dimensions:

$$\operatorname{Im}\beta_i \subset \beta(\ker\beta_i + \ker\beta_j, V) \subseteq U_i + U_j.$$

The vector $X_j \notin \operatorname{span}\{X_i\} \oplus (\ker\beta_i + \ker\beta_j)$. Otherwise,

$$\operatorname{Im}\beta_j \subset \beta(\operatorname{span}\{X_i\} \oplus (\ker\beta_i + \ker\beta_j), V) \subseteq \operatorname{Im}\beta_i + U_i + U_j = \operatorname{Im}\beta_i$$

which is in contradiction with (1.12). It hold that

$$\operatorname{Im}\beta_i + \operatorname{Im}\beta_j = U_{ij}^{\perp} \quad \text{and} \quad \operatorname{Im}\beta_i \cap \operatorname{Im}\beta_j = (U_i + U_j)^{\perp} \tag{1.20}$$

by (1.11) and (1.16). We assert that

$$\operatorname{Im}\beta_k \not\subset \operatorname{Im}\beta_i + \operatorname{Im}\beta_j. \tag{1.21}$$

On the contrary, $\operatorname{Im}\beta_k + \operatorname{Im}\beta_j \subseteq \operatorname{Im}\beta_i + \operatorname{Im}\beta_j$ and, since both spaces have dimension $d+3$, then the equality holds. But this jointly to (1.20) contradicts the hypothesis in (c).2. Also $X_k \notin \operatorname{span}\{X_i, X_j\} \oplus (\ker\beta_i + \ker\beta_j)$ due to (1.21). It follows from (1.19) that

$$V = \operatorname{span}\{X_1, X_2, X_3\} \oplus (\ker\beta_i + \ker\beta_j).$$

The above equality gives that $S(\beta) = \sum_{h=1}^{3} \operatorname{Im}\beta_h$. This fact, together with (1.11), (1.16) and $(\bigcap_{h=1}^{3} U_h) \subseteq S(\beta)$ prove that $S(\beta) \cap S(\beta)^{\perp} = (\bigcap_{h=1}^{3} U_h)$. Therefore, $\dim S(\beta) \cap S(\beta)^{\perp} = \dim(\bigcap_{h=1}^{3} U_h) = d-2$.

(c).2.2.3. Finally, suppose that $\rho = 1$ and $\theta_i^j = 0$. Then $\dim(\operatorname{Im}\beta_i)^{\perp} = d+1$ by (1.4). Thus, $\dim \operatorname{Im}\beta_i = d+1$ by (1.1). From (1.3), we have $U_j = \operatorname{Im}\Gamma_{ij} \subset \operatorname{Im}\beta_i$ since $\theta_i^j = 0$. These facts and $U_i \neq U_j$ imply that $U_i + U_j = \operatorname{Im}\beta_i$. Arguing as in (c).2 we obtain that $U_i + U_j$ is isotropic. So $U_i = \operatorname{Im}\beta_i \cap (\operatorname{Im}\beta_i)^{\perp} = \operatorname{Im}\beta_i$, and we get a contradiction due to dimensions.

Assertion 2. There exist an orthogonal decomposition

$$W = W_1^{(\ell,\ell)} \oplus W_2^{(d-\ell+1,\, d-\ell+1)}, \ \ \ell \geq d-2,$$

and symmetric bilinear forms $\omega_j\colon\ V \times V \to W_j$, $1 \leq j \leq 2$, satisfying

$$\beta = \omega_1 \oplus \omega_2$$

such that:

i) ω_1 is nonzero and null with respect to $\langle\ ,\ \rangle$ and
ii) ω_2 is flat with $\dim N(\omega_2) \geq n - \dim W_2$.

Let $v_1, \dots, v_\ell$ a basis of $S(\beta) \cap S(\beta)^\perp$. There exists (see [**Da**], p. 83) a pseudo-orthonormal basis $v_1, \dots, v_\ell, \hat{v}_1, \dots, \hat{v}_\ell, \theta_1, \dots, \theta_{2(d-\ell+1)}$ of W satisfying that $\langle v_i, v_j\rangle = \langle \hat{v}_i, \hat{v}_j\rangle = \langle \theta_i, v_j\rangle = \langle \theta_i, \hat{v}_j\rangle = 0, \langle v_i, \hat{v}_j\rangle = \delta_{ij}$ and that $\langle \theta_i, \theta_j\rangle = \pm\delta_{ij}$. Defining

$$W_1 = \operatorname{span}\{v_1, \dots, v_\ell,\, \hat{v}_1, \dots, \hat{v}_\ell\}, \quad W_2 = \operatorname{span}\{\theta_1, \dots, \theta_{2(d-\ell+1)}\}$$

and

$$\beta = \sum_{i=1}^{\ell} \phi_i v_i + \sum_{i=1}^{\ell} \psi_i \hat{v}_i + \sum_{i=1}^{2(d-\ell+1)} \kappa_i \theta_i,$$

we have that $\psi_i = \langle \beta, v_i\rangle = 0$. Set

$$\omega_1 = \sum_{i=1}^{\ell} \phi_i v_i \quad \text{and} \quad \omega_2 = \sum_{i=1}^{2(d-\ell+1)} \kappa_i \theta_i.$$

