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We give necessary and sufficient conditions for simplicity of Cuntz-Krieger algebras corresponding to infinite 0–1 matrices and of C^* -algebras corresponding to countable directed graphs. We show that simple algebras within these two classes are either purely infinite or AF .

0. Since their invention about twenty years ago Cuntz-Krieger algebras \mathcal{O}_A corresponding to finite, square 0–1 matrices [3] have attracted immense interest. It was remarked in their original paper by Cuntz and Krieger that the theory may be extended to infinite matrices as well. Unfortunately, no details were provided at that time. This quite non-trivial task has been successfully carried out recently by Exel and Laca [5], by means of some heavy-duty machinery. It turns out that the resulting class of C^* -algebras is rich enough to encompass, at least up to Morita equivalence, all graph C^* -algebras [10, 9, 1, 11], as shown in [6], as well as AF -algebras, as shown in [9, 4]. Despite a lot of interest in the subject and a number of well-aimed attempts the fundamental question of simplicity of these algebras had not been settled. Only partial results in this direction, covering various special cases, have been obtained in [10, 9, 5, 8, 6, 7, 1]. In this article we provide an easy to check necessary and sufficient condition for simplicity of Cuntz-Krieger algebras built on infinite matrices. This in turn implies an analogous criterion for graph algebras. Then, following the ideas of [9], we show that all simple Cuntz-Krieger algebras are either purely infinite or AF . This final result gives a good promise of the possibility of classification of these interesting C^* -algebras.

1. Let N be a countable non-empty set and let $A = [A(i, j)]_{i, j \in N}$ be a matrix with entries in $\{0, 1\}$ whose no row is identically zero (which we always assume in what follows). Ruy Exel and Marcelo Laca define in [5, Theorem 8.6] a Cuntz-Krieger algebra \mathcal{O}_A corresponding to A as the universal C^* -algebra generated by a family of partial isometries $\{S_i \mid i \in N\}$, subject to the following relations:

- [EL1] $S_i^* S_i$ and $S_j^* S_j$ commute for all $i, j \in N$,
- [EL2] $S_i^* S_j = 0$ for all $i \neq j$ in N ,
- [EL3] $(S_i^* S_i) S_j = A(i, j) S_j$ for all $i, j \in N$,

[EL4] for all finite subsets X, Y of N such that

$$A(X, Y, j) \stackrel{\text{def}}{=} \prod_{x \in X} A(x, j) \prod_{y \in Y} (I - A(y, j))$$

is non-zero for all but finitely many j 's we have

$$\prod_{x \in X} S_x^* S_x \prod_{y \in Y} (I - S_y^* S_y) = \sum_{j \in N} A(X, Y, j) S_j S_j^*.$$

I in the formulae above is the identity of the multiplier algebra. This definition generalizes the one given for finite matrices by Cuntz and Krieger [3]. For X, Y finite subsets of N we denote

$$P_{X,Y} = \prod_{x \in X} S_x^* S_x \prod_{y \in Y} (I - S_y^* S_y).$$

2. Following [5, Definition 10.5] we associate with A a directed graph $E_A = (E_A^0, E_A^1, s, r)$ (cf. [10, 9]), defined as follows: $E_A^0 = N$,

$$E_A^1 = \{(i, j) \in N \times N \mid A(i, j) = 1\},$$

with $s(i, j) = i$ and $r(i, j) = j$. For $\alpha = (\alpha_1, \dots, \alpha_r) \in E_A^*$ we write $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_r}$. One can show that

$$\mathcal{O}_A = \overline{\text{span}}\{S_\alpha P_{X,\emptyset} S_\beta^* \mid \alpha, \beta \in E_A^*, X \subseteq N \text{ finite}\}.$$

Here we allow α or β to be empty, i.e., $S_\alpha = I$ or $S_\beta = I$.

3. Universality of \mathcal{O}_A implies existence of the canonical gauge action $\gamma : \mathbf{T} \rightarrow \text{Aut}(\mathcal{O}_A)$, defined on the generators by $\gamma_t(S_i) = tS_i$, $t \in \mathbf{T}$, $i \in N$. We denote by Γ the corresponding faithful conditional expectation of \mathcal{O}_A onto the fixed point algebra \mathcal{O}_A^γ , given by integration over \mathbf{T} with respect to the normalized Haar measure

$$\Gamma(x) = \int_{t \in \mathbf{T}} \gamma_t(x) dt.$$

It is not difficult to verify that for all $\alpha, \beta \in E_A^*$ and finite $X \subseteq N$ we have

$$\Gamma(S_\alpha P_{X,\emptyset} S_\beta^*) = \delta_{|\alpha|, |\beta|} S_\alpha P_{X,\emptyset} S_\beta^*$$

with δ the Kronecker symbol and $|\alpha|$ the length of α .