Since $\ell = \dim S(\beta) \cap S(\beta)^\perp \geq d-2 \geq 3$, then ω_1 is nonzero. It is easy to verify that ω_1, ω_2 are symmetric bilinear forms such that ω_1 is null and ω_2 is flat. In order to see that $S(\omega_2)$ is non-degenerate, let $\sum_i \omega_2(X_i, Y_i) \in W_2$ be an arbitrary element in $S(\omega_2) \cap S(\omega_2)^\perp$. For all $v, w \in V$, we get

$$\left\langle \sum_i \omega_2(X_i, Y_i), \beta(v,w) \right\rangle = \left\langle \sum_i \omega_2(X_i, Y_i), \omega_2(v,w) \right\rangle = 0.$$

Therefore, $\sum_i \omega_2(X_i, Y_i) \in S(\beta) \cap S(\beta)^\perp$. Hence, $\sum_i \omega_2(X_i, Y_i) \in W_1$. Thus,

$$\sum_i \omega_2(X_i, Y_i) \in W_1 \cap W_2 = \{0\}.$$

Since the subspace $S(\omega_2)$ is non-degenerate and $d-\ell+1 \leq 3$, the inequality $\dim N(\omega_2) \geq n - \dim W_2$ is a consequence of the following result whose proof is part of the arguments for the Main Lemma 2.2 in ([**C-D**], pp. 968-974).

Sublemma 1.2. Let $\sigma\colon\ V_1 \times V_1 \to W^{(r,r)}$ be a nonzero flat symmetric bilinear form. Assume $r \leq 5$ and $\dim N(\sigma) < \dim V_1 - 2r$. Then $S(\sigma)$ is degenerate.

Now Lemma 1.1 is a consequence of the following fact.

Assertion 3. The bilinear form ω_2 is zero.

Suppose on the contrary that $\ell \leq d$. Set $v_i = \gamma_i + b_i\xi + c_i\eta + \delta_i$, $1 \leq i \leq \ell$, where $\gamma_i \in T_f^\perp M$ and $\delta_i \in T_g^\perp M$. Let $\hat{L}$ be the orthogonal complement in $T_{f(p)}^\perp M$ of the subspace $L = \operatorname{span}\{\gamma_i : 1 \leq i \leq \ell\}$. If the vectors γ_i are linearly dependent, then $\dim \hat{L} \geq d - \ell + 1 \geq 1$. For any $n \in N(\omega_2)$, $v \in V$ and $u \in \hat{L}$, it holds that

$$\left\langle \alpha^f(n,v), u \right\rangle = \langle \beta(n,v), u\rangle = \langle \omega_1(n,v), u\rangle = 0.$$

Hence,

$$\nu^c_{\dim \hat{L}} \geq \dim N\left(\alpha^f_{\hat{L}}\right) \geq \dim N(\omega_2) \geq n - 2\left(d - \ell + 1\right) \geq n - 2\dim \hat{L},$$

which is in contradiction with the hypothesis on the conformal $(\dim \hat{L})$-nullity when $1 \leq \dim \hat{L} \leq 3$ and with the Proposition 1.1 when $\dim \hat{L} \geq 4$.

Now, since the vectors γ_i are linearly independent, then $\dim \hat{L} = d - \ell$. The definition of β gives

$$\left\langle \alpha^G(n,v), \eta \right\rangle = \langle \beta(n,v), \eta\rangle = \langle \omega_1(n,v), \eta\rangle = \sum_{i=1}^{\ell} c_i\phi_i$$

and

$$\left\langle \alpha^G(n,v), \xi \right\rangle = \langle \beta(n,v), \xi\rangle = \langle \omega_1(n,v), \xi\rangle = -\sum_{i=1}^{\ell} b_i\phi_i.$$

Therefore,

$$-\langle n, v\rangle = \left\langle n, A^G_{\eta+\xi} v \right\rangle = \sum_{i=1}^{\ell} (c_i - b_i)\phi_i. \tag{1.22}$$

Fix $j \in \{1, \ldots, \ell\}$ and consider the hyperplane $L_j \subset L$ given by

$$L_j = \operatorname{span}\{\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_\ell\}.$$

Let μ_j be the orthogonal projection of γ_j onto the orthogonal complement of L_j in L. Observing that $\langle \mu_i, \gamma_j\rangle = \delta_{ij}|\mu_i|^2$, it is not difficult to see that the vectors $\mu_1, \ldots, \mu_\ell$ are linearly independent. Let $\gamma \in L \subset T_{f(p)}^\perp M$ be defined as

$$\gamma = \sum_{i=1}^{\ell} \frac{c_i - b_i}{|\mu_i|^2}\mu_i.$$

Then (1.22) yields that $\gamma \neq 0$. Set $U^{d-\ell+1} = \operatorname{span}\{\gamma\} \oplus \hat{L}$. It holds that

$$\left\langle \alpha^f(n,v) + \frac{\gamma}{|\gamma|^2}\langle n, v\rangle, \gamma \right\rangle = 0 \quad \text{and} \quad \left\langle \alpha^f(n,v) + \frac{\gamma}{|\gamma|^2}\langle n, v\rangle, u \right\rangle = 0$$

for all $n \in N(\omega_2)$, $v \in V$ and $u \in \hat{L}$. Then,

$$\nu^c_{d-\ell+1} \geq \dim N\left(\alpha^f_{U^{d-\ell+1}} + \frac{\gamma}{|\gamma|^2}\langle\ ,\ \rangle\right) \geq \dim N(\omega_2) \geq n - 2\,(d-\ell+1).$$

This is in contradiction with the hypothesis on the conformal $(d-\ell+1)$-nullity.

Assertion 4. There exists an orthonormal basis $\gamma_1, \dots, \gamma_d$ of $T^\perp_{f(p)}M$ and a pseudo-orthonormal basis $G, \mu_1, \dots, \mu_{d+1}$ of $T^\perp_{G(p)}M$, with $\langle G, \mu_1\rangle = 1$, $\langle \mu_1, \mu_1\rangle = 0$, such that

$$\alpha^G = -\langle\ ,\ \rangle\mu_1 + \sum_{j=1}^{d}\left\langle \alpha^f, \gamma_j\right\rangle \mu_{j+1}. \tag{1.23}$$

Define

$$\tilde{\beta} = \alpha^f - \left\langle \alpha^G, \xi\right\rangle \xi, \quad \bar{\beta} = \left\langle \alpha^G, \eta\right\rangle \eta + \left(\alpha^G\right)^*.$$

Since β is null, we conclude that

$$\langle \tilde{\beta}(X,Y), \tilde{\beta}(Z,U)\rangle = \langle \bar{\beta}(X,Y), \bar{\beta}(Z,U)\rangle, \quad \forall\, X,Y,Z,U \in T_pM\,,$$

where we have changed the sign of the metric in $T^\perp_{f(p)}M \oplus \operatorname{span}\{\xi\}$. It follows that there exists an orthogonal map

$$\tilde{T}\colon\ T^\perp_{f(p)}M \oplus \operatorname{span}\{\xi\} \to \operatorname{span}\{\eta\} \oplus T^\perp_{g(p)}M$$

such that $\tilde{T}\tilde{\beta} = \bar{\beta}$. Set

$$\tilde{T}\xi = \eta\cos\varphi + \delta_1 \sin\varphi,$$

where $\delta_1 \in \operatorname{span}\{\eta, \tilde{T}\xi\}$ satisfies $|\delta_1| = 1$ and $\langle \delta_1, \eta\rangle = 0$. Let $\gamma_1 \in T^\perp_{f(p)}M$ be chosen so that

$$\tilde{T}\gamma_1 = -\eta\sin\varphi + \delta_1\cos\varphi.$$

We extend γ_1 to an orthonormal basis $\gamma_1, \dots, \gamma_d$ of $T^\perp_{f(p)}M$ and define $\delta_j = \tilde{T}\gamma_j$, $j \geq 2$. If $\tilde{T}\xi = -\eta$, then we take $\gamma_1, \dots, \gamma_d$ as being any orthonormal basis in $T^\perp_{f(p)}M$ and $\delta_j = \tilde{T}\gamma_j$. If $\tilde{T}\xi = \eta$, then the equality $\tilde{T}\tilde{\beta} = \bar{\beta}$ yields that $\left\langle \alpha^G, \eta\right\rangle = -\left\langle \alpha^G, \xi\right\rangle$ which is in contradiction with $\left\langle \alpha^G, \eta + \xi\right\rangle = -\langle\ ,\ \rangle$. We write

$$\tilde{\beta} = -\left\langle \alpha^G, \xi\right\rangle \xi + \left\langle \alpha^f, \gamma_1\right\rangle \gamma_1 + \sum_{j=2}^{d}\left\langle \alpha^f, \gamma_j\right\rangle \gamma_j$$

and

$$\bar{\beta} = \left\langle \alpha^G, \eta\right\rangle \eta + \left\langle \left(\alpha^G\right)^*, \delta_1\right\rangle \delta_1 + \sum_{j=2}^{d}\left\langle \left(\alpha^G\right)^*, \delta_j\right\rangle \delta_j.$$

Thus, $\tilde{T}\tilde{\beta} = \bar{\beta}$ implies that

$$\langle \alpha^G, \eta \rangle = -\langle \alpha^G, \xi \rangle \cos\varphi - \left\langle \alpha^f, \gamma_1 \right\rangle \sin\varphi$$
$$\left\langle \left(\alpha^G\right)^*, \delta_1 \right\rangle = -\langle \alpha^G, \xi \rangle \sin\varphi + \left\langle \alpha^f, \gamma_1 \right\rangle \cos\varphi$$
$$\left\langle \left(\alpha^G\right)^*, \delta_j \right\rangle = \left\langle \alpha^f, \gamma_j \right\rangle, \quad \forall\, j \geq 2.$$

From the first equation, we get

$$-\langle \alpha^G, \xi \rangle \cos\varphi - \left\langle \alpha^f, \gamma_1 \right\rangle \sin\varphi = \langle \alpha^G, \eta \rangle = \langle \alpha^G, \eta + \xi \rangle - \langle \alpha^G, \xi \rangle$$
$$= -\langle\ ,\ \rangle - \langle \alpha^G, \xi \rangle.$$