4. In order to give a necessary and sufficient condition for simplicity of \mathcal{O}_A we need the concept of a hereditary and saturated subset of the index set N of A , defined as follows. For $j \in N$ we define $H_0(j)$ as the union of $\{j\}$ and the collection of all those $i \in N$ for which there exists a path in E_A from j to i . Then, by induction, we define $H_{n+1}(j)$ as the union of $H_n(j)$ and the collection of all those $i \in N$ for which there exists a finite $K \subseteq H_n(j)$ such that $A(i, t) \leq \max_{k \in K} A(k, t)$ for all $t \in N \setminus K$. It follows from conditions

[EL1–EL4] that this is equivalent to $S_i^* S_i \leq \sum_{k \in K} (S_k^* S_k + S_k S_k^*)$. Finally, we set $H(j) = \bigcup_{n=0}^\infty H_n(j)$ and call it the hereditary and saturated subset of N generated by j .

Before giving our main result, Theorem 8 below, we need three lemmas. A tedious but routine proof of Lemma 5 is omitted, while the proof of Lemma 7 is modelled after [9, Section 2].

Lemma 5. *If $A = [A(i, j)]_{i, j \in N}$ is a 0–1 matrix with no zero rows, $k \in N$, and J is the closed 2-sided ideal of \mathcal{O}_A generated by S_k , then*

$$J = \overline{\text{span}}\{S_\alpha P_{X, \emptyset} S_\beta^* \mid (\exists t \in H_0(k)) P_{X, \emptyset} \leq S_t^* S_t\}.$$

Lemma 6. *If $A = [A(i, j)]_{i, j \in N}$ is a 0–1 matrix with no zero rows, $i, j \in N$, and J is the closed 2-sided ideal of \mathcal{O}_A generated by S_j , then $S_i \in J$ if and only if $i \in H(j)$.*

Proof. It follows immediately from the inductive definition of $H(j)$ that $i \in H(j)$ implies $S_i \in J$. Conversely, suppose that $S_i \in J$ and, by way of contradiction, that $i \notin H(j)$. By Lemma 5 there exist $\alpha_k = (\alpha_k^1, \dots, \alpha_k^{a_k})$, $\beta_k = (\beta_k^1, \dots, \beta_k^{b_k}) \in E_A^*$, $\lambda_k \in \mathcal{C}$, and finite $X_k \subseteq N$ for $k = 1, \dots, n$, such that

$$\left\| \sum_{k=1}^n \lambda_k S_{\alpha_k} P_{X_k, \emptyset} S_{\beta_k}^* - S_i S_i^* \right\| < 1,$$

$\alpha_k^1 = \beta_k^1 = i$, and $(\exists t \in H_0(j)) P_{X_k, \emptyset} \leq S_t^* S_t$ for all $k = 1, \dots, n$. Applying the conditional expectation Γ we may assume that $a_k = b_k$ for $k = 1, \dots, n$. Indeed,

$$\begin{aligned} & \left\| \sum_{a_k=b_k} \lambda_k S_{\alpha_k} P_{X_k, \emptyset} S_{\beta_k}^* - S_i S_i^* \right\| \\ &= \left\| \Gamma \left(\sum_{k=1}^n \lambda_k S_{\alpha_k} P_{X_k, \emptyset} S_{\beta_k}^* - S_i S_i^* \right) \right\| \\ &\leq \left\| \sum_{k=1}^n \lambda_k S_{\alpha_k} P_{X_k, \emptyset} S_{\beta_k}^* - S_i S_i^* \right\| < 1. \end{aligned}$$

Let $Q = I - \bigvee_{k=1}^n S_{\alpha_k} S_{\alpha_k}^*$. Since

$$\begin{aligned} \left\| Q \left(\sum_{a_k=1} P_{X_k, \emptyset} - S_i^* S_i \right) Q \right\| &= \left\| Q S_i^* \left(\sum_{k=1}^n \lambda_k S_{\alpha_k} P_{X_k, \emptyset} S_{\beta_k}^* - S_i S_i^* \right) S_i Q \right\| \\ &< 1 \end{aligned}$$

it follows that

$$S_i^* S_i \leq \sum_{k=1}^n (S_{\alpha_k^2} S_{\alpha_k^2}^* + P_{X_k, \emptyset}).$$