Hence

$$\langle \alpha^G, \xi \rangle = \frac{1}{\cos\varphi - 1} \left(\langle\ ,\ \rangle - \left\langle \alpha^f, \gamma_1 \right\rangle \sin\varphi \right).$$

Furthermore,

$$\left\langle \left(\alpha^G\right)^*, \delta_1 \right\rangle = -\left\langle \alpha^f, \gamma_1 \right\rangle + \frac{1}{1 - \cos\varphi} \sin\varphi \langle\ ,\ \rangle.$$

We conclude that (1.23) holds for

$$\mu_1 = -\frac{1}{1 - \cos\varphi}(\xi + \eta\cos\varphi + \delta_1 \sin\varphi),$$
$$\mu_2 = \frac{-\sin\varphi}{1 - \cos\varphi}(\xi + \eta) - \delta_1 \quad \text{and} \quad \mu_{j+1} = \delta_j, \quad j \geq 2.$$

Now let $F\colon\ M^n \to \mathbb{V}^{n+d+1}$ be defined by $F = J_\zeta \circ f$, where $\zeta \in \mathbb{V}^{n+d+1}$ is arbitrary. The second fundamental form of F in $\mathbb{L}^{n+d+2}$ is given by

$$\alpha^F = \alpha^f - \langle\ ,\ \rangle\zeta. \tag{1.24}$$

As previously discussed, the proof of Theorem 1.1 will be completed once we show the following fact.

Assertion 5. There exists a smooth vector bundle isometry $\hat{T}\colon\ T_G^\perp M \to T_F^\perp M$ which preserves the second fundamental forms and normal connections.

Set $\xi_1 = \zeta$, $\xi_{j+1} = \gamma_j$, $1 \leq j \leq d$, and define

$$\hat{T}(\mu_j) = \xi_j, \ \ 1 \leq j \leq d+1, \quad \hat{T}(G) = F.$$

Clearly, $\hat{T}$ is isometric. We have that $\hat{T}\alpha^G(X,Y) = \alpha^F(X,Y)$ by (1.23) and (1.24). We claim that $\dim S(\alpha^F) = d+1$. In fact, let L be the orthogonal complement of $S(\alpha^F)$ in $T_F^\perp M = T_{f(p)}^\perp M \oplus \operatorname{span}\{\zeta, F\}$. Due to the fact that the inner product in $T_F^\perp M$ is non-degenerate, it holds that $\dim S(\alpha^F) + \dim L = \dim T_F^\perp M = d+2$. Take $\Gamma = \gamma + a\zeta + bF$ an arbitrary vector in L with $\gamma \in T_{f(p)}^\perp M$. Since $\left\langle \alpha^F, \Gamma \right\rangle = 0$ and $\langle \zeta, F \rangle = 1$ we conclude that

$\langle\alpha^f,\gamma\rangle - b\langle\ ,\ \rangle = 0$. If $\gamma \neq 0$, then we obtain that A^f_γ is umbilical and this contradicts the hypothesis on the conformal 1-nullity of f. Then $\gamma = 0$ and, consequently, $b = 0$. Hence $L = \operatorname{span}\{\zeta\}$ and the claim follows. By a similar argument, we also deduce that $\dim S(\alpha^G) = d+1$ due to (1.23). Thus, we have

$$T^\perp_G M = S(\alpha^G) \oplus \operatorname{span}\{G\} \quad \text{and} \quad T^\perp_F M = S(\alpha^F) \oplus \operatorname{span}\{F\}.$$

These facts easily imply that $\hat{T}$ is smooth. In particular, the vector field μ_1 is smooth, because $\hat{T}(\mu_1) = \zeta$. It remains to be shown that $\hat{T}$ preserves the normal connections. For any vector field $\xi \in T^\perp_G M$, define $\Phi_\xi \colon TM \to T^\perp_F M$ by setting $\Phi_\xi(X) = \hat{T}\left(\nabla^\perp_X \xi\right) - \nabla^\perp_X \hat{T}(\xi)$. It follows easily from the Codazzi equations for F and G that

$$\left\langle \alpha^F(Z,Y), \Phi_\xi(X)\right\rangle = \left\langle \alpha^F(Z,X), \Phi_\xi(Y)\right\rangle, \quad \forall\, X,Y,Z \in T_pM. \tag{1.25}$$

In particular, for $\xi = \mu_1$ this yields that

$$\left\langle \alpha^f(Z,Y), \Phi_{\mu_1}(X)\right\rangle = \left\langle \alpha^f(Z,X), \Phi_{\mu_1}(Y)\right\rangle, \quad \forall\, X,Y,Z \in T_pM,$$

because $\langle \Phi_{\mu_1}(X), \zeta\rangle = \left\langle \nabla^\perp_X \mu_1, \mu_1\right\rangle - \left\langle \nabla^\perp_X \zeta, \zeta\right\rangle = 0$.

We claim that $\Phi_{\mu_1} = 0$, that is, μ_1 is parallel in the normal connection. Suppose otherwise that $\dim(\operatorname{Im}\Phi_{\mu_1}) = r \geq 1$. In this case, we have that

$$\langle \alpha^f(Z,X), \Phi_{\mu_1}(Y)\rangle = 0, \quad \forall\, X \in \ker \Phi_{\mu_1},\ Y,Z \in T_pM.$$

Hence,

$$\nu^c_r(p) \geq \dim\ker\Phi_{\mu_1} = n - r.$$

But this is in contradiction with the hypothesis on the conformal 1-nullity of f when $r = 1$ and with the Proposition 1.1 when $r \geq 2$.

Now, we obtain that $\langle \Phi_\xi(X), \zeta\rangle = \left\langle \nabla^\perp_X \xi, \mu_1\right\rangle - \left\langle \nabla^\perp_X \hat{T}(\xi), \zeta\right\rangle = 0$ for any vector $\xi \in T^\perp_G M$. Hence, (1.24) and (1.25) imply that

$$\left\langle \alpha^f(Z,Y), \Phi_\xi(X)\right\rangle = \left\langle \alpha^f(Z,X), \Phi_\xi(Y)\right\rangle, \quad \forall\, X,Y,Z \in T_pM.$$

Arguing as before we conclude that $\Phi_\xi = 0$. According to observations made previously, Theorem 1.1 has been proved. ∎

Corollary 1.1. *Theorem 1.3 holds for $d = 5$.*

Proof. The form β defined on p. 976 in [**C-D**] always satisfies $\beta(Z,Z) \neq 0$ for all $0 \neq Z \in TM$. Given $X \in RE(\beta)$, since $n > 2d+2$, there exists $Z \neq 0$ such that $Z \in \ker\beta(X)$. Since β is flat, $\beta(Z,Y) \in U(X)$ for all Y and $U(X)$ is isotropic, we deduce that $\beta(Z,Z) \in S(\beta) \cap S(\beta)^\perp$. Thus β admits a decomposition as in the Main Lemma 2.2, p. 967. The inequality on the dimension of $N(\beta_2)$ follows from Sublemma 1.2 in this paper because $\dim W_2 \leq 10$ for $d = 5$. The remainder of the proof is identical.

Remark 1.1. In Theorem 1.1, the hypothesis $\tau_f^c(p) \geq 3$ implies that $n \geq 3d$. For $d = 1$ and $n < 2d + 3$, we have that either $n = 3$ or $n = 4$. An immersion of M^3 into $\mathbb{R}^4$ always satisfies $\nu_1^c \geq 1$. Cartan ([**Ca2**]) gave examples of immersions M^4 into $\mathbb{R}^5$ which have four distinct principal curvatures at each point, hence $\tau_f^c = 4$ and $\nu_1^c = 1$, which are not conformally rigid. For $d = 2$ and $n < 2d + 3$, we have that $n = 6$ and it is not known whether an immersion of M^6 into $\mathbb{R}^8$ with $\tau_f^c = 3$, $\nu_1^c \leq 3$, and $\nu_2^c \leq 1$ is conformally rigid.

Remark 1.2. The beginning of the argument to prove that the form β_2 is zero in Assertion 3.3 of [**C-D**] says that there exist orthonormal bases of $T_{f(p)}^{\perp}M$ and $T_{g(p)}^{\perp}M$ with some special properties which, in fact, may not be satisfied under the conditions there. The argument in Assertion 3 to prove that the form ω_2 is zero corrects the one in [**C-D**].

To finish this section, we point out that in Theorem 1.1 the requirement on $\nu_s^c(p)$ can not be dropped. First, we prove that any product of spheres is conformally deformable. Fix positive integers $d \geq 2$ and $k_i, 1 \leq i \leq d$, and arbitrary positive real numbers $\lambda_i, 1 \leq i \leq d$. Denote by $\mathbb{S}_{\lambda_i}^{k_i}$ the k_i-dimensional sphere centered in the origin and radius λ_i. Let M^n, $n = \sum_{i=1}^{d} k_i$, be the riemannian product with factors $\mathbb{S}_{\lambda_i}^{k_i}$, $1 \leq i \leq d$. Define the product immersion $f\colon\ M^n \to \mathbb{R}^{n+d}$ by setting $f = I_1 \times I_2 \times \cdots \times I_d$ where $I_i\colon\ \mathbb{S}_{\lambda_i}^{k_i} \to \mathbb{R}^{k_i+1}$ is the inclusion. We have that $f(M) \subset \mathbb{S}_{\lambda}^{n+d-1}$ where $\lambda = \sqrt{\sum_{i=1}^{d} \lambda_i^2}$. The immersion f is conformally deformable, that is, it is not conformally rigid. In fact, the induced metric on M by f is the same one induced by $\mathbb{S}_{\lambda}^{n+d-1}$, namely, the natural product metric. Identify $\mathbb{R}^{n+d-1}$ with the hyperplane of $\mathbb{R}^{n+d}$ whose points $x = (x_1, x_2, \ldots, x_{n+d})$ satisfy $x_{n+d} = 0$. Let $\Pi\colon\ \mathbb{S}_{\lambda}^{n+d-1} - \{p_\lambda\} \to \mathbb{R}^{n+d-1}$ be the stereographic projection raised from the point $p_\lambda = (0, \ldots, 0, \lambda)$ which is given by

$$\Pi(p) = p_\lambda + \frac{\lambda}{\lambda - p_{n+d}}(p - p_\lambda) \quad \text{where} \quad p = (p_1, p_2, \ldots, p_{n+d}).$$