Since $i \notin H(j)$ there is a $t \in \{1, \dots, n\}$ such that $\alpha_t^2 \notin H(j)$. We have

$$\begin{aligned} & \left\| \sum_{\alpha_k^2 = \alpha_t^2 = \beta_k^2} \lambda_k S_{\alpha_k^2} \cdots S_{\alpha_k^{a_k}} P_{X_k, \emptyset} S_{\beta_k^{b_k}}^* \cdots S_{\beta_k^2}^* - S_{\alpha_t^2} S_{\alpha_t^2}^* \right\| \\ &= \left\| S_{\alpha_t^2} S_{\alpha_t^2}^* \left(\sum_{k=1}^n \lambda_k S_{\alpha_k^2} \cdots S_{\alpha_k^{a_k}} P_{X_k, \emptyset} S_{\beta_k^{b_k}}^* \cdots S_{\beta_k^2}^* - S_i^* S_i \right) S_{\alpha_t^2} S_{\alpha_t^2}^* \right\| \\ &\leq \left\| \sum_{k=1}^n \lambda_k S_{\alpha_k^2} \cdots S_{\alpha_k^{a_k}} P_{X_k, \emptyset} S_{\beta_k^{b_k}}^* \cdots S_{\beta_k^2}^* - S_i^* S_i \right\| \\ &= \left\| \sum_{k=1}^n \lambda_k S_{\alpha_k} P_{X_k, \emptyset} S_{\beta_k}^* - S_i S_i^* \right\| < 1. \end{aligned}$$

Thus, we can repeat the same argument again with i replaced by α_t^2 . Continuing inductively in the like manner we conclude that there exists an $m \in \{1, \dots, n\}$ such that $\alpha_m^{a_m} \notin H(j)$ but $S_{\alpha_m^{a_m}}^* S_{\alpha_m^{a_m}} \leq \sum_{k=1}^n P_{X_k, \emptyset}$. This contradiction completes the proof. \square

Lemma 7. *If $A = [A(i, j)]_{i, j \in N}$ is a 0 – 1 matrix with no zero rows and E_A has a loop with no exits then \mathcal{O}_A contains a closed 2-sided ideal, Morita equivalent to $C(\mathbf{T})$.*

Proof. Let $i_1, \dots, i_n \in N$ (and we also write $i_{n+1} = i_1$) be such that $L = ((i_1, i_2), (i_2, i_3), \dots, (i_n, i_1))$ is a simple loop without exits in E_A , and let J be a closed 2-sided ideal of \mathcal{O}_A generated by S_{i_1}, \dots, S_{i_n} . Since L has no exits, we have $A(i_m, j) = \delta_{j, i_{m+1}}$ for all $j \in N$, $m = 1, \dots, n$ and, hence, [EL4] implies that $S_{i_m}^* S_{i_m} = S_{i_{m+1}} S_{i_{m+1}}^*$. From this and Lemma 5 we conclude that

$$J = \overline{\text{span}}\{S_{\alpha} S_{\beta}^* \mid r(\alpha) = r(\beta) \in \{i_1, \dots, i_n\}\}.$$

We define K , a closed right ideal of \mathcal{O}_A , as

$$K = \overline{\text{span}}\{S_{i_m} S_{\beta}^* \mid m \in \{1, \dots, n\}, r(\beta) = i_{m+1}\}.$$

Since $KK^* = C^*(S_{i_1}, \dots, S_{i_n})$ and $K^*K = J$ we see that $C^*(S_{i_1}, \dots, S_{i_n})$ and J are Morita equivalent [12]. By [11, Lemma 2.2] $C^*(S_{i_1}, \dots, S_{i_n})$ is isomorphic to the C^* -algebra of a directed graph F which consists of a single loop with n vertices. An easy argument as in the proof of [9, Theorem 2.4] shows that $C^*(F)$ is isomorphic to $M_n \otimes C(\mathbf{T})$. Consequently, J is Morita equivalent to $C(\mathbf{T})$, as required. \square

Theorem 8. *If $A = [A(i, j)]_{i, j \in N}$ is a 0–1 matrix with no zero rows then \mathcal{O}_A is simple if and only if $H(k) = N$ for all $k \in N$ and all loops in E_A have exits.*

Proof. Sufficiency. Let $J \neq 0$ be a closed 2-sided ideal in \mathcal{O}_A . Since all loops in E_A have exits, [5, Theorem 13.1] implies that there exists a $k \in N$ such that $S_k \in J$. Since $H(k) = N$, by hypothesis, it follows from Lemma 6 that $S_i \in J$ for all $i \in N$ and, hence, $J = \mathcal{O}_A$, as required.