The map Π is a conformal diffeomorphism with conformal factor $\frac{\lambda}{\lambda - p_{n+d}}$, that is,

$$|\,(\Pi_*)_p\, v|^2 = \frac{\lambda^2}{(\lambda - p_{n+d})^2}|v|^2$$

for any $p \in \mathbb{S}_{\lambda}^{n+d-1} - \{p_\lambda\}$ and $v \in T_p\left(\mathbb{S}_{\lambda}^{n+d-1} - \{p_\lambda\}\right)$. Observe also that $p_\lambda \notin M^n$ since $d \geq 2$. Let $\varphi\colon\ M^n \to \mathbb{R}^{n+d-1} \subset \mathbb{R}^{n+d}$ be the restriction of Π to M^n. Consider also the isometric immersion $\phi\colon\ \mathbb{R}^{n+d-1} \to \mathbb{S}_1^1 \times \mathbb{R}^{n+d-2} \subset \mathbb{R}^{n+d}$ defined by

$$\phi(x_1, x_2, \ldots, x_{n+d-1}, 0) = (\cos x_1, \sin x_1, x_2, \ldots, x_{n+d-1}).$$

Define $g\colon\ M^n \to \mathbb{R}^{n+d}$ as $g = \phi \circ \varphi$. It follows easily that f and g are conformal. Recall that the inversion with respect to the unit sphere centered at p_o is the conformal transformation $I_{p_o}(q) = p_o + \left[(q-p_o)/|q-p_o|^2\right]$, $q \in \mathbb{R}^{n+d} - \{p_o\}$, an isometry of $\mathbb{R}^{n+d}$ is a map $\Im$ such that $\Im(q) = O(q) + w$, where O is an orthogonal map of $\mathbb{R}^{n+d}$ and w is a fixed vector in $\mathbb{R}^{n+d}$, and a dilatation D_α is a transformation of $\mathbb{R}^{n+d}$ such that $D_\alpha(q) = \alpha\, q$ for some positive real constant α. Recall further that by Liouville's theorem (see [**dC**]) every conformal diffeomorphism Υ from an open subset U of $\mathbb{R}^{n+d}$ to an open subset V of $\mathbb{R}^{n+d}$ is the restriction to U of a composition of inversions, dilatations and isometries, at most one of each. We claim that there is not a such conformal diffeomorphism Υ satisfying $g = \Upsilon \circ f$. In fact, it is not difficult to see that there exist one inversion I_{p_o}, one dilatation D_α and one isometry $\Im$ such that Υ is the restriction to U either of the composition $\Im \circ D_\alpha \circ I_{p_o}$ or the composition $\Im \circ D_\alpha$. Now suppose on the contrary that $g = \Upsilon \circ f$ for some conformal diffeomorphism Υ. We have to analyze two cases.

Case i). The conformal map Υ is the restriction to U of the composition $\Im \circ D_\alpha \circ I_{p_o}$. In this case, extending Υ if necessary we can assume that $U = \mathbb{R}^{n+d} - \{p_o\}$. The conformal factor β of Υ is $\beta(q) = \alpha/|q-p_o|^2$. Since $g = \Upsilon \circ f$ on M^n we have that

$$(\phi_*)_{\varphi(p)}(\varphi_*)_p v = (g_*)_p v = (\Upsilon_*)_p v$$

and, consequently,

$$\frac{\lambda^2}{(\lambda - p_{n+d})^2}|v|^2 = |(g_*)_p v|^2 = |(\Upsilon_*)_p v|^2 = \beta(p)^2 |v|^2$$

for all $v \in T_pM$. This yields that

$$\frac{\lambda}{\lambda - p_{n+d}} = \beta(p) = \frac{\alpha}{|p-p_o|^2} \tag{1.26}$$

for all $p \in M^n$. Therefore,

$$\lambda\, |\gamma(t) - p_o|^2 = \alpha\,(\lambda - \gamma_{n+d}(t))$$

for any curve $\gamma(t)$ in M^n with $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+d}(t))$. Taking derivatives in the last equality, we have that

$$2\,\lambda\,\langle \gamma^{'}(t), \gamma(t) - p_o\rangle = -\alpha\, \gamma^{'}_{n+d}(t). \tag{1.27}$$

The vector $\gamma(t)$ is orthogonal to $\gamma^{'}(t)$ since $\gamma(t) \in \mathbb{S}^{n+d-1}_\lambda$ for all t. Then, we obtain that

$$\langle \gamma^{'}(t), 2\,\lambda p_o - \alpha\, e_{n+d}\rangle = 0$$

from (1.27) where $e_{n+d} = (0, \dots, 0, 1)$. So the vector $\gamma^{'}(t)$ belongs to the hyperplane through the origin and orthogonal to the vector $2\,\lambda p_o - \alpha\, e_{n+d}$, for any curve $\gamma(t)$ in M^n. Since the vectors tangent to M^n span $\mathbb{R}^{n+d}$ we have that $2\,\lambda p_o - \alpha\, e_{n+d} = 0$. Thus, $p_{oi} = 0, 1 \leq i \leq n+d-1$, and $\alpha = 2\,\lambda p_{o(n+d)}$. At a point such that $p_{n+d} = 0$ we deduce that $\alpha = 2\,\lambda^2$ from (1.26). Consequently, $p_o = p_\lambda$. Hence,

$$\Upsilon(q) = \left(\Im \circ D_\alpha \circ I_{p_o}\right)(q) = 2\,\lambda^2 O(p_\lambda) + \frac{2\lambda^2}{|\, q - p_\lambda\,|^2} O(q - p_\lambda) + w$$

for all $q \in \mathbb{R}^{n+d} - \{p_\lambda\}$. The equality $\Upsilon(p) - \Upsilon(-p) = g(p) - g(-p)$ on M^n implies that

$$2\,O(p) = \phi(p) - \phi(-p)$$

at a point such that $p_{n+d} = 0$ since the restriction to $M^n \cap \mathbb{R}^{n+d-1}$ of φ is the identity. Taking length in the last equality, we conclude that $\sin^2 p_1 = p_1^2$ if $p = (p_1, \ldots, p_{n+d-1}, 0)$. Choosing a point such that $p_1 \neq 0$ we have obtained a contradiction.

Case ii). The conformal map Υ is the restriction to U of $\Im \circ D_\alpha$. In this case, we obtain that $\lambda/(\lambda - p_{n+d}) = \alpha$ for all $p \in M^n$ by (1.26). Consequently, p_{n+d} is constant through M^n. But this is a contradiction.

Now we compute ν_s^c for $f = I \times I \times \cdots \times I$, where $I\colon\ \mathbb{S}_1^r \to \mathbb{R}^{r+1}$ is the inclusion. In this case, $M = \mathbb{S}_1^r \times \mathbb{S}_1^r \times \cdots \times \mathbb{S}_1^r$ and $M = f(M) \subset \mathbb{S}^{n+d-1}_{\sqrt{d}}$ with $n = r\,d$. Given $p = (p_1, p_2, \ldots, p_d) \in M^n$ and s, $1 \leq s \leq d$, consider at $T^\perp_{f(p)}M$ the points

$$q_1 = p,\ q_i = (0, \ldots, 0, p_i, p_{i+1}, \ldots, p_d),\ \ 2 \leq i \leq s,$$

and U^s defined as $U^s = \operatorname{span}\{\, q_1, \ldots, q_s\,\}$. It is not difficult to see that there exists $\eta \in U^s$ such that $\dim N\left(\alpha^f_{U^s} - \langle\ ,\ \rangle\eta\right) = n - r(s-1)$. Fix an orthonormal basis E_i^k, $1 \leq i \leq r$, on the tangent space of $\mathbb{S}_1^r$ at p for each k, $1 \leq k \leq d$. The vectors $E_i = (E_i^1, E_i^2, \ldots, E_i^d)$, $1 \leq i \leq r$, in $T^\perp_{f(p)}M$ are linearly independent. The normal space $T^\perp_{f(p)}M$ is spanned by p_k, $1 \leq k \leq d$. The second fundamental form of f satisfies $A_{p_k}E_i = (0, \ldots, 0, -E_i^k, 0, \ldots, 0)$, $1 \leq i \leq r$, $1 \leq k \leq d$. This show that $\tau_f^c(p) = r$. Consequently, we conclude that $\nu_s^c(p) = n - (s-1)r$ by Proposition 1.1. Observe that always $\nu_1^c = n$. If we take $r = 3$ and $d \geq 3$, we obtain that $\nu_s^c(p) > n - 2s - 1$, $1 \leq s \leq 3$. This fact shows that the assumption on $\nu_s^c(p)$ is necessary.

2. Proof of Theorem 1.2.

First, we recall from [**C-D**] the following concept.

Definition 2.1. Given s, $1 \leq s \leq d$, the *s-nullity* of an isometric immersion $f\colon\ M^n \to \tilde{M}^{n+d}$ at $p \in M^n$ is the integer

$$\nu_s(p) = \max_{U^s \subseteq T^\perp_{f(p)}M} \left\{\dim N\left(\alpha^f_{U^s}\right)\right\}.$$

Lemma 2.1. *If $\tau_f^k(p) \geq r$, then the dimension of the tangent subspace $L(U)$ defined as $L(U) = \operatorname{span}\{\, A_\mu X \,:\, \mu \in U^\ell,\ X \in T_pM \,\}$ is at least $(k+\ell-d)r$ for any ℓ-dimensional subspace $U^\ell \subset T^\perp_{f(p)}M$, $\ell \geq 1$.*

Proof. Take $\xi_1, \dots, \xi_k \in T^\perp_{f(p)}M$ and let $X_1, \dots, X_r \in T_pM$ be such that the vectors $A_{\xi_i}X_j$, $1 \leq i \leq k$, $1 \leq j \leq r$, are linearly independent. Recall that the vectors $\xi_1, \dots, \xi_k$, $X_1, \dots, X_r$ are linearly independent.