Necessity. Suppose that \mathcal{O}_A is simple. Then Lemma 7 implies that all loops in E_A have exits, and Lemma 6 implies that $H(k) = N$ for all $k \in N$, as required. \square

9. The above theorem generalizes a number of earlier results, most notably the Cuntz and Krieger simplicity criterion [3, Theorem 2.14]. For infinite matrices Exel and Laca show in [5, Theorem 14.1] that \mathcal{O}_A is simple if E_A is transitive. This is of course a very special case of one direction of our Theorem 8. Furthermore, it is shown in [11, Example 2.4] that the crossed product C^* -algebras corresponding to free products of cyclic groups Γ and trivial subgroups Γ_Λ , as considered by Zhang and the author [13], can be described as Cuntz-Krieger algebras with suitable infinite 0–1 matrices. Thus, for such crossed products [13, Theorem 3.1] is a consequence of our present result.

10. It is shown in [6, Theorem 10] that up to Morita equivalence all graph C^* -algebras may be realized as Cuntz-Krieger algebras in the sense of Exel and Laca. Thus, Theorem 8 implies a simplicity criterion for this important and widely investigated class of C^* -algebras. For reader's convenience we recall the definition of the C^* -algebra $C^*(E)$ of a directed graph E , as given in [6]. For explanations of basic terminology related to directed graphs and their path spaces we refer the reader to one of [9, 1, 11].

Let $E = (E^0, E^1, s, r)$ be a countable directed graph, with E^0 the set of vertices, E^1 the set of edges, and $s, r : E^1 \rightarrow E^0$ the source and range functions, respectively. $C^*(E)$ is the universal C^* -algebra generated by pairwise orthogonal projections $\{p_v \mid v \in E^0\}$ and partial isometries $\{s_e \mid e \in E^1\}$ with pairwise orthogonal ranges, subject to the following relations:

$$[\text{G1}] \quad s_e^* s_e = p_{r(e)},$$

$$[\text{G2}] \quad s_e s_e^* \leq p_{s(e)},$$

$$[\text{G3}] \quad p_v = \sum_{s(e)=v} s_e s_e^* \text{ if } s^{-1}(v) \text{ is finite and non-empty.}$$

11. In order to formulate our simplicity criterion for graph algebras we must recall the definition of a hereditary and saturated subset of E^0 (e.g., cf. [1]). For $v \in E^0$ we define $H_0(v)$ as the union of $\{v\}$ and all those vertices $w \in E^0$ for which there is a path from v to w . And then, inductively, $H_{n+1}(v)$ as the union of $H_n(v)$ and all those vertices $w \in E^0$ such that

$s^{-1}(w)$ is finite and non-empty and $r(e) \in H_n(v)$ for all $e \in s^{-1}(w)$. Finally, we set $H(v) = \bigcup_{n=0}^{\infty} H_n(v)$ and call it the hereditary and saturated subset of E^0 generated by v .

Theorem 12. *If E is a countable directed graph then $C^*(E)$ is simple if and only if $H(v) = E^0$ for all $v \in E^0$ and all loops in E have exits.*

Proof. At first we assume that E contains no sinks or sources. We consider the edge matrix A_E of E (cf. [10]) with rows and columns indexed by E^1 and such that $A_E(e, f) = \delta_{r(e), s(f)}$. It is shown in [6, Theorem 10] that $\{s_e \mid e \in E^1\}$ generate $C^*(E)$, satisfy the relations [EL1]-[EL4] for \mathcal{O}_{A_E} , and that identity gives an isomorphism between $C^*(E)$ and \mathcal{O}_{A_E} . For all $e \in E^1$ we have $s_e^*s_e = p_{r(e)}$ by [G1] and all vertices E^0 occur as ranges of the edges E^1 . It is not difficult to verify that $H(e) = E^1$ for A_E , in the sense of Section 4, if and only if $H(r(e)) = E^0$ for E , in the sense of Section 11. Furthermore, E_{A_E} is nothing but the dual graph \hat{E} (cf. [1, Corollary 2.5]) and, hence, all loops in E have exits if and only if the same holds true for E_{A_E} . Consequently, for graphs without sinks or sources Theorem 8 implies Theorem 12.