We claim that the vectors $A_{\gamma_i}X_j$, $1 \leq i \leq k'$, $1 \leq j \leq r$, are linearly independent when we take $\gamma_1, \dots, \gamma_{k'} \in \operatorname{span}\{\, \xi_1, \dots, \xi_k \,\}$, $k' \leq k$, linearly independent. In fact, consider a basis $\gamma_1, \dots, \gamma_k$ of $\operatorname{span}\{\, \xi_1, \dots, \xi_k \,\}$ which extends $\gamma_1, \dots, \gamma_{k'}$. Define the $k \times k$-matrix $B = (b_{ij})$ by setting $\xi_i = \sum_{h=1}^k b_{hi}\gamma_h$. It is not difficult to see that

$$\sum_{i,j=1}^{k,r} a_{ij} A_{\xi_i} X_j = \sum_{i,j=1}^{k,r} c_{ij} A_{\gamma_i} X_j$$

for arbitrary real numbers a_{ij}, $1 \leq i \leq k$, $1 \leq j \leq r$, being $C = (c_{ij})$ given by $C = BA$ with $A = (a_{ij})$. Thus, the vectors $A_{\xi_i}X_j$ and $A_{\gamma_i}X_j$ span the same subspace and the claim follows.

We can assume that $(k+\ell-d) \geq 1$. Otherwise, Lemma 2.1 is immediate. The subspace $\overline{L} = U \cap \operatorname{span}\{\, \xi_1, \dots, \xi_k \,\}$ satisfies $\dim \overline{L} \geq (k+\ell-d)$ by (1.13). If $\gamma_1, \dots, \gamma_{(k+\ell-d)}$ are vectors linearly independent in $\overline{L}$, then the $(k+\ell-d)r$ vectors $A_{\gamma_i}X_j \in L(U)$,$1 \leq i \leq (k+\ell-d)$, $1 \leq j \leq r$, are linearly independent from the claim. Hence, Lemma 2.1 has been proved.

Proposition 2.1. *Let $f\colon\ M^n \to \tilde{M}^{n+d}$ be an isometric immersion. If $\tau_f^k(p) \geq r$, then $\nu_s(p) \leq n-(k+s-d)r$ for $1 \leq s \leq d$.*

Proof. If $k+s-d \leq 0$ then the conclusion is immediate. Assume that $k+s-d \geq 1$. Let $U^s \subseteq T^\perp_{f(p)}M$ be such that $\nu_s(p) = \dim N\left(\alpha^f_{U^s}\right)$. We have that $\langle A_\mu X, v\rangle = 0$ for all $\mu \in U^s$, $v \in N\left(\alpha^f_{U^s}\right)$ and $X \in T_pM$, that is, $L(U) \subseteq N\left(\alpha^f_{U^s}\right)^\perp$. Therefore, using Lemma 2.1, we deduce that

$$n - \nu_s(p) = \dim N\left(\alpha^f_{U^s}\right)^\perp \geq \dim L(U) \geq (k+s-d)r,$$

and Proposition 2.1 follows.

Proof of Theorem 1.2. We only have to deal with the case of codimension $d \geq 6$ since the proof follows from a result in [**C-D**] for $d \leq 5$. If $\tau_f^{d-1}(p) \geq 3$, then $n \geq 3d-3$. We can assume that $n \geq 2d+1$. Otherwise, we obtain that $2 \leq d \leq 3$. For $d = 2$ and $d = 3$, we conclude that $n = 3, 4$ and $n = 6$, respectively. But both cases are not possible because we are assuming

$\nu_2 \leq n-5$ and $\nu_3 \leq n-7$. It follows from Proposition 2.1 for $k = d-1$ that $\nu_s(p) \leq n - 2s - 1$ when $s \geq 4$. Hence, their result applies for $d \leq 5$.

Let $g\colon M^n \to \mathbb{R}^{n+d}$ be another immersion isometric to f. Given $p \in M$, let

$$W = T^{\perp}_{f(p)}M \oplus T^{\perp}_{g(p)}M$$

be endowed with the natural metric of type (d,d) who is negative definite in $T^{\perp}_{f(p)}M$, and define the symmetric bilinear form $\beta\colon TM \times TM \to W$ by

$$\beta = \alpha^f + \alpha^g.$$

The Gauss equations for f and g imply that β is flat. Since $\tau_f^{d-1}(p) \geq 3$, we can fix vectors $\xi_1, \dots, \xi_{d-1}$ of $T^{\perp}_{f(p)}M$ and $X_1, X_2, X_3 \in V$ so that the vectors $A_{\xi_i}X_j$, $1 \leq i \leq d-1$, $1 \leq j \leq 3$, are linearly independent. To see that β is null we proceed exactly as in Lemma 1.1 with the $(n-3d+3)$-dimensional subspace

$$\tilde{L} = (\operatorname{span}\{\, A_{\xi_i}X_j\,,\ 1 \leq i \leq d-1\,,\ 1 \leq j \leq 3\,\})^{\perp},$$

with d instead of $d+1$ and Proposition 2.1 instead of Proposition 1.1.

The remainder of the proof is part of the argument for Theorem 1.4 in [**C-D**]. □

Remark 2.1. The hypothesis in Theorem 1.2, as mentioned in the introduction, are less restrictive than those in Allendoerfer's theorem. In fact, the assumption $\tau_f^d(p) \geq 3$ implies that $\tau_f^{d-1}(p) \geq 3$ and that $\nu_s(p) \leq n-3s$ for $1 \leq s \leq d$; c.f. Proposition 4.6 in [**Da**].

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