Let now E be an arbitrary countable directed graph and let E' be the graph obtained from E by adding tails and heads to sinks and sources, respectively, as described in [1, Section 1], [6], or [11, Section 1]. The algebras $C^*(E)$ and $C^*(E')$ are Morita equivalent and, obviously, E satisfies the two conditions of our theorem if and only if E' does. Therefore the general case follows from the previously considered one. \square

13. The above theorem finally achieves a previously elusive goal, pursued by various authors with help of quite powerful and sophisticated tools. For example, with essential help of the groupoid machinery of Jean Renault an analogous simplicity criterion was given in [10, Corollary 6.8] for locally finite graphs satisfying a rather restrictive condition (K) . For locally finite graphs without sinks or sources and satisfying a restrictive condition $(I)'$ a similar result was obtained in [8, Theorem 18] by means of the Pimsner bimodule approach. For arbitrary row-finite graphs a simplicity criterion was given in [1, Proposition 5.1] by more elementary methods. One should also note that in view of [4, Theorem 1], which says that any AF -algebra is Morita equivalent to a graph algebra, Theorem 12 also includes Bratteli's characterization of simplicity of AF -algebras [2, Corollary 3.5] as a very special case.

It is shown in [11] that for an arbitrary 0–1 matrix $A = [A(i, j)]_{i, j \in \mathbb{N}}$ there is a natural imbedding $\phi_A : C^*(E_A) \rightarrow \mathcal{O}_A$ such that $S_i S_i^* \in \text{im} \phi_A$ for all $i \in \mathbb{N}$. The precise relationship between these two algebras is not yet clear, though the following proposition may shed some light on this problem.

Proposition 14. *If $A = [A(i, j)]_{i, j \in \mathbb{N}}$ is a 0 – 1 matrix with no zero rows and $C^*(E_A)$ is simple then so is \mathcal{O}_A .*

Proof. Suppose that $C^*(E_A)$ is simple and $J \neq 0$ is a closed 2-sided ideal of \mathcal{O}_A . Since all loops of E_A have exits by Theorem 12, [5, Theorem 13.1] implies that J contains at least one of the generators, say S_i . Thus $S_i S_i^* \in J \cap \text{im} \phi_A$. Since $C^*(E_A)$ is simple this implies that $\text{im} \phi_A \subseteq J$. Consequently all $S_j S_j^*$, and hence S_j , are in J . Thus $J = \mathcal{O}_A$, as required. \square

15. We observe that most previously obtained simplicity criteria for Cuntz-Krieger algebras \mathcal{O}_A , e.g., [5, Theorem 14.1], were in fact results about simplicity of the corresponding graph algebras $C^*(E_A)$. However, the implication from Proposition 14 cannot be reversed. For example, if A is an infinite matrix with zero first column and all other entries 1 then \mathcal{O}_A is simple while $C^*(E_A)$ is not.

16. In view of Theorem 12 it seems reasonable to divide vertices of a directed graph into two types. Namely, we say that a vertex v is of type *I* if v emits finitely many (and at least one) edges, none of which ends in v itself. Otherwise we say that v is of type *II*. The following sharpening of [7, Corollary 4.5] follows immediately from Section 11 and Theorem 12.

Corollary 17. *Let E be a countable directed graph. If $C^*(E)$ is simple and $v \in E^0$ is of type *II* then for all $w \in E^0$ there exists a path from w to v . If all vertices of E are of type *II* then $C^*(E)$ is simple if and only if the graph E is transitive (and different from a single loop).*

It was observed in [9, Corollary 3.10] for locally finite graphs without sinks or sources and in [1, Remark 5.6] for arbitrary row-finite graphs that a simple graph C^* -algebra must be either *AF* or purely infinite. Our simplicity criteria allow us to extend this dichotomy to all Cuntz-Krieger algebras and, hence, to all graph algebras as well.

Theorem 18. *Any simple Cuntz-Krieger algebra corresponding to a 0 – 1 matrix without zero rows and any simple graph C^* -algebra is either *AF* or purely infinite.*

Proof. First consider a 0–1 matrix $A = [A(i, j)]_{i, j \in \mathbb{N}}$ without zero rows, and suppose that \mathcal{O}_A is simple. If there are no loops in E_A then \mathcal{O}_A is *AF* by the approximation argument of [11]. Otherwise all loops have exits by Theorem 8. Furthermore, in the latter case, if $i, j \in \mathbb{N}$ and j lies on a loop in E_A then it follows from Section 4 that $j \in H(i)$ if and only if there is a path in E_A from i to j . Thus, Theorem 8 implies in this case that every vertex of E_A connects to a loop. Consequently, \mathcal{O}_A is purely infinite by [5, Theorem 16.2]. The claim for graph algebras is a corollary of the above and [6, Theorem 10]. \square

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